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ON THE CONGRUENCE $f(x)+g(y)+c \equiv 0 \pmod{xy}$

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Abstract. Four theorems of the author on the subject are given without proofs.

L. J. Mordell [4] stated the following theorem:

The congruence

$$ax^3+by^3+c \equiv 0 \pmod{xy},$$

where a,b,c are given integers, has an infinite number of solutions in which $(cx,y)=1$ and we can give x,y as polynomials in a,b,c.

and outlined a proof. He also stated:

The same method proves the existence of an infinity of solutions of

$$ax^m+by^n+c \equiv 0 \pmod{xy},$$

where a,b,c are given integers, and also of

$$f(x)+g(y)+c \equiv 0 \pmod{xy}, \tag{1}$$

where

$$f(x)=a_0x^m+a_1x^{m-1}+\ldots+a_{m-1}x$$

and

$$g(y)=b_0y^n+b_1y^{n-1}+\ldots+b_{n-1}y$$

and the a's and b's are integers.

This is moreless repeated in Mordell's book [5], pp. 293–295.

Mordell was to a certain extent anticipated by Jacobsthal [2], who assumed $g=f$ and required only $f(x)+c \equiv 0 \pmod{y}$, $f(y)+c \equiv 0 \pmod{x}$.

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We have the following theorems.

THEOREM 1. *The congruence*

$$aX^3 + a_1X^2 + a_2X + bY + c \equiv 0 \pmod{XY}, \tag{2}$$

where $a, a_1, a_2, b, c \in \mathbb{Z}$, *has infinitely many solutions in integers if and only if the equation*

$$aX^3 + a_1X^2 + a_2X + bY + c = 0 \tag{3}$$

is soluble in integers.

THEOREM 2. *If* $f(x) = ax^2 + a_1x \in \mathbb{Z}[x]$, $g(y) = by^2 + b_1y \in \mathbb{Z}[y]$, $c \in \mathbb{Z} \setminus \{0\}$, $\operatorname{Rad} c \mid (a_1, b_1)$ *and* $|ab| \ge 9$, *then the congruence* (1) *has infinitely many solutions in integers* x, y *such that* $(y, c) = 1$. *If* $0 < |ab| < 9$ *and the remaining assumptions of the theorem are satisfied, there are only finitely many exceptions.*

$\operatorname{Rad} c$ means here $\prod_{p|c, p \text{ prime}} p$.

Jacobsthal ([2, §2, Theorem 4]) has shown that if $a = b = 1$, $a_1 = b_1$, $c = \pm 1$, the only exceptions are $a_1 = b_1 = \pm 1$.

COROLLARY 1. *The congruence*

$$ax^2 + by^2 + c \equiv 0 \pmod{xy},$$

where $a, b, c \in \mathbb{Z} \setminus \{0\}$, *has infinitely many solutions in integers* x, y *such that* $(y, c) = 1$ *except for* $a = b = \pm 1$, $c = \mp 2, \mp 3$.

THEOREM 3. *If* $m \ge 4$, $n = 1$, $a \in \mathbb{Z} \setminus \{0\}$, $a_1 = a_{m-1} = 0$, $b \in \mathbb{Z} \setminus \{0\}$, $c \in \mathbb{Z} \setminus \{0\}$, *then there exist infinitely many solutions of the congruence* (1) *in integers* x, y *such that* $(y, c) = 1$.

THEOREM 4. *Let* $m, n \in \mathbb{Z}$, $(m-1)(n-1) > 1$,

$$f(x) = ax^m + \sum_{i=1}^{m-1} a_i x^{m-i} \in \mathbb{Z}[x], \quad g(y) = by^n + \sum_{i=1}^{n-1} b_i y^{n-i} \in \mathbb{Z}[y], \quad c \in \mathbb{Z} \setminus \{0\},$$

$\operatorname{Rad} c \mid a_1$ *and either* $|abc| > 1$, *or* $a > 0$, $b > 0$, $c > 0$, $a_i \ge 0$ $(1 \le i \le m-1)$, $b_j \ge 0$ $(1 \le j \le n-1)$. *Then the congruence* (1) *has infinitely many solutions in integers* x, y *such that* $(y, c) = 1$.

COROLLARY 2. *The congruence*

$$ax^m + by^n + c \equiv 0 \pmod{xy},$$

where $a, b, c, m, n \in \mathbb{Z} \setminus \{0\}$, $(m-1)(n-1) > 1$, *has infinitely many solutions in integers* x, y *such that* $(y, c) = 1$.

Theorems 2–4 and Corollaries 1–2 vindicate for $mn > 3$ to a certain extent Mordell's claims. But only to a certain extent. Already I. Niven [6], the reviewer of Mordell's paper in *Math. Reviews*, pointed out that certain coefficients are assumed in the proofs to be non-zero without formal hypothesis in the statement of the theorem. Exceptions mentioned in Corollary 1 have been found by Jacobsthal [2] and independently by Barnes [1]. Finally, for $mn \le 3$ Mordell's assertion is false, as shown by Theorem 1.

Proofs of these theorems appear in a paper of mine [8]. Let me add a few remarks. Ramasamy and Mohanty [7] have found all solutions in positive integers of the equation $ax^3 + by + c = xyz$, which even in this special case does not prove our Theorem 1. In the case of Theorem 2 W. H. Mills [3] found all integer solutions of the equation $x^2 + xyz + \varepsilon y^2 + ax + by + c = 0$, where $\varepsilon = \pm 1$.

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