STUDIA MATHEMATICA 234 (2) (2016)

Topological radical of a Banach module

by

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Abstract. We introduce the concept of the topological radical of a Banach module. This closed submodule has two descriptions: as the intersection of the ranges of maximal contractive monomorphisms and as the union of the ranges of small morphisms. The topological radical is an analytic analogue of the radical of a module over a unital ring and has similar categorical properties.

1. Introduction. Consideration of projective covers in [A] prompted us to seek some analogue for the notion of small submodule in the Banach module context. Recall that a submodule Y in a module X over a ring is called *small* (other terms are 'superfluous' and 'coessential') if for a submodule Z in X, $Y+Z=X$ implies $Z=X$. A generalization of Dixon's theorem on topologically nilpotent Banach algebras (see Theorem 2.2) leads us to the definition of a *small morphism*. The range of a small morphism is a submodule in a Banach module and can be considered as a functional-analytic analogue of small submodule.

Our main aim is to extend the concept of Jacobson radical from Banach algebras to Banach modules. As a pattern we take the notion of the radical of a module from ring theory. But our approach offers some functional-analytic modifications. The Jacobson radical of a unital ring can be described as the *intersection of all maximal left ideals* or as the set of *all r such that $1+ar$ is invertible for every a*. This concept applies as well to a unital Banach algebra A because every maximal left ideal is closed, and $1+ar$ is invertible for every $a\in A$ iff ar is *topologically nilpotent* (i.e. $\|(ar)^n\|^{1/n}\to 0$ for every $a\in A$).

On the other hand, it is well known that the notion of radical can be extended to modules. The radical of a unital module X over a unital ring is

2010 *Mathematics Subject Classification*: Primary 46H25; Secondary 16D.
Key words and phrases: Banach algebra, topological radical, topologically nilpotent, small morphism.
Received 5 November 2015; revised 24 May 2016.
Published online 2 August 2016.

DOI: 10.4064/sm8443-5-2016

the *intersection of all maximal submodules* and coincides with the *union of all small submodules* (the notation is $\operatorname{rad} X$). Note that for an element r of a ring A, the submodule Ar is small iff $1+ar$ is invertible for every a. The pure algebraic notion of the radical of a module is useful in Banach module theory only in particular cases, for example, for finitely generated modules [A]. In general, neither a maximal submodule nor a small submodule of a Banach module has to be closed. But then again we cannot restrict ourselves to some classes of closed submodules because submodules of the form $A \cdot x$ (which need not be closed) play an important role in the basic theory of module radicals. As we see below, the right way is to consider ranges of bounded module morphisms as an intermediate class between closed submodules and all submodules. But it seems more appropriate from the ideological and technical point of view to work with morphisms themselves instead of their ranges.

In this article we introduce the concept of the *topological radical* of a Banach module. This closed submodule has two descriptions: as the intersection of the ranges of maximal contractive monomorphisms or as the union of the ranges of small morphisms.

2. Small morphisms of Banach modules. Let A be a Banach algebra. We suppose that the norm of multiplication in A is not greater than 1. For $n \in \mathbb{N}$ set

$$S(n) := \sup \|r_1 \cdots r_n\|^{1/n}, \tag{2.1}$$

where $r_1, \dots, r_n$ run over the unit ball of A. If $\lim_{n\to\infty} S(n) = 0$ then A is called *topologically nilpotent*. Note that A is topologically nilpotent if and only if for every bounded sequence $(r_n) \subset A$,

$$\lim_{n\to\infty} \|r_1 \cdots r_n\|^{1/n} = 0.$$

Obviously, a topologically nilpotent Banach algebra is radical.

Recall that $C[0,1]$ and $L^1[0,1]$ are radical Banach algebras with respect to the cut-off convolution. The former algebra is topologically nilpotent but the latter is not [P, Section 4.8.8].

In [D] P. G. Dixon shows that $A \cdot X \neq X$ for every non-trivial left Banach module X over a topologically nilpotent Banach algebra A (see also the proof in [P, Theorem 4.8.9]). But in fact his argument gives a stronger assertion. Set

$$\pi_X^A : A \mathbin{\widehat{\otimes}} X \to X : r \otimes x \mapsto r \cdot x$$

for a left Banach A-module X, where $\widehat{\otimes}$ denotes the projective tensor product of Banach spaces. (We also suppose that the norm of multiplication in X is not greater than 1.) Below, "a module" means an A-module.

THEOREM 2.1 (Dixon). *If X is a non-trivial left Banach module over a topologically nilpotent Banach algebra A, then* $\operatorname{Im}\pi_X^A \neq X$.

We need a more general result.

THEOREM 2.2. *Let A be a topologically nilpotent Banach algebra, and let $\phi\colon Y \to X$ be a morphism of left Banach modules such that $X = \operatorname{Im}\phi + \operatorname{Im}\pi_X^A$. Then ϕ is surjective.*

Proof. The assumption of the theorem means that the morphism

$$Y \oplus (A \mathbin{\widehat{\otimes}} X) \to X\colon (y,u) \mapsto \phi(y) + \pi_X^A(u)$$

is surjective. (Here the sum is endowed with the ℓ^1-norm.) By the open mapping theorem there is $C>0$ with the following property. For every x in X there exist $y \in Y$, $r_i \in A$, and $x_i \in X$ such that

$$x = \phi(y) + \sum_{i=1}^{\infty} r_i \cdot x_i \quad \text{and} \quad \|y\| + \sum_i \|r_i\|\,\|x_i\| \le C\|x\|. \tag{2.2}$$

Now we fix x in X and choose by induction sequences $(y_n) \subset Y$ and $(v_n) \subset X$ such that

$$x = \phi(y_n) + v_n, \tag{2.3}$$

where v_n can be represented as

$$v_n = \sum_{i=1}^{\infty} r_{1,i} \cdots r_{n,i} \cdot x_i \text{ for some } r_{1,i}, \dots, r_{n,i} \in A \text{ and } x_i \in X,\ i \in \mathbb{N}, \tag{2.4}$$

and the following two conditions are satisfied:

$$\|y_{n+1} - y_n\| \le C \sum_i \|r_{1,i} \cdots r_{n,i}\|\,\|x_i\|, \tag{2.5}$$

$$\sum_i \|r_{1,i}\| \cdots \|r_{n,i}\|\,\|x_i\| \le C^n \|x\|. \tag{2.6}$$

Suppose that for $n \in \mathbb{N}$ we have elements $y_1, \dots, y_n$ and $v_1, \dots, v_n$ that satisfy the above conditions, in particular, the condition (2.5) holds up to $n-1$. Fix decompositions in (2.3) and (2.4). Set $t_i := r_{1,i} \cdots r_{n,i}$. Applying (2.2) we can write every x_i as

$$x_i = \phi(y_i') + \sum_j s_{ji} \cdot x_{ji}', \quad \text{where} \quad \|y_i'\| + \sum_j \|s_{ji}\|\,\|x_{ji}'\| \le C\|x_i\|. \tag{2.7}$$

Then

$$x = \phi(y_n) + v_n = \phi(y_n) + \sum_i t_i \cdot \phi(y_i') + \sum_{i,j} t_i s_{ji} \cdot x_{ji}'.$$

Now set $y_{n+1} := y_n + \sum_i t_i \cdot y_i'$ and $v_{n+1} := \sum_{i,j} t_i s_{ji} \cdot x_{ji}'$. It follows from (2.7) that $\|y_i'\| \le C\|x_i\|$. Hence,

$$\|y_{n+1} - y_n\| \le \sum_i \|t_i\|\,\|y_i'\| \le C\sum_i \|t_i\|\,\|x_i\|,$$

i.e. we obtain (2.5). By (2.6) and (2.7) we get

$$\sum_{i,j} \|r_{1,i}\| \cdots \|r_{n,i}\|\,\|s_{ji}\|\,\|x_{ji}'\| \le C^{n+1}\|x\|,$$

i.e. after an obvious change of notation we have (2.4) and (2.6) for $n+1$. By induction, there exist sequences with the desired properties.

Note that (2.6) implies (see (2.1))

$$\sum_i \|r_{1,i}\cdots r_{n,i}\|\,\|x_i\| \le \sum_i S(n)^n\|r_{1,i}\|\cdots\|r_{n,i}\|\,\|x_i\| \le S(n)^n C^n\|x\|$$

for every n. Therefore $\|v_n\| \le S(n)^n C^n\|x\|$ and $\|y_{n+1} - y_n\| \le S(n)^n C^{n+1}\|x\|$ by (2.5). Hence for $m > n$ we have

$$\|y_m - y_n\| \le \sum_{k=n}^{m-1} S(k)^k C^{k+1}\|x\|.$$

Since $S(n) \to 0$, it follows that y_n is a fundamental sequence and $v_n \to 0$. Finally, from $x = \phi(y_n) + v_n$ we get $x = \phi(\lim_n y_n)$, i.e. $x \in \operatorname{Im}\phi$. ∎

Notation 2.3. Let $\phi : Y \to X$ and $\psi : Z \to X$ be morphisms of Banach modules. Denote by $\phi \dotplus \psi$ the morphism

$$Y \oplus Z \to X : (y,z) \mapsto \phi(y) + \psi(z).$$

Definition 2.4. We say that a morphism $\psi : X_0 \to X$ of Banach modules is *small* if for every morphism $\phi : Y \to X$ such that $\phi \dotplus \psi$ is surjective, ϕ is also surjective, i.e., $\operatorname{Im}\phi + \operatorname{Im}\psi = X$ implies $\operatorname{Im}\phi = X$.

Thus, Theorem 2.2 asserts that *for every left Banach module X over a topologically nilpotent Banach algebra A the morphism π_X^A is small.*

Proposition 2.5. *If $\psi : X_0 \to X$ is a small morphism then $\tau\psi$ is small for each module V and each morphism $\tau : X \to V$.*

Proof. Let $\tau : X \to V$ and $\phi : Y \to V$ be morphisms of Banach modules such that $\phi \dotplus \tau\psi$ is surjective. Consider the pullback diagram

$$\begin{tikzcd} Y \times_V X \arrow[r] \arrow[d, "\phi'"'] & Y \arrow[d, "\phi"] \\ X \arrow[r, "\tau"] & V \end{tikzcd}$$

For every $x \in X$ there are $y \in Y$ and $z \in X_0$ such that $\tau(x) = \phi(y) + \tau\psi(z)$. Then $\phi(y) = \tau(x - \psi(z))$. By explicit construction of $Y \times_V X$ this means that $w = (y, x - \psi(z)) \in Y \times_X V$ and $\phi'(w) = x - \psi(z)$. Hence, $\phi' \dotplus \psi$ is surjective. Since ψ is small, ϕ' is also surjective. Therefore for every $x \in X$ there exists $y \in Y$ such that $\phi(y) = \tau(x)$. The assumption that $\phi \dotplus \tau\psi$ is surjective implies that $\phi \dotplus \tau$ is surjective. Thus, so is ϕ. ∎

PROPOSITION 2.6. *Let $\psi : X_1 \to X$ be a morphism of Banach modules, and let $\varepsilon : X_0 \to X_1$ be a surjective morphism of Banach modules such that $\psi\varepsilon$ is small. Then ψ is small.*

Proof. Suppose that $\phi : Y \to X$ is a morphism such that $\phi \dotplus \psi$ is surjective. Then $\phi \dotplus \psi\varepsilon$ is also surjective. Since $\psi\varepsilon$ is small, ϕ is surjective. ∎

Recall that a left Banach A-module P is called *strictly projective* if for each surjective morphism $\varepsilon : Y \to P$ of Banach A-modules there exists a morphism $\rho : P \to Y$ such that $\varepsilon\rho = 1$. Denote by ℓ^1 the infinite-dimensional Banach ℓ^1-space with a countable basis.

THEOREM 2.7 (cf. [K, Th. 11.5.5]). *Let I be a closed left ideal in a unital Banach algebra A, and let $\iota : I \to A$ be the natural inclusion. The following conditions are equivalent:*

(A) *I is topologically nilpotent.*
(B) *For every unital left Banach A-module X the morphism $I \mathbin{\widehat{\otimes}}_A X \to X : a \otimes_A x \mapsto a \cdot x$ of Banach A-modules is small.*
(C) *For every strictly projective unital left Banach A-module P the morphism $I \mathbin{\widehat{\otimes}}_A P \to P : a \otimes_A x \mapsto a \cdot x$ of Banach A-modules is small.*
(D) *The morphism $(\iota \otimes 1) : I \mathbin{\widehat{\otimes}} \ell^1 \to A \mathbin{\widehat{\otimes}} \ell^1$ of left Banach A-modules is small.*

Proof. (A)⇒(B). If I is topologically nilpotent and X is a left Banach A-module then by Theorem 2.2, π^I_X is small as a morphism of left Banach I-modules. Hence, it is small as a morphism of left Banach A-modules. Since π^I_X is the composition of a surjective morphism $I \mathbin{\widehat{\otimes}} X \to I \mathbin{\widehat{\otimes}}_A X$ and a morphism $I \mathbin{\widehat{\otimes}}_A X \to X$, Proposition 2.6 implies (B).

(B)⇒(C). This it is obvious.

(C)⇒(D). It is easy to see that $A \mathbin{\widehat{\otimes}} \ell^1$ is strictly projective. By assumption

$$I \mathbin{\widehat{\otimes}}_A A \mathbin{\widehat{\otimes}} \ell^1 \to A \mathbin{\widehat{\otimes}} \ell^1 : a \otimes_A b \otimes x \mapsto ab \otimes x \tag{2.8}$$

is a small morphism of left Banach A-modules. Since A is unital, $I \mathbin{\widehat{\otimes}}_A A \cong I$, and we have (D).

(D)⇒(A). Let (a_n) be a bounded sequence in I, and let $\{e_i\}_{i\in\mathbb{N}}$ be the canonical basis in ℓ^1. Consider

$$\phi : A \mathbin{\widehat{\otimes}} \ell^1 \to A \mathbin{\widehat{\otimes}} \ell^1 : \sum_{i=1}^{\infty} b_i \otimes e_i \mapsto \sum_{i=1}^{\infty} b_i a_i \otimes e_{i+1}. \tag{2.9}$$

It is obvious that ϕ is a morphism of left Banach modules. Fix $\lambda \in \mathbb{C}$. Since

$$\sum_{i=1}^{\infty} b_i a_i \otimes e_{i+1} \in I \mathbin{\widehat{\otimes}} \ell^1,$$

we have $I \mathbin{\widehat{\otimes}} \ell^1 + \operatorname{Im}(1+\lambda\phi) = A \mathbin{\widehat{\otimes}} \ell^1$. Since $\iota \otimes 1$ is small, $1+\lambda\phi$ is surjective. If $(1+\lambda\phi)(u) = 0$ for some $u = \sum_i b_i \otimes e_i$, then $b_1 = 0$ and $b_{i+1} - \lambda b_i a_i = 0$ for all i. It follows that $b_i = 0$ for all i, so that $1+\lambda\phi$ is injective. Thus, $1+\lambda\phi$ is an isomorphism for every $\lambda \in \mathbb{C}$. This implies that ϕ is a topologically nilpotent operator, i.e. $\lim_{n\to\infty} \|\phi^n\|^{1/n} = 0$.

It is clear that

$$\|\phi^n(1 \otimes e_1)\| = \|a_1 \cdots a_n \otimes e_{n+1}\| = \|a_1 \cdots a_n\|.$$

Therefore $\|a_1 \cdots a_n\| \le \|\phi^n\|$. The rest is obvious. ∎

Considering every Banach algebra as an ideal in the unitization we have

Corollary 2.8. *A Banach algebra A is topologically nilpotent if and only if for every Banach A-module X the morphism $A \mathbin{\widehat{\otimes}}_A X \to X$ is small if and only if for every strictly projective left Banach A-module P the morphism $A \mathbin{\widehat{\otimes}}_A P \to P$ is small.*

Note that the definition of $S(n)$ is invariant under replacement of left multiplication by right multiplication. So all results above can be applied to right Banach modules.

If X and Y are left Banach A-modules we denote by ${}_A\mathrm{h}(X,Y)$ the set of all bounded A-module morphisms from X to Y. Recall that a left A-module X is called *unital* if $1 \cdot x = x$ for all $x \in X$.

Proposition 2.9. *Let A be a Banach algebra, X and Y unital left Banach A-modules, and α in ${}_A\mathrm{h}(X,Y)$. The following conditions are equivalent:*

(1) *$1 - \alpha\phi$ is right invertible in the unital algebra ${}_A\mathrm{h}(Y)$ for every $\phi \in {}_A\mathrm{h}(Y,X)$.*
(2) *$\alpha \circ {}_A\mathrm{h}(Y,X)$ is a small right ideal in ${}_A\mathrm{h}(Y)$.*

Proof. (1)⇒(2). Let L be a right ideal in ${}_A\mathrm{h}(Y)$ such that $\alpha \circ {}_A\mathrm{h}(Y,X) + L = {}_A\mathrm{h}(Y)$. Then there are $\phi \in {}_A\mathrm{h}(Y,X)$ and $\psi \in L$ satisfying $\alpha\phi + \psi = 1$. By assumption ψ has a right inverse ψ_1, hence, as L is a right ideal in ${}_A\mathrm{h}(Y)$, we have $1 = \psi\psi_1 \in L$, so that $L = {}_A\mathrm{h}(Y)$.

(2)⇒(1). Let $\phi \in {}_A\mathrm{h}(Y,X)$. Set $L := (1-\alpha\phi) \circ {}_A\mathrm{h}(Y)$. Then $1-\alpha\phi \in L$; so that $1 \in \alpha \circ {}_A\mathrm{h}(Y,X) + L$. Therefore $\alpha \circ {}_A\mathrm{h}(Y,X) + L = {}_A\mathrm{h}(Y)$. Since $\alpha \circ {}_A\mathrm{h}(Y,X)$ is small, $L = {}_A\mathrm{h}(Y)$. This implies that $1-\alpha\phi$ is right invertible. ∎

THEOREM 2.10. *Let X and P be unital left Banach A-modules. Suppose that P is strictly projective and $\alpha \in {}_A\mathrm{h}(X,P)$. The following conditions are equivalent:*

(1) *α is small.*
(2) *$1-\alpha\phi$ is right invertible in ${}_A\mathrm{h}(P)$ for all $\phi \in {}_A\mathrm{h}(P,X)$.*
(3) *$\alpha \circ {}_A\mathrm{h}(P,X)$ is a small right ${}_A\mathrm{h}(P)$-submodule in ${}_A\mathrm{h}(P)$.*

Proof. (1)⇒(2). Let $\phi \in {}_A\mathrm{h}(P,X)$. Then $(1-\alpha\phi) \dotplus \alpha : P \oplus X \to P$ is obviously surjective. Since α is small, $1-\alpha\phi$ is surjective. Since P is strictly projective, $1-\alpha\phi$ admits a right inverse.

(2)⇒(1). Suppose $\eta \in {}_A\mathrm{h}(Y,P)$ for some Y and $\eta \dotplus \alpha$ is surjective. Since P is strictly projective, $\eta \dotplus \alpha$ is right invertible, i.e. there exist $\psi_1 \in {}_A\mathrm{h}(P,Y)$ and $\psi_2 \in {}_A\mathrm{h}(P,X)$ such that $\eta\psi_1 + \alpha\psi_2 = 1$. By assumption $\eta\psi_1$ is right invertible. Hence, η is surjective.

(3)⇔(2) follows from Proposition 2.9. ∎

A surjective morphism $\varepsilon : X \to V$ of Banach A-modules is said to be a *cover* if a morphism $\phi : Y \to X$ of Banach A-modules is surjective whenever $\varepsilon\phi$ is [A].

PROPOSITION 2.11. *A surjective morphism $\varepsilon : X \to V$ of Banach modules is a cover if and only if the embedding* $\operatorname{Ker}\varepsilon \to X$ *is a small morphism.*

Proof. Denote the embedding $\operatorname{Ker}\varepsilon \to X$ by $\ker\varepsilon$. Suppose that ε is a cover. Let $\phi : Y \to X$ be a morphism of Banach modules such that $\phi \dotplus \ker\varepsilon$ is surjective. Note that $\varepsilon(\phi \dotplus \ker\varepsilon) = \varepsilon\phi$ is also surjective. Since ε is a cover, ϕ is surjective. Thus, $\ker\varepsilon$ is a small morphism.

Conversely, suppose that $\ker\varepsilon$ is small. Let $\phi : Y \to X$ be a morphism of Banach modules such that $\varepsilon\phi$ is surjective. Then for every $x \in X$ there is $y \in Y$ such that $\varepsilon(x) = \varepsilon\phi(y)$. Hence $x = \phi(y) + (x - \phi(y))$ where $x - \phi(y) \in \operatorname{Ker}\varepsilon$. Therefore $\phi \dotplus \ker\varepsilon$ is surjective. Since $\ker\varepsilon$ is small, ϕ is surjective. Thus, ε is a cover. ∎

3. Maximal contractive monomorphisms. Fix a unital Banach algebra A and a left unital Banach A-module X. Consider a pre-order on the set of contractive monomorphisms with range in X, defined by $\beta \succeq \gamma$ if there exists a contractive morphism κ such that $\gamma = \beta\kappa$. We say that β and γ are *equivalent* if κ is an isometric isomorphism. The pre-order induces an order on the set of equivalence classes of contractive monomorphisms.

REMARK 3.1. If X is unital and $\beta : Y \to X$ is a monomorphism then Y is also unital. To see this, consider the decomposition $Y = Y_0 \oplus Y_1$, where $Y_0 = \{y \in Y : 1 \cdot y = 0\}$ and $Y_1 = \{y \in Y : 1 \cdot y = y\}$. Since X is unital, ${}_A\mathrm{h}(Y_0, X) = 0$. Therefore $Y_0 = 0$. Thus, we do not need the restriction on the initial module of a monomorphism.

Definition 3.2. Let $\beta : Y \to X$ and $\gamma : Z \to X$ be contractive monomorphisms.

(1) Denote by $\beta \vee \gamma$ the natural morphism $(Y \oplus Z)/\mathrm{Ker}(\beta \dotplus \gamma) \to X$ associated with $\beta \dotplus \gamma$.
(2) Denote by $\beta \wedge \gamma$ the natural morphism $Y \times_X Z \to X$, where $Y \times_X Z$ is the pullback of β and γ.

It is not hard to check that $\beta \vee \gamma$ and $\beta \wedge \gamma$ are contractive monomorphisms. For equivalence classes $[\beta]$ and $[\gamma]$ we set $[\beta] \vee [\gamma] := [\beta \vee \gamma]$ and $[\beta] \wedge [\gamma] := [\beta \wedge \gamma]$. It is easy to see that these operations are well-defined.

Proposition 3.3. *Let β and γ be contractive monomorphisms. Then, with respect to the order define above, $[\beta] \vee [\gamma]$ and $[\beta] \wedge [\gamma]$ are the supremum and the infimum of $[\beta]$ and $[\gamma]$, respectively.*

The proof is standard.

Definition 3.4. We say that a contractive monomorphism $\alpha : Y \to X$ of left unital Banach A-modules is *maximal* if α is not surjective and for every non-surjective contractive monomorphism β and every contractive morphism κ the equality $\alpha = \beta\kappa$ implies that κ is an isometric isomorphism.

Thus, α is maximal iff $[\alpha]$ is maximal in the set of equivalence classes of all non-surjective monomorphisms with range in X.

Recall that a morphism $\varepsilon : Y \to X$ is called a *C-epimorphism* for some $C \geq 1$ if for every $x \in X$ there exist $y \in Y$ such that $x = \phi(y)$ and $\|y\| \leq C\|x\|$.

Proposition 3.5. *For $x_0 \in X$, set*

$$\tau : A \to X : a \mapsto a \cdot x_0.$$

Suppose that $\phi : Y \to X$ is a morphism such that $x_0 \notin \mathrm{Im}\,\phi$ and $\phi \dotplus \tau$ is a C-epimorphism for $C \geq 1$. Then $\mathrm{dist}(x_0, \mathrm{Im}\,\phi) \geq 1/C$.

Proof. Assume that $\|x_0 - \phi(y)\| < 1/C$ for some $y \in Y$. Since $\phi \dotplus \tau$ is a C-epimorphism, there exist $y' \in Y$ and $a \in A$ such that $x_0 - \phi(y) = \phi(y') + a \cdot x_0$ and

$$\|y'\| + \|a\| \leq C\|x_0 - \phi(y)\| < 1.$$

Thus, $\|a\| < 1$, hence $1 - a$ is invertible in A. Therefore

$$x_0 = \phi((1-a)^{-1} \cdot (y' + y)).$$

Hence, $x_0 \in \mathrm{Im}\,\phi$. We get a contradiction. ∎

Theorem 3.6. *Every maximal contractive monomorphism is an isometry.*

Proof. Let $\alpha : Y \to X$ be a maximal contractive monomorphism, so $\mathrm{Im}\,\alpha \neq X$. Suppose that $x_0 \in X \setminus \mathrm{Im}\,\alpha$. Define τ as in Proposition 3.5.

Denote by β the natural monomorphism $(Y \oplus A)/\mathrm{Ker}(\alpha \dotplus \tau) \to X$ and by κ the composition

$$Y \to Y \oplus A \to (Y \oplus A)/\mathrm{Ker}(\alpha \dotplus \tau).$$

We claim that β is surjective. Indeed, assume otherwise. Since α is maximal, κ is an isometric isomorphism. In particular, there is $y \in Y$ such that

$$(y, 0) - (0, 1) \in \mathrm{Ker}(\alpha \dotplus \tau).$$

Hence, $\alpha(y) = x_0$, a contradiction.

Since β is surjective, so is $\alpha \dotplus \tau$. By the open mapping theorem, $\alpha \dotplus \tau$ is a C-epimorphism for some $C \geq 1$. It follows Proposition 3.5 that $\mathrm{dist}(x_0, \mathrm{Im}\,\phi) \geq 1/C$. Since x_0 is arbitrary, $\mathrm{Im}\,\alpha$ is closed. Let $\gamma : \mathrm{Im}\,\alpha \to X$ be the natural embedding. Since $\alpha = \gamma\alpha$ and α is maximal, α is an isometry. ∎

THEOREM 3.7. *Let X' be a closed submodule of X. Then the natural embedding $\iota : X' \to X$ is a maximal contractive monomorphism if and only if X/X' is an irreducible module.*

Proof. ($\Rightarrow$) Assume that ι is maximal. Let $x_0 \in X \setminus X'$ and $x_1 \in X$. Since α is maximal, $X' + A \cdot x_0 = X$. In particular, there is $a \in A$ such that $x_1 - a \cdot x_0 \in X'$. Therefore $x_0 + X'$ is a cyclic element of X/X'. Hence, X/X' is irreducible.

($\Leftarrow$) Assume that X/X' is irreducible. Suppose that there are a non-surjective contractive monomorphism β and a contractive morphism κ such that $\beta\kappa = \iota$.

Since $X' \subset \mathrm{Im}\,\beta \neq X$ and X/X' is irreducible, $\mathrm{Im}\,\beta = X'$. Therefore, $\beta\kappa = 1$. Since β is a monomorphism, it is an isomorphism. Since β and κ are contractive, κ is isometric. ∎

Note that X/X' is irreducible iff X' is a maximal submodule in the algebraic sense. Thus, maximal monomorphisms can be described as embeddings of closed maximal submodules.

LEMMA 3.8. *Let Z be a closed submodule of X and $\alpha : Y \to X/Z$ a maximal contractive monomorphism. Denote the projection $X \to X/Z$ by σ. Then there exists a commutative diagram*

$$\begin{tikzcd} W \arrow[r, "\mu"] \arrow[d, "\beta"'] & Y \arrow[d, "\alpha"] \\ X \arrow[r, "\sigma"] & X/Z \end{tikzcd}$$

where β is a maximal contractive monomorphism.

Proof. Set $W := Y \times_{X/Z} X$ or, more precisely, $W = \{(y, x) \in Y \times X : \alpha(y) = \sigma(x)\}$. Denote by β and μ the morphisms $(y, x) \mapsto x$ and $(y, x) \mapsto y$, respectively. Note that μ is surjective and $Z \subset \operatorname{Im}\beta$. It is obvious that β is a contractive monomorphism.

Suppose that $\gamma : V \to X$ is a non-surjective contractive monomorphism and $\kappa : W \to V$ is a contractive morphism such that $\beta = \gamma\kappa$.

Assume that $\operatorname{Im}\alpha + \operatorname{Im}(\sigma\gamma) = X/Z$. Since $\operatorname{Im}\alpha = \operatorname{Im}(\alpha\mu) = \operatorname{Im}(\sigma\beta) = \operatorname{Im}(\sigma\gamma\kappa)$, we have $\operatorname{Im}(\sigma\gamma) = X/Z$. It follows from $Z \subset \operatorname{Im}\beta \subset \operatorname{Im}\gamma$ that γ is surjective, a contradiction. Hence, $\operatorname{Im}\alpha + \operatorname{Im}(\sigma\gamma) \neq X/Z$.

Since α is maximal, $\operatorname{Im}(\sigma\gamma) \subset \operatorname{Im}\alpha$. By Theorem 3.6, α is an isometry, therefore there is a well-defined contractive morphism $\delta : V \to Y$ such that $\alpha\delta = \sigma\gamma$. The pull-back property implies that there is a contractive morphism $\rho : V \to W$ such that $\beta\rho = \gamma$. Then $[\beta] = [\gamma]$. Thus, β is maximal. ∎

Lemma 3.9. *Let α and β be non-surjective contractive monomorphisms with ranges in X such that $\beta \succeq \alpha$, and let ϕ be a morphism such that $\alpha \dotplus \phi$ is a C-epimorphism for some $C \geq 1$. Then $\beta \dotplus \phi$ is a C-epimorphism.*

Proof. Suppose that $\kappa : Y \to Z$ is a contractive morphism such that $\alpha = \beta\kappa$. Since $\alpha \dotplus \phi$ is a C-epimorphism, for every $x \in X$ there exist $x_0 \in X_0$ and $y \in Y$ such that $x = \phi(x_0) + \alpha(y)$ and $\|x_0\| + \|y\| \leq C\|x\|$. Denote $\kappa(y)$ by z. Then $x = \phi(x_0) + \beta(z)$ and $\|x_0\| + \|z\| \leq \|x_0\| + \|y\| \leq C\|x\|$. Therefore, $\phi \dotplus \beta$ is a C-epimorphism. ∎

Lemma 3.10. *Let $C \geq 1$, and let ϕ be a contractive morphism with range in X. Denote by Γ the family of all contractive monomorphisms α with range in X such that*

(1) *α is not surjective;*
(2) *$\alpha \dotplus \phi$ is a C-epimorphism.*

Suppose that there are $\delta > 0$ and $x_0 \in X$ such that $\operatorname{dist}(x_0, \operatorname{Im}\alpha) \geq \delta$ *for every $\alpha \in \Gamma$. Then for every $\alpha_0 \in \Gamma$ there exists a maximal contractive monomorphism γ such that $\gamma \in \Gamma$ and $\gamma \succeq \alpha_0$.*

Proof. Set $\Gamma' := \{\alpha \in \Gamma : \alpha \geq \alpha_0\}$. Suppose that Γ_0 is a linearly ordered subset of Γ'. We claim that Γ_0 has an upper bound.

Denote by Y_α the initial module of $\alpha \in \Gamma_0$ and by $\kappa_{\alpha\alpha'}$ the connecting contractive morphism for α and α' in Γ_0 such that $\alpha' \succeq \alpha$. Then the family $(\kappa_{\alpha\alpha'})$ has an inductive limit Y in the category of contractive morphisms. In particular, there is a family $(\kappa_\alpha : Y_\alpha \to Y)$ of contractive morphisms and $\beta : Y \to X$ such that $\alpha = \beta\kappa_\alpha$ for every α. Note that $\bigcup_{\Gamma_0} \operatorname{Im}\kappa_\alpha$ is dense in Y, hence $\bigcup_{\Gamma_0} \operatorname{Im}\alpha$ is dense in $\operatorname{Im}\beta$. Since $\operatorname{dist}(x_0, \operatorname{Im}\alpha) \geq \delta$ for all α, we have $\operatorname{dist}(x_0, \operatorname{Im}\beta) \geq \delta$. Hence, β is not surjective. Applying Lemma 3.9 we find that $\beta \dotplus \phi$ is a C-epimorphism. Therefore, $\beta \in \Gamma'$ and $\beta \succeq \alpha$ for every $\alpha \in \Gamma_0$.

Since Γ' is not empty and every linearly ordered subset in Γ' has an upper bound, there is a maximal element γ in Γ'. Now we claim that γ is a maximal contractive monomorphism. Suppose that $\gamma' : Z \to X$ is a non-surjective contractive monomorphism and $\kappa : Y \to Z$ is a contractive morphism such that $\gamma = \gamma'\kappa$. It follows from Lemma 3.9 that $\gamma' \dotplus \phi$ is a C-epimorphism. Hence, $\gamma' \in \Gamma'$. Since γ is maximal in Γ', κ is an isometric isomorphism. Thus γ is a maximal contractive monomorphism. By construction $\gamma \succeq \alpha_0$. ∎

Remark 3.11. In the proof we have found the supremum of a directed set of contractive monomorphisms implicitly. It is not hard to see that the constructions of $\vee$ and $\wedge$ from Definition 3.2 can be applied to arbitrary sets of monomorphisms.

Proposition 3.12. *Suppose that X is finitely generated. Then for every non-surjective contractive monomorphism α_0 with range in X there exists a maximal contractive monomorphism γ such that $\gamma \succeq \alpha_0$.*

Proof. Let $x_1, \dots, x_n$ be generators of X. Consider the morphisms

$$\tau_i : A \to X : a \to a \cdot x_i \quad (i = 1, \dots, n).$$

Since α_0 is not surjective and $\alpha_0 \dotplus \tau_1 \dotplus \cdots \dotplus \tau_n$ is surjective, there exists a minimal k in $\{1, \dots, n\}$ such that $\alpha_0 \dotplus \tau_1 \dotplus \cdots \dotplus \tau_k$ is surjective. This implies that there exists $C \geq 1$ such that $\alpha_0 \dotplus \tau_1 \dotplus \cdots \dotplus \tau_k$ is a C-epimorphism.

Denote by Γ the family of all contractive monomorphisms β with range in X such that β is not surjective and $\beta \dotplus \tau_k$ is a C-epimorphism. Set $\beta_0 = \alpha_0 \vee \alpha_1 \vee \cdots \vee \alpha_{k-1}$, where α_i is a contractive monomorphism such that $\operatorname{Im} \alpha_i = \operatorname{Im} \tau_i$. Note that $x_k \notin \operatorname{Im} \beta$ for every $\beta \in \Gamma$. It follows from Proposition 3.5 that $\operatorname{dist}(x_k, \operatorname{Im} \beta) \geq 1/C$ for every $\beta \in \Gamma$. Thus, the conditions of Lemma 3.10 are satisfied. Hence, there exists a maximal contractive monomorphism γ such that $\gamma \succeq \beta_0$. Therefore $\gamma \succeq \alpha_0$. ∎

4. Topological radical of a Banach module. Note that the equivalence classes of contractive morphisms form a lattice with respect to the operations $\vee$ and $\wedge$. Under some conditions there is a standard way to define a radical in a lattice using small and maximal elements (see, for example, [K, Ch. 9, Exercises]). But on the way we meet two difficulties. First, we define small and maximal morphisms in different categories of Banach modules (the topological and the metric categories). Second, there are not sufficiently many compact elements in our lattice. However, using Proposition 3.5 and its corollaries we can find the desired topological interplay between small and maximal morphisms.

Proposition 4.1. *Let X be a left unital Banach A-module and let $x_0 \in X$. If*

$$\tau : A \to X : a \to a \cdot x_0$$

is not small then there exists a maximal contractive monomorphism γ such that $x_0 \notin \operatorname{Im}\gamma$.

Proof. Since τ is not small, there exists a non-surjective morphism $\alpha_0 : Y \to X$ such that $\alpha_0 \dotplus \tau$ is surjective. By the open mapping theorem, there is $C \geq 1$ such that $\alpha_0 \dotplus \tau$ is a C-epimorphism. We can assume α_0 is a contractive monomorphism.

Denote by Γ the family of all contractive monomorphisms α with range in X such that α is not surjective and $\alpha \dotplus \tau$ is a C-epimorphism for C chosen above. Note that $x_0 \notin \operatorname{Im}\alpha$ for every $\alpha \in \Gamma$. Proposition 3.5 implies that $\operatorname{dist}(x_0, \operatorname{Im}\alpha) \geq 1/C$ for every $\alpha \in \Gamma$. It follows from Lemma 3.10 that there exists a maximal contractive monomorphism γ such that $\gamma \in \Gamma$. Hence, $x_0 \notin \operatorname{Im}\gamma$. ■

THEOREM 4.2. *Let X be a left unital Banach A-module. Set $X_1 = \bigcup \operatorname{Im}\psi$, where ψ runs all small morphisms with range in X, and set $X_2 = \bigcap \operatorname{Im}\gamma$, where γ runs all maximal contractive monomorphisms with range in X. Then $X_1 = X_2$ and this submodule of X is closed.*

Proof. (1) Let $x_0 \in X_2$. Assume that $\tau : A \to X : a \to a \cdot x_0$ is not small. By Proposition 4.1 there exists a maximal contractive monomorphism γ such that $x_0 \notin \operatorname{Im}\gamma$. Therefore $x_0 \notin X_2$. This contradiction implies that τ is small. Thus, $X_2 \subset X_1$.

(2) Suppose that ψ is a small morphism with range in X. We can assume ψ is a contractive monomorphism. Suppose that there is a maximal contractive monomorphisms γ such that $\operatorname{Im}\psi$ is not a subset of $\operatorname{Im}\gamma$. Then $\gamma \vee \psi = 1$. Therefore, $\operatorname{Im}\gamma + \operatorname{Im}\psi = \operatorname{Im}(\gamma \vee \psi) = X$. Since ψ is small, γ is surjective. This contradiction implies that $\operatorname{Im}\psi \subset \operatorname{Im}\gamma$. Thus, $X_1 \subset X_2$.

It follows from Theorem 3.6 that X_2 is closed. ■

DEFINITION 4.3. Let X be a left unital Banach A-module. We say that the closed submodule of X from Theorem 4.2 is the *topological radical* of X, and we denote it by $\text{t-rad}\, X$.

PROPOSITION 4.4. *The topological radical of an irreducible Banach module is trivial.*

Proof. Let X be an irreducible Banach module, and let ϕ be a small morphism with range in X. Then $\operatorname{Im}\phi = X$ or $\operatorname{Im}\phi = 0$. Since ϕ is small and $X \neq 0$, ϕ is not surjective. Hence, $\phi = 0$. This implies that $\text{t-rad}\, X = 0$. ■

PROPOSITION 4.5. *Let X be a finitely-generated Banach module. Then the natural embedding $\iota : \text{t-rad}\, X \to X$ is a small morphism.*

Proof. Let ϕ be a contractive monomorphism such that $\phi \dotplus \iota$ is surjective. If ϕ is not surjective it follows from Proposition 3.12 that there is a maximal contractive monomorphism γ such that $\gamma \succeq \phi$. Then $\gamma \dotplus \iota$ is surjective and

$\gamma \succeq \iota$ (by the definition of the topological radical). This contradiction implies that ϕ is surjective. ■

Proposition 4.6. *If X is a unital finitely generated Banach module over a unital Banach algebra A, then* $\text{t-rad}\, X = \text{rad}\, X$. *In particular,* $\text{t-rad}\, A$ *coincides with the Jacobson radical of A.*

The proposition follows immediately from Theorem 3.6 and the following lemma.

Lemma 4.7. *Every algebraically maximal submodule in a finitely generated Banach module is closed.*

Proof. Let X_0 be an algebraically maximal submodule in a finitely generated Banach A-module X. Let k be the minimal number such that for any finite set generating X only k generators are not contained in X_0. Note that $k > 0$.

Fix generators $x_1, \dots, x_n$ of X such that $x_1, \dots, x_k \in X \setminus X_0$. Then we have $x_{k+1}, \dots, x_n \in X_0$. Denote by U the set of all elements of the form

$$x = \sum_{i>1} a_i \cdot x_i + (1 - a_1) \cdot x_1,$$

where $\sum_i \|a_i\| < 1$.

If there exists $x_0 \in X_0 \cap U$, then $x_0 = \sum_{i>1} a_i \cdot x_i + (1 - a_1) \cdot x_1$, where $\|a_1\| < 1$. Therefore, $1 - a_1$ is invertible and

$$x_1 = (1 - a_1)^{-1} \Big(x_0 - \sum_{i>1} a_i \cdot x_i \Big).$$

Hence, $x_0, x_2, \dots, x_n$ are generators of X but only $k-1$ generators are not in X_0. This contradicts the minimality of k, so implies that $X_0 \cap U = \emptyset$.

It follows from the open mapping theorem that the surjective map

$$A \widehat{\otimes} \ell^1_n \to X : e_i \mapsto x_i$$

is open. Therefore U is open. Since $x_1 \in U$, we have $x_1 \notin \overline{X_0}$.

Now assume that $\overline{X_0} \neq X_0$ and take $y \in \overline{X_0} \setminus X_0$. Since X_0 is maximal, we have $X_0 + A \cdot y = X$. Therefore there are $a \in A$ and $x_0 \in X_0$ such that $x_0 + a \cdot y = x_1$. Note that $x_1 + a \cdot (y' - y) = x_0 + a \cdot y' \in X_0$ for every $y' \in X_0$. However, since $x_1 \notin \overline{X_0}$, we can take $y' \in X_0$ sufficiently close to y to satisfy $x_1 + a \cdot (y' - y) \notin X_0$. This contradiction implies that X_0 is closed. ■

Now we can establish the main properties of the topological radical, which are similar to the algebraic case (cf. [K, Secs. 9.1, 9.2]).

Theorem 4.8. *Let X be a unital left Banach A-module.*

(1) *If $\phi \in {}_A\mathrm{h}(X, Y)$ then* $\phi(\text{t-rad}(X)) \subset \text{t-rad}\, Y$.
(2) $\tau : A \to X : a \to a \cdot x_0$ *is small iff* $x_0 \in \text{t-rad}\, X$.
(3) $\overline{R \cdot X} \subset \text{t-rad}\, X$, *where R is the Jacobson radical of A.*

(4) $\text{t-rad}(X/\text{t-rad}\,X) = 0$.

(5) *If Z is a closed submodule in X such that* $\text{t-rad}(X/Z) = 0$ *then* $\text{t-rad}\,X \subset Z$.

Proof. (1) follows from the definition and Proposition 2.5.

(2) See the proof of Theorem 4.2.

(3) Let $x_0 \in X$. It is sufficient to show that $\tau' : R \to X : r \mapsto r \cdot x_0$ is small. Since τ' is the composition of $R \to A$, which is small by Propositions 4.5 and 4.6, and $\tau : A \to X$, Proposition 2.5 implies that τ' is small.

(4) Suppose that $x \in X$ is such that $x + \text{t-rad}\,X \in \text{t-rad}(X/\text{t-rad}\,X)$. By the definition $x + \text{t-rad}\,X \in \operatorname{Im}\alpha$ for every maximal contractive monomorphism $\alpha : Y \to X/\text{t-rad}\,X$. By Lemma 3.8 there exists a maximal contractive monomorphism $\beta : W \to X$ such that $x \in \operatorname{Im}\beta$. This implies that $x \in \text{t-rad}\,X$.

(5) Denote by σ the projection $X \to X/Z$. It follows from (1) that $\sigma(\text{t-rad}\,X) = 0$. Therefore $\text{t-rad}\,X \subset Z$. ∎

Corollary 4.9. $\operatorname{rad} X \subset \text{t-rad}\,X$ *for each unital left Banach A-module X.*

Proof. Suppose that $x_0 \in \operatorname{rad} X$. Then $A \cdot x_0$ is a small submodule in X [K, Sec. 9.1.3(a)]. Consider $\tau : A \to X : a \to a \cdot x_0$. If $\phi : Y \to X$ is such that $\phi \dotplus \tau$ is surjective then $X = A \cdot x_0 + \operatorname{Im}\phi$. Since $A \cdot x_0$ is small, ϕ is surjective. Thus τ is a small morphism. By Theorem 4.8(2), $x_0 \in \text{t-rad}\,X$. ∎

If A is not unital we can treat each Banach A-module X as a unital Banach module over the unitization A_+ and consider the topological radical of X.

Lemma 4.10. *Let A be a radical Banach algebra. Then*

(1) $\operatorname{rad} A = A^2$ *and* $\overline{A^2} \subset \text{t-rad}\,A$*;*

(2) *if A admits a right b.a.i. then* $\operatorname{rad} A = A^2 = \overline{A^2} = \text{t-rad}\,A$.

Proof. (1) Since A_+/A is classically semisimple, A is a left good ring [K, 9.7.2, 9.7.3(a)]. Therefore $\operatorname{rad} X = A \cdot X$ for every left unital A_+-module X [K, 9.7.1]. In particular, $\operatorname{rad} A = A^2$. The inclusion $\overline{A^2} \subset \text{t-rad}\,A$ follows from Theorem 4.8(3).

The second statement follows from the Cohen factorization theorem. ∎

Consider $C[0,1]$ and $L^1[0,1]$ as Banach algebras with respect to the cut-off convolution

$$(f * g)(s) := \int_0^s f(t) g(s-t)\,dt.$$

It is well known that both algebras are radical.

Proposition 4.11.

(1) *If $A = (L^1[0,1], *)$, then* $\text{t-rad}\,A = \operatorname{rad} A$.

(2) *If $A = (C[0,1], *)$, then* $\text{t-rad}\,A \neq \operatorname{rad} A$.

Proof. (1) Since $A = L^1[0,1]$ admits a b.a.i., Lemma 4.10 implies that $\text{t-rad}\, A = \text{rad}\, A$.

(2) It is easy to see that $I_0 := \{f \in C[0,1] : f(0) = 0\}$ is a closed ideal in $A = C[0,1]$ and $A^2 \subset I_0$. Since smooth functions vanishing at 0 are dense in I_0 and every such function is the convolution of a derivative and a constant, we have $\overline{A^2} = I_0$. Note that A/I_0 is one-dimensional. This implies that $I_0 \to A$ is a maximal contractive monomorphism. Therefore $\text{rad}\, A \subset \text{t-rad}\, A \subset I_0$. By Lemma 4.10, $\text{rad}\, A = A^2$ and $I_0 = \overline{A^2} = \text{t-rad}\, A$. To see that $A^2 \neq I_0$ note that every function in A^2 is majorized by a linear function, therefore $f(s) = \sqrt{s}$ is not in A^2. ∎

Recall that a left Banach A-module P is called *projective* if a morphism of Banach A-modules with range in P admits a right inverse morphism provided it admits a right inverse bounded operator.

PROPOSITION 4.12. *If P is a unital projective module with the approximation property, then* $\text{t-rad}\, P = \overline{R \cdot P}$, *where R is the Jacobson radical of A.*

Proof. By Theorem 4.8(3), $\overline{R \cdot P} \subset \text{t-rad}\, P$.

On the other hand, suppose that $x_0 \in \text{t-rad}\, P$. Since P is projective and has the approximation property, [S, Theorem 1(3)] implies that x_0 can be approximated in the norm topology by elements of the form $\sum_{i=1}^{n} \chi_i(x_0) \cdot y_i$ where $\chi_1, \ldots, \chi_n \in {}_A\mathrm{h}(P, A)$ and $y_1, \ldots, y_n \in P$. It follows from Theorem 4.8(1) that $\chi_i(x_0) \in R$. Hence, $x_0 \in \overline{R \cdot P}$. ∎

REMARK 4.13. It is not hard to check that in the case when P is free, i.e. has the form $A \widehat{\otimes} E$ for some Banach space E, the argument of Proposition 4.12 can be applied to the case when A or E has the approximation property.

Acknowledgements. This research was partially supported by RFBR under grant no. 12-01-00577.

The author would like to thank the referee for the valuable comments which helped to improve the manuscript.

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