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Amalgamations of classes of Banach spaces with a monotone basis

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Abstract. It was proved by Argyros and Dodos that, for many classes $\mathcal{C}$ of separable Banach spaces which share some property P, there exists an isomorphically universal space that satisfies P as well. We introduce a variant of their amalgamation technique which provides an isometrically universal space in the case that $\mathcal{C}$ consists of spaces with a monotone Schauder basis. For example, we prove that if $\mathcal{C}$ is a set of separable Banach spaces which is analytic with respect to the Effros Borel structure and every $X \in \mathcal{C}$ is reflexive and has a monotone Schauder basis, then there exists a separable reflexive Banach space that is isometrically universal for $\mathcal{C}$.

1. Introduction and the main result. Let $\mathcal{C}$ be a class of Banach spaces. We say that a Banach space X is *isomorphically* [*isometrically*] *universal for* $\mathcal{C}$ if it contains an isomorphic [isometric] copy of every member of $\mathcal{C}$.

The present paper deals with universality questions in separable Banach space theory. Our aim is to find an isometric version of the amalgamation theory of S. A. Argyros and P. Dodos [1] and provide a method of constructing small isometrically universal spaces for small families of Banach spaces. Many of the results considered in this paper employ methods from descriptive set theory. The connection of universality problems and descriptive set theory, discovered by J. Bourgain [4, 5], deepened the theory and enabled several intrinsic questions to be understood. (See also [3], [11], [9], [13]; for an introduction, see [19].)

In 1968, W. Szlenk [33] proved that the class of separable reflexive spaces has no isomorphically universal element. (It had been shown some time earlier by J. Lindenstrauss [28] that it has no isometrically universal element.)

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Szlenk proved that a Banach space which is isomorphically universal for separable reflexive spaces has non-separable dual. His proof led to the famous Szlenk index which will also be useful in our proofs.

Later, J. Bourgain [4] proved that if a separable Banach space is isomorphically universal for separable reflexive spaces, then it is actually isomorphically universal for all separable Banach spaces. A somewhat different proof of this result was provided by B. Bossard [3] who showed that if an analytic set of separable Banach spaces contains all separable reflexive spaces up to isomorphism, then it contains a space which is isomorphically universal for all separable Banach spaces (an analytic set of Banach spaces is defined in Section 2). For a separable Banach space X, the set of all Banach spaces with an isomorphic copy in X is analytic. Therefore, Bourgain's result follows from Bossard's.

Bossard's approach consists in constructing a tree space such that every infinite branch supports a universal space and every tree without infinite branches supports a reflexive space. One can apply this approach to analogous questions concerning isometries as well. It was shown in [21] that if a separable Banach space is isometrically universal for separable strictly convex spaces, then it is actually isometrically universal for all separable Banach spaces. The same result holds for the class of reflexive spaces [25].

In the work of S. A. Argyros and P. Dodos [1], the concept of a tree space also turned out to be a powerful tool for constructing universal spaces (see also [10]). When a set $\mathcal{C}$ of separable Banach spaces is simple (in the sense that $\mathcal{C}$ is analytic and every member has a Schauder basis), then one can construct a tree space such that the spaces supported by infinite branches are isomorphic copies of all members of $\mathcal{C}$. If the tree space is constructed properly, the properties of spaces from $\mathcal{C}$ can be preserved.

Some results of the Argyros–Dodos amalgamation theory are summed up in the following theorem (by a *basis* we mean a Schauder basis).

THEOREM 1.1 ([1]). *Let $\mathcal{P}$ be one of the following classes of separable Banach spaces:*

- *the class of spaces with a shrinking basis,*
- *the class of reflexive spaces with a basis,*
- *the class of spaces with a basis which are not isomorphically universal for all separable Banach spaces.*

Let $\mathcal{C}$ be an analytic set of spaces from $\mathcal{P}$. Then there exists a Banach space E which belongs to $\mathcal{P}$ and which contains a complemented isomorphic copy of every member of $\mathcal{C}$.

Reliance on a basis was soon dropped in work of P. Dodos and V. Ferenczi [11] and P. Dodos [9]. They proved that Theorem 1.1 also holds (without the property that the copies are complemented) for the following classes:

- the class of spaces with separable dual [11] (see also [14] for a quantitative version achieved by different methods),
- the class of separable reflexive spaces [11] (see also [29] for a quantitative version obtained by different methods),
- the class of separable spaces which are not isomorphically universal for all separable Banach spaces [9].

In the present work, we study the problem of whether these results have an isometric version (see also [20, Problem 9]). We establish an isometric variant of Theorem 1.1.

A basis $x_1, x_2, \ldots$ is said to be *monotone* if the associated partial sum operators $P_n : \sum_{k=1}^{\infty} a_k x_k \mapsto \sum_{k=1}^{n} a_k x_k$ satisfy $\|P_n\| \leq 1$.

THEOREM 1.2. *Let $\mathcal{P}$ be one of the following classes of separable Banach spaces:*

- *the class of spaces with a monotone shrinking basis,*
- *the class of reflexive spaces with a monotone basis,*
- *the class of spaces with a monotone basis which are not isometrically universal for all separable Banach spaces,*
- *the class of strictly convex spaces with a monotone basis.*

Let $\mathcal{C}$ be an analytic set of spaces from $\mathcal{P}$. Then there exists a Banach space E which belongs to $\mathcal{P}$ and which contains a 1*-complemented isometric copy of every member of $\mathcal{C}$.*

We do not know ([1]) whether reliance on a basis can be dropped, similarly to the isomorphic setting. The monotone basis requirement is a weak point of Theorem 1.2, but hopefully the theorem will help to obtain more powerful results in the future.

We make several remarks concerning Theorem 1.2.

(I) For the class of spaces with a shrinking basis and the class of reflexive spaces with a basis, it is not difficult to show that Theorem 1.1 follows from Theorem 1.2.

(II) The theorem remains valid if we consider monotone finite-dimensional decompositions instead of monotone bases. Consequently, a variant of the space constructed by S. Prus [30] can be provided. Indeed, since the class of superreflexive spaces is analytic, there exists a separable reflexive space which contains a 1-complemented isometric copy of every superreflexive space with a monotone finite-dimensional decomposition. Since $F \oplus_2 \ell_2$ is superreflexive for each finite-dimensional F, we also obtain the result of

([1]) We already know. After the submission of this work, a version of Theorem 1.2 not requiring the existence of a basis has been developed [26, 27].

A. Szankowski [32] which states that there exists a separable reflexive space, isometrically universal for all finite-dimensional spaces.

(III) The methods of this paper can be used to construct a Pełczyński universal space which contains a 1-complemented isometric copy of every Banach space with a monotone basis (see Definition 9.3). Similar examples have been constructed by J. Garbulińska-Węgrzyn [15, 16].

(IV) Theorem 1.2 holds for more general classes than the class of non-universal spaces. Let Z be a separable Banach space for which there are $a \in Z$ and a subset $H \subset Z$ whose closed linear span contains an isometric copy of Z and such that, for every $h \in H$, there is an $\varepsilon > 0$ with $\|a \pm \varepsilon h\| = \|a\|$. Then the theorem holds for the class of spaces with a monotone basis not containing an isometric copy of Z. Besides the universal space $Z = C(\{0,1\}^{\mathbb{N}})$, the required property is fulfilled e.g. by the spaces $Z = c_0$ and $Z = \ell_1$. (For $Z = \ell_1$, we can consider $a \in \ell_1$ whose coordinates are all positive and $H = \{(1/2, -1/2, 0, 0, 0, \dots), (0, 0, 1/2, -1/2, 0, \dots), \dots\}$.)

(V) If a separable Banach space X is isomorphically universal for separable Schur spaces, then it is actually isomorphically universal for all separable Banach spaces. This follows from methods in [3] (see [6, Corollary 51]). We are able to prove the isometric version of this statement (see Remark 3.7).

It is not known if the class of Schur spaces with a basis has the property from Theorem 1.1. It is not clear whether the tree space method can be used in this case. However, the property is fulfilled by the related class of ℓ_1-saturated spaces with a basis (see [1, Theorem 91]).

2. Preliminaries. We denote by $\Lambda^{<\mathbb{N}}$ the set of all finite sequences of elements of a set Λ, including the empty sequence $\emptyset$. That is,

$$\Lambda^{<\mathbb{N}} = \bigcup_{l=0}^{\infty} \Lambda^l$$

where $\Lambda^0 = \{\emptyset\}$. The length of $\eta \in \Lambda^{<\mathbb{N}}$ is denoted by $|\eta|$. If $\eta \in \Lambda^{<\mathbb{N}}$ and $\nu \in \Lambda^{<\mathbb{N}} \cup \Lambda^{\mathbb{N}}$, then by writing $\eta \subset \nu$ we mean that η is an initial segment of ν, i.e., $|\eta| \leq |\nu|$ and $\eta(i) = \nu(i)$ for $1 \leq i \leq |\eta|$. By $(n_1, \dots, n_k)^\wedge n$ we mean $(n_1, \dots, n_k, n)$. A subset T of $\Lambda^{<\mathbb{N}}$ is called a *tree on* Λ if

$$\eta \subset \nu \,\&\, \nu \in T \;\Rightarrow\; \eta \in T.$$

Moreover, a set $T \subset \Lambda^{<\mathbb{N}} \setminus \{\emptyset\}$ is called an *unrooted tree on* Λ if $T \cup \{\emptyset\}$ is a tree on Λ. An (unrooted) tree T is called *pruned* if every $\eta \in T$ has a proper extension $\nu \supsetneq \eta$ with $\nu \in T$. The set of all infinite branches of T, i.e., sequences $\nu \in \Lambda^{\mathbb{N}}$ such that T contains all non-empty initial segments of ν, is denoted by $[T]$. An (unrooted) tree T is called *well-founded* if it does not have an infinite branch.

A *Polish space* [*topology*] means a separable completely metrizable space [topology]. A set P equipped with a σ-algebra is called a *standard Borel space* if the σ-algebra is generated by a Polish topology on P. A subset of a standard Borel space is called *analytic* if it is a Borel image of a Polish space.

The following lemma can be found e.g. in [23, (25.2)].

LEMMA 2.1. *A subset $A \subset \mathbb{N}^{\mathbb{N}}$ is analytic if and only if there is a pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $A = p[T]$ where $p : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ denotes the projection on the first coordinate.*

For a topological space X, the set $\mathcal{F}(X)$ of all closed subsets of X is equipped with the *Effros Borel structure*, defined as the σ-algebra generated by the sets

$$\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$$

where U varies over open subsets of X. If X is Polish, then, equipped with this σ-algebra, $\mathcal{F}(X)$ forms a standard Borel space.

The *standard Borel space of separable Banach spaces* is defined by

$$\mathcal{SE}(C([0,1])) = \big\{F \in \mathcal{F}(C([0,1])) : F \text{ is linear}\big\}$$

and considered as a subspace of $\mathcal{F}(C([0,1]))$.

For a separable Banach space X and $F \subset B_{X^*}$, let

$$F'_\varepsilon = F \setminus \bigcup \big\{U \subset X^* : U \text{ is } w^*\text{-open}, \operatorname{diam}(U \cap F) < \varepsilon\big\}, \quad \varepsilon > 0,$$

and recursively

$$F_\varepsilon^{(0)} = F, \quad F_\varepsilon^{(\alpha)} = \bigcap_{\beta<\alpha} (F_\varepsilon^{(\beta)})'_\varepsilon, \quad \varepsilon > 0.$$

We define

$$\begin{aligned} \mathrm{Sz}_\varepsilon(F) &= \min\big(\{\omega_1\} \cup \{\alpha < \omega_1 : F_\varepsilon^{(\alpha)} = \emptyset\}\big), \qquad \varepsilon > 0, \\ \mathrm{Sz}(F) &= \sup\{\mathrm{Sz}_\varepsilon(F) : \varepsilon > 0\}. \end{aligned}$$

The *Szlenk index* of X is defined by $\mathrm{Sz}(X) = \mathrm{Sz}(B_{X^*})$.

For an (unrooted) tree T and a system $\{x_\eta : \eta \in T\}$ of elements of a Banach space, we define

$$\sum_{\eta \in T} x_\eta = \lim_{S \to T} \sum_{\eta \in S} x_\eta \qquad \text{(if the limit exists)},$$

where the limit is taken over all finite subtrees $S \subset T$ directed by inclusion.

The notions and notation we use but do not introduce here are classical and well explained e.g. in [12] and [23].

3. The initial tree space construction. In this section, we introduce our basic tool for constructing tree spaces. Basically, two ways have been developed of extracting the norm of a tree space from the norms of the subspaces supported by infinite branches (excluding the norm constructed in [25]). The first way, based on the well known James tree space [22], was employed mainly in works of B. Bossard [3] and S. A. Argyros and P. Dodos [1].

However, we follow the second way which is more suitable for isometric problems. The method was introduced by B. Bossard [2] and employed later by G. Godefroy [18] and G. Godefroy and N. J. Kalton [21]. In fact, the tree space in the following definition is a simplified version of the original tree space from [2], which will be introduced later in Definition 5.1 nevertheless.

DEFINITION 3.1. Let Λ be a countable set and let T be a pruned unrooted tree on Λ. For every $\sigma \in [T]$, let $(F_\sigma, \|\cdot\|_\sigma)$ be a Banach space with a monotone basis $f_1^\sigma, f_2^\sigma, \ldots$ and suppose these bases have the property that $f_1^\sigma, \ldots, f_l^\sigma$ and $f_1^\varphi, \ldots, f_l^\varphi$ are 1-equivalent whenever σ and φ have the same initial segment of length l.

Let a norm on $c_{00}(T)$ be defined by

$$\|x\| = \sup_{\sigma \in [T]} \Big\| \sum_{\eta \subset \sigma} x(\eta) f_{|\eta|}^\sigma \Big\|_\sigma \tag{1}$$

and for every unrooted subtree $S \subset T$ consider the projection

$$P_S x = \mathbf{1}_S \cdot x. \tag{2}$$

From the monotonicity of the bases f_n^σ, we obtain

$$\|P_S x\| \le \|x\|. \tag{3}$$

Finally, we define E as a completion of $(c_{00}(T), \|\cdot\|)$. The members of the canonical basis of $c_{00}(T)$ will be denoted by e_η (i.e., $e_\eta = \mathbf{1}_{\{\eta\}}$). We note that the system $\{e_\eta : \eta \in T\}$ is a basis of E, which follows from the observation that the property $x = \lim_{S \to T} P_S x$ extends from $c_{00}(T)$ to its closure E, due to the uniform boundedness of the projections P_S. The basis is monotone in the sense of (3).

Since $\{e_\eta : \eta \in T\}$ is a basis of E, we are allowed to consider all elements of E as systems $x = \{x(\eta)\}_{\eta \in T}$ of scalars. In this way, formulae (1)–(3) remain valid for every $x \in E$. We will denote the members of the corresponding dual system by e_η^* (i.e., $e_\eta^*(x) = x(\eta)$).

For every $\sigma \in [T]$, we further define spaces

$$\begin{aligned} E_\sigma &= \{x \in E : \eta \not\subset \sigma \Rightarrow x(\eta) = 0\}, \\ E_\sigma^* &= \{x^* \in E^* : \eta \not\subset \sigma \Rightarrow x^*(e_\eta) = 0\} \end{aligned} \tag{4}$$

and a projection

$$P_\sigma = P_{\{(\sigma_1),(\sigma_1,\sigma_2),\dots\}}. \tag{5}$$

We also denote

$$\varPhi = \bigcup_{\sigma\in[T]} B_{E_\sigma} \quad\text{and}\quad \varPsi = \bigcup_{\sigma\in[T]} B_{E^*_\sigma}. \tag{6}$$

FACT 3.2. *For every $\sigma \in [T]$, the basis $f^\sigma_1, f^\sigma_2, \dots$ of F_σ is 1-equivalent to the basis $e_{(\sigma_1)}, e_{(\sigma_1,\sigma_2)}, \dots$ of E_σ. In particular, the space E contains a 1-complemented isometric copy of F_σ for every $\sigma \in [T]$.*

Proof. Let $f = \sum_{n=1}^\infty r_n f^\sigma_n$ and $x = \sum_{n=1}^\infty r_n e_{(\sigma_1,\dots,\sigma_n)}$ where $r_n \neq 0$ for finitely many n only. We have

$$\Big\|\sum_{\eta\subset\sigma} x(\eta) f^\sigma_{|\eta|}\Big\|_\sigma = \Big\|\sum_{n=1}^\infty r_n f^\sigma_n\Big\|_\sigma = \|f\|_\sigma,$$

and so it remains to check that

$$\Big\|\sum_{\nu\subset\tau} x(\nu) f^\tau_{|\nu|}\Big\|_\tau \le \|f\|_\sigma$$

for each $\tau \in [T]\setminus\{\sigma\}$. Let η be the longest segment such that $\eta \subset \sigma$ and $\eta \subset \tau$, and let l be its length. Then

$$\Big\|\sum_{\nu\subset\tau} x(\nu) f^\tau_{|\nu|}\Big\|_\tau = \Big\|\sum_{n=1}^l r_n f^\tau_n\Big\|_\tau = \Big\|\sum_{n=1}^l r_n f^\sigma_n\Big\|_\sigma \le \|f\|_\sigma.$$

The second part of the assertion follows from $E_\sigma = P_\sigma E$. ■

FACT 3.3. *For $x \in E$, we have*

$$\|P_\sigma x\| = \sup_{x^*\in B_{E^*_\sigma}} |x^*(x)|.$$

For $x^* \in E^*$, we have

$$\|P^*_\sigma x^*\| = \sup_{x\in B_{E_\sigma}} |x^*(x)|.$$

Proof. The fact follows directly from the observation that $P_\sigma B_E = B_{E_\sigma}$ and $P^*_\sigma B_{E^*} = B_{E^*_\sigma}$. ■

LEMMA 3.4. *The set $\varPsi$ is compact in the weak* topology of E^* and its convex hull is w^*-dense in B_{E^*}.*

Proof. To show that $\varPsi$ is w^*-compact, we just write

$$\varPsi = B_{E^*} \setminus \bigcup \{x^* \in E^* : x^*(e_\eta) \neq 0 \,\&\, x^*(e_\nu) \neq 0\}$$

where the union is taken over all couples η, ν of incomparable segments in T. Using (1) in combination with Facts 3.2 and 3.3, we deduce for $x \in E$ that

$$\|x\| = \sup_{\sigma \in [T]} \|P_\sigma x\| = \sup_{\sigma \in [T]} \sup_{x^* \in B_{E^*_\sigma}} |x^*(x)| = \sup_{x^* \in \Psi} |x^*(x)|.$$

Now, to prove that the convex hull of Ψ is w^*-dense in B_{E^*}, it is sufficient to apply the Hahn–Banach theorem. ∎

PROPOSITION 3.5. *If the basis $f_1^\sigma, f_2^\sigma, \ldots$ is shrinking for every $\sigma \in [T]$, then the basis $\{e_\eta : \eta \in T\}$ is also shrinking.*

Proof. Let us fix an increasing sequence $T_1, T_2, \ldots$ of finite unrooted trees with $\bigcup_{n=1}^\infty T_n = T$. We show first that

$$x^* \in \Psi \ \Rightarrow \ P^*_{T_n} x^* \to x^*.$$

Given a $\sigma \in [T]$, we check the implication for the elements of $B_{E^*_\sigma}$. By Fact 3.2, the sequence $e_{(\sigma_1)}, e_{(\sigma_1,\sigma_2)}, \ldots$ is a shrinking basis of E_σ. By Fact 3.3, the elements of E^*_σ satisfy

$$\|x^*\| = \sup_{x \in B_{E_\sigma}} |x^*(x)|, \quad x^* \in E^*_\sigma.$$

Hence E^*_σ is (isometric to) the dual of E_σ indeed. The dual sequence $e^*_{(\sigma_1)}, e^*_{(\sigma_1,\sigma_2)}, \ldots$ is a basis of E^*_σ. It follows that $P^*_{T_n} x^* \to x^*$ for each $x^* \in E^*_\sigma$.

Now, let $y^* \in B_{E^*}$. By Lemma 3.4 and a standard integral representation argument (see e.g. [31, Theorem 3.28]), there exists a probability measure μ on Ψ such that

$$y^* = \int_\Psi x^* \, d\mu(x^*).$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \|y^* - P^*_{T_n} y^*\| &= \lim_{n\to\infty} \Big\| \int_\Psi (x^* - P^*_{T_n} x^*) \, d\mu(x^*) \Big\| \\
&\le \lim_{n\to\infty} \int_\Psi \|x^* - P^*_{T_n} x^*\| \, d\mu(x^*) \\
&= \int_\Psi \lim_{n\to\infty} \|x^* - P^*_{T_n} x^*\| \, d\mu(x^*) = \int_\Psi 0 \, d\mu(x^*) = 0.
\end{aligned}$$

This proves that y^* belongs to the closed linear span of the functionals e^*_η, $\eta \in T$. ∎

LEMMA 3.6. *If the space F_σ is reflexive for every $\sigma \in [T]$, then the set Φ is compact in the weak topology of E.*

Proof. Let $x_1, x_2, \ldots$ be a sequence in Φ. We want to find a subsequence x_{n_k} which converges weakly to an $x \in \Phi$. By Proposition 3.5, it is sufficient to check that

$$x_{n_k}(\eta) \to x(\eta), \quad \eta \in T.$$

Using a diagonal argument, we choose the subsequence x_{n_k} so that

$$x_{n_k}(\eta) \to u(\eta), \quad \eta \in T,$$

for a system $u = \{u(\eta)\}_{\eta \in T}$ of scalars. It remains to show that this system forms the coordinates of an $x \in \Phi$.

First, we observe that there is a $\sigma \in [T]$ such that u is supported by the branch $\{(\sigma_1), (\sigma_1, \sigma_2), \dots\}$. Indeed, if $u(\eta) \neq 0 \neq u(\nu)$ for some incomparable $\eta, \nu \in T$, then $x_{n_k}(\eta) \neq 0 \neq x_{n_k}(\nu)$ for a large enough k, which is not allowed by the definition of Φ.

By Fact 3.2 and our assumption, E_σ is reflexive. A subsequence of $P_\sigma x_{n_k}$ converges weakly to an $x \in B_{E_\sigma}$, and this limit satisfies $x(\eta) = u(\eta)$ for every $\eta \in T$. ∎

Remark 3.7. (a) If $S \subset T$ is a well-founded unrooted subtree, then the subspace

$$H(S) = \overline{\operatorname{span}}\{e^*_\eta : \eta \in S\} \tag{7}$$

of E^* has the Schur property. Let us assume that $H(S)$ is not Schur and denote $H_\nu(S) = \overline{\operatorname{span}}\{e^*_\eta : \eta \in S \,\&\, \nu \subset \eta\}$ for $\nu \in T \cup \{\emptyset\}$. It is sufficient to prove that

$$H_\nu(S) \text{ is not Schur } \Rightarrow\ H_{\nu^\wedge n}(S) \text{ is not Schur for some } n,$$

as this allows us to find an infinite branch of S. One can show that

$$\Big\| \sum_{n=1}^{m} x^*_n \Big\| \geq \frac{1}{2} \sum_{n=1}^{m} \|x^*_n\|, \quad x^*_n \in H_{\nu^\wedge n}(S), \quad n = 1, \dots, m.$$

Therefore, a hyperplane of $H_\nu(S)$ (or $H_\nu(S)$ itself when $\nu = \emptyset$) is isomorphic to the ℓ_1-sum of $H_{\nu^\wedge 1}(S), H_{\nu^\wedge 2}(S), \dots$, and the implication follows.

(b) If a separable Banach space X contains an isometric copy of every separable Schur space, then it contains an isometric copy of every separable Banach space. To show this, we follow the method of B. Bossard [3]. Let $x_1, x_2, \dots$ be a monotone basis of $C([0,1])$ (see e.g. [8, p. 34]) and $f_1, f_2, \dots$ be the dual basic sequence in $C([0,1])^*$. Let $T = \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$, $f^\sigma_n = f_n$ and $F_\sigma = \overline{\operatorname{span}}\{f_n : n \in \mathbb{N}\}$ for every $\sigma \in \mathbb{N}^{\mathbb{N}}$. In this setting, let $H(S)$ be given by (7). Let Tr be the subspace of 2^T consisting of all unrooted trees on $\mathbb{N}$ and let WF be the set of all well-founded $S \in \mathrm{Tr}$. Consider the set

$$\mathcal{A} = \{S \in \mathrm{Tr} : X \text{ contains an isometric copy of } H(S)\}.$$

Then $\mathcal{A}$ is analytic (see [18, Lemmas 7 and 8]) and it contains WF, due to our assumption. Since WF is not analytic (see e.g. [23, (27.1)]), there is an $S \in \mathcal{A} \setminus \mathrm{WF}$. So, X contains an isometric copy of $H(S)$ for some $S \notin \mathrm{WF}$, which contains an isometric copy of $C([0,1])$.

(c) Assume that, for every $\sigma \in [T]$, the space $\overline{\mathrm{span}}\{f_n^* : n \in \mathbb{N}\}$ has the Schur property, where $f_1^*, f_2^*, \ldots$ is the dual basic sequence of the basis $f_1^\sigma, f_2^\sigma, \ldots$. We do not know whether $H = \overline{\mathrm{span}}\{e_\eta^* : \eta \in T\}$ necessarily has the Schur property in that case. One can show that $B_H = \overline{\mathrm{co}}(H \cap \Psi)$ and that every weakly convergent sequence in $H \cap \Psi$ is convergent, but this does not seem to be sufficient for H to be Schur.

4. The interpolation method. The aim of this section is to provide a reflexive variant of the tree space from Definition 3.1. Just as the authors of [1], we apply the Davis–Figiel–Johnson–Pełczyński interpolation method.

Definition 4.1 ([7]). Let W be a bounded, closed, convex and symmetric subset of a Banach space X. For each $n \in \mathbb{N}$, let $\|\cdot\|_n$ be the equivalent norm given by

$$B_{(X,\|\cdot\|_n)} = \overline{2^n W + 2^{-n} B_X}.$$

The 2-*interpolation space of the pair* (X, W) is defined as the space $(Y, |||\cdot|||)$ where

$$|||x||| = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{1/2}, \quad x \in X,$$
$$Y = \{x \in X : |||x||| < \infty\}.$$

Lemma 4.2. *Let $P : X \to X$ be a projection such that $\|P\| \le 1$ and $PW \subset W$. Then*

$$|||Px||| \le |||x|||, \quad x \in X.$$

If, moreover, $PW = PB_X$, then there is a constant $c > 0$ such that

$$|||x||| = c\|x\|, \quad x \in PX.$$

In particular, Y contains a 1-complemented isometric copy of PX.

Proof. The inequality $|||Px||| \le |||x|||$ (which can be proven quite easily actually) follows from [7, p. 316, Lemma 1(viii)]. To provide a suitable constant $c > 0$, it is sufficient to show that

$$\|x\|_n = \frac{1}{2^n + 2^{-n}}\|x\|, \quad x \in PX.$$

Let $x \in PX$. We will assume that $\|x\| = 1$. Since $x \in B_X$ and $x = Px \in PB_X = PW \subset W$, we have $(2^n + 2^{-n})x \in B_{(X,\|\cdot\|_n)}$. Therefore, $(2^n + 2^{-n})\|x\|_n \le 1 = \|x\|$.

Choose $0 < \theta < 1/\|x\|_n$. We have $\|\theta x\|_n < 1$, and so $\theta x \in 2^n W + 2^{-n} B_X$. Thus $\theta x = 2^n w + 2^{-n} y$ for some $w \in W$ and $y \in B_X$. Since $Pw \in PW = PB_X \subset B_X$, we have $\theta x = \theta Px = 2^n Pw + 2^{-n} Py$ and $\theta\|x\| \le 2^n + 2^{-n}$, so $\|x\| \le (2^n + 2^{-n})\|x\|_n$ due to the choice of θ. ∎

Definition 4.3. Adopting the notation from Definition 3.1, we define A as the 2-interpolation space of the pair $(E, \overline{\mathrm{co}}\,\Phi)$.

Fact 4.4. *The system $\{e_\eta : \eta \in T\}$ is a monotone basis of A.*

Proof. The associated projections P_S satisfy $|||P_S x||| \le |||x|||$ by (3) and Lemma 4.2. The fact thus follows from [7, p. 316, Lemma 1(ix)]. ∎

Fact 4.5. *A contains a 1-complemented isometric copy of F_σ for every $\sigma \in [T]$.*

Proof. Recall that F_σ is isometric to $E_\sigma = P_\sigma E$ by Fact 3.2. The assumptions of Lemma 4.2 are met for $P = P_\sigma$, since $P_\sigma(\overline{\mathrm{co}}\,\Phi) \subset \overline{\mathrm{co}}\,\Phi$ and $P_\sigma(\overline{\mathrm{co}}\,\Phi) = B_{E_\sigma} = P_\sigma B_E$. ∎

Proposition 4.6. *If the space F_σ is reflexive for every $\sigma \in [T]$, then the space A is also reflexive.*

Proof. By Lemma 3.6 and the Krein–Smulian theorem, the set $\overline{\mathrm{co}}\,\Phi$ is weakly compact. To show that A is reflexive, it is sufficient to apply [7, p. 313, Lemma 1(iv)]. ∎

5. A rotund version of the tree space. The following definition of a tree space is based on a construction from [2] which was also applied in [18] and [21]. Regarding the results from these papers, it is not surprising that this tree space preserves strict convexity of the norm. However, it turns out that the method is also suitable for amalgamating spaces which are not isometrically universal (see Proposition 5.5).

Definition 5.1. Let $\Lambda, T, (F_\sigma, \|\cdot\|_\sigma)$ and f_n^σ be as in Definition 3.1. Suppose moreover that there are positive constants $c_1, c_2, \ldots$ such that, for every $\sigma \in [T]$,

$$\|\pi_n f\|_\sigma^2 \ge \|\pi_{n-1} f\|_\sigma^2 + c_n^2 |f_n^*(f)|^2, \qquad f \in F_\sigma,\ n \in \mathbb{N}, \tag{8}$$

where $f_1^*, f_2^*, \ldots$ is the dual basic sequence and $\pi_0, \pi_1, \ldots$ is the sequence of partial sum operators associated with the basis $f_1^\sigma, f_2^\sigma, \ldots$.

For every $x \in c_{00}(T)$, let us consider the formulae

$$|||x|||_\sigma^2 = \Big\| \sum_{\eta \subset \sigma} x(\eta) f_{|\eta|}^\sigma \Big\|_\sigma^2 + \sum_{\eta \not\subset \sigma} c_{|\eta|}^2 |x(\eta)|^2, \qquad \sigma \in [T], \tag{9}$$

$$|||x||| = \sup_{\sigma \in [T]} |||x|||_\sigma \tag{10}$$

and

$$P_S x = \mathbf{1}_S \cdot x, \tag{11}$$

where $S \subset T$ is an unrooted subtree. From the monotonicity of the bases f_n^σ, we obtain

$$|||P_S x||| \leq |||x|||. \tag{12}$$

Finally, we define B as a completion of $(c_{00}(T), |||\cdot|||)$. Again, the system $\{b_\eta = \mathbf{1}_{\{\eta\}} : \eta \in T\}$ is a basis of B which is monotone in the sense of (12). Therefore, we are allowed to consider all elements of B as systems $x = \{x(\eta)\}_{\eta \in T}$ of scalars. In this way, formulae (9)–(12) remain valid for every $x \in B$.

For every $\sigma \in [T]$, we further denote

$$B_\sigma = \{x \in B : \eta \not\subset \sigma \Rightarrow x(\eta) = 0\}, \tag{13}$$

$$P_\sigma = P_{\{(\sigma_1),(\sigma_1,\sigma_2),\ldots\}}. \tag{14}$$

Fact 5.2. *For every $\sigma \in [T]$, the basis $f_1^\sigma, f_2^\sigma, \ldots$ of F_σ is 1-equivalent to the basis $b_{(\sigma_1)}, b_{(\sigma_1,\sigma_2)}, \ldots$ of B_σ. In particular, B contains a 1-complemented isometric copy of F_σ for every $\sigma \in [T]$.*

We do not prove this fact, as an analogous statement appeared in [18] and [21]. Actually, the fact can be proven similarly to Fact 3.2, with the difference that (8) is applied. The proof of the following lemma, which is essentially contained in [21, p. 186], is also skipped.

Lemma 5.3 ([21]). *For every $x \in B$, the supremum in (10) is attained.*

Lemma 5.4. *Let $[u, v]$ be a non-degenerate line segment in B such that $|||\cdot|||$ is constant on $[u, v]$. Let $w = \frac{1}{2}(u + v)$ and suppose the supremum in (10) for $x = w$ is attained at a $\sigma \in [T]$. Then $v - u \in B_\sigma$ and $[P_\sigma u, P_\sigma v]$ is also a non-degenerate line segment on which $|||\cdot|||$ is constant.*

Proof. Consider the seminorm

$$|x|_\sigma^2 = \sum_{\eta \not\subset \sigma} c_{|\eta|}^2 |x(\eta)|^2, \quad x \in B.$$

Using Fact 5.2, we obtain

$$|||x|||_\sigma^2 = |||P_\sigma x|||^2 + |x|_\sigma^2, \quad x \in B.$$

We can compute

$$|||w||| = |||w|||_\sigma \leq \tfrac{1}{2}(|||u|||_\sigma + |||v|||_\sigma) \leq \tfrac{1}{2}(|||u||| + |||v|||) = |||w|||,$$

and it is clear that all these norms must be equal. Thus,

$$\begin{aligned}
0 &= 2|||u|||_\sigma^2 + 2|||v|||_\sigma^2 - 4|||w|||_\sigma^2 \\
&= 2|||P_\sigma u|||^2 + 2|||P_\sigma v|||^2 - 4|||P_\sigma w|||^2 + 2|u|_\sigma^2 + 2|v|_\sigma^2 - 4|w|_\sigma^2 \\
&= (|||P_\sigma u||| - |||P_\sigma v|||)^2 + (|||P_\sigma u||| + |||P_\sigma v|||)^2 - |||P_\sigma(u+v)|||^2 \\
&\quad + (|u|_\sigma - |v|_\sigma)^2 + (|u|_\sigma + |v|_\sigma)^2 - |u+v|_\sigma^2.
\end{aligned}$$

It follows that

$$\|\!|P_\sigma u|\!\|=\|\!|P_\sigma v|\!\|,\qquad \|\!|P_\sigma(u+v)|\!\|=\|\!|P_\sigma u|\!\|+\|\!|P_\sigma v|\!\|, \tag{15}$$

$$|u|_\sigma=|v|_\sigma,\qquad |u+v|_\sigma=|u|_\sigma+|v|_\sigma. \tag{16}$$

By (15), the norm $\|\!|\cdot|\!\|$ is constant on $[P_\sigma u, P_\sigma v]$. By (16), the points u and v satisfy $u(\eta)=v(\eta)$ for every $\eta\not\subset\sigma$. That is, $u-P_\sigma u=v-P_\sigma v$. Therefore, $v-u=P_\sigma v-P_\sigma u\in B_\sigma$ and the segment $[P_\sigma u,P_\sigma v]$ is non-degenerate. ∎

Proposition 5.5.

(a) *If no F_σ, $\sigma\in[T]$, is isometrically universal for all separable Banach spaces, then B is also non-universal.*
(b) *If every F_σ, $\sigma\in[T]$, is strictly convex, then so is B.*

Proof. (a) Assume that B is isometrically universal for all separable Banach spaces. Set

$$\Delta=\{0,1\}^{\mathbb N},\quad \Delta(i)=\{\gamma\in\Delta:\gamma(1)=i\},\quad i=0,1,$$
$$Z=C(\Delta),\quad Z(i)=\{h\in Z:\gamma\notin\Delta(i)\Rightarrow h(\gamma)=0\},\quad i=0,1.$$

Considering an isometry $I:Z\to B$, we denote

$$x=I(\mathbf 1_{\Delta(0)}).$$

By Lemma 5.3, the supremum in (10) is attained at some $\sigma\in[T]$. We claim that B_σ (and therefore F_σ by Fact 5.2) is universal, showing that I maps $Z(1)$ into B_σ.

Given an $h\in Z(1)$ with $\|h\|\le 1$, we observe that $\|\mathbf 1_{\Delta(0)}\|=\|\mathbf 1_{\Delta(0)}\pm h\|$ $=1$, and so $\|\!|x|\!\|=\|\!|x\pm Ih|\!\|=1$. By Lemma 5.4, we have $Ih\in B_\sigma$.

(b) Assume that B is not strictly convex. This means that $\|\!|\cdot|\!\|$ is constant on a non-degenerate line segment $[u,v]$. Let $x=\frac12(u+v)$ and let the supremum in (10) be attained at $\sigma\in[T]$ (Lemma 5.3). By Lemma 5.4, the space B_σ is not strictly convex. Since B_σ and F_σ are isometric (see Fact 5.2), the proof is finished. ∎

6. Construction of branches. In the isomorphic setting, it is possible to construct a tree space such that isomorphic copies of the spaces we want to amalgamate are placed on the infinite branches (as mentioned in the introduction). In the isometric setting, we are not allowed to renorm the spaces, and an additional embedding result is needed.

We prove that a Banach space X with a monotone basis can be embedded into another (not much bigger) Banach space F with a monotone basis $f_1,f_2,\ldots$ such that the subspaces $\operatorname{span}\{f_1,\ldots,f_d\}$ are chosen from a countable family of spaces. To this end, we employ the following notion which was also useful in [15, 16, 17, 24].

Definition 6.1. A Banach space Z is called *rational* if $Z = \mathbb{R}^d$ with a norm whose unit ball is generated by finitely many points with rational coordinates.

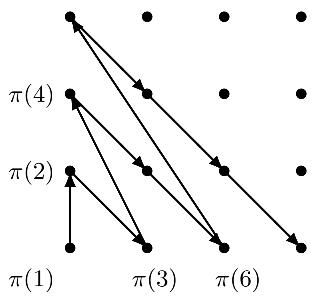
Spaces which have a basis consisting of d elements will often be identified with $\mathbb{R}^d$ in the obvious way.

The main goal of this section is to prove the following result. Its proof is based on a construction provided in [25, Section 4] (which in turn was based on a construction from [21]), but the present method is considerably simpler.

Proposition 6.2. *Let X be a Banach space and $e_1, e_2, \dots$ be a monotone basis of X. Then there exists a Banach space F with a monotone basis $f_1, f_2, \dots$ such that:*

(1) *F is isomorphic to $\ell_2(X)$.*
(2) *If the basis $e_1, e_2, \dots$ is shrinking, then so is $f_1, f_2, \dots$.*
(3) *For every $d \in \mathbb{N}$, the space $\operatorname{span}\{f_1, \dots, f_d\}$, identified with $\mathbb{R}^d$, is rational.*
(4) *F contains a 1-complemented isometric copy of X.*

Definition 6.3. We will denote by π the bijection $\mathbb{N} \to \mathbb{N}^2$ given by $\pi(1) = (1, 1)$, $\pi(2) = (1, 2)$, $\pi(3) = (2, 1)$, $\pi(4) = (1, 3)$ etc.



Definition 6.4. For every $d \in \mathbb{N}$, let us fix an ordering of all monotone rational norms on $\mathbb{R}^d$ into a sequence $|\cdot|_{d,1}, |\cdot|_{d,2}, \dots$.

Let $e_1, e_2, \dots$ be a normalized monotone basis of a Banach space $(X, \|\cdot\|_X)$. Let $f_i = e_{\pi(i)}$ where $e_{(n,k)}$ stands for the element of $\ell_2(X)$ which has e_k on the nth place and 0 elsewhere. Let us moreover denote

$$F_d = \operatorname{span}\{f_1, \dots, f_d\}. \tag{17}$$

For every $d \in \mathbb{N}$, let $l_d = l_d(e_1, e_2, \dots)$ be the least natural number such that the monotone rational norm $|\cdot|_d = |\cdot|_{d,l_d}$ satisfies

$$\Big(1 - \frac{1}{2^{2d+1}}\Big)\|f\|_{\ell_2(X)} \le |f|_d \le \Big(1 - \frac{1}{2^{2d+2}}\Big)\|f\|_{\ell_2(X)}, \quad f \in F_d. \tag{18}$$

This condition is valid for some monotone rational norm on F_d, as $f_1, f_2, \dots$ is a monotone basis of $\ell_2(X)$.

We define a space $F = F(e_1, e_2, \dots)$ with a norm $\|\cdot\|$ by

$$B_{(F,\|\cdot\|)} = \overline{\mathrm{co}} \bigcup_{d=1}^{\infty} B_{(F_d,|\cdot|_d)}. \tag{19}$$

We also define operators

$$T : \ell_2(X) \to X, \quad (x_1, x_2, \dots) \mapsto \frac{\sqrt{3}}{2}\big(x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{4}x_3 + \cdots\big), \tag{20}$$

$$U : X \to \ell_2(X), \quad x \mapsto \frac{\sqrt{3}}{2}\big(x, \tfrac{1}{2}x, \tfrac{1}{4}x, \cdots\big). \tag{21}$$

The sequence of partial sum operators associated with the basis $f_1, f_2, \dots$ will be denoted by $P_1, P_2, \dots$.

Lemma 6.5. *We have $F = \ell_2(X)$ and the norm $\|\cdot\|$ fulfills*

$$\tfrac{7}{8}\|f\|_{\ell_2(X)} \le \|f\| \le \|f\|_{\ell_2(X)}, \quad f \in F. \tag{22}$$

The basis $f_1, f_2, \dots$ is a monotone basis of $(F, \|\cdot\|)$, and it is shrinking if $e_1, e_2, \dots$ is shrinking. Finally, for every $d \in \mathbb{N}$,

$$B_{(F_d,\|\cdot\|)} = \mathrm{co} \bigcup_{j=1}^{d} B_{(F_j,|\cdot|_j)}. \tag{23}$$

In particular, the space $(F_d, \|\cdot\|)$ is rational.

Proof. By (18), we have $\frac{7}{8}\|f\|_{\ell_2(X)} \le |f|_d \le \|f\|_{\ell_2(X)}$ for $f \in F_d$. Thus,

$$B_{(F_d,|\cdot|_d)} \subset \tfrac{8}{7}B_{\ell_2(X)} \quad \text{and} \quad B_{\ell_2(X)} \cap F_d \subset B_{(F_d,|\cdot|_d)} \subset B_{(F,\|\cdot\|)},$$

and it follows that

$$B_{(F,\|\cdot\|)} \subset \tfrac{8}{7}B_{\ell_2(X)} \quad \text{and} \quad B_{\ell_2(X)} \subset B_{(F,\|\cdot\|)}.$$

Clearly, if $e_1, e_2, \dots$ is shrinking, then so is $f_1, f_2, \dots$. To show that the latter basis is monotone with respect to $\|\cdot\|$, it is sufficient to note that the associated partial sum operators $P_1, P_2, \dots$ map the unit ball of $(F_d, |\cdot|_d)$ into itself, and consequently the unit ball of $(F, \|\cdot\|)$ has the same property. To show (23), it is sufficient to prove that P_d maps the unit ball of $(F_j, |\cdot|_j)$, where $j > d$, into the unit ball of $(F_d, |\cdot|_d)$. Indeed, for $f \in F_j$,

$$|P_d f|_d \le \left(1 - \frac{1}{2^{2d+2}}\right)\|P_d f\|_{\ell_2(X)} \le \left(1 - \frac{1}{2^{2j+1}}\right)\|f\|_{\ell_2(X)} \le |f|_j. \ \blacksquare$$

Lemma 6.6. *We have $\|Tf\|_X \le \|f\|$ for $f \in F$.*

Proof. For $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$, we can write

$$\begin{aligned}\|T(x_1, \dots, x_n, 0, 0, \dots)\|_X &= \frac{\sqrt{3}}{2}\left\| x_1 + \frac{1}{2}x_2 + \dots + \frac{1}{2^{n-1}}x_n \right\|_X \\ &\le \frac{\sqrt{3}}{2}\left(\|x_1\|_X + \frac{1}{2}\|x_2\|_X + \dots + \frac{1}{2^{n-1}}\|x_n\|_X \right) \\ &\le \frac{\sqrt{3}}{2}\sqrt{1 + \frac{1}{4} + \dots + \frac{1}{4^{n-1}}}\,\sqrt{\|x_1\|_X^2 + \|x_2\|_X^2 + \dots + \|x_n\|_X^2} \\ &= \sqrt{1 - 1/4^n}\,\|(x_1, \dots, x_n, 0, 0, \dots)\|_{\ell_2(X)}.\end{aligned}$$

It follows that

$$\|Tf\|_X \le \sqrt{1 - 1/4^d}\,\|f\|_{\ell_2(X)}, \quad f \in F_d,\ d \in \mathbb{N},$$

as the elements of F_d are supported by the first d coordinates (obvious from the definition of π).

Now, given $d \in \mathbb{N}$, for $f \in F_d$ we obtain

$$\|Tf\|_X \le \sqrt{1 - 1/4^d}\,\|f\|_{\ell_2(X)} \le (1 - 1/2^{2d+1})\|f\|_{\ell_2(X)} \le |f|_d.$$

Therefore, the unit ball of $(F_d, |\cdot|_d)$, where $d \in \mathbb{N}$, and consequently the unit ball of $(F, \|\cdot\|)$, are subsets of $\{f \in F : \|Tf\|_X \le 1\}$. ∎

Lemma 6.7. *We have $\|Ux\| = \|x\|_X$ for $x \in X$ and the range of U is 1-complemented in $(F, \|\cdot\|)$.*

Proof. It can be easily shown that

$$TUx = x \quad \text{and} \quad \|Ux\|_{\ell_2(X)} = \|x\|_X$$

for $x \in X$. Using Lemmas 6.6 and 6.5, we can write

$$\|x\|_X = \|TUx\|_X \le \|Ux\| \le \|Ux\|_{\ell_2(X)} = \|x\|_X, \quad x \in X.$$

Moreover, $UT : F \to F$ is a projection onto UX with $\|UT\| \le 1$. ∎

The proof of Proposition 6.2 is complete. Nevertheless, we prove one more lemma which will be useful later.

Lemma 6.8. *We have*

$$\|f\| \ge \|P_n f\| + \frac{1}{2^{2n+4}}\|f - P_n f\|, \quad f \in F,\ n \in \mathbb{N}.$$

Proof. As in the proof of Lemma 6.6, it is sufficient to show that the unit ball of $(F_d, |\cdot|_d)$, where $d \in \mathbb{N}$, is a subset of $\{f \in F : \|P_n f\| + \frac{1}{2^{2n+4}}\|f - P_n f\| \le 1\}$. So, we just need to check that

$$\|P_n f\| + \frac{1}{2^{2n+4}}\|f - P_n f\| \le |f|_d, \quad f \in F_d.$$

The inequality is clear when $d \le n$, as $P_n f = f$. If $d \ge n+1$, then

$$\|P_n f\| \le |P_n f|_n \le \left(1 - \frac{1}{2^{2n+2}}\right)\|P_n f\|_{\ell_2(X)} \le \left(1 - \frac{1}{2^{2n+2}}\right)\|f\|_{\ell_2(X)},$$

and so

$$\begin{aligned}
&\|P_n f\| + \frac{1}{2^{2n+4}}\|f - P_n f\| \\
&\quad = \left(1 - \frac{1}{2^{2d+1}}\right)\|P_n f\| + \frac{1}{2^{2d+1}}\|P_n f\| + \frac{1}{2^{2n+4}}\|f - P_n f\| \\
&\quad \le \left(1 - \frac{1}{2^{2d+1}}\right)\left(1 - \frac{1}{2^{2n+2}}\right)\|f\|_{\ell_2(X)} + \frac{1}{2^{2n+3}}\|f\| + \frac{1}{2^{2n+4}} \cdot 2\|f\| \\
&\quad \le \left(1 - \frac{1}{2^{2n+2}}\right)|f|_d + \frac{1}{2^{2n+3}}|f|_d + \frac{1}{2^{2n+4}} \cdot 2|f|_d = |f|_d
\end{aligned}$$

for every $f \in F_d$. ∎

7. Renorming I. For the class of reflexive spaces and the class of spaces with a shrinking basis, the construction of the space F from Definition 6.4 is satisfactory. For the other two classes from Theorem 1.2, the space F has to be renormed so that the relevant isometric properties of the initial space X are preserved.

In fact, we renorm the space in two steps (renormings $\|\cdot\|_I$ and $\|\cdot\|_{II}$). For the class of non-universal spaces, one renorming is sufficient. For the class of strictly convex spaces, one more renorming is needed.

Let us emphasize two aspects of the renormings. Firstly, the new norm on $F_d = \operatorname{span}\{f_1, \ldots, f_d\}$ depends only on the old norm on F_d itself. In this way, only countably many possibilities for the norm of F_d may occur. Secondly, the norm is not changed on the subspace UX which is still a 1-complemented copy of X.

Definition 7.1. We define a seminorm by

$$\beta(f)^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n+1,k)}}|e^*_{(n,k)}(f) - 2e^*_{(n+1,k)}(f)|^2, \quad f \in F, \tag{24}$$

where $e^*_{(n,k)}$ is the system biorthogonal to the basic system $e_{(n,k)}$.

The proof of the following observation is skipped.

Fact 7.2. *For $f \in F$, the following assertions are equivalent:*

(i) $\beta(f) = 0$,
(ii) $e^*_{(n,k)}(f) - 2e^*_{(n+1,k)}(f) = 0$ *for all* n, k,
(iii) $f \in UX$.

LEMMA 7.3. *Let $d \in \mathbb{N} \cup \{0\}$. Then every $f \in \overline{\operatorname{span}}\{f_{d+1}, f_{d+2}, \dots\}$ satisfies*

$$\beta(f) \leq \frac{2}{2^{2d}}\|f\|.$$

In particular, $\beta(f) \leq 2\|f\|$ for every $f \in F$.

Proof. We can compute

$$|e^*_{(n,k)}(f)| \leq \|e^*_{(n,k)}\|_{\ell_2(X)}\|f\|_{\ell_2(X)} \leq 2 \cdot \tfrac{8}{7}\|f\|, \tag{25}$$

$$|e^*_{(n,k)}(f) - 2e^*_{(n+1,k)}(f)| \leq 3 \cdot 2 \cdot \tfrac{8}{7}\|f\| \leq 2\sqrt{15}\|f\|. \tag{26}$$

Moreover, from $f \in \overline{\operatorname{span}}\{f_{d+1}, f_{d+2}, \dots\}$ we obtain

$$\pi^{-1}(n,k) \leq d \ \Rightarrow\ e^*_{(n,k)}(f) = 0,$$

and consequently

$$\pi^{-1}(n+1,k) \leq d \ \Rightarrow\ e^*_{(n,k)}(f) - 2e^*_{(n+1,k)}(f) = 0.$$

Therefore,

$$\begin{aligned}\beta(f)^2 &\leq \sum_{\pi^{-1}(n+1,k)>d} \frac{1}{2^{4\pi^{-1}(n+1,k)}} \cdot (2\sqrt{15}\|f\|)^2 \\ &\leq \sum_{j>d} \frac{1}{2^{4j}} \cdot 4 \cdot 15\|f\|^2 = \frac{4}{2^{4d}}\|f\|^2. \ \blacksquare\end{aligned}$$

DEFINITION 7.4. We define

$$\|f\|_I^2 = \|f\|^2 + \frac{1}{2^7}\beta(f)^2, \quad f \in F. \tag{27}$$

A simple application of Lemma 7.3 gives

$$\|f\| \leq \|f\|_I \leq 2\|f\|. \tag{28}$$

LEMMA 7.5. *We have $\|Ux\|_I = \|x\|_X$ for $x \in X$ and the range of U is 1-complemented in $(F, \|\cdot\|_I)$.*

Proof. Using Fact 7.2 and Lemma 6.7, we can write $\|Ux\|_I = \|Ux\| = \|x\|_X$ for $x \in X$. The projection UT works as in the proof of Lemma 6.7, because $\|UTf\|_I = \|UTf\| \leq \|f\| \leq \|f\|_I$ for $f \in F$. $\blacksquare$

LEMMA 7.6. *Let $[u, v]$ be a non-degenerate line segment in F such that $\|\cdot\|_I$ is constant on $[u, v]$. Then $v - u \in UX$.*

Proof. By the same argument as in the proof of Lemma 5.4, we arrive at

$$\begin{aligned}\|u\| = \|v\|, &\qquad \|u+v\| = \|u\| + \|v\|, \\ \beta(u) = \beta(v), &\qquad \beta(u+v) = \beta(u) + \beta(v),\end{aligned}$$

and consequently

$$e^*_{(n,k)}(u) - 2e^*_{(n+1,k)}(u) = e^*_{(n,k)}(v) - 2e^*_{(n+1,k)}(v), \quad n,k \in \mathbb{N}.$$

It follows that $v - u \in UX$ by Fact 7.2. ∎

PROPOSITION 7.7. *If X is not isometrically universal for all separable Banach spaces, then neither is $(F, \|\cdot\|_I)$.*

Proof. Assume that $(F, \|\cdot\|_I)$ is isometrically universal for all separable Banach spaces. Again, set

$$\Delta = \{0,1\}^{\mathbb{N}}, \quad \Delta(i) = \{\gamma \in \Delta : \gamma(1) = i\}, \quad i = 0,1,$$
$$Z = C(\Delta), \quad Z(i) = \{h \in Z : \gamma \notin \Delta(i) \Rightarrow h(\gamma) = 0\}, \quad i = 0,1.$$

Considering an isometry $I : Z \to F$, we denote

$$f = I(\mathbf{1}_{\Delta(0)}).$$

We claim that UX (and thus X by Lemma 7.5) is universal, showing that I maps $Z(1)$ into UX.

Given an $h \in Z(1)$ with $\|h\| \le 1$, we observe that $\|\mathbf{1}_{\Delta(0)}\| = \|\mathbf{1}_{\Delta(0)} \pm h\| = 1$, and so $\|f\|_I = \|f \pm Ih\|_I = 1$. By Lemma 7.6, we have $Ih \in UX$. ∎

LEMMA 7.8. *We have*

$$\|f\|_I \ge \|P_d f\|_I + \frac{1}{2^{2d+7}}\|f - P_d f\|_I, \quad f \in F,\ d \in \mathbb{N}.$$

Proof. By Lemma 6.8,

$$\begin{aligned}\|f\|^2 - \|P_d f\|^2 &= (\|f\| + \|P_d f\|)(\|f\| - \|P_d f\|)\\ &\ge (\|f\| + \|P_d f\|) \cdot \frac{1}{2^{2d+4}}\|f - P_d f\|.\end{aligned}$$

At the same time, by Lemma 7.3,

$$\begin{aligned}\beta(P_d f)^2 - \beta(f)^2 &= \big(\beta(P_d f) + \beta(f)\big)\big(\beta(P_d f) - \beta(f)\big)\\ &\le \big(\beta(P_d f) + \beta(f)\big) \cdot \beta(f - P_d f)\\ &\le 2(\|P_d f\| + \|f\|) \cdot \frac{2}{2^{2d}}\|f - P_d f\|.\end{aligned}$$

Thus, using (28), we can compute

$$\begin{aligned}\|f\|_I^2 - \|P_d f\|_I^2 &= \|f\|^2 - \|P_d f\|^2 + \frac{1}{2^7}\big(\beta(f)^2 - \beta(P_d f)^2\big)\\ &\ge \left(\frac{1}{2^{2d+4}} - \frac{1}{2^7}\cdot\frac{4}{2^{2d}}\right)(\|f\| + \|P_d f\|) \cdot \|f - P_d f\|\\ &= \frac{1}{2^{2d+5}}(\|f\| + \|P_d f\|) \cdot \|f - P_d f\|\\ &\ge \frac{1}{2^{2d+5}} \cdot \frac{1}{2}(\|f\|_I + \|P_d f\|_I) \cdot \frac{1}{2}\|f - P_d f\|_I.\end{aligned}$$

Now, it is sufficient to divide both sides by $\|f\|_I + \|P_d f\|_I$. ∎

8. Renorming II

Definition 8.1. We define a seminorm by

$$\alpha(f)^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n,k)}} |e^*_{(n,k)}(f)|^2, \quad f \in F, \tag{29}$$

where $e^*_{(n,k)}$ is the system biorthogonal to the basic system $e_{(n,k)}$.

Lemma 8.2. *We have*

$$\alpha(f) < \|f\|, \quad 0 \neq f \in F.$$

Proof. Using (25), we can compute

$$\alpha(f)^2 \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n,k)}} \left(2 \cdot \frac{8}{7}\|f\|\right)^2 = \frac{1}{15}\left(2 \cdot \frac{8}{7}\|f\|\right)^2 < \|f\|^2. \blacksquare$$

Fact 8.3. *There is a norm ϱ on $\mathbb{R}^3$ such that*

- $\frac{1}{2}(|r| + |s|) \le \varrho(r, s, t) \le \max\{|r|, |s|, |t|\}$ *and, in particular, the unit sphere contains the line segment* $[(1, 1, -1), (1, 1, 1)]$,
- $\varrho(r', s', t') \ge \varrho(r, s, t)$ *for* $0 \le r \le r', 0 \le s \le s', 0 \le t \le t'$,
- $\varrho(r, s, t') > \varrho(r, s, t)$ *for* $0 < r < s, 0 < t < t'$,
- $\varrho(r', s, t) \ge \varrho(r, s, t) + \frac{1}{4}(r' - r)$ *for* $r, r', s, t > 0$, $0 < r < r'$.

Proof (sketch). Let a norm ϱ_0 be given by

$$B_{(\mathbb{R}^3, \varrho_0)} = \mathrm{co}\big(\{(\pm 1, \pm 1, \pm 1)\} \cup \sqrt{2}\, B\big),$$

where B stands for the Euclidean unit ball of $\mathbb{R}^3$. This norm satisfies the first three properties, and the norm

$$\varrho(r, s, t) = \tfrac{1}{4}(|r| + |s|) + \tfrac{1}{2}\varrho_0(r, s, t)$$

also satisfies the fourth one. $\blacksquare$

Definition 8.4. We define

$$\|f\|_{II} = \varrho\big(\|f\|, \|f\|_I, \alpha(f)\big), \quad f \in F. \tag{30}$$

A simple application of (28) and Lemma 8.2 gives

$$\|f\| \le \|f\|_{II} \le 2\|f\|, \tag{31}$$

since

$$\begin{aligned} \|f\| \le \tfrac{1}{2}(\|f\| + \|f\|_I) &\le \varrho\big(\|f\|, \|f\|_I, \alpha(f)\big) \\ &\le \max\big\{\|f\|, \|f\|_I, \alpha(f)\big\} = \|f\|_I \le 2\|f\|. \end{aligned}$$

Lemma 8.5. *We have $\|Ux\|_{II} = \|x\|_X$ for $x \in X$ and the range of U is 1-complemented in $(F, \|\cdot\|_{II})$.*

Proof. Let $x \in X$ with $\|x\|_X = 1$. By Lemmas 8.2, 6.7 and 7.5, we have $\alpha(Ux) < \|Ux\| = 1 = \|Ux\|_I$. Since the unit sphere $S_{(\mathbb{R}^3, \varrho)}$ contains the line segment $[(1,1,-1),(1,1,1)]$, we obtain $\|Ux\|_{II} = 1 = \|x\|_X$. The projection UT still works, because $\|UTf\|_{II} = \|Tf\|_X = \|UTf\| \le \|f\| \le \|f\|_{II}$ for $f \in F$. ∎

LEMMA 8.6. *Let $[u,v]$ be a non-degenerate line segment in F such that $\|\cdot\|_{II}$ is constant on $[u,v]$. Then u and v belong to UX.*

Proof. It is enough to show that $w = \frac{1}{2}(u+v) \in UX$ (the argument can be repeated for any subsegment of $[u,v]$). Assume the opposite, i.e., $w \notin UX$. We have $\beta(w) > 0$ by Fact 7.2, and so $\|w\| < \|w\|_I$. Using the inequality

$$\alpha(w) < \tfrac{1}{2}\big(\alpha(u) + \alpha(v)\big),$$

a property of ϱ provides

$$\varrho\big(\|w\|, \|w\|_I, \tfrac{1}{2}(\alpha(u)+\alpha(v))\big) > \varrho\big(\|w\|, \|w\|_I, \alpha(w)\big) = \|w\|_{II}.$$

The computation

$$\begin{aligned}\tfrac{1}{2}(\|u\|_{II} + \|v\|_{II}) &= \tfrac{1}{2}\big(\varrho(\|u\|, \|u\|_I, \alpha(u)) + \varrho(\|v\|, \|v\|_I, \alpha(v))\big)\\ &\ge \varrho\big(\tfrac{1}{2}(\|u\| + \|v\|), \tfrac{1}{2}(\|u\|_I + \|v\|_I), \tfrac{1}{2}(\alpha(u)+\alpha(v))\big)\\ &\ge \varrho\big(\|w\|, \|w\|_I, \tfrac{1}{2}(\alpha(u)+\alpha(v))\big) > \|w\|_{II}\end{aligned}$$

concludes the proof. ∎

PROPOSITION 8.7. *If X is strictly convex, then so is $(F, \|\cdot\|_{II})$.*

Proof. This follows from Lemmas 8.5 and 8.6. ∎

LEMMA 8.8. *We have*

$$\|f\|_{II} \ge \|P_d f\|_{II} + \frac{1}{2^{2d+7}}\|f - P_d f\|_{II}, \quad f \in F,\ d \in \mathbb{N}.$$

Proof. Using Lemmas 6.8 and 7.8, we can compute

$$\begin{aligned}\|f\|_{II} &= \varrho\big(\|f\|, \|f\|_I, \alpha(f)\big)\\ &\ge \varrho\bigg(\|P_d f\| + \frac{1}{2^{2d+4}}\|f - P_d f\|, \|P_d f\|_I + \frac{1}{2^{2d+7}}\|f - P_d f\|_I, \alpha(f)\bigg)\\ &\ge \varrho\bigg(\|P_d f\| + \frac{1}{2^{2d+4}}\|f - P_d f\|, \|P_d f\|_I, \alpha(P_d f)\bigg)\\ &\ge \varrho\big(\|P_d f\|, \|P_d f\|_I, \alpha(P_d f)\big) + \frac{1}{4}\cdot\frac{1}{2^{2d+4}}\|f - P_d f\|\\ &\ge \|P_d f\|_{II} + \frac{1}{4}\cdot\frac{1}{2^{2d+4}}\cdot\frac{1}{2}\|f - P_d f\|_{II}.\end{aligned}$$ ∎

9. Amalgamations of Asplund and reflexive spaces. In the final stage of the proof of Theorem 1.2, we need some further notation. We introduce a coding of all rational Banach spaces whose basis is monotone. This enables us to provide a version of the Pełczyński universal space.

Definition 9.1. We fix a system $\{(Z_\eta, \|\cdot\|_\eta)\}_{\eta\in\mathbb{N}^{<\mathbb{N}}}$ of rational Banach spaces which satisfies the following requirements.

(a) For every η, the basis of Z_η, denoted by $z^\eta_1, \dots, z^\eta_{|\eta|}$, is monotone, consists of $|\eta|$ members, and whenever $\eta \subset \nu$, the space Z_ν is an extension of Z_η in the sense that the basis $z^\eta_1, \dots, z^\eta_{|\eta|}$ is 1-equivalent to $z^\nu_1, \dots, z^\nu_{|\eta|}$.

(b) Every monotone rational extension of Z_η is Z_ν for some $\nu \supset \eta$. More precisely, if Z is a rational space whose basis $z_1, \dots, z_d$ is monotone and such that $z^\eta_1, \dots, z^\eta_{|\eta|}$ is 1-equivalent to $z_1, \dots, z_{|\eta|}$, then there is a $\nu \supset \eta$ with $|\nu| = d$ such that $z^\nu_1, \dots, z^\nu_{|\nu|}$ is 1-equivalent to $z_1, \dots, z_d$.

Definition 9.2. For every $\varphi \in \mathbb{N}^\mathbb{N}$, let $(Z_\varphi, \|\cdot\|_\varphi)$ be a Banach space with a monotone basis $z^\varphi_1, z^\varphi_2, \dots$ such that, for every $\eta \subset \varphi$, the basis $z^\eta_1, \dots, z^\eta_{|\eta|}$ of Z_η is 1-equivalent to $z^\varphi_1, \dots, z^\varphi_{|\eta|}$.

Definition 9.3. Let U be a completion of $c_{00}(\mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\})$ with the norm defined by one of the equivalent formulae

$$\|x\| = \sup_{\nu\in\mathbb{N}^{<\mathbb{N}}} \Big\| \sum_{\eta\subset\nu} x(\eta) z^\nu_{|\eta|} \Big\|_\nu, \tag{32}$$

$$\|x\| = \sup_{\varphi\in\mathbb{N}^{\mathbb{N}}} \Big\| \sum_{\eta\subset\varphi} x(\eta) z^\varphi_{|\eta|} \Big\|_\varphi. \tag{33}$$

Further, let $\varpi : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ be a fixed non-decreasing bijection and let

$$u_i = \mathbf{1}_{\{\varpi(i)\}}, \quad i \in \mathbb{N}. \tag{34}$$

As U is defined according to Definition 3.1, several remarkable properties follow. First of all, $u_1, u_2, \dots$ is a monotone basis of U. If we denote

$$\Delta : \varphi \in \mathbb{N}^\mathbb{N} \mapsto \{\varpi^{-1}((\varphi_1)) < \varpi^{-1}((\varphi_1, \varphi_2)) < \dots\} \subset \mathbb{N}, \tag{35}$$

then, by Fact 3.2, the sequences $\{z^\varphi_n : n \in \mathbb{N}\}$ and $\{u_i : i \in \Delta(\varphi)\}$ are 1-equivalent for every $\varphi \in \mathbb{N}^\mathbb{N}$. The copy $\overline{\mathrm{span}}\{u_i : i \in \Delta(\varphi)\}$ of Z_φ is 1-complemented in U. Moreover, due to Proposition 6.2, every Banach space X with a monotone basis has a 1-complemented isometric copy in Z_φ for some $\varphi \in \mathbb{N}^\mathbb{N}$. It follows that X also has a 1-complemented isometric copy in U.

We note that the space U, including its construction and properties, is fairly similar to the space constructed and studied in [15].

Lemma 9.4. *Let $\mathcal{C}$ be an analytic set of Banach spaces with separable dual. Then there is a $\beta < \omega_1$ such that* $\mathrm{Sz}(\ell_2(X)) \le \beta$ *for every $X \in \mathcal{C}$.*

Proof. It follows from [3, Theorem 4.11 and Proposition 0.1(ii)] that $\sup\{\mathrm{Sz}(X) : X \in \mathcal{C}'\} < \omega_1$ for any analytic set $\mathcal{C}'$ of Banach spaces with separable dual. So, it is sufficient to find an analytic set $\mathcal{C}'$ which contains an isomorphic copy of $\ell_2(X)$ for every $X \in \mathcal{C}$, and every $Y \in \mathcal{C}'$ is isomorphic to $\ell_2(X)$ for some $X \in \mathcal{C}$.

Let us consider an isometry $I : \ell_2(C([0,1])) \to C([0,1])$ and let $\kappa : \mathcal{SE}(C([0,1])) \to \mathcal{SE}(C([0,1]))$ be defined by $\kappa(X) = I(\ell_2(X))$ where $\ell_2(X)$ is considered as a subspace of $\ell_2(C([0,1]))$. As κ is a Borel mapping, $\mathcal{C}' = \kappa(\mathcal{C})$ works. ∎

LEMMA 9.5. *For every $\beta < \omega_1$, the set*

$$\mathcal{A} = \{\varphi \in \mathbb{N}^{\mathbb{N}} : \mathrm{Sz}(Z_\varphi) \le \beta \text{ and } z_1^\varphi, z_2^\varphi, \dots \text{ is shrinking}\} \tag{36}$$

is Borel in $\mathbb{N}^{\mathbb{N}}$.

Proof. By [3, Theorem 5.4(i) and Proposition 0.1(i)], the set

$$\mathcal{B} = \big\{\{i_1 < i_2 < \cdots\} \subset \mathbb{N} : \mathrm{Sz}(\overline{\mathrm{span}}\{u_{i_1}, u_{i_2}, \dots\}) \le \beta \\ \text{and } u_{i_1}, u_{i_2}, \dots \text{ is shrinking}\big\}$$

is Borel in the space of all subsets of $\mathbb{N}$. As Δ is a continuous mapping, it remains to observe that $\mathcal{A} = \Delta^{-1}(\mathcal{B})$. ∎

Proof of Theorem 1.2, part 1 (shrinking basis case). Let $\mathcal{C}$ be an analytic set of Banach spaces such that every member admits a monotone shrinking basis. Let $\beta < \omega_1$ be as in Lemma 9.4 and let $\mathcal{A}$ be given by (36). By Lemma 9.5, $\mathcal{A}$ is Borel, and thus analytic. Notice that Proposition 6.2 guarantees that every $X \in \mathcal{C}$ has a 1-complemented isometric copy in Z_φ for some $\varphi \in \mathcal{A}$.

By Lemma 2.1, there is an unrooted pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $\mathcal{A} = p[T]$ where p denotes the projection on the first coordinate. Let us consider the collection

$$(F_\sigma, \|\cdot\|_\sigma) = (Z_{p(\sigma)}, \|\cdot\|_{p(\sigma)}), \quad f_n^\sigma = z_n^{p(\sigma)}, \quad \sigma \in [T],\ n \in \mathbb{N}.$$

In this way, the collection F_σ, $\sigma \in [T]$, consists of the same spaces as the collection Z_φ, $\varphi \in \mathcal{A}$.

Finally, let E be the space constructed in Definition 3.1 for this collection. This space has the required properties, due to Fact 3.2 and Proposition 3.5. ∎

LEMMA 9.6. *For an analytic set $\mathcal{C}$ of Banach spaces, the set*

$$\mathcal{A} = \{\varphi \in \mathbb{N}^{\mathbb{N}} : Z_\varphi \text{ is isomorphic to } \ell_2(X) \text{ for some } X \in \mathcal{C}\} \tag{37}$$

is analytic in $\mathbb{N}^{\mathbb{N}}$.

Proof. It is easy to show (see the proof of Lemma 9.4) that there is an analytic set $\mathcal{C}'$ which contains an isomorphic copy of $\ell_2(X)$ for every $X \in \mathcal{C}$, and every $Y \in \mathcal{C}'$ is isomorphic to $\ell_2(X)$ for some $X \in \mathcal{C}$. By [3, Theorem 2.3(i)], the saturation

$$\begin{aligned}\mathcal{C}'' &= \{Z \in \mathcal{SE}(C([0,1])) : Z \text{ is isomorphic to some } Y \in \mathcal{C}'\}\\ &= \{Z \in \mathcal{SE}(C([0,1])) : Z \text{ is isomorphic to } \ell_2(X) \text{ for an } X \in \mathcal{C}\}\end{aligned}$$

is analytic.

Let $I : U \to C([0,1])$ be an isometry. It is easy to show that the mapping

$$\zeta : \mathbb{N}^{\mathbb{N}} \to \mathcal{SE}(C([0,1])), \quad \varphi \mapsto \overline{\operatorname{span}}\{I(\mathbf{1}_{\{(\varphi_1)\}}), I(\mathbf{1}_{\{(\varphi_1,\varphi_2)\}}), \ldots\},$$

is Borel. Due to Fact 3.2, the spaces Z_φ and $\zeta(\varphi)$ are isometric. It follows that $\mathcal{A} = \zeta^{-1}(\mathcal{C}'')$, and so $\mathcal{A}$ is analytic. ∎

Proof of Theorem 1.2, part 2 (reflexive case). Let $\mathcal{C}$ be an analytic set of reflexive Banach spaces such that every member has a monotone basis. Let $\mathcal{A}$ be given by (37). By Lemma 9.6, $\mathcal{A}$ is analytic. Proposition 6.2 guarantees that every $X \in \mathcal{C}$ has a 1-complemented isometric copy in Z_φ for some $\varphi \in \mathcal{A}$. At the same time, Z_φ is reflexive for every $\varphi \in \mathcal{A}$.

By Lemma 2.1, there is an unrooted pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $\mathcal{A} = p[T]$ where p denotes the projection on the first coordinate. Let us consider the collection

$$(F_\sigma, \|\cdot\|_\sigma) = (Z_{p(\sigma)}, \|\cdot\|_{p(\sigma)}), \quad f_n^\sigma = z_n^{p(\sigma)}, \quad \sigma \in [T],\ n \in \mathbb{N}.$$

In this way, the collection F_σ, $\sigma \in [T]$, consists of the same spaces as the collection Z_φ, $\varphi \in \mathcal{A}$.

Finally, let A be the space established in Definition 4.3 for this collection. This space admits the required properties, due to Facts 4.4 and 4.5 and Proposition 4.6. ∎

10. Amalgamations of non-universal and rotund spaces

Definition 10.1. Let $\varphi \in \mathbb{N}^{\mathbb{N}}$ and let $z_1^*, z_2^*, \ldots$ denote the dual basic sequence of $z_1^\varphi, z_2^\varphi, \ldots$. Let us define seminorms

$$\alpha(z)^2 = \sum_{i=1}^{\infty} \frac{1}{2^{4i}} |z_i^*(z)|^2 \quad \Bigg(= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n,k)}} |z^*_{\pi^{-1}(n,k)}(z)|^2 \Bigg), \tag{38}$$

$$\beta(z)^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n+1,k)}} |z^*_{\pi^{-1}(n,k)}(z) - 2z^*_{\pi^{-1}(n+1,k)}(z)|^2, \tag{39}$$

where π is introduced in Definition 6.3. Let us further define

$$Z_\varphi^I = \{z \in Z_\varphi : \beta(z) < \infty\}, \quad Z_\varphi^{II} = \{z \in Z_\varphi^I : \alpha(z) < \infty\}, \tag{40}$$

$$\|z\|_{\varphi,I}^2 = \|z\|_\varphi^2 + \frac{1}{2^7}\beta(z)^2, \qquad z \in Z_\varphi^I, \tag{41}$$

$$\|z\|_{\varphi,II} = \varrho\big(\|z\|_\varphi, \|z\|_{\varphi,I}, \alpha(z)\big), \quad z \in Z_\varphi^{II}, \tag{42}$$

where ϱ is the norm given by Fact 8.3.

Definition 10.2. The subspace of $(S_{C([0,1])})^{\mathbb{N}}$ consisting of all normalized monotone basic sequences will be denoted by $\mathcal{M}$.

The following proposition summarizes most of the results of Sections 6–8.

Proposition 10.3. *There exists a Borel mapping $\Theta : \mathcal{M} \to \mathbb{N}^{\mathbb{N}}$ such that, for every $(e_1, e_2, \dots) \in \mathcal{M}$, if we denote $\varphi = \Theta(e_1, e_2, \dots)$ and $X = \overline{\operatorname{span}}\{e_1, e_2, \dots\}$, then:*

(1) *$Z_\varphi^I = Z_\varphi^{II} = Z_\varphi$ and the norms fulfill*
$$\|z\|_\varphi \le \|z\|_{\varphi,I} \le 2\|z\|_\varphi, \quad \|z\|_\varphi \le \|z\|_{\varphi,II} \le 2\|z\|_\varphi, \quad z \in Z_\varphi.$$
(2) *Z_φ^I and Z_φ^{II} each contain a 1-complemented isometric copy of X.*
(3) *If X is not isometrically universal for all separable Banach spaces, then neither is Z_φ^I.*
(4) *If X is strictly convex, then so is Z_φ^{II}.*
(5) *We have*
$$\|P_n z\|_{\varphi,I}^2 \ge \|P_{n-1} z\|_{\varphi,I}^2 + \left(\frac{7}{2^{2n+8}}\right)^2 |z_n^*(z)|^2, \qquad z \in Z_\varphi^I,\ n \in \mathbb{N},$$
$$\|P_n z\|_{\varphi,II}^2 \ge \|P_{n-1} z\|_{\varphi,II}^2 + \left(\frac{7}{2^{2n+8}}\right)^2 |z_n^*(z)|^2, \qquad z \in Z_\varphi^{II},\ n \in \mathbb{N},$$
where $z_1^*, z_2^*, \dots$ is the dual basic sequence and $P_0, P_1, \dots$ is the sequence of partial sum operators associated with $z_1^\varphi, z_2^\varphi, \dots$.

Proof. First, the functions $l_d : \mathcal{M} \to \mathbb{N}$ from Definition 6.4 are Borel. If $l \in \mathbb{N}$, then the set of basic sequences $e_1, e_2, \dots$ for which (18) holds with $|\cdot|_d = |\cdot|_{d,l}$ is closed. Therefore, the set of sequences with $l_d = l$ is the difference of two closed sets.

Now, if a monotone basic sequence $e_1, e_2, \dots$ is given, the properties of the system $\{(Z_\varphi, \|\cdot\|_\varphi)\}_{\varphi \in \mathbb{N}^{\mathbb{N}}}$ together with Lemma 6.5 guarantee that there is a $\varphi = (\varphi_1, \varphi_2, \dots)$ such that $(Z_\varphi, \|\cdot\|_\varphi)$ and $(F, \|\cdot\|)$ coincide, together with their bases. To show that the choice of φ can be Borel, it is sufficient to note that φ can be constructed recursively in such a way that φ_d depends only on $l_1, \dots, l_d$. This is allowed by formula (23) which implies that the norm on $\operatorname{span}\{f_1, \dots, f_d\}$ is determined by the values $l_1, \dots, l_d$.

Let us check the required properties. Notice that the spaces Z_φ^I and Z_φ^{II} coincide with $(F, \|\cdot\|_I)$ and $(F, \|\cdot\|_{II})$. So, the properties easily follow from lemmata and propositions proven above.

Property (1) follows from (28) and (31), and property (2) follows from Lemmas 7.5 and 8.5. Property (3) follows from Proposition 7.7, and property (4) follows from Proposition 8.7. Finally, property (5) needs a little calculation. By Lemma 6.5, we have
$$\tfrac{7}{8}|z_n^*(z)| \le |z_n^*(z)|\, \|z_n^\varphi\|_\varphi = \|P_n z - P_{n-1} z\|_\varphi \le \|P_n z - P_{n-1} z\|_{\varphi,I}.$$

Using Lemma 7.8, we obtain

$$\begin{aligned}\|P_n z\|_{\varphi,I}^2 &\geq \left(\|P_{n-1}z\|_{\varphi,I} + \frac{1}{2^{2(n-1)+7}}\|P_n z - P_{n-1}z\|_{\varphi,I}\right)^2 \\ &\geq \left(\|P_{n-1}z\|_{\varphi,I} + \frac{7}{2^{2n+8}}|z_n^*(z)|\right)^2 \\ &\geq \|P_{n-1}z\|_{\varphi,I}^2 + \left(\frac{7}{2^{2n+8}}\right)^2 |z_n^*(z)|^2.\end{aligned}$$

The proof of the analogous inequality for Z_φ^{II} is the same: we just use Lemma 8.8 instead of Lemma 7.8. ∎

Proof of Theorem 1.2, part 3. Suppose that $\mathcal{C}$ is an analytic set of Banach spaces such that every member has a monotone basis. Let $\mathcal{M}_\mathcal{C}$ be the subset of $\mathcal{M}$ consisting of all normalized monotone bases of members of $\mathcal{C}$. Since the mapping

$$(e_1, e_2, \dots) \mapsto \overline{\operatorname{span}}\{e_1, e_2, \dots\}$$

is Borel, the pre-image $\mathcal{M}_\mathcal{C}$ of $\mathcal{C}$ is analytic. Let Θ be the mapping from Proposition 10.3. Then $\Theta(\mathcal{M}_\mathcal{C})$ is an analytic subset of $\mathbb{N}^\mathbb{N}$. By Lemma 2.1, there is an unrooted pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $\Theta(\mathcal{M}_\mathcal{C}) = p[T]$ where p denotes the projection on the first coordinate. Let us consider the collections

$$(F_\sigma^I, \|\cdot\|_{\sigma,I}) = (Z_{p(\sigma)}^I, \|\cdot\|_{p(\sigma),I}), \quad (F_\sigma^{II}, \|\cdot\|_{\sigma,II}) = (Z_{p(\sigma)}^{II}, \|\cdot\|_{p(\sigma),II}),$$
$$f_n^\sigma = z_n^{p(\sigma)}, \quad \sigma \in [T],\, n \in \mathbb{N}.$$

Finally, let B^I and B^{II} be the spaces constructed in Definition 5.1 for these collections. Note that Proposition 10.3(5) guarantees that the requirement (8) is fulfilled.

B^I and B^{II} each contain a 1-complemented isometric copy of every $X \in \mathcal{C}$. Indeed, a monotone basis of X is contained in $\mathcal{M}_\mathcal{C}$, and so property (2) from Proposition 10.3 is satisfied for some $\varphi \in \Theta(\mathcal{M}_\mathcal{C}) = p[T]$. If we choose a $\sigma \in [T]$ with $p(\sigma) = \varphi$, then X has a 1-complemented isometric copy in F_σ^I and in F_σ^{II}, and it is sufficient to apply Fact 5.2.

If every $X \in \mathcal{C}$ is non-universal [strictly convex], then B^I is non-universal [B^{II} is strictly convex]. Indeed, in that case, property (3) [property (4)] from Proposition 10.3 implies that the spaces Z_φ^I, $\varphi \in \Theta(\mathcal{M}_\mathcal{C})$, [$Z_\varphi^{II}, \varphi \in \Theta(\mathcal{M}_\mathcal{C})$], and so F_σ^I, $\sigma \in [T]$, [$F_\sigma^{II}, \sigma \in [T]$] are non-universal [strictly convex], and it remains to apply Proposition 5.5. ∎

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