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A note on global regularity results for 2D Boussinesq equations with fractional dissipation

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Abstract. We study the Cauchy problem for the two-dimensional (2D) incompressible Boussinesq equations with fractional Laplacian dissipation and thermal diffusion. Invoking the energy method and several commutator estimates, we get a global regularity result for the 2D Boussinesq equations as long as $1-\alpha<\beta<\min\left\{\frac{\alpha}{2},\frac{3\alpha-2}{2\alpha^2-6\alpha+5},\frac{2-2\alpha}{4\alpha-3}\right\}$ with $0.77963\approx\alpha_0<\alpha<1$. This improves on some previous work.

1. Introduction. In this paper we study the Cauchy problem for the 2D incompressible Boussinesq equations with fractional Laplacian dissipation in $\mathbb{R}^2$,

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+\nu\Lambda^\alpha u+\nabla p=\theta e_2, \\ \partial_t\theta+(u\cdot\nabla)\theta+\kappa\Lambda^\beta\theta=0, \\ \nabla\cdot u=0, \\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), \end{cases} \tag{1.1}$$

where $u(x,t)=(u_1(x,t),\,u_2(x,t))$ is a vector field denoting the velocity, $\theta=\theta(x,t)$ is a scalar function denoting the temperature in the context of thermal convection and the density in the modeling of geophysical fluids, p is the scalar pressure and $e_2=(0,1)$. The numbers $\nu,\kappa,\alpha,\beta\geq 0$ are real parameters. The fractional Laplacian operator Λ^α, $\Lambda:=(-\Delta)^{1/2}$, denotes the Zygmund operator which is defined through the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi)=|\xi|^\alpha\hat{f}(\xi), \quad \text{where} \quad \hat{f}(\xi)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{-ix\cdot\xi}f(x)\,dx.$$

The fractional Laplacian models many physical phenomena such as overdriven detonations in gases [Cla]. It is also used in some mathematical mod-

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els in hydrodynamics, molecular biology and finance mathematics (see for instance [DI]).

Actually, the standard 2D Boussinesq equations (that is, $\alpha = \beta = 2$) model geophysical flows such as atmospheric fronts and oceanic circulation, and play an important role in the study of Rayleigh–Benard convection (see for example [MB, Ped] and references therein). Moreover, there are some geophysical circumstances related to the Boussinesq equations with fractional Laplacian (see [Cap, Ped] for details). The Boussinesq equations with fractional Laplacian are also closely related to equations such as the surface quasi-geostrophic equation modeling important geophysical phenomena (see, e.g., [CMT]). The standard 2D Boussinesq equations and their fractional Laplacian generalizations have attracted considerable attention recently due to their physical applications and mathematical significance. Obviously, for $\nu = \kappa = 0$, system (1.1) reduces to the inviscid Boussinesq equations, for which global well-posedness of smooth solutions is an outstanding open problem in fluid dynamics (except if θ_0 is a constant, of course) which may be formally compared to the similar problem for the three-dimensional axisymmetric Euler equations with swirl (see [MB]). In contrast, in the case $\alpha = \beta = 2$, global well-posedness has been shown previously (see for example, [CD]). There are a large number of works devoted to intermediate cases, such as fractional dissipation, partial anisotropic dissipation and so on. Global regularity results for system (1.1) for the cases when $\alpha = 2$ and $\kappa = 0$ or $\beta = 2$ and $\mu = 0$ were established by Chae [Cha] and by Hou and Li [HL] independently. By deeply developing the structures of the coupling system, Hmidi, Keraani and Rousset [HKR1, HKR2] were able to establish global well-posedness for (1.1) in two special critical cases, namely [$\alpha = 1$ and $\kappa = 0$] and [$\beta = 1$ and $\mu = 0$]. The more general critical case $\alpha + \beta = 1$ with $0 < \alpha, \beta < 1$ is extremely difficult. Very recently, global regularity of the general critical case $\alpha + \beta = 1$ with $\alpha > (23 - \sqrt{145})/12 \approx 0.9132$ and $0 < \beta < 1$ was examined by Jiu, Miao, Wu and Zhang [JMWZ]. This result was further improved by Stefanov and Wu [SW], which requires $\alpha + \beta = 1$ with $\alpha > (\sqrt{1777} - 23)/24 \approx 0.798$ and $0 < \beta < 1$.

Here we mention that even in the subcritical ranges, $\alpha + \beta > 1$ with $0 < \alpha, \beta < 1$, the global regularity of (1.1) is also definitely nontrivial and quite difficult. Actually, to the best of our knowledge there are only a few works concerning the subcritical cases [CV, MX, YJW, YX1, YX2, YXX]. More precisely, Miao and Xue [MX] obtained global regularity for system (1.1) for $\nu, \kappa > 0$ and

$$\frac{6 - \sqrt{6}}{4} < \alpha < 1, \quad 1 - \alpha < \beta < \min\left\{\frac{7 + 2\sqrt{6}}{5}\alpha - 2, \frac{\alpha(1 - \alpha)}{\sqrt{6} - 2\alpha}, 2 - 2\alpha\right\}.$$

This range was improved in [YXX] to $\nu, \kappa > 0$ and

$$\frac{21-\sqrt{217}}{8} < \alpha < 1, \quad 1-\alpha < \beta < \min\left\{\frac{\alpha}{2}, \frac{(3\alpha-2)(\alpha+2)}{10-7\alpha}, \frac{2-2\alpha}{4\alpha-3}\right\}.$$

In addition, Constantin and Vicol [CV] verified global regularity for (1.1) in the case when thermal diffusion dominates, namely

$$\nu, \kappa > 0, \quad 0 < \alpha, \beta < 2, \quad \beta > \frac{2}{2+\alpha}.$$

Recently, Yang, Jiu and Wu [YJW] proved global regularity for (1.1) with $\nu, \kappa > 0$ and

$$0 < \alpha, \beta < 1, \quad \beta > 1 - \frac{\alpha}{2}, \quad \beta \geq \frac{2+\alpha}{3}, \quad \beta > \frac{10-5\alpha}{10-4\alpha}.$$

The results of [CV, YJW] have been improved by two recent manuscripts [YX1, YX2]. In particular, in [YX2], we proved global well-posedness for (1.1) with $n, \kappa > 0$, $0 < \alpha, \beta < 1$, and

$$\beta > \beta^*,$$

where

$$\beta^* := \begin{cases} \max\left\{\frac{2}{3}, \frac{4-\alpha^2}{4+3\alpha}, \frac{1}{1+\alpha}\right\}, & 0 < \alpha \leq 2/3, \\ \frac{2-\alpha}{2}, & 2/3 \leq \alpha < 1. \end{cases}$$

It is also worth mentioning that there are numerous studies concerning the Boussinesq equations with partial anisotropic dissipation (see for example [ACW1, ACW2, AC$^+$, DP, CaW, LLT]). Many other interesting recent results on the Boussinesq equations can be found in e.g. [ACWX, ChW, CDJ, Dan, HH, Hmi, JPL, JWY, KRTW, LT, LMZ, WX, WXY, Xu, Ye2, Ye3, Ye1] and in the references therein (the list with no intention to be complete).

2. Theorem. This paper continues the previous two works [MX, YXX]. Since the concrete values of the constants ν, κ play no role in our discussion, we shall assume $\nu = \kappa = 1$ throughout. The following theorem is our main result.

THEOREM 2.1. *Suppose that* $0.77963 \approx \alpha_0 < \alpha < 1$ *and* $0 < \beta < 1$ *obey*

$$1-\alpha < \beta < \min\left\{\frac{\alpha}{2}, \frac{3\alpha-2}{2\alpha^2-6\alpha+5}, \frac{2-2\alpha}{4\alpha-3}\right\}. \tag{2.1}$$

Let $(u_0, \theta_0) \in H^\sigma(\mathbb{R}^2) \times H^\sigma(\mathbb{R}^2)$ *with* $\sigma > 2$. *Then system* (1.1) *admits a unique global solution such that for any* $T > 0$,

$$u \in C([0,T]; H^\sigma(\mathbb{R}^2)) \cap L^2([0,T]; H^{\sigma+\alpha/2}(\mathbb{R}^2)),$$
$$\theta \in C([0,T]; H^\sigma(\mathbb{R}^2)) \cap L^2([0,T]; H^{\sigma+\beta/2}(\mathbb{R}^2)).$$

REMARK 2.2. Here the number α_0 is explicitly given as

$$\alpha_0 = \frac{8 - (\sqrt[3]{6\sqrt{609}+118} - \sqrt[3]{6\sqrt{609}-118})}{6} \approx 0.77963.$$

According to the well-known Shengjin's formulas [Fan], it is easy to show that α_0 is a unique real solution to the cubic equation

$$2\alpha^3 - 8\alpha^2 + 14\alpha - 7 = 0.$$

REMARK 2.3. The condition $\alpha > \alpha_0 \approx 0.77963$ is weaker than in [MX, YXX], where the corresponding conditions are $\alpha > (6-\sqrt{6})/4 \approx 0.887627$ and $\alpha > (21-\sqrt{217})/8 \approx 0.783635$, respectively.

REMARK 2.4. For technical reasons, the β should be smaller than a complicated explicit function. As a matter of fact, it is strongly believed that the diffusion term is always good and the larger the power β, the better effect it produces. Therefore, we conjecture that the above theorem should hold for all the cases $\alpha_0 < \alpha < 1$ and $1-\alpha < \beta < 1$.

3. The proof of Theorem 2.1. First, the local well-posedness of system (1.1) for smooth initial data is well-known (see for example [MB]), and therefore it suffices to prove the global in time *a priori* estimate on $[0,T]$ for any given $T > 0$. In this paper, all constants will be denoted by C, which is a generic constant depending only on the quantities specified in the context.

Thanks to the basic energy estimates, we obtain immediately

$$\sup_{0\le t\le T} \|\theta(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{\beta/2}\theta(\tau)\|_{L^2}^2\,d\tau \le \|\theta_0\|_{L^2}, \tag{3.1}$$

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}, \quad \forall p \in [2,\infty],$$

$$\sup_{0\le t\le T} \|u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{\alpha/2}u(\tau)\|_{L^2}^2\,d\tau \le C(T,u_0,\theta_0). \tag{3.2}$$

Applying the curl operator to $(1.1)_1$, we can show that the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ satisfies

$$\partial_t\omega + (u\cdot\nabla)\omega + \Lambda^\alpha\omega = \partial_x\theta. \tag{3.3}$$

The "vortex stretching" term $\partial_x\theta$ prevents us from proving any global bound for ω. To overcome this difficulty, a natural idea is to eliminate the term $\partial_x\theta$ from the vorticity equation. This method was first introduced by Hmidi, Keraani and Rousset [HKR1, HKR2] to treat the Boussinesq equations in critical cases. Now we set $\mathcal{R}_\alpha$ to be the singular integral operator

$$\mathcal{R}_\alpha := \partial_x\Lambda^{-\alpha}.$$

Then we can show that the new quantity $G = \omega - \mathcal{R}_\alpha\theta$ satisfies

$$\partial_t G + (u\cdot\nabla)G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u\cdot\nabla]\theta + \Lambda^{\beta-\alpha}\partial_x\theta; \tag{3.4}$$

here and below the standard commutator notation is used,

$$[\mathcal{R}_\alpha, u\cdot\nabla]\theta := \mathcal{R}_\alpha(u\cdot\nabla\theta) - u\cdot\nabla\mathcal{R}_\alpha\theta.$$

The above equation is very important in our analysis in order to derive some crucial *a priori* estimates. Moreover, the velocity field u can be decomposed into the following two parts:

$$u = \nabla^\perp\Delta^{-1}\omega = \nabla^\perp\Delta^{-1}G + \nabla^\perp\Delta^{-1}\mathcal{R}_\alpha\theta =: u_G + u_\theta.$$

To prove our main result, we need some lemmas. The first lemma gives a commutator estimate.

LEMMA 3.1 (see [YX2]). *Let* $p \in [2,\infty)$, $r \in [1,\infty]$, $\delta \in (0,1)$, *and* $s \in (0,1)$ *with* $s+\delta < 1$. *Then*

$$\|[\Lambda^\delta, f]g\|_{B^s_{p,r}} \le C(p,r,\delta,s)(\|\nabla f\|_{L^p}\|g\|_{B^{s+\delta-1}_{\infty,r}} + \|f\|_{L^2}\|g\|_{L^2}). \tag{3.5}$$

Here and in what follows, $B^s_{p,r}$ *denotes the standard Besov space.*

We also need the following commutator estimate involving $\mathcal{R}_\alpha$, which was established by Stefanov and Wu [SW].

LEMMA 3.2. *Assume that* $1/2 < \alpha < 1$ *and* $1 < p_2 < \infty$, $1 < p_1$, $p_3 \le \infty$ *with* $1/p_1 + 1/p_2 + 1/p_3 = 1$. *Then for* $0 \le s_1 < 1-\alpha$ *and* $s_1 + s_2 > 1-\alpha$,

$$\Big|\int_{\mathbb{R}^2} F[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\,dx\Big| \le C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}}\|G\|_{L^{p_3}}. \tag{3.6}$$

Similarly, for $0 \le s_1 < 1-\alpha$ *and* $s_1+s_2 > 2-2\alpha$,

$$\Big|\int_{\mathbb{R}^2} F[\mathcal{R}_\alpha, u_\theta\cdot\nabla]H\,dx\Big| \le C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}}\|H\|_{L^{p_3}}. \tag{3.7}$$

Here and in what follows, $W^{s,p}$ *denotes the standard Sobolev space.*

The following lemma contains bilinear estimates.

LEMMA 3.3. *Let* $2 < m < \infty$, $0 < s < 1$ *and* $p,q,r \in (1,\infty)^3$ *with* $1/p = 1/q + 1/r$. *Then*

$$\begin{aligned}\|\Lambda^s(|f|^{m-2}f)\|_{L^p} &\le C\|f\|_{\dot B^s_{q,p}}\|f\|^{m-2}_{L^{r(m-2)}},\\ \big\||f|^{m-2}f\big\|_{W^{s,p}} &\le C\|f\|_{B^s_{q,p}}\|f\|^{m-2}_{L^{r(m-2)}}.\end{aligned} \tag{3.8}$$

Proof. One can find the proof in [YXX]; we only sketch it for convenience. Let us recall the following characterization of $\dot W^{s,p}$ with $0 < s < 1$:

$$\big\||f|^{m-2}f\big\|^p_{\dot W^{s,p}} \approx \int_{\mathbb{R}^2} \frac{\big\||f|^{m-2}f(x+\cdot) - |f|^{m-2}f(\cdot)\big\|^p_{L^p}}{|x|^{2+sp}}\,dx.$$

Note that the simple inequality

$$\big||a|^{m-2}a-|b|^{m-2}b\big| \le C(m)|a-b|(|a|^{m-2}+|b|^{m-2})$$

and the Hölder inequality result in

$$\begin{aligned}\big\||f|^{m-2}f(x+\cdot)-|f|^{m-2}f(\cdot)\big\|_{L^p} &\le C\|f(x+\cdot)-f(\cdot)\|_{L^q}\big\||f|^{m-2}\big\|_{L^r}\\ &\le C\|f(x+\cdot)-f(\cdot)\|_{L^q}\|f\|_{L^{r(m-2)}}^{m-2}.\end{aligned}$$

Thus, it follows from the characterization of Besov space that

$$\begin{aligned}\|\Lambda^s(|f|^{m-2}f)\|_{L^p}^p &\le C\int_{\mathbb{R}^2}\frac{\|f(x+\cdot)-f(\cdot)\|_{L^q}^p\|f\|_{L^{r(m-2)}}^{(m-2)p}}{|x|^{2+sp}}\,dx\\ &\le C\|f\|_{L^{r(m-2)}}^{(m-2)p}\int_{\mathbb{R}^2}\frac{\|f(x+\cdot)-f(\cdot)\|_{L^q}^p}{|x|^{2+sp}}\,dx\\ &\le C\|f\|_{L^{r(m-2)}}^{(m-2)p}\|f\|_{\dot{B}_{q,p}^s}^p.\end{aligned}$$

The Hölder inequality directly gives

$$\big\||f|^{m-2}f\big\|_{L^p} \le C\|f\|_{L^q}\big\||f|^{m-2}\big\|_{L^r} = C\|f\|_{L^q}\|f\|_{L^{r(m-2)}}^{m-2}.$$

This concludes the proof of the lemma. ∎

With the above lemmas in hand, we turn to the proof of the main result. First we will derive the following estimate concerning the temperature θ and G, which plays an important role in proving the main theorem and is also the main difference from [YXX].

Lemma 3.4. *Under the assumptions of Theorem* 2.1, *let* (u,θ) *be the corresponding solution of* (1.1). *If* $\beta > 1-\alpha$ *and* $\alpha > 2/3$, *then for* $\max\{(2-2\alpha-\beta)/2,(2+\beta-3\alpha)/2\} < \delta < \beta/2$,

$$\sup_{0\le t\le T}(\|G(t)\|_{L^2}^2+\|\Lambda^\delta\theta(t)\|_{L^2}^2) + \int_0^T(\|\Lambda^{\alpha/2}G\|_{L^2}^2+\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2)(\tau)\,d\tau \le C(T,u_0,\theta_0), \tag{3.9}$$

where $C(T,u_0,\theta_0)$ *is a constant depending on* T *and the initial data.*

Remark 3.5. Although the above estimate (3.9) is stated with the restriction $\max\{(2-2\alpha-\beta)/2,(2+\beta-3\alpha)/2\} < \delta < \beta/2$, by the energy estimate (3.1) and classical interpolation, we find that (3.9) is actually true for any $0\le\delta<\beta/2$.

Proof of Lemma 3.4. Applying Λ^δ ($\delta>0$ to be fixed later) to $(1.1)_2$, then multiplying it by $\Lambda^\delta\theta$, after integration by parts, we find that

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\delta\theta(t)\|_{L^2}^2+\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2 = -\int_{\mathbb{R}^2}\Lambda^\delta(u\cdot\nabla\theta)\Lambda^\delta\theta\,dx. \tag{3.10}$$

Hence, an application of the divergence-free condition and commutator estimate (3.5) directly gives rise to

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} \Lambda^\delta(u\cdot\nabla\theta)\Lambda^\delta\theta\,dx\Big| &= \Big|\int_{\mathbb{R}^2}[\Lambda^\delta, u\cdot\nabla]\theta\Lambda^\delta\theta\,dx\Big| = \Big|\int_{\mathbb{R}^2}\nabla\cdot[\Lambda^\delta,u]\theta\Lambda^\delta\theta\,dx\Big|\\ &\le C\|\Lambda^{1-\beta/2}[\Lambda^\delta,u]\theta\|_{L^2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2} \le C\|[\Lambda^\delta,u]\theta\|_{H^{1-\beta/2}}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\le C\|[\Lambda^\delta,u]\theta\|_{B^{1-\beta/2}_{2,2}}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\le C\big(\|\nabla u\|_{L^2}\|\theta\|_{B^{\delta-\beta/2}_{\infty,2}} + \|u\|_{L^2}\|\theta\|_{L^2}\big)\|\Lambda^{\delta+\beta/2}\theta\|_{L^2} := I,\end{aligned}$$

where in the last inequality, the number δ should satisfy $\delta < \beta/2$. Making use of the Besov embedding and the Gagliardo–Nirenberg inequality, we get

$$\begin{aligned} I &\le C\|\omega\|_{L^2}\|\theta\|_{L^\infty}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2} + C\|u\|_{L^2}\|\theta\|_{L^2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\le C(\|G\|_{L^2}+\|\mathcal{R}_\alpha\theta\|_{L^2})\|\theta\|_{L^\infty}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2} + C\|u\|_{L^2}\|\theta\|_{L^2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\le C\|\theta\|_{L^\infty}\|G\|_{L^2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2} + C\|\theta\|_{L^\infty}\|\Lambda^{1-\alpha}\theta\|_{L^2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\quad + C\|u\|_{L^2}\|\theta\|_{L^2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\le C\|\theta\|_{L^\infty}\|G\|_{L^2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\quad + C\|\theta\|_{L^\infty}\|\theta\|_{L^2}^{2\delta+\beta+2\alpha-2/2\delta+\beta}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^{2-2\alpha/2\delta+\beta}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\quad + C\|u\|_{L^2}\|\theta\|_{L^2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\\ &\le \tfrac{1}{2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2 + C(\|\theta\|_{L^\infty}^{2(2\delta+\beta)/2\delta+\beta+2\alpha-2} + \|u\|_{L^2}^2)\|\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|G\|_{L^2}^2.\end{aligned}$$

Here we have applied the following facts:

$$L^\infty \hookrightarrow B^{\delta-\beta/2}_{\infty,2} \quad \text{and} \quad \|\Lambda^{1-\alpha}\theta\|_{L^2(\mathbb{R}^2)} \le C\|\theta\|_{L^2(\mathbb{R}^2)}^{2\delta+\beta+2\alpha-2/2\delta+\beta}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2(\mathbb{R}^2)}^{\frac{2-2\alpha}{2\delta+\beta}},$$

which hold true for $\delta < \beta/2$ and $\delta > (2-2\alpha-\beta)/2$ ($\delta > (2-2\alpha-\beta)/2 \Rightarrow \frac{2-2\alpha}{2\delta+\beta} < 1$), respectively. Consequently,

$$\begin{aligned}(3.11)\qquad \Big|\int_{\mathbb{R}^2}\Lambda^\delta(u\cdot\nabla\theta)\Lambda^\delta\theta\,dx\Big| &\le \tfrac{1}{2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2\\ &\quad + C(\|\theta\|_{L^\infty}^{\frac{2(2\delta+\beta)}{2\delta+\beta+2\alpha-2}} + \|u\|_{L^2}^2)\|\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|G\|_{L^2}^2.\end{aligned}$$

Substituting the above estimate into (3.10), we arrive at

$$\begin{aligned}(3.12)\qquad \frac{d}{dt}\|\Lambda^\delta\theta(t)\|_{L^2}^2 &+ \|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2\\ &\le C(\|\theta\|_{L^\infty}^{\frac{2(2\delta+\beta)}{2\delta+\beta+2\alpha-2}} + \|u\|_{L^2}^2)\|\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|G\|_{L^2}^2.\end{aligned}$$

Now we test equation (3.4) with G, integrate the resulting inequality with respect to x and make use of the divergence-free condition to obtain

$$\frac{1}{2}\frac{d}{dt}\|G(t)\|_{L^2}^2+\|\Lambda^{\alpha/2}G\|_{L^2}^2 = \int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u\cdot\nabla]\theta\, G\,dx+\int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_x\theta\, G\,dx. \tag{3.13}$$

We easily deduce from the Gagliardo–Nirenberg inequality and the Young inequality that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_x\theta\, G\,dx\Big| &\le C\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\|\Lambda^{1+\beta/2-\alpha-\delta}G\|_{L^2}\\ &\le C\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}\|G\|_{L^2}^{(3\alpha+2\delta-2-\beta)/\alpha}\|\Lambda^{\alpha/2}G\|_{L^2}^{(2+\beta-2\alpha-2\delta)/\alpha}\\ &\le \tfrac{1}{4}\|\Lambda^{\alpha/2}G\|_{L^2}^2+\tfrac{1}{4}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2+C\|G\|_{L^2}^2,\end{aligned} \tag{3.14}$$

where in the second line, we have used the Gagliardo–Nirenberg inequality

$$\|\Lambda^{1+\beta/2-\alpha-\delta}G\|_{L^2}\le C\|G\|_{L^2}^{(3\alpha+2\delta-2-\beta)/\alpha}\|\Lambda^{\alpha/2}G\|_{L^2}^{(2+\beta-2\alpha-2\delta)/\alpha},$$

for $(2+\beta-3\alpha)/2<\delta<(2+\beta-2\alpha)/2$.

Observing the decomposition $u=u_G+u_\theta$, we get

$$\int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u\cdot\nabla]\theta\, G\,dx=\int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\, G\,dx+\int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u_\theta\cdot\nabla]\theta\, G\,dx.$$

Let us use the estimate (3.6) with $s_1=0$ to control the above first term as

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\, G\,dx\Big| &\le C\|\theta\|_{L^\infty}\|G\|_{L^2}\|G\|_{H^{s_2}} \quad (s_2>1-\alpha)\\ &\le C\|\theta\|_{L^\infty}\|G\|_{L^2}\|G\|_{H^{\alpha/2}} \quad (s_2\le\alpha/2)\\ &\le \tfrac{1}{8}\|\Lambda^{\alpha/2}G\|_{L^2}^2+C(1+\|\theta\|_{L^\infty}^2)\|G\|_{L^2}^2.\end{aligned} \tag{3.15}$$

To estimate the second term, we can apply (3.7) with $s_2=\alpha/2$ to conclude that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u_\theta\cdot\nabla]\theta\, G\,dx\Big| &\le C\|\theta\|_{L^\infty}\|\theta\|_{H^{s_1}}\|G\|_{H^{\alpha/2}}\\ &\le C\|\theta\|_{L^\infty}\|\theta\|_{L^2}^{\frac{2\delta+\beta-2s_1}{2\delta+\beta}}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^{\frac{2s_1}{2\delta+\beta}}\|G\|_{H^{\alpha/2}}\\ &\le \tfrac{1}{8}\|\Lambda^{\alpha/2}G\|_{L^2}^2+\tfrac{1}{4}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2+C\|\theta\|_{L^\infty}^{\frac{4\delta+2\beta}{2\delta+\beta-2s_1}}\|\theta\|_{L^2}^2,\end{aligned} \tag{3.16}$$

where in the first and second lines, the number s_1 should satisfy

$$\max\{0,(4-5\alpha)/2\}<s_1<\min\{1-\alpha,\delta+\beta/2\},$$

which can be ensured by choosing $\delta>(4-5\alpha-\beta)/2$ and $\alpha>2/3$.

Inserting (3.14)–(3.16) into (3.13), we conclude

$$(3.17)\qquad \frac{1}{2}\frac{d}{dt}\|G(t)\|_{L^2}^2+\frac{1}{2}\|\Lambda^{\alpha/2}G\|_{L^2}^2 \le \frac{1}{2}\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2+C(1+\|\theta\|_{L^\infty}^2)\|G\|_{L^2}^2+C\|\theta\|_{L^\infty}^{\frac{4\delta+2\beta}{2\delta+\beta-2s_1}}\|\theta\|_{L^2}^2.$$

By putting (3.12) and (3.17) together, we finally get

$$(3.18)\qquad \frac{d}{dt}(\|G(t)\|_{L^2}^2+\|\Lambda^{\delta}\theta(t)\|_{L^2}^2)+\|\Lambda^{\alpha/2}G\|_{L^2}^2+\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2 \le C(1+\|\theta\|_{L^\infty}^{\frac{2(2\delta+\beta)}{2\delta+\beta+2\alpha-2}}+\|\theta\|_{L^\infty}^{\frac{4\delta+2\beta}{2\delta+\beta-2s_1}}+\|u\|_{L^2}^2)\|\theta\|_{L^2}^2 + C(1+\|\theta\|_{L^\infty}^2)\|G\|_{L^2}^2$$

for any δ satisfying

$$\max\{(2-2\alpha-\beta)/2,(2+\beta-3\alpha)/2,(4-5\alpha-\beta)/2\} < \delta < \min\{\beta/2,(\beta+2-2\alpha)/2\}=\beta/2.$$

Observing that $\alpha>2/3 \Rightarrow (4-5\alpha-\beta)/2<(2-2\alpha-\beta)/2$ and $\beta/2<(\beta+2-2\alpha)/2$, the range of δ becomes

$$\max\{(2-2\alpha-\beta)/2,(2+\beta-3\alpha)/2\}<\delta<\beta/2.$$

By the standard Gronwall inequality, we can easily see from (3.18) that

$$\sup_{0\le t\le T}(\|G(t)\|_{L^2}^2+\|\Lambda^{\delta}\theta(t)\|_{L^2}^2)+\int_0^T(\|\Lambda^{\alpha/2}G\|_{L^2}^2+\|\Lambda^{\delta+\beta/2}\theta\|_{L^2}^2)(\tau)\,d\tau\le C.$$

Thus the conclusion is proved. ■

Next we establish the following global *a priori* bound of the L^m norm for G, based on Lemma 3.4. This bound plays a crucial role in proving the main theorem.

Lemma 3.6. *Let $\alpha_0<\alpha<1$ and*

$$1-\alpha<\beta<\min\left\{\frac{\alpha}{2},\frac{3\alpha-2}{2\alpha^2-6\alpha+5},\frac{2-2\alpha}{4\alpha-3}\right\}.$$

Assume that (u_0,θ_0) satisfies the assumptions of Theorem 2.1. Then the function G in (3.4) admits the following bound for any $0\le t\le T$:

$$(3.19)\qquad \|G(t)\|_{L^m}^m+\int_0^T\|G(\tau)\|_{L^{\frac{2m}{2-\alpha}}}^m\,d\tau\le C(T,u_0,\theta_0),$$

where $m=2/(2\alpha-1)+\epsilon$ for some $\epsilon>0$ small enough, which may depend on α and β.

REMARK 3.7. It follows from [YXX] (see also [MX]) that we need the key requirement $m > 2/(2\alpha-1)$, but m can be arbitrarily close to $2/(2\alpha-1)$. Thus it is sufficient to select $m = 2/(2\alpha-1)+\epsilon$ with any $\epsilon > 0$ small enough.

Proof of Lemma 3.6. Recall the fractional version of the Gagliardo–Nirenberg inequality, due to Hajaiej–Molinet–Ozawa–Wang [HMOW]:

$$\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}} \le C\|\Lambda^{\beta/2}\theta\|_{L^2}^{2\gamma}\|\theta\|_{L^\infty}^{1-2\gamma}, \quad 0<\gamma<1/2.$$

In fact, the above inequality is a direct consequence of [HMOW, Theorem 1.2] as well as the equivalence $\dot{W}^{s,p} \approx \dot{B}^s_{p,p}$ for $0 < s \ne \mathbb{N}$ and $1 < p < \infty$. Thanks to (3.9), for any $0<\gamma<1/2$ and $2 \le q < \infty$ we have

$$\begin{aligned}
(3.20)\qquad \int_0^T \|\Lambda^{\gamma\beta}\theta(t)\|_{L^{1/\gamma}}^q\,dt &\le C\|\theta_0\|_{L^\infty}^{(1-2\gamma)q}\int_0^T\|\Lambda^{\beta/2}\theta(t)\|_{L^2}^{2\gamma q}\,dt \\
&\le C\|\theta_0\|_{L^\infty}^{(1-2\gamma)q}\int_0^T\|\Lambda^{\delta}\theta(t)\|_{L^2}^{4\delta\gamma q/\beta}\|\Lambda^{\delta+\beta/2}\theta(t)\|_{L^2}^{2(\beta-2\delta)\gamma q/\beta}\,dt \\
&\le C\|\theta_0\|_{L^\infty}^{(1-2\gamma)q}\sup_{0\le t\le T}\|\Lambda^{\delta}\theta(t)\|_{L^2}^{4\delta\gamma q/\beta}\int_0^T\|\Lambda^{\delta+\beta/2}\theta(t)\|_{L^2}^{2(\beta-2\delta)\gamma q/\beta}\,dt \\
&\le C(T,u_0,\theta_0),
\end{aligned}$$

where in the last line we just take δ such that $\min\{(\beta/2)(1-1/q\gamma),0\} \le \delta < \beta/2$. Multiplying (3.4) by $|G|^{m-2}G$ ($m = 2/(2\alpha-1)+\epsilon$ and $\epsilon > 0$ to be fixed later), after integrating by parts and using the divergence-free condition we obtain

$$\begin{aligned}
(3.21)\qquad \frac{1}{m}\frac{d}{dt}\|G(t)\|_{L^m}^m + \int_{\mathbb{R}^2}(\Lambda^\alpha G)|G|^{m-2}G\,dx
&= \int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u\cdot\nabla]\theta\,|G|^{m-2}G\,dx + \int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_x\theta\,|G|^{m-2}G\,dx \\
&= \int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\,|G|^{m-2}G\,dx + \int_{\mathbb{R}^2}[\mathcal{R}_\alpha, u_\theta\cdot\nabla]\theta\,|G|^{m-2}G\,dx \\
&\quad + \int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_x\theta\,|G|^{m-2}G\,dx.
\end{aligned}$$

We infer from the maximum principle and Sobolev embedding that

$$(3.22)\qquad \int_{\mathbb{R}^2}(\Lambda^\alpha G)|G|^{m-2}G\,dx \ge \widetilde{C}\|\Lambda^{\alpha/2}G^{m/2}\|_{L^2}^2 \ge \widetilde{C}\|G\|_{L^{2m/(2-\alpha)}}^m,$$

where $\widetilde{C} > 0$ is an absolute constant. Taking into account (3.8), we find that

$$(3.23)\quad \Big|\int_{\mathbb{R}^2} \Lambda^{\beta-\alpha}\partial_x\theta|G|^{m-2}G\,dx\Big| \le C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|\Lambda^{1-\alpha+(1-\gamma)\beta}(|G|^{m-2}G)\|_{L^{1/(1-\gamma)}} \le C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{B^{1-\alpha+(1-\gamma)\beta}_{2,\,1/(1-\gamma)}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}} \le C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\alpha/2}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}},$$

where we have used $H^{\alpha/2}\hookrightarrow B^{1-\alpha+(1-\gamma)\beta}_{2,1/(1-\gamma)}$ and $1-\alpha+(1-\gamma)\beta<\alpha/2$, that is,

$$(3.24)\quad \gamma > \frac{2\beta+2-3\alpha}{2\beta}.$$

Now the estimate (3.7) with $s_1=\gamma\beta$ implies that

$$(3.25)\quad \Big|\int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_\theta\cdot\nabla]\theta\,|G|^{m-2}G\,dx\Big| \le C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|\theta\|_{L^\infty}\big\||G|^{m-2}G\big\|_{W^{s_2,1/(1-\gamma)}} \quad (s_2>2-2\alpha-\gamma\beta,\ 0\le\gamma\beta<1-\alpha)$$
$$\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|\theta_0\|_{L^\infty}\|G\|_{B^{s_2}_{2,1/(1-\gamma)}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}} \le C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\alpha/2}}\|G\|^{m-2}_{L^{2(m-2)/(1-2\gamma)}} \quad (s_2<\alpha/2).$$

Now we verify that the number s_2 as above can be found. Indeed, it is sufficient to select γ as follows:

$$(3.26)\quad 0\le\gamma\beta<1-\alpha,\quad 2-2\alpha-\gamma\beta<\alpha/2.$$

According to inequality (3.6) with $s_1=0$ as well as inequality (3.8), this gives

$$(3.27)\quad \Big|\int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\,|G|^{m-2}G\,dx\Big| \le C\|G\|_{L^q}\|\theta\|_{L^\infty}\big\||G|^{m-2}G\big\|_{W^{s_2-\widetilde{\delta}/2,\,p}} \quad (\widetilde{\delta}>0 \text{ is small enough})$$
$$\le C\|\theta_0\|_{L^\infty}\|G\|_{L^q}\|G\|^{m-2}_{L^{(m-2)q/(m-2)}}\|G\|_{B^{s_2-\widetilde{\delta}/2}_{q/(q-(m-1)),\,p}} \le C\|\theta_0\|_{L^\infty}\|G\|^{m-1}_{L^q}\|G\|_{B^{s_2-\widetilde{\delta}/2}_{q/(q-(m-1)),\,p}} \le C\|\theta_0\|_{L^\infty}\|G\|^{m-1}_{L^q}\|G\|_{H^{s_2-1+2(m-1)/q}}.$$

Here the exponents should satisfy

$$s_2 - \widetilde{\delta}/2 > 1 - \alpha, \quad 1/p + 1/q = 1, \quad m - 1 < q \leq 2(m-1),$$

which leads to the embedding $H^{s_2-1+2(m-1)/q} \hookrightarrow B^{s_2-\widetilde{\delta}/2}_{q/(q-(m-1)),\,p}$. Thanks to the requirement $s_2 - \widetilde{\delta}/2 > 1 - \alpha$ in (3.27), we can choose a sufficiently small $\widetilde{\delta} > 0$ (in fact we can take $\widetilde{\delta} \leq (4\alpha - 3)/8$ for example to satisfy all the conditions) such that

$$s_2 = 1 - \alpha + \widetilde{\delta}.$$

By the interpolation inequality

$$\|G\|_{H^{-\alpha+\widetilde{\delta}+2(m-1)/q}} \leq C\|G\|_{L^2}^{1-\mu}\|G\|_{H^{\alpha/2}}^{\mu},$$

where

$$\mu = \frac{-2\alpha + 2\widetilde{\delta} + 4(m-1)/q}{\alpha}, \qquad \frac{4(m-1)}{3\alpha - 2\widetilde{\delta}} \leq q \leq \frac{2(m-1)}{\alpha - \widetilde{\delta}},$$

one can conclude that

$$\text{(3.28)} \quad \Big|\int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\,|G|^{m-2}G\,dx\Big| \leq C\|\theta_0\|_{L^\infty}\|G\|_{L^q}^{m-1}\|G\|_{L^2}^{1-\mu}\|G\|_{H^{\alpha/2}}^{\mu} \leq C\|G\|_{L^q}^{m-1}\|G\|_{H^{\alpha/2}}^{\mu}.$$

Substituting (3.22)–(3.25) and (3.28) into (3.21), one arrives at

$$\text{(3.29)} \quad \frac{d}{dt}\|G(t)\|_{L^m}^m + \|G\|_{L^{2m/2-\alpha}}^m \leq C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}} + C\|G\|_{L^q}^{m-1}\|G\|_{H^{\alpha/2}}^{\mu}.$$

By the Gagliardo–Nirenberg inequalities, we know that

$$\text{(3.30)} \quad \|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}} \leq C\|G\|_{L^m}^{1-\lambda_1}\|G\|^{\lambda_1}_{L^{\frac{2m}{2-\alpha}}}, \quad \lambda_1 = \frac{(1+2\gamma)m - 4}{\alpha(m-2)},$$

$$\text{(3.31)} \quad \|G\|_{L^q} \leq C\|G\|_{L^m}^{1-\lambda_2}\|G\|^{\lambda_2}_{L^{\frac{2m}{2-\alpha}}}, \quad \lambda_2 = \frac{2 - \frac{2m}{q}}{\alpha}.$$

Here we emphasize that the restrictions

$$\text{(3.32)} \quad \frac{4-m}{2m} \leq \gamma \leq \frac{m - (2-\alpha)(m-2)}{2m}, \quad m \leq q \leq \frac{2m}{2-\alpha}$$

imply $0 \leq \lambda_1 \leq 1$ and $0 \leq \lambda_2 \leq 1$, respectively.

In view of the interpolation inequalities (3.30) and (3.31), we can obtain

$$\text{(3.33)} \quad C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\alpha/2}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}} \leq C\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\alpha/2}}\|G\|_{L^m}^{(m-2)(1-\lambda_1)}\|G\|^{(m-2)\lambda_1}_{L^{\frac{2m}{2-\alpha}}} \leq \frac{1}{4}\|G\|^m_{L^{\frac{2m}{2-\alpha}}} + C(\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\alpha/2}})^{\frac{m}{m-(m-2)\lambda_1}}\|G\|_{L^m}^{\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1}},$$

and

$$C\|G\|_{L^q}^{m-1}\|G\|_{H^{\alpha/2}}^{\mu} \le C\|G\|_{L^m}^{(m-1)(1-\lambda_2)}\|G\|_{L^{\frac{2m}{2-\alpha}}}^{(m-1)\lambda_2}\|G\|_{H^{\alpha/2}}^{\mu} \le \tfrac{1}{4}\|G\|_{L^{2m/2-\alpha}}^{m} + C\|G\|_{H^{\alpha/2}}^{\frac{m\mu}{m-(m-1)\lambda_2}}\|G\|_{L^m}^{\frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2}}. \tag{3.34}$$

Inserting (3.33) and (3.34) into (3.29) yields

$$\frac{d}{dt}\|G(t)\|_{L^m}^m + \|G\|_{L^{\frac{2m}{2-\alpha}}}^m \le C(\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\alpha/2}})^{\frac{m}{m-(m-2)\lambda_1}}\|G\|_{L^m}^{\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1}} + C\|G\|_{H^{\alpha/2}}^{\frac{m\mu}{m-(m-1)\lambda_2}}\|G\|_{L^m}^{\frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2}}.$$

By direct calculation, we have

$$\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1} \le m, \qquad \frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2} \le m,$$
$$m \le \frac{2}{2-2\alpha+\widetilde{\delta}} \Rightarrow \frac{m\mu}{m-(m-1)\lambda_2} \le 2,$$

and

$$\gamma < \frac{8-(2-\alpha)m}{4m}\left(m < \frac{8}{2-\alpha}\right) \Rightarrow \frac{m}{m-(m-2)\lambda_1} < 2. \tag{3.35}$$

We thus get

$$\frac{d}{dt}\|G(t)\|_{L^m}^m + \|G\|_{L^{\frac{2m}{2-\alpha}}}^m \le C\Big\{(\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\alpha/2}})^{\frac{m}{m-(m-2)\lambda_1}} + \|G\|_{H^{\alpha/2}}^{\frac{m\mu}{m-(m-1)\lambda_2}}\Big\}(1+\|G\|_{L^m}^m). \tag{3.36}$$

Thanks to (3.35) as well as (3.20), we can deduce that

$$(\|\Lambda^{\gamma\beta}\theta\|_{L^{1/\gamma}}\|G\|_{H^{\alpha/2}})^{\frac{m}{m-(m-2)\lambda_1}} \in L^1(0,T), \qquad \|G\|_{H^{\alpha/2}}^{\frac{m\mu}{m-(m-1)\lambda_2}} \in L^1(0,T).$$

By the Gronwall inequality, we can deduce from (3.36) that

$$\|G(t)\|_{L^m}^m + \int_0^T \|G(\tau)\|_{L^{\frac{2m}{2-\alpha}}}^m\, d\tau \le C < \infty. \tag{3.37}$$

Finally, let us check that all the restrictions would work. Combining all the requirements on q yields

$$\max\Big\{m-1, \frac{4(m-1)}{3\alpha-2\widetilde{\delta}}, m\Big\} < q < \min\Big\{2(m-1), \frac{2(m-1)}{\alpha-\widetilde{\delta}}, \frac{2m}{2-\alpha}\Big\}.$$

Direct computations show that the number q exists if we select $\widetilde{\delta} < (3\alpha-2)/2$. Putting all the restrictions (3.24), (3.26), (3.32), (3.35) and $0 < \gamma < 1/2$ on γ, we have

$$\underline{\mathcal{B}}(\alpha) < \gamma < \overline{\mathcal{B}}(\alpha), \tag{3.38}$$

where

$$\underline{\mathcal{B}}(\alpha) = \max\left\{0, \frac{2\beta + 2 - 3\alpha}{2\beta}, \frac{4-5\alpha}{2\beta}, \frac{4-m}{2m}\right\},$$
$$\overline{\mathcal{B}}(\alpha) = \min\left\{\frac{1}{2}, \frac{1-\alpha}{\beta}, \frac{m-(2-\alpha)(m-2)}{2m}, \frac{8-(2-\alpha)m}{4m}\right\},$$

and

$$2 < m < \min\left\{4, \frac{2}{2-2\alpha+\widetilde{\delta}}, \frac{8}{2-\alpha}\right\} = 4.$$

According to $\beta > 1-\alpha$ and $m < 4$, the expressions for $\underline{\mathcal{B}}(\alpha)$ and $\overline{\mathcal{B}}(\alpha)$ can be reduced to

$$\underline{\mathcal{B}}(\alpha) = \max\left\{0, \frac{2\beta + 2 - 3\alpha}{2\beta}, \frac{4-m}{2m}\right\},$$
$$\overline{\mathcal{B}}(\alpha) = \min\left\{\frac{1}{2}, \frac{1-\alpha}{\beta}, \frac{m-(2-\alpha)(m-2)}{2m}\right\}.$$

Therefore the estimate (3.38) for γ would work if β satisfies

$$1-\alpha < \beta < \min\left\{\frac{\alpha}{2}, \frac{(3\alpha-2)m}{m+(2-\alpha)(m-2)}, \frac{2(1-\alpha)m}{4-m}\right\}. \tag{3.39}$$

Noticing that inequality (3.39) is strict and we have the key requirement

$$m > \frac{2}{2\alpha-1}, \tag{3.40}$$

we just need to verify (3.39) when $m = 2/(2\alpha-1)$. Then (3.39) reduces to

$$1-\alpha < \beta < \min\left\{\frac{\alpha}{2}, \frac{3\alpha-2}{2\alpha^2-6\alpha+5}, \frac{2-2\alpha}{4\alpha-3}\right\}. \tag{3.41}$$

It is not difficult to check that β can be found as long as

$$1-\alpha < \frac{3\alpha-2}{2\alpha^2-6\alpha+5} \Rightarrow \alpha > \alpha_0.$$

If (3.39) holds true when $m = 2/(2\alpha-1)$, then one may take $m = 2/(2\alpha-1) + \epsilon$ for some sufficiently small ϵ ($\epsilon > 0$ may depend on α and β) such that both (3.39) and (3.40) are fulfilled. Such a choice of $\epsilon > 0$ is possible because both (3.39) and (3.40) are strict. ∎

Proof of Theorem 2.1. In Lemma 3.6, we have proved that

$$\sup_{0\le t\le T} \|G(t)\|_{L^{\frac{2}{2\alpha-1}+\epsilon}} < \infty, \tag{3.42}$$

which is a key estimate in order to complete the proof of Theorem 2.1 (see for example [MX, YXX]). For convenience, we sketch it here. In fact, as detailed in [YXX, Step 2], the above estimate (3.42) implies

$$\int_0^T \|\omega(\tau)\|_{L^{\frac{2}{2\alpha-1}+\epsilon}}\, d\tau < \infty,$$

which further gives rise to

$$\int_0^T \|G(\tau)\|_{B^0_{\infty,1}}\,d\tau < \infty.$$

Finally, by [YXX, Lemma 3.3], we obtain

$$\int_0^T \|\omega(\tau)\|_{B^0_{\infty,1}}\,d\tau < \infty.$$

It follows from the Littlewood–Paley technique that

$$\int_0^T \|\nabla u(\tau)\|_{L^\infty}\,d\tau \le C\int_0^T (\|u(\tau)\|_{L^2} + \|\omega(\tau)\|_{B^0_{\infty,1}})\,d\tau < \infty.$$

This is sufficient to get the result of Theorem 2.1. The details can be found in [MX, YXX]. ■

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