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Spectrally isometric elementary operators

by

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Dedicated to Professor Richard V. Kadison on the occasion of his 90th birthday

Abstract. We present criteria for unital elementary operators (of small length) on unital semisimple Banach algebras to be spectral isometries. The surjective ones among them turn out to be algebra automorphisms.

1. Introduction. The study of isometries on Banach spaces is a vast and active area of research. See, for example, the proceedings volume [12] for some recent developments. Many of the results describe (linear, surjective) isometries between certain types of spaces in considerable detail; often one discovers a high degree of compatibility with algebraic or order-theoretic structure. For our purposes, we highlight Kadison's theorem [13] which states that when $T\colon A \to B$ is a surjective linear isometry between two unital C^*-algebras A and B, then $T1$ is a unitary in B and the mapping $x \mapsto (T1)^{-1}Tx$, $x \in A$, is a Jordan *-isomorphism (that is, it preserves selfadjoint elements and squares). A fortiori, if $T1 = 1$ (that is, T is *unital*), then T preserves invertibility of elements in both directions; hence the spectrum $\sigma(x)$ of each $x \in A$ agrees with $\sigma(Tx)$ and thus the spectral radius $r(x)$ remains unaltered. We will refer to a linear mapping T with the property $r(Tx) = r(x)$ for all x in the domain of T as a *spectral isometry.*

It has been an open question for some time (see [16] and [14]) whether the following non-selfadjoint version of Kadison's theorem holds: Every unital surjective spectral isometry between unital C^*-algebras is a Jordan isomorphism. (It is a fact that a unital surjective linear mapping is an isometry if and only if it is a selfadjoint spectral isometry.) As it stands, this conjecture

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is still open, though there has been substantial progress towards it: see, e.g., [1], [8], [15], [18] and the references contained therein. The present paper aims to contribute to these studies, but rather than putting additional conditions on the algebras involved, we investigate special spectral isometries on arbitrary semisimple Banach algebras, that is, we put the constraints on the operators.

Let A be a complex, unital Banach algebra. A linear mapping $S\colon A \to A$ is said to be an *elementary operator* if there exist $a_1, \dots, a_n, b_1, \dots, b_n \in A$ such that $Sx = \sum_{j=1}^{n} a_j x b_j$ for all $x \in A$. As such a representation of S is far from unique, we define the *length* $\ell(S)$ of S as follows. For $a, b \in A$, let $M_{a,b}$ stand for the two-sided multiplication $x \mapsto axb$. If $S = 0$ then $\ell(S) = 0$. If $S \neq 0$ then $\ell(S)$ is the smallest $n \in \mathbb{N}$ such that S can be written as a sum of n two-sided multiplications. We shall denote the algebra of all elementary operators on A by $\mathcal{E}\ell(A)$ and the space of all elementary operators of length at most n by $\mathcal{E}\ell_n(A)$.

Elementary operators on Banach algebras have been studied under a variety of aspects for many decades. Recent interest in elementary operators on C^*-algebras has been sparked by the fact that the completely positive ones describe the quantum channels in Quantum Information Theory. Several newer investigations have been compiled in [10]. Elementary operators that are *spectrally bounded*, that is, $r(Sx) \leq M r(x)$ for some $M \geq 0$ and all $x \in A$, are investigated in [4] and [6], extending earlier work in [9], for instance. General spectrally bounded operators do not allow for a detailed structure theory; for example, *every* bounded linear operator from a commutative C^*-algebra is spectrally bounded. Nevertheless, there are some surprisingly strong structural results; once again we refer to [15] for details and references. Our aim in this paper is to determine when elementary operators are spectrally isometric; this problem does not seem to have been attacked so far.

Suppose $S = \sum_{j=1}^{n} M_{a_j, b_j}$ for n-tuples $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n) \in A^n$. We will abbreviate this fact by $S = S_{\boldsymbol{a},\boldsymbol{b}}$ whenever convenient. With this notation, the two questions we pursue in the following are:

(i) Suppose $S = S_{\boldsymbol{a},\boldsymbol{b}}$; which conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$ ensure that S is a spectral isometry?
(ii) Suppose S is a spectral isometry; can we represent $S = S_{\boldsymbol{a},\boldsymbol{b}}$ with "nice" properties of $\boldsymbol{a}$ and $\boldsymbol{b}$?

Throughout, we will shall make the assumption that S is unital ($S1 = 1$) and at times that S is surjective, too. The main results of this paper are Theorems 4.2 and 4.4, which, somewhat surprisingly, state that conditions on $\boldsymbol{a}$ and $\boldsymbol{b}$ which imply that $S = S_{\boldsymbol{a},\boldsymbol{b}}$ is spectrally bounded together with the assumption that S is unital already entail that S is a spectral isometry. If, moreover, S is surjective, it turns out to be an inner automorphism.

In Corollary 4.6 we prove that every unital surjective spectrally isometric $S \in \mathcal{E}\ell_3(A)$ on a C^*-algebra A is an automorphism. On the other hand, we provide an example (Example 4.7) of a non-surjective spectrally isometric unital elementary operator of length three which is not a Jordan homomorphism.

2. Preliminaries. In the following, A and B will denote unital Banach algebras over the complex numbers $\mathbb{C}$. We let $\mathrm{rad}(A)$ stand for the Jacobson radical of the algebra A, while $Z(A)$ denotes its centre.

The following basic properties of spectral isometries are by now standard; see, e.g., [16], [18].

LEMMA 2.1. *Let $T\colon A \to B$ be a surjective spectral isometry. Then $T\mathrm{rad}(A) = \mathrm{rad}(B)$.*

As a result, if A is *semisimple* (i.e., $\mathrm{rad}(A) = 0$) then B is semisimple, and we can without loss of generality always assume that our algebras are semisimple (in order to avoid formulating the results "modulo the radical").

LEMMA 2.2. *Let $T\colon A \to B$ be a spectral isometry, where A is semisimple. Then T is injective.*

Consequently, in this case, a surjective spectral isometry has an inverse which is also a spectral isometry. Moreover, such a mapping is a linear topological isomorphism by [2, Theorem 5.5.2].

LEMMA 2.3. *Let $T\colon A \to B$ be a surjective spectral isometry, where A is semisimple. Then $TZ(A) = Z(B)$.*

This result has a number of very neat applications. Since $Z(A)$ is a commutative semisimple Banach algebra whenever A is a semisimple Banach algebra, we can apply Gelfand theory to it. As norm and spectral radius coincide for continuous functions on a compact Hausdorff space, a spectral isometry turns into an isometry (with respect to the spectral norm) when restricted to the centres. Thus one can apply the very rich theory of isometries, which has been successfully exploited in [19]. While it is difficult to control the behaviour of $T1$ when T is just a spectrally bounded operator (see the discussion in [15]), if T is a surjective spectral isometry, then $T1$ is central and $\sigma(T1)$ is always contained in the unit circle $\mathbb{T}$ [19, Proposition 2.3]. By the afore-mentioned method, this follows immediately from a description of non-unital isometries on subalgebras of algebras of continuous functions due to deLeeuw–Rudin–Wermer [11, Corollary 2.3.16]. Replacing T by $x \mapsto (T1)^{-1}Tx$, $x \in A$, if necessary, we can henceforth assume that our spectral isometries are unital. This will turn out to be an important simplification.

Before we move on to our main theme, we shall illustrate our techniques by an example of an isometric elementary operator.

EXAMPLE 2.4. Let $A \subseteq B(H)$ be a unital C^*-algebra acting faithfully on a Hilbert space H. Let s_1, s_2 be two isometries in A satisfying $s_1 s_1^* + s_2 s_2^* = 1$ (in particular, they have orthogonal ranges). Let $S \in \mathcal{E}\ell(A)$ be defined by $S = M_{s_1,s_1^*} + M_{s_2,s_2^*}$. Then S is unital and completely positive. Moreover, S is isometric, multiplicative and not surjective. The quickest way to check the isometric property is probably by observing that

$$\begin{aligned}
&\left\|\begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} s_1^* & 0 \\ s_2^* & 0 \end{pmatrix}\right\|^2 \\
&\qquad = \left\|\begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^* & 0 \\ 0 & x^* \end{pmatrix}\begin{pmatrix} s_1^* & 0 \\ s_2^* & 0 \end{pmatrix}\begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} s_1^* & 0 \\ s_2^* & 0 \end{pmatrix}\right\| \\
&\qquad = \left\|\begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^*x & 0 \\ 0 & x^*x \end{pmatrix}\begin{pmatrix} s_1^* & 0 \\ s_2^* & 0 \end{pmatrix}\right\| \\
&\qquad = \left\|\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} s_1^* & 0 \\ s_2^* & 0 \end{pmatrix}\begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x^* & 0 \\ 0 & x^* \end{pmatrix}\right\| \\
&\qquad = \left\|\begin{pmatrix} x^*x & 0 \\ 0 & x^*x \end{pmatrix}\right\| = \left\|\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right\|^2,
\end{aligned}$$

since $\left(\begin{smallmatrix} s_1^* & 0 \\ s_2^* & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} s_1 & s_2 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and that

$$\begin{aligned}
\|Sx\| &= \left\|\begin{pmatrix} Sx & 0 \\ 0 & 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} s_1 x s_1^* + s_2 x s_2^* & 0 \\ 0 & 0 \end{pmatrix}\right\| \\
&= \left\|\begin{pmatrix} s_1 & s_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} s_1^* & 0 \\ s_2^* & 0 \end{pmatrix}\right\|.
\end{aligned}$$

Let $x, y \in A$. Then

$$\begin{aligned}
(Sx)(Sy) &= (s_1 x s_1^* + s_2 x s_2^*)(s_1 y s_1^* + s_2 y s_2^*) \\
&= s_1 x s_1^* s_1 y s_1^* + s_2 x s_2^* s_1 y s_1^* + s_1 x s_1^* s_2 y s_2^* + s_2 x s_2^* s_2 y s_2^* \\
&= s_1 x s_1^* s_1 y s_1^* + s_2 x s_2^* s_2 y s_2^* = S(xy),
\end{aligned}$$

so that S is multiplicative.

Finally, suppose S is surjective and thus $s_1 x s_1^* + s_2 x s_2^* = s_1$ for some $x \in A$. Then $x = s_2^* s_2 x s_2^* s_2 = s_2^* s_1 s_2^* = 0$, which is impossible.

Similar arguments will be used regularly in the next two sections, with the norm replaced by the spectral radius.

3. Spectrally bounded elementary operators. Let A be a semisimple unital Banach algebra. The recent papers [4]–[6] by Boudi and Mathieu contain necessary and sufficient conditions for an elementary operator S on A to be spectrally bounded; some restrictions on the length of S had to be imposed too. We shall recall some of these results below, as we will need them in the discussion on spectral isometries in the next section. However, we will throughout restrict our attention to the unital case, that is, we assume that $S1 = 1$. This is justified by the properties of surjective spectral isometries as explained in the previous section and the fact that $x \mapsto uSx$, $x \in A$, is another elementary operator on A for any $u \in A$. We shall make this assumption on S even if S is not surjective (and it will at times help to find out whether S is surjective or not).

To begin with, the simple identity $r(M_{a,b}x) = r(bax) = r(M_{ba,1}x)$ together with Pták's description of spectrally bounded one-sided multiplications ([20], see also [9] for an alternative proof) tells us that $M_{a,b}$ is spectrally bounded if and only if $ba \in Z(A)$. Now if $ab = M_{a,b}1 = 1$ then $ba = baab = abab = 1$ too. As a result, $b = a^{-1}$ and we find that $M_{a,b} = M_{a,a^{-1}}$ is an inner automorphism of A, thus a surjective spectral isometry. In this way, we obtain our first observation.

PROPOSITION 3.1. *Let A be a unital semisimple Banach algebra and let $a, b \in A$. The following conditions are equivalent:*

(a) *$M_{a,b}$ is unital and spectrally bounded;*
(b) *$M_{a,b}$ is a unital spectral isometry;*
(c) *a is invertible with $b = a^{-1}$.*

In each case, $M_{a,b}$ is automatically surjective.

When the length of the elementary operator is greater than 1, the situation becomes of course more involved. This is due to the different choices for the coefficients representing the same elementary operator we may have. From [6, Corollary 2.6] we immediately obtain the following result.

PROPOSITION 3.2. *Let A be a semisimple unital Banach algebra. Let $S \in \mathcal{E}\ell_n(A)$ be unital. Suppose that $S = S_{\boldsymbol{a},\boldsymbol{b}}$ with $b_i a_i \in Z(A)$ for all $1 \le i \le n$ and $b_i a_j = 0$ for all $i < j$. Then S is a spectral contraction, that is, $r(Sx) \le r(x)$ for all $x \in A$.*

Proof. Note that the different convention "$i < j$" we use here simply amounts to a re-enumeration of the coefficient n-tuples in comparison with [6, Corollary 2.6]. From $\sum_{i=1}^{n} a_i b_i = 1$ we obtain $b_k = \sum_{i=1}^{n} b_k a_i b_i = \sum_{i=1}^{k} b_k a_i b_i$ for each $1 \le k \le n$. Hence $b_k a_k = \sum_{i=1}^{k} b_k a_i b_i a_k = (b_k a_k)^2$, so that each $e_k = b_k a_k$ is a central idempotent in A. Moreover, $b_1 = b_1 e_1$ and $a_n = a_n e_n$. In particular, for each $1 \le i \le n$, $r(M_{a_i,b_i} x) \le r(x)$ for all $x \in A$.

Following the argument in [6, proof of Corollary 2.6, and the end of the proof of Proposition 2.5], we find that $r(Sx) \leq r(x)$ for all $x \in A$. ■

It was shown in [4, Proposition 2.3] that $S = M_{a,b} + M_{c,d}$ is spectrally bounded if $ba, dc \in Z(A)$ and $bc = 0$. Note that this condition is however not necessary as $M_{a,1} + M_{1,d}$ is spectrally bounded if and only if $a, d \in Z(A)$ [9, Theorem B]. Under the assumption that S is unital, we obtain a stronger result.

Corollary 3.3. *Let A be a semisimple unital Banach algebra. Suppose $S = M_{a,b} + M_{c,d}$ is unital and $e = ba$, $f = dc \in Z(A)$ and $bc = 0$. Then S is an injective spectral contraction, and S is surjective if and only if $e + f = 1$. In the latter case, there is an invertible element $w \in A$ such that $S = M_{w,w^{-1}}$.*

Proof. From the above proposition we know that both e and f are central idempotents and that S is a spectral contraction. Moreover, $b = be$ and $c = cf$.

Take $x \in A$ with $axb + cxd = 0$. Then $0 = cxdc = cxf = cx$ and hence $fx = 0$. Substituting this back yields $0 = baxb = exb = xb$ and hence $xe = 0$. From $S1 = 1$ we conclude that $x = (ab + cd)x = abex + cdfx = 0$ and thus S is injective.

Suppose that $e + f = 1$. From $ef = 0$ we obtain $cxb = cfxbe = 0$ for all x and thus, setting $w = ae + c$, it is straightforward to check that w is invertible with inverse $b + fd$. Since

$$\begin{aligned} wxw^{-1} &= (ae + c)x(b + fd) = axb + cxb + aexfd + cxfd \\ &= axb + cxd = Sx \end{aligned} \tag{3.1}$$

for all $x \in A$, we find that S is an inner automorphism, in particular surjective.

Suppose that $e + f \neq 1$. Then there must be a primitive ideal P in A such that $e_P + f_P \neq 1_P$ (where $x_P = x + P$ denotes the coset in A/P). As $Z(A/P) = \mathbb{C}1_P$ this implies that $e_P = f_P = 1_P$ ($e_P = f_P = 0$ is ruled out by $S1 = 1$). From

$$d_P a_P = d_P(a_P b_P + c_P d_P)a_P = (e_P + f_P)d_P a_P = 2\, d_P a_P \tag{3.2}$$

we obtain $d_P a_P = 0$. If $x \in A$ satisfies $a_P x_P b_P + c_P x_P d_P = a_P$ then $x_P = d_P c_P x_P d_P c_P = d_P a_P c_P = 0$, which is impossible since $e_P = b_P a_P = 1$. Therefore S cannot be surjective. ■

4. Spectrally isometric elementary operators. In Example 2.4 we determined that a certain unital elementary operator of length 2 is isometric and multiplicative while not surjective. Using similar ideas, we will now obtain a more general result for spectral isometries which strengthens Corollary 3.3 above.

We first need to have a look at the behaviour of an elementary operator with respect to primitive quotients. Let A be a semisimple unital Banach algebra and let $P \subseteq A$ be a primitive ideal in A. Let S be an elementary operator on A with $\ell(S) = n > 0$. As $SP \subseteq P$ we obtain an induced elementary operator $S_P \in \mathcal{E}\ell_n(A/P)$ via $S_P x_P = (Sx)_P$, where $x_P = x + P$ denotes the coset of $x \in A$. Clearly, if $S = S_{\boldsymbol{a},\boldsymbol{b}}$ then $S_P = S_{\boldsymbol{a}_P,\boldsymbol{b}_P}$, and S is unital if and only if S_P is unital for every primitive ideal P.

Denote by $\mathrm{Prim}(A)$ the set of all primitive ideals of A. If S_P is spectrally bounded for each $P \in \mathrm{Prim}(A)$, say $r(S_P x_P) \le M_P\, r(x_P)$ for some $M_P \ge 0$ and all $x \in A$, and if $M = \sup_P M_P < \infty$, then S is spectrally bounded with $r(Sx) \le M\, r(x)$ for all $x \in A$. However, assuming that S is spectrally bounded we cannot conclude that each S_P is spectrally bounded in general.

From [4, Theorem 3.5] (see also [6, Corollary 3.7]), we can deduce the following characterisation for spectral boundedness of a unital elementary operator $S \in \mathcal{E}\ell_2(A)$. Note that the exceptional case pointed out in [6, Corollary 3.7] cannot occur if $S1 = 1$.

LEMMA 4.1. *Let A be a semisimple unital Banach algebra. Let $S \in \mathcal{E}\ell_2(A)$ be unital. Then S is spectrally bounded if and only if, for each $P \in \mathrm{Prim}(A)$, there exist $a_P, b_P, c_P, d_P \in A/P$ such that $S_P = M_{a_P,b_P} + M_{c_P,d_P}$ and $e_P = b_P a_P$, $f_P = d_P c_P$ are central idempotents in A/P and $b_P c_P = 0$. In particular, S is a spectral contraction.*

Proof. By Corollary 3.3, the conditions on the coefficients imply that each S_P is a spectral contraction. Hence so is S, which proves the "if" part.

To obtain the "only if" part suppose that $S = M_{u,v} + M_{s,t}$ for some $u, v, s, t \in A$ is unital and spectrally bounded. Let $P \in \mathrm{Prim}(A)$. By [4, Theorem 3.5], there is $\beta_P \in \mathbb{C}$ such that

$$(v_P + \beta_P t_P)u_P \in \mathbb{C}1_P \quad \text{and} \quad t_P(s_P - \beta_P u_P) \in \mathbb{C}1_P$$

and either $(v_P + \beta_P t_P)(s_P - \beta_P u_P) = 0$, or $\beta_P = 0$ and $t_P u_P = 0$. In the first case, we set $b_P = v_P + \beta_P t_P$, $a_P = u_P$, $c_P = s_P - \beta_P u_P$ and $d_P = t_P$. Then

$$M_{a_P,b_P} + M_{c_P,d_P} = M_{u_P,v_P+\beta_P t_P} + M_{s_P-\beta_P u_P,t_P} = M_{u_P,v_P} + M_{s_P,t_P} = S_P$$

and $b_P c_P = 0$. From $S_P 1_P = 1_P$ it follows as in Corollary 3.3 that $e_P = b_P a_P$ and $f_P = d_P c_P$ are central idempotents in A/P.

In the second case, set $a_P = s_P$, $b_P = t_P$, $c_P = u_P$ and $d_P = v_P$. Clearly, $S_P = M_{a_P,b_P} + M_{c_P,d_P}$ and the other conditions are satisfied as well.

In either case, each S_P, $P \in \mathrm{Prim}(A)$ is a spectral contraction, so S is too. ∎

THEOREM 4.2. *Let A be a semisimple unital Banach algebra. Suppose $S \in \mathcal{E}\ell_2(A)$ is unital. The following conditions are equivalent:*

(a) *S is spectrally bounded;*
(b) *S is spectrally isometric;*
(c) *S is multiplicative.*

Proof. If S is multiplicative and unital then $\sigma(Sx) \subseteq \sigma(x)$ for all $x \in A$; hence S is a spectral contraction, which proves (c)$\Rightarrow$(a). Evidently (b)$\Rightarrow$(a). We now show (a)$\Rightarrow$(b) and (a)$\Rightarrow$(c) simultaneously.

Let $P \in \mathrm{Prim}(A)$ and choose $a_P, b_P, c_P, d_P \in A/P$ such that $S_P = M_{a_P,b_P} + M_{c_P,d_P}$ and the other conditions in Lemma 4.1 are satisfied. As $e_P = b_P a_P$ and $f_P = d_P c_P$ are central idempotents in A/P they can only be either 0 or 1_P. We distinguish the following four cases.

CASE 1: $e_P = f_P = 0$. From $a_P b_P + c_P d_P = 1_P$ we obtain $c_P d_P c_P = c_P$ and $b_P a_P b_P = b_P$, that is,

$$c_P f_P = c_P \quad \text{and} \quad b_P e_P = b_P. \tag{4.1}$$

In the case under consideration this would imply $c_P = b_P = 0$, which contradicts $S_P 1_P = 1_P$; so this case cannot occur.

CASE 2: $e_P = 1$, $f_P = 0$. From (4.1) we obtain $S_P = M_{a_P,b_P}$, which is a spectral isometry as $b_P a_P = 1_P$. In fact, $b_P = a_P^{-1}$ (cf. Proposition 3.1), hence S_P is an inner automorphism in this case.

CASE 3: $e_P = 0$, $f_P = 1$. This case is treated analogously to the previous one, and $S_P = M_{c_P,c_P^{-1}}$.

CASE 4: $e_P = f_P = 1$. From (3.2) we get $d_P a_P = 0$ and a straightforward computation shows that S_P is multiplicative in this case. Moreover,

$$\begin{aligned} r(S_P x_P) &= r\left(\begin{pmatrix} S_P x_P & 0 \\ 0 & 0 \end{pmatrix}\right) = r\left(\begin{pmatrix} a_P & c_P \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_P & 0 \\ 0 & x_P \end{pmatrix}\begin{pmatrix} b_P & 0 \\ d_P & 0 \end{pmatrix}\right) \\ &= r\left(\begin{pmatrix} b_P & 0 \\ d_P & 0 \end{pmatrix}\begin{pmatrix} a_P & c_P \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_P & 0 \\ 0 & x_P \end{pmatrix}\right) \\ &= r\left(\begin{pmatrix} b_P a_P & 0 \\ 0 & d_P c_P \end{pmatrix}\begin{pmatrix} x_P & 0 \\ 0 & x_P \end{pmatrix}\right) = r\left(\begin{pmatrix} x_P & 0 \\ 0 & x_P \end{pmatrix}\right) = r(x_P) \end{aligned}$$

for all $x \in A$.

We conclude that S_P is a spectral isometry in each case, and therefore

$$r(Sx) = \sup_P r(S_P x_P) = \sup_P r(x_P) = r(x)$$

for all x. The fact that S_P is multiplicative for all P completes the argument. ∎

Note that Theorem 4.2 entails in particular that a unital spectrally bounded elementary operator of length at most two is injective. To determine when such an operator is surjective we employ a similar criterion to the one in Corollary 3.3; however, since we do not have a global condition on the coefficients, the argument is slightly more involved.

Following the notation used in [6], for $S \in \mathcal{E}\ell_n(A)$, $S = S_{\boldsymbol{a},\boldsymbol{b}}$, we write S^* for the elementary operator $S^* = S_{\boldsymbol{b},\boldsymbol{a}}$.

Proposition 4.3. *Let A be a semisimple unital Banach algebra. Suppose $S \in \mathcal{E}\ell_2(A)$ is unital and spectrally bounded. Then S is surjective if and only if $S^*1 = 1$.*

Proof. We first note the following. If $S = M_{u,v} + M_{s,t}$ with $u, v, s, t \in A$ then $S^* = M_{v,u} + M_{t,s}$ and therefore

$$(S^*)_P = M_{v_P,u_P} + M_{t_P,s_P} = (S_P)^*;$$

thus it is legitimate to simply write S^*_P. As A is semisimple, $S^*1 = 1$ if and only if $S^*_P 1_P = 1_P$ for all $P \in \mathrm{Prim}(A)$. Continuing to use the notation in the proof of Lemma 4.1, we have

$$S^*_P 1_P = v_P u_P + t_P s_P = b_P a_P + d_P c_P = e_P + f_P$$

with $b_P c_P = 0$ and $b_P e_P = b_P$ (where $e_P = b_P a_P$) and $c_P f_P = c_P$ (where $f_P = d_P c_P$) whichever case for $P \in \mathrm{Prim}(A)$ occurs in this lemma.

Suppose now that $S^*1 \neq 1$. Pick $P \in \mathrm{Prim}(A)$ such that $S^*_P 1_P \neq 1_P$ and write $S_P = M_{a_P,b_P} + M_{c_P,d_P}$ in a representation as in Lemma 4.1. Then $e_P + f_P = S^*_P 1_P \neq 1_P$, and since both e_P and f_P are central idempotents in A/P, we must have $e_P = f_P = 1$. It follows from (3.2) and the subsequent argument that $d_P a_P = 0$ and S_P is not surjective. As a result, S is not surjective.

Assuming on the other hand that $S^*1 = 1$, which means $e_P + f_P = S^*_P 1_P = 1_P$ for all $P \in \mathrm{Prim}(A)$, we can follow the argument in the proof of Corollary 3.3 to show that S_P is an inner automorphism of A/P in this case. Indeed, setting $w_P = a_P e_P + c_P$ we find that w_P is invertible with inverse $b_P + f_P d_P$, because $e_P f_P = 0$. The same calculation as in identity (3.1) entails that $S_P = M_{w_P,w_P^{-1}}$.

Define $T\colon A \to A$ by $(Tx)_P = M_{w_P^{-1},w_P} x_P$ for $x \in A$. Since A is semisimple, it is easily verified that T is well defined and that $TS = ST = \mathrm{id}_A$. Consequently, $T = S^{-1}$; in particular, S is surjective. ∎

Next we extend this condition for surjectivity to elementary operators of arbitrary length. However, in the absence of an if-and-only-if condition for spectral boundedness we have to use the slightly stronger assumptions of Proposition 3.2 instead, which in fact implies that the operator is spectrally isometric, thus extending Corollary 3.3 and part of Theorem 4.2.

THEOREM 4.4. *Let A be a semisimple unital Banach algebra. Let $S \in \mathcal{E}\ell_n(A)$ be unital. Suppose that $S = S_{\boldsymbol{a},\boldsymbol{b}}$ with $e_i = b_i a_i \in Z(A)$ for all $1 \le i \le n$ and $b_i a_j = 0$ for all $i < j$. Then S is a spectral isometry. Moreover, the following are equivalent:*

(a) *S is surjective;*
(b) *$\sum_{i=1}^n e_i = 1$;*
(c) *$S = M_{w,w^{-1}}$ for an invertible element $w \in A$.*

Proof. First recall from the proof of Proposition 3.2 that each $e_i = b_i a_i$ is a central idempotent in A and that $b_1 = b_1 e_1$, $a_n = a_n e_n$. We shall obtain more complicated relations for the other coefficients of S below.

By Proposition 3.2, $r(Sx) \le r(x)$ for all $x \in A$ so it suffices to show that $r(Sx) \ge r(x^P)$ for all $x \in A$ and $P \in \operatorname{Prim}(A)$ (where we now changed the notation to $x^P = x + P$ in order to avoid the conflict with subscripts). We accomplish this by induction on n. The cases $n = 1$ and $n = 2$ are Proposition 3.1 and Theorem 4.2, respectively. Thus suppose that $S = \sum_{j=1}^{n+1} M_{a_j,b_j}$ with $n \ge 2$ and $e_i = b_i a_i \in Z(A)$ and $b_i a_j = 0$ for all $1 \le i < j \le n+1$, and that the statement holds for elementary operators of length at most n. Let $\boldsymbol{a} = (a_1, \dots, a_{n+1})$, $\boldsymbol{b} = (b_1, \dots, b_{n+1})$ and $\boldsymbol{x} = \operatorname{diag}(x, \dots, x)$. Then, with $\boldsymbol{b}^t$ denoting the obvious column, we have

$$Sx = \boldsymbol{a}\boldsymbol{x}\boldsymbol{b}^t$$

and hence

$$\begin{aligned} r(Sx) &= r(\boldsymbol{b}^t \boldsymbol{a}\boldsymbol{x}) \\ &= r\left(\begin{pmatrix} b_1a_1 & 0 & \dots & 0 \\ b_2a_1 & b_2a_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_{n+1}a_1 & b_{n+1}a_2 & \dots & b_{n+1}a_{n+1} \end{pmatrix} \begin{pmatrix} x & 0 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & x \end{pmatrix}\right). \end{aligned}$$

Applying the same reasoning in the primitive quotient A/P we have

$$r(S_P x^P) = r\left(\begin{pmatrix} e_1^P & 0 & \dots & 0 \\ (b_2a_1)^P & e_2^P & \dots & 0 \\ \vdots & \vdots & & \vdots \\ (b_{n+1}a_1)^P & (b_{n+1}a_2)^P & \dots & e_{n+1}^P \end{pmatrix} \begin{pmatrix} x^P & 0 & \dots & 0 \\ 0 & x^P & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & x^P \end{pmatrix}\right),$$

where $e_i^P \in \{0, 1^P\}$, $1 \le i \le n+1$. If $e_{n+1}^P = 0$ then $a_{n+1}^P = a_{n+1}^P e_{n+1}^P = 0$ and therefore $\ell(S_P) \le n$, so that we can apply the induction hypothesis to get $r(Sx) \ge r(x^P)$. Otherwise, for $\lambda \in \mathbb{C}$ and $y \in A$,

$$\left(\lambda - \begin{pmatrix} e_1^P & 0 & \dots & 0 \\ (b_2a_1)^P & e_2^P & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (b_{n+1}a_1)^P & (b_{n+1}a_2)^P & \dots & 1^P \end{pmatrix} \begin{pmatrix} x^P & 0 & \dots & 0 \\ 0 & x^P & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & x^P \end{pmatrix}\right) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ y^P \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - e_1^P x^P & 0 & \dots & 0 \\ -(b_2a_1)^P x^P & \lambda - e_2^P x^P & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(b_{n+1}a_1)^P x^P & -(b_{n+1}a_2)^P x^P & \dots & \lambda - x^P \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ y^P \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (\lambda - x^P)y^P \end{pmatrix}.$$

Let $\lambda \in \sigma(x^P)$ be such that $|\lambda| = r(x^P)$. Then λ belongs to the left approximate point spectrum of x_P, and thus we can take a sequence $(y_n^P)_{n\in\mathbb{N}}$ of unit elements in A/P with $(\lambda - x^P)y_n^P \to 0$ $(n \to \infty)$. The above calculations show that $r(S_P x^P) \geq |\lambda| = r(x^P)$.

Since this argument yields $r(Sx) \geq r(x_P)$ for every primitive ideal P, we conclude that S is a spectral isometry.

As for the equivalence of the three conditions listed, evidently (c)$\Rightarrow$(a). In order to establish (a)$\Rightarrow$(b), suppose that $\sum_{i=1}^n e_i \neq 1$. Then there is $P \in \mathrm{Prim}(A)$ such that $\sum_{i=1}^n e_i^P \neq 1^P$. As $Z(A/P) = \mathbb{C}1^P$ this entails that there are $k, \ell \in \{1, \dots, n\}$, $k < \ell$, such that $e_k^P = e_\ell^P = 1^P$. (It is easy to verify that the assumption $S1 = 1$ rules out the possibility that all $e_i^P = 0$, $1 \leq i \leq n$.) Take $x \in A$. Upon multiplying the identity

$$S_P x^P = \sum_{j=1}^n a_j^P x^P b_j^P$$

first on the left by b_k^P and then on the right by a_k^P and noting that $b_i^P a_j^P = 0$ for all $i < j$ we obtain in succession

$$b_k^P(S_P x^P) = \sum_{j=1}^{k-1} b_k^P a_j^P x^P b_j^P + b_k^P a_k^P x^P b_k^P, \qquad b_k^P(S_P x^P)a_k^P = e_k^P x^P e_k^P = x^P.$$

Consequently, no $x \in A$ can satisfy $S_P x^P = a_\ell^P$ as the last identity on the left hand side yields $b_k^P a_\ell^P a_k^P = 0$ but $x_P = 0$ is incompatible with $b_\ell^P a_\ell^P = 1^P$. We conclude that S_P cannot be surjective, so S cannot be surjective either.

Finally, to show (b)⇒(c) we note first that $\sum_{i=1}^{n} e_i = 1$ implies that all e_i's are mutually orthogonal (which follows at once from the fact that $Z(A)$ is a commutative semisimple Banach algebra, or purely algebraically). Consequently, any sum of the form $\sum_{i\in I}\sum_{j\in J} e_i e_j$ with $I, J \subseteq \{1,\dots,n\}$ reduces to $\sum_{k\in I\cap J} e_k$ (which we interpret as 0 if $I\cap J=\emptyset$). This fact will be used repeatedly in the following.

From $S1 = \sum_{j=1}^{n} a_j b_j = 1$ we obtain

$$b_k = b_k \sum_{j=1}^{n} a_j b_j = b_k \Bigl(\sum_{j=1}^{k-1} a_j b_j + e_k\Bigr), \tag{4.2}$$

$$a_k = \sum_{j=1}^{n} a_j b_j a_k = \Bigl(\sum_{j=k+1}^{n} a_j b_j + e_k\Bigr) a_k \tag{4.3}$$

for each $1 \le k \le n$. Our next claim is that

$$b_k = b_k \sum_{j=1}^{k} e_j \quad \text{and} \quad a_k = a_k \sum_{j=k}^{n} e_j \qquad (1 \le k \le n), \tag{4.4}$$

which we shall prove by induction and "induction from the top", respectively. We have $b_1 = b_1 e_1$ and thus assume that $b_\ell = b_\ell \sum_{j=1}^{\ell} e_j$ for all $1 \le \ell < k$. This entails

$$b_\ell = b_\ell \sum_{j=1}^{\ell} e_j = b_\ell \sum_{j=1}^{\ell} e_j \sum_{i=1}^{k} e_i = b_\ell \sum_{i=1}^{k} e_i.$$

Putting this identity together with (4.2) we find that

$$\begin{aligned} b_k \sum_{i=1}^{k} e_i &= b_k \Bigl(\sum_{\ell=1}^{k-1} a_\ell b_\ell + e_k\Bigr) \sum_{i=1}^{k} e_i \\ &= b_k \Bigl(\sum_{\ell=1}^{k-1} a_\ell \Bigl(b_\ell \sum_{i=1}^{k} e_i\Bigr) + e_k\Bigr) \\ &= b_k \Bigl(\sum_{\ell=1}^{k-1} a_\ell b_\ell + e_k\Bigr) = b_k, \end{aligned}$$

which proves the claim for the b_k. We also know that $a_n = a_n e_n$, and thus assume that $a_\ell = a_\ell \sum_{j=\ell}^{n} e_j$ for all $k < \ell \le n$. It follows that

$$a_\ell = a_\ell \sum_{j=\ell}^{n} e_j = a_\ell \sum_{j=\ell}^{n} e_j \sum_{i=k}^{n} e_i = a_\ell \sum_{i=k}^{n} e_i.$$

This identity together with (4.3) gives

$$a_k \sum_{i=k}^{n} e_i = \Big(\sum_{\ell=k+1}^{n} a_\ell b_\ell + e_k \Big) a_k \sum_{i=k}^{n} e_i = \Big(\sum_{\ell=k+1}^{n} \Big(a_\ell \sum_{i=k}^{n} e_i \Big) b_\ell + e_k \Big) a_k$$
$$= \Big(\sum_{\ell=k+1}^{n} a_\ell b_\ell + e_k \Big) a_k = a_k,$$

proving the second half of our claim.

The identities in (4.4) immediately yield the following information on the M_{a_k,b_k}:

$$M_{a_k,b_k} = \sum_{i=k}^{n} \sum_{j=1}^{k} e_i e_j M_{a_k,b_k} = e_k M_{a_k,b_k} \quad (1 \le k \le n). \tag{4.5}$$

Set $w = \sum_{k=1}^{n} a_k e_k$ and $v = \sum_{j=1}^{n} b_j e_j$. Then

$$wv = \sum_{k=1}^{n} \sum_{j=1}^{n} e_k e_j a_k b_j = \sum_{k=1}^{n} e_k a_k b_k = \sum_{k=1}^{n} a_k b_k = 1,$$
$$vw = \sum_{j=1}^{n} \sum_{k=1}^{n} e_j e_k b_j a_k = \sum_{j=1}^{n} e_j = 1.$$

Therefore w is invertible with $w^{-1} = v$. Finally, from (4.5),

$$M_{w,w^{-1}} = \sum_{k=1}^{n} \sum_{j=1}^{n} e_k e_j M_{a_k,b_j} = \sum_{k=1}^{n} e_k M_{a_k,b_k} = S,$$

which completes the proof of Theorem 4.4. ∎

In the remainder of this paper we shall discuss the state of our knowledge concerning the gap between a full description of spectrally isometric elementary operators of length (at most) two (Theorem 4.2) and the somewhat more restricted conclusion in the general case (Theorem 4.4).

In contrast to length two elementary operators there is (at present) no necessary condition for spectral boundedness of a length three elementary operator, such as in Lemma 4.1, without further assumptions. Firstly, starting with a spectrally bounded elementary operator S, the induced operator S_P may or may not be spectrally bounded. Even if it is, we only have a complete description for $S_P \in \mathcal{E}\ell_3(A/P)$ under the assumption that the representation space has dimension at least 4. We can slightly improve this description, which was obtained in [6, Theorem 4.3], under the hypothesis that $S_P 1^P = 1^P$, which we record here for completeness.

PROPOSITION 4.5. *Let A be a unital Banach algebra acting irreducibly as bounded linear operators on a Banach space E of dimension at least 4.*

Let $S \in \mathcal{E}\ell_3(A)$ be unital. Then S is spectrally bounded if and only if there exist $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ in $\mathscr{L}(E)^3$ such that $S = S_{\boldsymbol{a},\boldsymbol{b}}$, $e_i = b_i a_i$, $1 \le i \le 3$, are central idempotents and $b_i a_j = 0$ for $1 \le i < j \le 3$. In this case, S is in fact a spectral contraction.

Proof. Clearly we can extend $S\colon A \to A$ to an elementary operator on $\mathscr{L}(E)$, the algebra of all bounded linear operators on E, by the same formula. The conditions on the coefficients imply that the extended operator is spectrally bounded, indeed a spectral contraction, by Proposition 3.2. As the spectral radius of an element is independent of the surrounding Banach algebra, it follows that $S\colon A \to A$ is a spectral contraction. This proves the "if" part.

For the "only if" part we only have to deal with the exceptional cases that are listed in [6, Theorem 4.3], as this result will then imply the statement. By hypothesis, if S is not already represented as claimed in the above result, then cases (ii) and (iii) in [6, Theorem 4.3] can be summarised as follows:

$$S = \sum_{j=1}^{3} M_{u_j,v_j} \quad \text{where} \quad (v_i u_j)_{1\le i,j\le 3} = \begin{pmatrix} \lambda & r & 0 \\ s & \lambda & r \\ 0 & -s & \lambda \end{pmatrix} \tag{4.6}$$

for some $\lambda \in \mathbb{C}$ and rank one operators $r, s \in \mathscr{L}(E)$. We can read off the following identities:

$$v_1 u_3 = v_3 u_1 = 0, \qquad v_1 u_2 = v_2 u_3, \qquad v_2 u_1 = -v_3 u_2.$$

Upon multiplying $S1 = 1$ on the left by v_1 and on the right by u_3 and using $v_1 u_3 = 0$ we find that $v_1 u_2 v_2 u_3 = 0$. As $v_1 u_2 = v_2 u_3$ it follows that the rank one operator $r = v_2 u_3$ has square zero and therefore must be zero. We conclude that $v_i u_j = 0$ whenever $i < j$, as desired. Finally, multiplying $S1 = 1$ simply on the right by u_3 yields $u_3 v_3 u_3 = u_3$, thus $(\lambda - 1)u_3 = 0$, so $\lambda = 1$, which finishes the argument. ∎

A Banach algebra A is called an *SR-algebra* if the spectral radius formula holds in every quotient of A; that is, if I is a closed ideal of A then, for each $x \in A$, $r(x + I) = \inf_{y\in I} r(x + y)$. Every C^*-algebra has this property.

COROLLARY 4.6. *Let A be a unital semisimple SR-algebra. Then every unital surjective $S \in \mathcal{E}\ell_3(A)$ which is spectrally isometric is an algebra automorphism of A.*

Proof. Let $P \in \mathrm{Prim}(A)$. By assumption and as $SP \subseteq P$, $S_P \in \mathcal{E}\ell_3(A/P)$ is unital, surjective and a spectral contraction, by [6, Proposition 2.2] or [17, Proposition 9], and is spectrally isometric if $SP = P$. Suppose first that $\dim A/P < \infty$. Take $y \in P$ and write $y = Sx$ for a (unique) $x \in A$. Then $0 = Sx + P = S_P(x + P)$, and therefore $x + P = 0$ as S_P is injective (it is surjective on the finite-dimensional space A/P). Consequently, $x \in P$ and hence $P = SP$. Since $A/P \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, we conclude that S_P is a Jordan automorphism by [3, Proposition 2] (see also [7, Corollary 1.4] and [15, Example 5.4] for independent proofs).

It is well known that every Jordan automorphism of $M_n(\mathbb{C})$, $n > 1$, is either of the form $x \mapsto wxw^{-1}$ or $x \mapsto wx^tw^{-1}$ for some invertible $w \in M_n(\mathbb{C})$, where x^t denotes the transpose of x (see, e.g., [21, Corollary 1.4]). Note that $x \mapsto x^t$ is the elementary operator $T = \sum_{i,j=1}^{n} M_{e_{ji},e_{ji}}$, where e_{ij}, $1 \le i, j \le n$, denotes the usual set of matrix units. As this set is linearly independent, $\ell(T) = n^2$. Since $\{we_{ji} \mid 1 \le i, j \le n\}$ and $\{e_{ji}w^{-1} \mid 1 \le i, j \le n\}$ are linearly independent too, whenever $w \in M_n(\mathbb{C})$ is invertible, $\ell(M_{w,w^{-1}}T) = n^2 > 3$ for all $n > 1$. Hence, $S \neq T$, and therefore S is multiplicative.

Suppose next that $\dim A/P = \infty$. Applying Proposition 4.5 together with Theorem 4.4 to S_P (and its extension to $\mathscr{L}(E)$) we find that S_P is an inner automorphism. As a result, S_P is multiplicative in either case, and therefore S is an algebra automorphism of A. ∎

We will now discuss an example illustrating that Theorem 4.2 cannot be entirely extended to length three elementary operators.

Example 4.7. Let $A = \mathscr{L}(E)$ for an infinite-dimensional Banach space E. Let $S \in \mathscr{E}\ell(A)$ with $S1 = 1$ and $\ell(S) = 3$. Suppose S is a spectral isometry. By the results above, there exist two linearly independent subsets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ of A such that $S = \sum_{j=1}^{3} M_{a_j,b_j}$, $e_i = b_ia_i \in \{0, 1\}$, $1 \le i \le 3$, and $b_ia_j = 0$ for $1 \le i < j \le 3$. Suppose that $e_1 = e_3 = 1$ and $e_2 = 0$ (and thus S is non-surjective by Theorem 4.4). For a concrete realisation of this situation, we can, e.g., take the isometries s_1, s_2 from Example 2.4 and a non-zero operator $z \in \mathscr{L}(E)$ with $z^2 = 0$ and let $a_1 = s_1$, $a_2 = s_2z$, $a_3 = s_2$, $b_1 = s_1^*$, $b_2 = zs_1^*$ and $b_3 = s_2^*$. It is easily verified that these choices satisfy the above conditions. We claim that S is not a Jordan homomorphism.

To show this, we first observe that neither $\{1, b_2a_1\}$ nor $\{1, b_3a_2\}$ can be linearly dependent. For example, using $a_1b_1 + a_2b_2 + a_3b_3 = 1$ and multiplying this identity on the left by b_2 yields $b_2a_1b_1 = b_2$, thus, if $b_2a_1 = \lambda 1$ for some $\lambda \in \mathbb{C}$, we obtain $\lambda b_1 = b_2$ violating the above assumption of linear independence. Similarly, linear dependence of $\{1, b_3a_2\}$ would result in linear dependence of $\{a_2, a_3\}$.

We can therefore find $\zeta, \eta \in E$ such that $\{\zeta, b_2a_1\zeta\}$ and $\{\eta, b_3a_2\eta\}$ are linearly independent. Since $\zeta = b_1a_1\zeta$, setting $\xi = a_1\zeta$, we see $\{b_1\xi, b_2\xi\}$ is linearly independent, and since $\eta = b_3a_3\eta$, it follows that $\{a_3\eta, a_2\eta\}$ is linearly independent too. Take $x \in A$ such that $xb_1\xi = 0$ and $xb_2\xi = \eta$. If $\eta \in \operatorname{lin}\mu\{b_1\xi, b_2\xi\}$ then $x\eta = \beta\eta$ for some $\beta \in \mathbb{C}$. If $b_3a_2\eta \in \operatorname{lin}\mu\{b_1\xi, b_2\xi\}$ then $xb_3a_2\eta = \beta'\eta$ for some $\beta' \in \mathbb{C}$. Suppose that both cases occur together and $\beta = \beta' = 0$. Then $\eta = \alpha b_1\xi$ and $b_3a_2\eta = \alpha' b_1\xi$ for some $\alpha, \alpha' \in \mathbb{C} \setminus \{0\}$. However, this violates the linear independence of $\{\eta, b_3a_2\eta\}$, thus it cannot happen. Consequently, if both cases occur together, we have

$$x\eta = \beta\eta \quad \text{and} \quad xb_3a_2\eta = \beta'\eta \quad \text{with } |\beta|^2 + |\beta'|^2 \neq 0.$$

In the case when $\{b_1\xi, b_2\xi, \eta\}$ is linearly independent, we can also require of x that $x\eta = \eta$, and in the case when $\{b_1\xi, b_2\xi, b_3a_2\eta\}$ is linearly independent, we can additionally require that $xb_3a_2\eta = \eta$. It follows that, in any of the cases, we have $x \in A$ satisfying

$$\begin{aligned} &xb_1\xi = 0, \quad xb_2\xi = \eta, \\ &x\eta = \beta\eta, \quad xb_3a_2\eta = \beta'\eta \quad \text{with } |\beta|^2 + |\beta'|^2 \neq 0. \end{aligned} \tag{4.7}$$

We will complete the argument by showing that, for this x, $(Sx)^2 \neq S(x^2)$, so S is not a Jordan homomorphism.

The initial assumptions on the coefficients of S reduce the left hand side in the first line below to the right hand side, and then we apply the special choice of x as given in (4.7). We have

$$\begin{aligned} ((Sx)^2 - S(x^2))\xi &= a_2xb_2a_1xb_1\xi + a_3xb_3a_1xb_1\xi + a_3xb_3a_2xb_2\xi - a_2x^2b_2\xi \\ &= a_3xb_3a_2\eta - a_2x\eta = \beta' a_3\eta - \beta a_2\eta \end{aligned}$$

with $|\beta|^2 + |\beta'|^2 \neq 0$. As $\{a_3\eta, a_2\eta\}$ is linearly independent, it follows that $((Sx)^2 - S(x^2))\xi \neq 0$, as desired.

We conclude this paper by noting that a unital spectrally bounded elementary operator of length four and above which is not surjective need not be a spectral isometry. As an example the trace on $M_2(\mathbb{C})$ can serve which can be represented as

$$x \mapsto \frac{1}{2}\sum_{j=1}^{4} e_{ij}xe_{ji}$$

where e_{ij} denotes the standard matrix units.

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