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SOLVABILITY OF SINGULAR FRACTIONAL ORDER ELASTIC BEAM SYSTEMS WITH NONLINEARITIES DEPENDING ON LOWER DERIVATIVES

Abstract. In this article, existence results for positive solutions of a class of boundary-value problems for singular fractional order elastic beam systems with nonlinearities depending on lower derivatives are established. The nonlinearities in fractional differential equations may be singular at $t = 0$ and $t = 1$. A weighted function space is constructed and complete continuity of a nonlinear operator is proved. Our analysis relies on a well known fixed point theorem.

1. Introduction. Fractional differential equations have many applications in modeling of physical and chemical processes and in engineering, and have been of great interest recently. Moreover, mathematical aspects of fractional differential equations were discussed by many authors; see the textbooks [27], the survey papers [1, 17], papers [3, 6, 7, 8, 9, 10, 18, 24, 26, 30, 32, 33] and the references therein.

The study of coupled systems of fractional differential equations is also important, as such systems occur in various problems of applied nature; see for instance [4], [5], [11], [12], [13], [16], [19], [25].

In [4], the authors studied the solvability of the following boundary value problem for a fractional differential system:

$$\text{(I)}\qquad \begin{cases} D_{0^+}^{\alpha}u(t) + f(t, v(t), D_{0^+}^{\beta-1}v(t)) = 0, \quad t \in (0,1), 1 < \alpha < 2, \\ D_{0^+}^{\beta}u(t) + g(t, u(t), D_{0^+}^{\alpha-1}u(t)) = 0, \quad t \in (0,1), 1 < \beta < 2, \\ u(0) = 0, \quad u(1) = \gamma u(\eta), \quad v(0) = 0, \quad v(1) = \gamma u(\eta), \end{cases}$$

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where D^{*}_{0+} is the Riemann–Liouville fractional derivative, $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions, and $\gamma\in\mathbb{R}$ and $\eta\in(0,1)$ are such that the homogeneous problem

$$\begin{cases} D^{\alpha}_{0+}u(t)=0, \quad t\in(0,1),\ 1<\alpha<2,\\ D^{\beta}_{0+}u(t)=0, \quad t\in(0,1),\ 1<\beta<2,\\ u(0)=0, \quad u(1)=\gamma u(\eta), \quad v(0)=0, \quad v(1)=\gamma u(\eta) \end{cases}$$

has nontrivial solutions $(u(t),v(t))=(c_1t^{\alpha-1},c_2t^{\beta-1})$. The methods used in [4] are based upon coincidence degree theory.

In [28], Su studied the existence of solutions of the boundary value problem

$$\text{(II)} \qquad \begin{cases} D^{\alpha}_{0+}u(t)+f(t,v(t),D^{p}_{0+}v(t))=0, \quad t\in(0,1),\ 1<\alpha<2,\\ D^{\beta}_{0+}u(t)+g(t,u(t),D^{q}_{0+}u(t))=0, \quad t\in(0,1),\ 1<\beta<2,\\ u(0)=0, \quad u(1)=0, \quad v(0)=0, \quad v(1)=0, \end{cases}$$

where $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions.

In [5], existence of solutions was studied for the boundary value problem

$$\text{(III)} \qquad \begin{cases} D^{\alpha}_{0+}u(t)+f(t,v(t),D^{p}_{0+}v(t))=0, \quad t\in(0,1),\ 1<\alpha<2,\\ D^{\beta}_{0+}u(t)+g(t,u(t),D^{q}_{0+}u(t))=0, \quad t\in(0,1),\ 1<\beta<2,\\ u(0)=0, \quad u(1)=\gamma u(\eta), \quad v(0)=0, \quad v(1)=\gamma u(\eta), \end{cases}$$

where $1<\alpha,\beta<2$, $p,q,\gamma>0$, $0<\eta<1$, $\alpha-q\geq 1$, $\beta-p\geq 1$, $\gamma\eta^{\alpha-1}<1$ and $\gamma\eta^{\beta-1}<1$, and $f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

The main assumptions imposed on f,g in [4, 5, 28] are as follows:

(BA1) there exists a nonnegative function $a\in L^1(0,1)$ and numbers $\epsilon_1,\epsilon_2>0$, $\rho_1,\rho_2\in(0,1)$ such that

$$|f(t,x,y)|\leq a(t)+\epsilon_1|x|^{\rho_1}+\epsilon_2|y|^{\rho_2},$$

(BA2) there exists a nonnegative function $b\in L(0,1)$ and numbers $\delta_1,\delta_2>0$, $\sigma_1,\sigma_2\in(0,1)$ such that

$$|g(t,x,y)|\leq b(t)+\delta_1|x|^{\sigma_1}+\delta_2|y|^{\sigma_2}.$$

One sees that (BA1) and (BA2) imply that both f and g are at most of linear growth. A natural question arises: Do BVP(I), BVP(II) and BVP(III) have solutions if (BA1) and (BA2) do not hold?

On the other hand, existence of positive solutions of fractional elastic beam equations was studied in [14, 20, 21]. To the best of our knowledge, there has been no paper discussing the solvability of singular elastic beam systems with nonlinearities depending on lower derivatives.

With this motivation we discuss the following new boundary value problem for a singular fractional order elastic beam system:

(1)
$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, v(t), v'(t), v''(s), D_{0+}^{n}v(t)), \quad t \in (0,1), \\ D_{0+}^{\beta}v(t) = g(t, u(t), u'(t), u''(t), D_{0+}^{m}u(t)), \quad t \in (0,1), \\ \lim_{t\to 0} t^{4-\alpha}u(t) = 0, \quad D_{0+}^{\alpha-3}u(0) = 0, \quad D_{0+}^{\alpha-2}u(1) = 0, \quad D_{0+}^{\alpha-1}u(1) = 0, \\ \lim_{t\to 0} t^{4-\alpha}v(t) = 0, \quad D_{0+}^{\alpha-3}v(0) = 0, \quad D_{0+}^{\beta-2}v(1) = 0, \quad D_{0+}^{\alpha-1}v(1) = 0, \end{cases}$$

where $\alpha, \beta \in (3,4]$ and $m \in (2, \alpha-1)$, $n \in (2, \beta-1)$, $f, g : (0,1) \times \mathbb{R}^4 \to [0,\infty)$ are continuous, maybe singular at $t = 0$ and $t = 1$, and such that $|x|, |y|, |z|, |w| \le r > 0$ implies that on each subinterval $(0, c]$ with $c \in (0,1)$,

$$g(t, t^{\alpha-2}x, t^{\alpha-3}y, t^{\alpha-4}z, t^{\alpha-m-2}w) \not\equiv 0,$$
$$f(t, t^{\beta-2}x, t^{\beta-3}y, t^{\beta-4}z, t^{\beta-m-2}w) \not\equiv 0.$$

A couple of functions (x, y) with $x, y : (0,1] \to \mathbb{R}$ is called a positive solution of BVP(1) if $x, x', x'', x''', D_{0+}^{\alpha}x$ and $y, y', y'', y''', D_{0+}^{\beta}y$ are in $C^0(0,1]$ and all equations in (1) are satisfied and $[x(t)]^2 + [y(t)]^2 > 0$ on $(0,1)$. The purpose of this paper is to establish existence results for BVP(1) by using a fixed point theorem in Banach spaces.

Our methods are also useful to establish existence results for BVP(I), BVP(II) and BVP(III). We omit the details. But we point out that our assumptions imposed on f, g are weaker than (BA1) and (BA2) (see Remark 3.1).

We note that fourth order two-point boundary value problems are useful for material mechanics because this kind of problems usually characterize the deflection of an elastic beam [14]. The boundary conditions are given according to the controls at the ends of the beam. For example, the nonlinear fourth order problem

$$\begin{cases} x^{(4)}(t) = \lambda f(t, x(t), x'(t)), \quad t \in [0,1], \\ x(0) = x'(0) = x''(1) = x'''(1) = 0 \end{cases} \tag{2}$$

describes the deformations of an elastic beam with one end fixed and the other free. The existence of positive solutions of various boundary value problems for elastic beam equations was extensively studied (see [2, 14, 22, 23, 29, 31]).

Few paper deals with the situation where all lower order derivatives are involved in the nonlinear term explicitly. In fact, the derivatives are of great importance in the problem in some cases, for example, in the linear elastic beam equation (Euler–Bernoulli equation)

$$(EIu''(t))'' = f(t), \qquad t \in (0, L),$$

where $u(t)$ is the deformation function, L is the length of the beam, $f(t)$ is the load density, E is the Young modulus of elasticity and I is the moment of inertia of the cross-section of the beam. In this problem, the physical meaning of the derivatives of $u(t)$ is as follows: $u^{(4)}(t)$ is the load density stiffness, $u'''(t)$ is the shear force stiffness, $u''(t)$ is the bending moment stiffness and $u'(t)$ is the slope. If the payload depends on the shear force stiffness, bending moment stiffness or slope, the cases where the derivatives of the unknown function are involved in the nonlinear term f explicitly occur [15]. In applications, coupled elastic beam systems occur. This leads us to study the solvability of the following BVP for coupled elastic beam equations:

$$\begin{cases} x^{(4)}(t) = f(t, y(t), y'(t), y''(t), y'''(t)), & t \in [0,1], \\ y^{(4)}(t) = g(t, x(t), x'(t), x''(t), x'''(t)), & t \in [0,1], \\ x(0) = x'(0) = x''(1) = x'''(1) = 0, \\ y(0) = y'(0) = y''(1) = y'''(1) = 0. \end{cases} \tag{3}$$

It is easy to see that BVP(1) is a generalization of BVP(3). Hence BVP(1) is a fractional elastic beam system model, and our studies on BVP(1) may be useful.

The remainder of the paper is divided into two sections. In Section 2, we present some preliminary results, i.e., we construct a Banach space and prove complete continuity of a nonlinear operator. In Section 3, we establish sufficient conditions for the existence of positive solutions of BVP(1) by using a fixed point theorem.

2. Preliminary results. For the convenience of the readers, we present some necessary definitions from fractional calculus theory [27]. Let $\sigma_i > 0$, $i = 1, 2, 3$. Denote the Gamma function and Beta function, respectively, by

$$\Gamma(\sigma_1) = \int_0^\infty s^{\sigma_1 - 1} e^{-s}\, ds, \qquad \mathbf{B}(\sigma_2, \sigma_3) = \int_0^1 (1-x)^{\sigma_2 - 1} x^{\sigma_3 - 1}\, dx.$$

Definition 2.1. The *Riemann–Liouville fractional integral* of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha - 1} f(s)\, ds,$$

provided that the right-hand side exists.

Definition 2.2. The *Riemann–Liouville fractional derivative* of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n - 1 \leq \alpha < n$, provided that the right-hand side exists.

One sees from [27], for $\sigma_1 > 0$, $\sigma_2 \geq 0$, that

$$I_{0+}^{\sigma_1} t^{\sigma_2} = \frac{\Gamma(\sigma_2+1)}{\Gamma(\sigma_1+\sigma_2+1)} t^{\sigma_1+\sigma_2}, \qquad D_{0+}^{\sigma_1} t^{\sigma_2} = \frac{\Gamma(\sigma_2+1)}{\Gamma(\sigma_2-\sigma_1+1)} t^{\sigma_2-\sigma_1}.$$

Furthermore for $n - 1 \leq \alpha < n$, $u \in C^0(0,1) \cap L^1(0,1)$, we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where $C_i \in \mathbb{R}$, $i = 1, \ldots, n$ [27].

For our construction, we choose

$$X = \left\{ x : (0,1] \to \mathbb{R} : \begin{array}{l} x, x', x'', D_{0+}^{m} x \in C^0(0,1] \\ \text{and the limits} \\ \lim\limits_{t\to 0} t^{2-\alpha} x(t),\ \lim\limits_{t\to 0} t^{3-\alpha} x'(t), \\ \lim\limits_{t\to 0} t^{4-\alpha} x''(t),\ \lim\limits_{t\to 0} t^{2+m-\alpha} D_{0+}^{m} x(t) \text{ exist} \end{array} \right\}$$

with the norm

$$\|u\|_X = \max\Big\{ \sup_{t\in(0,1]} t^{2-\alpha}|u(t)|,\ \sup_{t\in(0,1]} t^{3-\alpha}|u'(t)|,\ \sup_{t\in(0,1]} t^{4-\alpha}|u''(t)|,\ \sup_{t\in(0,1]} t^{2+m-\alpha}|D_{0+}^{m} u(t)| \Big\}, \qquad u \in X.$$

Lemma 2.0. *Suppose that $\alpha \in (3,4)$ and $m \in (2, \alpha-1)$. Then X is a real Banach space.*

Proof. It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X. Then $\|x_u - x_v\| \to 0$ as $u, v \to \infty$. It follows that

$$\sup_{t\in(0,1]} t^{2-\alpha}|x_u(t) - x_v(t)| \to 0,$$

$$\sup_{t\in(0,1]} t^{3-\alpha}|x_u'(t) - x_v'(t)| \to 0,$$

$$\sup_{t\in(0,1]} t^{4-\alpha}|x_u''(t) - x_v''(t)| \to 0,$$

$$\sup_{t\in(0,1]} t^{2+m-\alpha}|D_{0+}^{m} x_u(t) - D_{0+}^{m} x_v(t)| \to 0, \qquad u, v \to \infty.$$

Define

$$t^{2-\alpha}\overline{x}(t) = \begin{cases} \lim_{t\to 0^+} t^{2-\alpha}x(t), & t=0,\\ t^{2-\alpha}x(t), & t\in(0,1], \end{cases}$$

$$t^{3-\alpha}\overline{x}'(t) = \begin{cases} \lim_{t\to 0^+} t^{3-\alpha}x'(t), & t=0,\\ t^{3-\alpha}x'(t), & t\in(0,1], \end{cases}$$

$$t^{4-\alpha}\overline{x}''(t) = \begin{cases} \lim_{t\to 0^+} t^{4-\alpha}x''(t), & t=0,\\ t^{4-\alpha}x''(t), & t\in(0,1], \end{cases}$$

$$t^{2+m-\alpha}D_{0^+}^m\overline{x}(t) = \begin{cases} \lim_{t\to 0^+} t^{2+m-\alpha}D_{0^+}^m x(t), & t=0,\\ t^{2+m-\alpha}D_{0^+}^m x(t), & t\in(0,1]. \end{cases}$$

We know that

$$t\mapsto t^{2-\alpha}\overline{x}(t),\quad t\mapsto t^{3-\alpha}\overline{x}'(t),\quad t\mapsto t^{4-\alpha}\overline{x}''(t),\quad t\mapsto t^{2+m-\alpha}D_{0^+}^m\overline{x}(t)$$

are continuous on $[0,1]$. Thus

$$t\mapsto t^{2-\alpha}\overline{x}_u(t),\quad t\mapsto t^{3-\alpha}\overline{x}_u'(t),\quad t\mapsto t^{4-\alpha}\overline{x}_u''(t),\quad t\mapsto t^{2+m-\alpha}D_{0^+}^m\overline{x}_u(t)$$

are Cauchy sequences in $C[0,1]$. Hence as $u\to\infty$, $t^{2-\alpha}\overline{x}_u(t)$ uniformly converges to some $x_0\in C[0,1]$, $t^{3-\alpha}\overline{x}_u'(t)$ uniformly converges to some $y_0\in C[0,1]$, $t^{4-\alpha}\overline{x}_u''(t)$ uniformly converges to some $z_0\in C[0,1]$, and $t^{2+m-\alpha}D_{0^+}^m\overline{x}_u(t)$ uniformly converges to some $w_0\in C[0,1]$.

For some $c_u\in\mathbb{R}$ we have

$$\begin{aligned}\Big|\overline{x}_u(t)-c_u-\int_0^t s^{\alpha-3}y_0(s)\,ds\Big| &\le \int_0^t |\overline{x}_u'(s)-s^{\alpha-3}y_0(s)|\,ds\\ &= \int_0^t s^{\alpha-3}|s^{3-\alpha}\overline{x}_u'(s)-y_0(s)|\,ds \le \int_0^t s^{\alpha-3}\,ds \sup_{t\in[0,1]}|t^{3-\alpha}\overline{x}_u'(t)-y_0(t)|\\ &= \frac{t^{\alpha-2}}{\alpha-2}\sup_{t\in[0,1]}|t^{3-\alpha}\overline{x}_u'(t)-y_0(t)| \le \sup_{t\in[0,1]}|t^{3-\alpha}\overline{x}_u'(t)-y_0(t)|\\ &\to 0 \quad \text{uniformly as } n\to\infty.\end{aligned}$$

So $\lim_{u\to\infty}(\overline{x}_u(t)-c_u)=\int_0^t s^{\alpha-3}y_0(s)\,ds$. It follows that $t^{2-\alpha}x_0(t)-c_0=\int_0^t s^{\alpha-3}y_0(s)\,ds$. Thus $y_0(t)=t^{3-\alpha}[t^{2-\alpha}x_0(t)]'$.

For some $c_u,d_u\in\mathbb{R}$ we have

$$\begin{aligned}&\Big|\overline{x}_u(t)-c_u-d_ut-\int_0^t(t-s)s^{\alpha-4}z_0(s)\,ds\Big|\\ &\quad= \Big|\int_0^t(t-s)[\overline{x}_u''(s)\,ds-s^{\alpha-4}z_0(s)]\,ds\Big| = \int_0^t(t-s)s^{\alpha-4}|s^{4-\alpha}\overline{x}_u''(s)-z_0(s)|\,ds\\ &\quad\le \int_0^t(t-s)s^{\alpha-4}\,ds\sup_{t\in[0,1]}|t^{4-\alpha}\overline{x}_u''(t)-z_0(t|\end{aligned}$$

$$= t^{\alpha-2}\int_0^1 (1-w)w^{\alpha-4}\,dw \sup_{t\in[0,1]} |t^{4-\alpha}\overline{x}_u''(t)-z_0(t)| \le \sup_{t\in[0,1]} |t^{4-\alpha}x_n''(t)-z_0(t)|$$
$$= t^{\alpha-2}\mathbf{B}(2,\alpha-3) \sup_{t\in[0,1]} |t^{4-\alpha}\overline{x}_u''(t)-z_0(t)| \le \sup_{t\in[0,1]} |t^{4-\alpha}\overline{x}_u''(t)-z_0(t)| \to 0$$

uniformly as $u\to\infty$. So $\lim_{u\to\infty}(\overline{x}_u(t)-c_u-d_ut) = \int_0^t (t-s)s^{\alpha-4}z_0(s)ds$. It follows that $t^{2-\alpha}x_0(t)-c_0-d_0t = \int_0^t (t-s)s^{\alpha-4}z_0(s)\,ds$. Thus $z_0(t) = t^{4-\alpha}[t^{2-\alpha}x_0(t)]''$.

For some $c_u, d_u, e_u \in \mathbb{R}$ we have

$$\begin{aligned}
&|\overline{x}_u(t)+c_ut^{m-1}+d_ut^{m-2}+e_ut^{m-3}-I_{0+}^m t^{\alpha-m-2}w_0(t)|\\
&\quad= |I_{0+}^m D_{0+}^m \overline{x}_u(t) - -I_{0+}^m t^{\alpha-m-2}w_0(t)|\\
&\quad= \left|\int_0^t \frac{(t-s)^{m-1}}{\Gamma(m)} D_{0+}^m\overline{x}_u(s)\,ds - \int_0^t \frac{(t-s)^{m-1}}{\Gamma(m)} s^{\alpha-m-2}w_0(s)\,ds\right|\\
&\quad\le \int_0^t \frac{(t-s)^{m-1}}{\Gamma(m)} |D_{0+}^m\overline{x}_u(s) - s^{\alpha-m-2}w_0(s)|\,ds\\
&\quad\le \int_0^t \frac{(t-s)^{m-1}}{\Gamma(m)} s^{\alpha-m-2}|s^{2+m-\alpha}D_{0+}^m\overline{x}_u(s) - w_0(s)|\,ds\\
&\quad\le \int_0^t \frac{(t-s)^{m-1}}{\Gamma(m)} s^{\alpha-m-2}\,ds \sup_{t\in[0,1]} |t^{2+m-\alpha}D_{0+}^m\overline{x}_u(t) - w_0(t)|\\
&\quad= t^{\alpha-2}\int_0^1 \frac{(1-w)^{m-1}}{\Gamma(m)} w^{\alpha-m-2}\,dw \sup_{t\in[0,1]} |t^{2+m-\alpha}D_{0+}^m\overline{x}_u(t) - w_0(t)|\\
&\quad= t^{\alpha-2}\frac{\mathbf{B}(m,\alpha-m-1)}{\Gamma(m)} \sup_{t\in[0,1]} |t^{2+m-\alpha}D_{0+}^m\overline{x}_u(t) - w_0(t)| \to 0
\end{aligned}$$

uniformly as $u\to\infty$. So $\lim_{u\to\infty}(\overline{x}_u(t)+c_ut^{m-1}+d_ut^{m-2}+e_ut^{m-3}) = I_{0+}^\beta t^{\alpha-m-2}w_0(t)$. It follows that $t^{2-\alpha}x_0(t)+c_0t^{m-1}+d_0t^{m-2}+e_0t^{m-3}) = I_{0+}^\beta t^{\alpha-m-2}w_0(t)$. Thus $w_0(t) = t^{2+m-\alpha}D_{0+}^m[t^{2-\alpha}x_0(t)]$.

So $x_n\to x_0$ as $n\to\infty$ in X and $t^{2-\alpha}x_0\in X$. It follows that X is a Banach space. ■

Choose

$$Y = \left\{ y:(0,1]\to\mathbb{R} : \begin{array}{l} y, y', y'', D_{0+}^n y \in C^0(0,1] \\ \text{and the limits} \\ \lim_{t\to 0} t^{2-\beta}y(t),\ \lim_{t\to 0} t^{3-\beta}y'(t), \\ \lim_{t\to 0} t^{4-\beta}y''(t),\ \lim_{t\to 0} t^{2+n-\beta}y'''(t) \text{ exist} \end{array} \right\}$$

with the norm

$$\|v\|_Y = \max\Big\{\sup_{t\in(0,1]} t^{2-\beta}|v(t)|,\ \sup_{t\in(0,1]} t^{3-\beta}|v'(t)|,\ \sup_{t\in(0,1]} t^{4-\beta}|v''(t)|,\ \sup_{t\in(0,1]} t^{2+n-\beta}|v'''(t)|\Big\}, \qquad v\in Y.$$

One can show similarly that Y is a real Banach space.

Thus, $(X\times Y, \|\cdot\|)$ is Banach space with the norm defined by

$$\|(x,y)\| = \max\{\|x\| = \|x\|_X,\ \|y\| = \|y\|_Y\} \quad \text{for } (x,y)\in X\times Y.$$

Lemma 2.1. *Suppose that*

(B0) $h\in C^0(0,1)$ *and there exist* $M>0$, $k>-1$ *and* $\sigma\le 0$ *such that* $2+\sigma+k>0$ *and* $|h(t)|\le Mt^k(1-t)^\sigma$ *for all* $t\in(0,1)$.

Then $x\in X$ *is a solution of*

$$\begin{cases} D^\alpha x(t) = h(t), \quad 0<t<1,\\ \lim_{t\to 0} t^{4-\alpha}x(t) = 0,\ \lim_{t\to 0} D_{0+}^{\alpha-3}x(t) = 0,\ D_{0+}^{\alpha-2}x(1) = 0,\ D_{0+}^{\alpha-1}x(1) = 0\end{cases} \tag{4}$$

if and only if x *satisfies*

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds - \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\right]\int_0^1 h(s)\,ds - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)h(s)\,ds = \int_0^1 G(t,s)h(s)\,ds, \tag{5}$$

where G *is defined by*

$$G(t,s) = \begin{cases} -\frac{t^{\alpha-2}(1-s)}{\Gamma(\alpha-1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & s\le t,\\ -\frac{t^{\alpha-2}(1-s)}{\Gamma(\alpha-1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}, & s>t.\end{cases}$$

Proof. Since $h\in C^0(0,1)$ and there exist $M>0$, $k>-1$ and $\sigma\in(-2-k,0]$ such that $|h(t)|\le Mt^k(1-t)^\sigma$ for all $t\in(0,1)$, we have

$$\begin{aligned}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds\right| &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}Ms^k(1-s)^\sigma\,ds\\ &\le M\int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha)}s^k\,ds = Mt^{\alpha+\sigma+k}\int_0^1 \frac{(1-w)^{\alpha+\sigma-1}}{\Gamma(\alpha)}w^k\,dw\\ &= Mt^{\alpha+\sigma+k}\frac{\mathbf{B}(\alpha+\sigma,k+1)}{\Gamma(\alpha)},\end{aligned}$$

$$\left|\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)\,ds\right| \le Mt^{\alpha+\sigma+k-1} \frac{\mathbf{B}(\alpha+\sigma-1, k+1)}{\Gamma(\alpha-1)},$$

$$\left|\int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} h(s)\,ds\right| \le Mt^{\alpha+\sigma+k-2} \frac{\mathbf{B}(\alpha+\sigma-2, k+1)}{\Gamma(\alpha-2)},$$

$$\left|\int_0^t \frac{(t-s)^{\alpha-4}}{\Gamma(\alpha-3)} h(s)\,ds\right| \le Mt^{\alpha+\sigma+k-3} \frac{\mathbf{B}(\alpha+\sigma-3, k+1)}{\Gamma(\alpha-3)},$$

$$\begin{aligned}\left|\int_0^t (t-s)h(s)\,ds\right| &\le \int_0^t (t-s)Ms^k(1-s)^\sigma\,ds \\ &\le M\int_0^t (t-s)^{1+\sigma}s^k\,ds = Mt^{2+\sigma+k}\int_0^1 (1-w)^{1+\sigma}w^k\,dw \\ &= Mt^{2+\sigma+k}\mathbf{B}(2+\sigma, k+1),\end{aligned}$$

$$\left|\int_0^t h(s)\,ds\right| \le Mt^{1+\sigma+k}\mathbf{B}(1+\sigma, k+1).$$

So the integrals

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds,\ \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)\,ds,\ \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} h(s)\,ds,$$

$$\int_0^t \frac{(t-s)^{\alpha-4}}{\Gamma(\alpha-3)} h(s)\,ds \text{ and } \int_0^t (t-s)h(s)\,ds,\ \int_0^t h(s)\,ds, \qquad t \in (0,1],$$

are convergent.

For $t \in (0,1]$, $D_{0^+}^\alpha u(t) = -h(t)$ implies that there exist constants c_i $(i = 1,2,3,4)$ such that

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} + c_4t^{\alpha-4} \tag{6}$$

with

$$D_{0^+}^{\alpha-3}x(t) = \int_0^t \frac{(t-s)^2}{2} h(s)\,ds + c_1\frac{\Gamma(\alpha)}{2}t^2 + c_2\Gamma(\alpha-1)t + c_3\Gamma(\alpha-2),$$

$$D_{0^+}^{\alpha-2}x(t) = \int_0^t (t-s)h(s)\,ds + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1),$$

$$D_{0^+}^{\alpha-1}x(t) = \int_0^t h(s)\,ds + c_1\Gamma(\alpha).$$

One sees that

$$t^{4-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds \to 0 \quad \text{as } t\to 0 \text{ (note } 4+\sigma+k>0), \tag{7}$$

$$\int_0^t (t-s)^2 h(s)\,ds \to 0 \quad \text{as } t\to 0 \text{ (note } 2+\sigma+k>0). \tag{8}$$

Now, (6) and (7) with $\lim_{t\to 0} t^{4-\alpha}x(t)=0$ imply that $c_4=0$. Furthermore, (8) with $\lim_{t\to 0} D_{0+}^{\alpha-3}x(t)=0$ implies $c_3=0$. Using $D_{0+}^{\alpha-2}x(1)=0$ and $D_{0+}^{\alpha-1}x(1)=0$, we get

$$c_1\Gamma(\alpha)+c_2\Gamma(\alpha-1) = -\int_0^1 (1-s)h(s)\,ds, \qquad c_1\Gamma(\alpha) = -\int_0^1 h(s)\,ds.$$

It follows that

$$c_1 = -\frac{1}{\Gamma(\alpha)}\int_0^1 h(s)\,ds,$$

$$c_2 = \frac{1}{\Gamma(\alpha-1)}\Big[-\int_0^1 (1-s)h(s)\,ds + \int_0^1 h(s)\,ds\Big].$$

Then

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds - \Big[\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\Big]\int_0^1 h(s)\,ds - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)h(s)\,ds = \int_0^1 G(t,s)h(s)\,ds,$$

where G is defined above. Hence x satisfies (5).

Conversely, if x satisfies (5), then

$$x'(t) = -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}h(s)\,ds + \Big[\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{t^{\alpha-3}}{\Gamma(\alpha-2)}\Big]\int_0^1 h(s)\,ds + \frac{t^{\alpha-3}}{\Gamma(\alpha-2)}\int_0^1 (1-s)h(s)\,ds,$$

$$x''(t) = -\int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}h(s)\,ds + \Big[\frac{t^{\alpha-3}}{\Gamma(\alpha-2)} - \frac{t^{\alpha-4}}{\Gamma(\alpha-3)}\Big]\int_0^1 h(s)\,ds + \frac{t^{\alpha-4}}{\Gamma(\alpha-3)}\int_0^1 (1-s)h(s)\,ds,$$

$$\begin{aligned}D_{0^+}^m x(t) = & -\int\limits_0^t \frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} h(s)\,ds \\ & + \left[\frac{t^{\alpha-m-1}}{\Gamma(\alpha-m)} - \frac{t^{\alpha-m-2}}{\Gamma(\alpha-m-1)}\right] \int\limits_0^1 h(s)\,ds \\ & + \frac{t^{\alpha-m-2}}{\Gamma(\alpha-m-1)} \int\limits_0^1 (1-s)h(s)\,ds.\end{aligned}$$

It is easy to see that $x, x', x'', x''' \in C^0(0,1]$ and

$$t^{2-\alpha}\left|\int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds\right| \le M t^{2-\alpha} t^{\alpha+\sigma+k} \frac{\mathbf{B}(\alpha+\sigma, k+1)}{\Gamma(\alpha)},$$

$$t^{3-\alpha}\left|\int\limits_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)\,ds\right| \le M t^{3-\alpha} t^{\alpha+\sigma+k-1} \frac{\mathbf{B}(\alpha+\sigma-1, k+1)}{\Gamma(\alpha-1)},$$

$$t^{4-\alpha}\left|\int\limits_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} h(s)\,ds\right| \le M t^{4-\alpha} t^{\alpha+\sigma+k-2} \frac{\mathbf{B}(\alpha+\sigma-2, k+1)}{\Gamma(\alpha-2)},$$

$$t^{2+m-\alpha}\left|\int\limits_0^t \frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} h(s)\,ds\right| \le M t^{2+m-\alpha} t^{\alpha+\sigma+k-m} \frac{\mathbf{B}(\alpha+\sigma-m, k+1)}{\Gamma(\alpha-3)}.$$

So

$$\lim_{t\to0} t^{2-\alpha}x(t), \quad \lim_{t\to0} t^{3-\alpha}x'(t), \quad \lim_{t\to0} t^{4-\alpha}x''(t), \quad \lim_{t\to0} t^{2+m-\alpha}D_{0^+}^m x(t)$$

exist. We can show that $x \in X$ and x is a solution of BVP(4).

So $x \in X$ is a solution of problem (4) if and only if x satisfies (5). The proof is complete. ■

Remark 2.1. $G(t,s) \ge 0$ for all $(t,s) \in [0,1]\times[0,1]$. In fact, for $s > t$, we have

$$\begin{aligned}-\frac{t^{\alpha-2}(1-s)}{\Gamma(\alpha-1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} &= \frac{t^{\alpha-2}}{\Gamma(\alpha)}\big(-(\alpha-1)(1-s) - t + (\alpha-1)\big) \\ &= (\alpha-1)s - t > (\alpha-1)s - s = (\alpha-2)s > 0.\end{aligned}$$

For $0 \le s \le t$, set

$$\phi(s) = -\frac{t^{\alpha-2}(1-s)}{\Gamma(\alpha-1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \qquad s \le t.$$

One finds $\phi(0) = 0$ and

$$\phi'(s) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \ge 0.$$

Hence $\phi(s) \geq 0$ for $s \in [0,t]$. From the above discussion, we have $G(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$.

REMARK 2.2. If h is nonnegative on $(0,1)$ and $h(t) \not\equiv 0$ on each subinterval $(0,c]$ with $c \in (0,1)$, then $x(t)$ is positive on $(0,1)$. In fact, by Remark 2.1, $x(t) \geq 0$ on $(0,1)$. Suppose that there exists $t_0 \in (0,1)$ such that $x(t_0) = 0$. From $-\frac{t^{\alpha-2}(1-s)}{\Gamma(\alpha-1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \geq 0$ for every $t,s \in [0,1]$, we have

$$\begin{aligned} 0 = x(t_0) &= \int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds - \left[\frac{t_0^{\alpha-1}}{\Gamma(\alpha)} - \frac{t_0^{\alpha-2}}{\Gamma(\alpha-1)}\right] \int_0^1 h(s)\,ds \\ &\quad - \frac{t_0^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s)h(s)\,ds \\ &\geq \int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds \geq 0. \end{aligned}$$

As $h \in C^0(0,1)$ and h is nonnegative, we have $h(t) \equiv 0$ on $(0,t_0)$, which is a contradiction. Hence x is positive on $(0,1)$.

LEMMA 2.2. *Suppose that* (B0) *holds. Then $y \in Y$ is a solution of*

$$\text{(9)} \quad \begin{cases} D^{\beta} y(t) = h(t), \quad 0 < t < 1, \\ \lim_{t\to 0} t^{4-\alpha} y(t) = 0, \ \lim_{t\to 0} D_{0+}^{\alpha-3} y(t) = 0, \ D_{0+}^{\beta-2} y(1) = 0, \ D_{0+}^{\alpha-1} y(1) = 0 \end{cases}$$

if and only if y satisfies

$$\text{(10)} \quad \begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s)\,ds - \left[\frac{t^{\beta-1}}{\Gamma(\beta)} - \frac{t^{\beta-2}}{\Gamma(\beta-1)}\right] \int_0^1 h(s)\,ds \\ &\quad - \frac{t^{\beta-2}}{\Gamma(\beta-1)} \int_0^1 (1-s)h(s)\,ds = \int_0^1 H(t,s)h(s)\,ds, \end{aligned}$$

where H is defined by

$$H(t,s) = \begin{cases} -\frac{t^{\beta-2}(1-s)}{\Gamma(\beta-1)} - \frac{t^{\beta-1}}{\Gamma(\beta)} + \frac{t^{\beta-2}}{\Gamma(\beta-1)} + \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}, & s \leq t, \\ -\frac{t^{\beta-2}(1-s)}{\Gamma(\beta-1)} - \frac{t^{\beta-1}}{\Gamma(\beta)} + \frac{t^{\beta-2}}{\Gamma(\beta-1)}, & s > t. \end{cases}$$

Proof. The proof is similar to that of Lemma 2.1 and is omitted. ∎

REMARK 2.3. Similarly to Remark 2.1, we can prove that $H(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$.

REMARK 2.4. If h is nonnegative on $(0,1)$ and $h(t) \not\equiv 0$ on each subinterval $(0,c]$ with $c \in (0,1)$, then $y(t)$ is positive on $(0,1)$.

Define an operator T on $X \times Y$, for $(x, y) \in X \times Y$, by $T(x, y)(t) = ((T_1 y)(t), (T_2 x)(t))$ with

$$(T_1 x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), y'(s), y''(s), D_{0+}^n y(s))\, ds - \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\right] \int_0^1 f(s, y(s), y'(s), y''(s), D_{0+}^n y(s))\, ds - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s) f(s, y(s), y'(s), y''(s), D_{0+}^n y(s))\, ds$$

and

$$(T_2 y)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s, x(s), x'(s), x''(s), D_{0+}^m x(s))\, ds - \left[\frac{t^{\beta-1}}{\Gamma(\beta)} - \frac{t^{\beta-2}}{\Gamma(\beta-1)}\right] \int_0^1 g(s, x(s), x'(s), x''(s), D_{0+}^m x(s))\, ds - \frac{t^{\beta-2}}{\Gamma(\beta-1)} \int_0^1 (1-s) g(s, x(s), x'(s), x''(s), D_{0+}^m x(s))\, ds.$$

REMARK 2.5. If both f, g are nonnegative, and

$$f(t, t^{\beta-2}x, t^{\beta-3}y, t^{\beta-4}z, t^{\beta-n-2}w) \not\equiv 0,$$
$$g(t, t^{\alpha-2}x, t^{\alpha-3}y, t^{\alpha-4}z, t^{\alpha-n-2}w) \not\equiv 0$$

on each subinterval $(0, c]$ with $c \in (0, 1)$, we know from Lemmas 2.1 and 2.2 and Remarks 2.1–2.4 that $(x, y) \in X \times Y$ is a positive solution of BVP(1) if and only if (x, y) is a fixed point of T.

LEMMA 2.3. *Suppose that*

(B1) *$f(t, x, y, z, w)$ is continuous on $(0, 1) \times \mathbb{R}^4$ and for each $r > 0$ there exist $M_r > 0$, $k_1 > -1$, $\sigma_1 \le 0$ such that $2 + \sigma_1 + k_1 > 0$ and*

$$|f(t, t^{\beta-2}x, t^{\beta-3}y, t^{\beta-4}z, t^{\beta-n-2}w)| \le M_r t^{k_1} (1-t)^{\sigma_1}$$

for all $t \in (0, 1)$ and $|x|, |y|, |z|, |w| \le r$.

Then $T_1 : Y \to X$ is completely continuous.

Proof. We divide the proof into four steps.

STEP 1. *$T_1 : Y \to X$ is well defined.*

For $y \in Y$, there exists $r > 0$ such that

$$\max\Big\{\sup_{t\in(0,1]} t^{2-\beta}|y(t)|,\ \sup_{t\in(0,1]} t^{3-\beta}|y'(t)|,\ \sup_{t\in(0,1]} t^{4-\beta}|y''(t)|,\ \sup_{t\in(0,1]} t^{2+n-\beta}|D_{0+}^{n} y(t)|\Big\} < r.$$

Then there exist $k_1 > -1$, $\sigma_1 \le 0$ and $M_r \ge 0$ such that $2 + k_1 + \sigma_1 > 0$ and

$$\begin{aligned}(11)\quad &|f(t, y(t), y'(t), y''(t), D_{0+}^{n} y(t))|\\ &= |f(t, t^{\beta-2}t^{2-\beta}y(t), t^{\beta-3}t^{3-\beta}y'(t), t^{\beta-4}t^{4-\beta}y''(t), t^{\beta-n-2}t^{2+n-\beta}D_{0+}^{n}y(t))|\\ &\le M_r t^{k_1}(1-t)^{\sigma_1} \quad \text{for all } t \in (0,1).\end{aligned}$$

On the other hand, by direct computation, we can get the formulas for $(T_1y)'(t)$, $(T_2x)''(t)$, $D_{0+}^{m}(T_1y)(t)$. Similarly to Lemmas 2.1, from (11), we can prove that $T_1y \in X$. So $T_1 : Y \to X$ is well defined.

STEP 2. *T_1 is continuous.*

Let $\{y_u \in Y\}$ be a sequence such that $y_u \to y_0$ in Y as $u \to \infty$. Then there exists $r > 0$ such that

$$\max\Big\{\sup_{t\in(0,1]} t^{2-\beta}|y_u(t)|,\ \sup_{t\in(0,1]} t^{3-\beta}|y_u'(t)|,\ \sup_{t\in(0,1]} t^{4-\beta}|y_u''(t)|,\ \sup_{t\in(0,1]} t^{2+n-\beta}|D_{0+}^{n} y_u(t)|\Big\} \le r$$

for $n = 0, 1, 2, \ldots$. Then there exist $M_r > 0$, $k_1 > -1$ and $\sigma_1 \le 0$ such that $2 + \sigma_1 + k_1 > 0$ and

$$|f(t, y_u(t), y_u'(t), y_u''(t), D_{0+}^{n} y_u(t))| \le M_r t^{k_1}(1-t)^{\sigma_1}$$

for all $t \in (0,1), n = 0, 1, 2, \ldots$.

Denote $f_y(t) = f(t, y(t), y'(t), y''(t), D_{0+}^{n} y(t))$. We have

$$\begin{aligned}t^{2-\alpha}|(T_1y_u)(t) - (T_1y_0)(t)| &\le t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_{y_u}(s) - f_{y_0}(s|\,ds\\ &\quad + \left|\frac{t}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha-1)}\right|\int_0^1 |f_{y_u}(s) - f_{y_0}(s|\,ds\\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)|f_{y_u}(s) - f_{y_0}(s|\,ds\end{aligned}$$

$$\leq t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} 2M_r s^{k_1}(1-s)^{\sigma_1}\,ds$$
$$+\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right]\int_0^1 2M_r s^{k_1}(1-s)^{\sigma_1}\,ds$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)2M_r s^{k_1}(1-s)^{\sigma_1}\,ds$$
$$\leq 2M_r t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(t-s)^{\sigma_1}\,ds$$
$$+2M_r\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right]\int_0^1 s^{k_1}(1-s)^{\sigma_1}\,ds$$
$$+\frac{2M_r}{\Gamma(\alpha-1)}\int_0^1 (1-s)s^{k_1}(1-s)^{\sigma_1}\,ds$$
$$= 2M_r t^{2+\sigma_1+k_1}\int_0^1 \frac{(1-w)^{\alpha+\sigma_1-1}}{\Gamma(\alpha)} s^{k_1}\,ds$$
$$+2M_r\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right]\mathbf{B}(\sigma_1+1,K_1+1)$$
$$+\frac{2M_r}{\Gamma(\alpha-1)}\mathbf{B}(2+\sigma_1,k_1+1)$$
$$\leq 2M_r\frac{\mathbf{B}(\alpha+\sigma_1,k_1+1)}{\Gamma(\alpha)}$$
$$+2M_r\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right]\mathbf{B}(\sigma_1+1,K_1+1)$$
$$+\frac{2M_r}{\Gamma(\alpha-1)}\mathbf{B}(2+\sigma_1,k_1+1)$$

and similarly

$$t^{3-\alpha}|(T_1y_u)'(t)-(T_1y_0)'(t)|\leq 2M_r\frac{\mathbf{B}(\alpha+\sigma_1-1,k_1+1)}{\Gamma(\alpha-1)}$$
$$+2M_r\left[\frac{1}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-2)}\right]\mathbf{B}(\sigma_1+1,K_1+1)+\frac{2M_r}{\Gamma(\alpha-2)}\mathbf{B}(2+\sigma_1,k_1+1),$$
$$t^{4-\alpha}|(T_1y_u)''(t)-(T_1y_0)''(t)|\leq 2M_r\frac{\mathbf{B}(\alpha+\sigma_1-2,k_1+1)}{\Gamma(\alpha-2)}$$
$$+2M_r\left[\frac{1}{\Gamma(\alpha-2)}+\frac{1}{\Gamma(\alpha-3)}\right]\mathbf{B}(\sigma_1+1,K_1+1)+\frac{2M_r}{\Gamma(\alpha-3)}\mathbf{B}(2+\sigma_1,k_1+1),$$

$$t^{2+m-\alpha}|D_{0^+}^m(T_1y_u)(t)-D_{0^+}^m(T_1y_0)(t)|\le 2M_r\frac{\mathbf{B}(\alpha+\sigma_1-m,k_1+1)}{\Gamma(\alpha-m)}$$
$$+2M_r\left[\frac{1}{\Gamma(\alpha-m)}+\frac{1}{\Gamma(\alpha-m-1)}\right]\mathbf{B}(\sigma_1+1,K_1+1)$$
$$+\frac{2M_r}{\Gamma(\alpha-m-1)}\mathbf{B}(2+\sigma_1,k_1+1).$$

By the dominant convergence theorem, we have

$$\lim_{n\to\infty}\sup_{t\in(0,1]}t^{2-\alpha}|(T_1y_u)(t)-(T_1y_0)(t)|$$
$$=\lim_{n\to\infty}\sup_{t\in(0,1]}t^{3-\alpha}|(T_1y_u)'(t)-(T_1y_0)'(t)|=0$$

and

$$\lim_{n\to\infty}\sup_{t\in(0,1]}t^{4-\alpha}|(T_1y_u)''(t)-(T_1y_0)''(t)|$$
$$=\lim_{n\to\infty}\sup_{t\in(0,1]}t^{2+m-\alpha}|D_{0^+}^m(T_1y_u)(t)-D_{0^+}^m(T_1y_0)(t)|=0.$$

Thus $\|T_1y_u-T_1y_0\|\to 0$ as $u\to\infty$, so T_1 is continuous.

Now we prove that T_1 maps bounded sets in Y into relatively compact sets in X. Let $\Omega\subset Y$ be a bounded subset. Then there exists $r>0$ such that $\|y\|_Y\le r$ for all $y\in\Omega$.

Hence there exist $M_r>0$, $k_1>-1$ and $\sigma_1\le 0$ such that $2+\sigma_1+k_1>0$ and (11) holds.

Step 3. *$\{T_1y : y\in\Omega\}$ is a bounded set in X for every bounded $\Omega\subset Y$.*

Similarly to Step 2, we can show that

$$t^{2-\alpha}|(T_1y)(t)|\le M_r\frac{\mathbf{B}(\alpha+\sigma_1,k_1+1)}{\Gamma(\alpha)}$$
$$+M_r\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right]\mathbf{B}(\sigma_1+1,K_1+1)+\frac{M_r}{\Gamma(\alpha-1)}\mathbf{B}(2+\sigma_1,k_1+1),$$

$$t^{3-\alpha}|(T_1y)'(t)|\le M_r\frac{\mathbf{B}(\alpha+\sigma_1-1,k_1+1)}{\Gamma(\alpha-1)}$$
$$+M_r\left[\frac{1}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-2)}\right]\mathbf{B}(\sigma_1+1,k_1+1)+\frac{M_r}{\Gamma(\alpha-2)}\mathbf{B}(2+\sigma_1,k_1+1),$$

$$t^{4-\alpha}|(T_1y)''(t)|\le M_r\frac{\mathbf{B}(\alpha+\sigma_1-2,k_1+1)}{\Gamma(\alpha-2)}$$
$$+M_r\left[\frac{1}{\Gamma(\alpha-2)}+\frac{1}{\Gamma(\alpha-3)}\right]\mathbf{B}(\sigma_1+1,K_1+1)+\frac{M_r}{\Gamma(\alpha-3)}\mathbf{B}(2+\sigma_1,k_1+1),$$

$$t^{2+m-\alpha}|D_{0^+}^m(T_1y)(t)| \le M_r\frac{\mathbf{B}(\alpha+\sigma_1-m,k_1+1)}{\Gamma(\alpha-m)} + M_r\left[\frac{1}{\Gamma(\alpha-m)}+\frac{1}{\Gamma(\alpha-m-1)}\right]\mathbf{B}(\sigma_1+1,K_1+1) + \frac{M_r}{\Gamma(\alpha-m-1)}\mathbf{B}(2+\sigma_1,k_1+1).$$

So T_1 maps bounded sets into bounded sets in X.

STEP 4. *$\{T_1y : y\in\Omega\}$ is a relatively compact set in X for every bounded set $\Omega\subset Y$.*

We prove that

$$\{t^{2-\alpha}(T_1y)(t) : y\in\Omega\},\quad \{t^{3-\alpha}(T_1y)'(t) : y\in\Omega\},$$
$$\{t^{4-\alpha}(T_1y)''(t) : y\in\Omega\},\quad \{t^{2+m-\alpha}D_{0^+}^m(T_1y)(t) : y\in\Omega\}$$

are equi-continuous on $(0,1]$.

First, let $t_1,t_2\in[e,d]\subset(0,1]$, $t_1<t_2$, $0<e<d\le1$, and $y\in\Omega$. Then

$$\begin{aligned}
&|t_1^{2-\alpha}(T_1y)(t_1)-t_2^{2-\alpha}(T_1y)(t_2)|\\
&\le\bigg|t_1^{2-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s),y'(s),y''(s),D_{0^+}^ny(s))\,ds\\
&\quad- t_2^{2-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s),y'(s),y''(s),D_{0^+}^ny(s))\,ds\bigg|\\
&\quad+\frac{1}{\Gamma(\alpha)}|t_1-t_2|\int_0^1|f(s,y(s),y'(s),y''(s),D_{0^+}^ny(s))|\,ds\\
&\le\bigg|t_1^{2-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s),y'(s),y''(s),D_{0^+}^ny(s))\,ds\\
&\quad- t_2^{2-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s),y'(s),y''(s),D_{0^+}^ny(s))\,ds\bigg|\\
&\quad+\frac{M_r}{\Gamma(\alpha)}|t_1-t_2|\int_0^1 s^{k_1}(1-s)^{\sigma_1}\,ds.
\end{aligned}$$

We find that

$$\bigg|t_1^{2-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s),y'(s),y''(s),D_{0^+}^ny(s))\,ds - t_2^{2-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s),y'(s),y''(s),D_{0^+}^ny(s))\,ds\bigg|$$

$$
\begin{aligned}
&\leq |t_2^{2-\alpha}-t_1^{2-\alpha}|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds\\
&\quad+t_2^{2-\alpha}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds\\
&\quad+t_2^{2-\alpha}\int_0^{t_2}\frac{|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|}{\Gamma(\alpha)}|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds\\
&\leq |t_2^{2-\alpha}-t_1^{2-\alpha}|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}M_r s^{k_1}(1-s)^{\sigma_1}\,ds\\
&\quad+t_2^{2-\alpha}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}M_r s^{k_1}(1-s)^{\sigma_1}\,ds\\
&\quad+t_2^{2-\alpha}\int_0^{t_2}\frac{|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|}{\Gamma(\alpha)}M_r s^{k_1}(1-s)^{\sigma_1}\,ds\\
&\leq M_r\bigg[e^{4-2\alpha}|t_2^{\alpha-2}-t_1^{\alpha-2}|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(t_1-s)^{\sigma_1}ds\\
&\quad+e^{2-\alpha}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(t_2-s)^{\sigma_1}\,ds\\
&\quad+t_2^{2-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{\sigma_1}\,ds\bigg]\\
&=M_r\bigg[e^{4-2\alpha}|t_2^{\alpha-2}-t_1^{\alpha-2}|t_1^{\alpha+k_1+\sigma_1}\int_0^1\frac{(1-w)^{\alpha+\sigma_1-1}}{\Gamma(\alpha)}w^{k_1}\,dw\\
&\quad+e^{2-\alpha}t_2^{\alpha+k_1+\sigma_1}\int_{t_1/t_2}^1\frac{(1-w)^{\alpha+\sigma_1-1}}{\Gamma(\alpha)}w^{k_1}\,dw\\
&\quad+e^{2-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{\sigma_1}\,ds\bigg].
\end{aligned}
$$

We can prove that $1-x^\tau\leq\frac{\tau}{\tau-2}(1-x)^{\tau-2}$ for all $x\in[0,1]$ and $\tau\in(2,3)$. In fact, let $g(x)=1-x^\tau-\frac{\tau}{\tau-2}(1-x)^{\tau-2}$. It is easy to see that $g(1)=0$ and

$$g'(x)=-\tau x^{\tau-1}+\tau(1-x)^{\tau-3}\geq 0,\qquad x\in(0,1).$$

Thus $g(x)\leq g(1)=0$ for all $x\in[0,1]$. Choose $\tau=\alpha-1$ and $x=v/u$. It follows that

$$u^{\alpha-2}-\frac{v^{\alpha-1}}{u}\leq\frac{\alpha-1}{\alpha-3}u(u-v)^{\alpha-3},\qquad u\geq v.$$

Hence

$$\begin{aligned}\Big|t_1^{2-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\\ -t_2^{2-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\Big|\\ \le M_r\Big[e^{4-2\alpha}|t_2^{\alpha-2}-t_1^{\alpha-2}|\frac{\mathbf{B}(\alpha+\sigma_1,k_1+1)}{\Gamma(\alpha)}\\ +e^{2-\alpha}\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-1}}{\Gamma(\alpha)}w^{k_1}\,dw\\ +e^{2-\alpha}\int_0^1\frac{\frac{\alpha-1}{\alpha-3}(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{\sigma_1}\,ds\Big]\\ \le M_r\Big[e^{4-2\alpha}|t_2^{\alpha-2}-t_1^{\alpha-2}|\frac{\mathbf{B}(\alpha+\sigma_1,k_1+1)}{\Gamma(\alpha)}\\ +e^{2-\alpha}\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-1}}{\Gamma(\alpha)}w^{k_1}\,dw\\ +e^{2-\alpha}\frac{\alpha-1}{\alpha-3}(t_2-t_1)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\mathbf{B}(\sigma_1+1,k_1+1)\Big].\end{aligned}$$

As $t_1\to t_2$, the right-hand side tends to zero uniformly. So

$$|t_1^{2-\alpha}(T_1y)(t_1)-t_2^{2-\alpha}(T_1y)(t_2)|\to 0 \quad \text{uniformly as } t_1\to t_2 \text{ on } [e,d]\subseteq(0,1]. \tag{12}$$

Furthermore,

$$\begin{aligned}\Big|t^{2-\alpha}(T_1y)(t)-\Big(-\frac{1}{\Gamma(\alpha-1)}\int_0^1 f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))d\\ +\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\Big)\Big|\\ \le t^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds\\ +\frac{1}{\Gamma(\alpha)}t\int_0^1|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds\\ \le M_rt^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{\sigma_1}\,ds+\frac{M_r}{\Gamma(\alpha)}t\int_0^1 s^{k_1}(1-s)^{\sigma_1}\,ds\\ \le M_rt^{2+\sigma_1+k_1}\int_0^1\frac{(1-w)^{\alpha+\sigma_1-1}}{\Gamma(\alpha)}w^{k_1}\,dw+\frac{M_r}{\Gamma(\alpha)}t\mathbf{B}(\sigma_1,k_1+1).\end{aligned}$$

It follows that

$$t^{2-\alpha}(T_1y)(t) \text{ is uniformly convergent as } t \to 0. \tag{13}$$

From (12) and (13), we know that $\{t^{2-\beta}(T_1y)(t) : y \in \Omega\}$ is equi-continuous on $(0,1]$.

Secondly, let $t_1, t_2 \in [e,d] \subset (0,1]$ with $t_1 < t_2$, $0 < e < d \leq 1$, and $y \in \Omega$. Then

$$\begin{aligned}
&|t_1^{3-\alpha}(T_1y)'(t_1) - t_2^{3-\alpha}(T_1y)'(t_2)| \\
&\quad \leq \Bigg| t_1^{3-\alpha}\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \\
&\qquad - t_2^{3-\alpha}\int_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\Bigg| \\
&\qquad + |t_1-t_2|\frac{1}{\Gamma(\alpha-1)}\int_0^1 |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds \\
&\quad \leq \Bigg| t_1^{3-\alpha}\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \\
&\qquad - t_2^{3-\alpha}\int_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\Bigg| \\
&\qquad + M_r|t_1-t_2|\frac{1}{\Gamma(\alpha-1)}\mathbf{B}(\sigma_1+1,k_1+1).
\end{aligned}$$

We find that

$$\begin{aligned}
&\Bigg| t_1^{3-\alpha}\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \\
&\quad - t_2^{3-\alpha}\int_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\Bigg| \\
&\quad \leq |t_2^{3-\alpha} - t_1^{3-\alpha}|\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds \\
&\quad + t_2^{3-\alpha}\int_{t_1}^{t_2} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds \\
&\quad + t_2^{3-\alpha}\int_0^{t_2} \frac{|(t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2}|}{\Gamma(\alpha-1)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds
\end{aligned}$$

$$\le M_r\Big[e^{6-2\alpha}|t_2^{\alpha-3}-t_1^{\alpha-3}|t_1^{\alpha+k_1+\sigma_1-1}\int_0^1\frac{(1-w)^{\alpha+\sigma_1-2}}{\Gamma(\alpha-1)}w^{k_1}\,dw$$

$$+e^{3-\alpha}t_2^{\alpha+k_1+\sigma_1-1}\int_{t_1/t_2}^1\frac{(1-w)^{\alpha+\sigma_1-2}}{\Gamma(\alpha-1)}w^{k_1}\,dw$$

$$+e^{3-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{k_1}(1-s)^{\sigma_1}\,ds\Big].$$

We can prove that $1-x^\tau\le\frac{\tau}{\tau-1}(1-x)^{\tau-1}$ for all $x\in[0,1]$ and $\tau\in(1,2)$. In fact, let $g(x)=1-x^\tau-\frac{\tau}{\tau-1}(1-x)^{\tau-1}$. It is easy to see that $g(1)=0$ and

$$g'(x)=-\tau x^{\tau-1}+\tau(1-x)^{\tau-2}\ge 0,\qquad x\in(0,1).$$

Thus $g(x)\le g(1)=0$ for all $x\in[0,1]$. Choose $\tau=\alpha-2$ and $x=v/u$. Then

$$u^{\alpha-2}-v^{\alpha-2}\le\frac{\alpha-2}{\alpha-3}(u-v)^{\alpha-3},\qquad u\ge v.$$

Hence

$$\Big|t_1^{3-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds$$

$$-t_2^{3-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\Big|$$

$$\le M_r\Big[e^{6-2\alpha}|t_2^{\alpha-3}-t_1^{\alpha-3}|\frac{\mathbf{B}(\alpha+\sigma_1-1,k_1+1)}{\Gamma(\alpha-1)}$$

$$+e^{3-\alpha}\int_{t_1/t_2}^1\frac{(1-w)^{\alpha+\sigma_1-2}}{\Gamma(\alpha-1)}w^{k_1}\,dw$$

$$+e^{3-\alpha}\frac{\alpha-2}{\alpha-3}(t_2-t_1)^{\alpha-3}\frac{1}{\Gamma(\alpha-1)}\int_0^1 s^{k_1}(1-s)^{\sigma_1}\,ds\Big].$$

As $t_1\to t_2$, the right-hand side tends to zero uniformly. So

$$|t_1^{3-\alpha}(T_1y)'(t_1)-t_2^{3-\alpha}(T_1y)'(t_2)|\to 0 \tag{14}$$

uniformly as $t_1\to t_2$ on $[e,d]\subseteq(0,1]$.

Furthermore,

$$\Big|t^{3-\alpha}(T_1y)'(t)-\Big(-\frac{1}{\Gamma(\alpha-2)}\int_0^1 f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds$$

$$+\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-s)f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\Big)\Big|$$

$$\le t^{3-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds$$
$$+\frac{1}{\Gamma(\alpha-1)}t\int_0^1 |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds$$
$$\le M_r t^{3-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{k_1}(1-s)^{\sigma_1}\,ds+\frac{M_r}{\Gamma(\alpha-1)}t\int_0^1 s^{k_1}(1-s)^{\sigma_1}\,ds$$
$$\le M_r t^{2+\sigma_1+k_1}\int_0^1 \frac{(1-w)^{\alpha+\sigma_1-2}}{\Gamma(\alpha-1)}w^{k_1}\,dw+\frac{M_r}{\Gamma(\alpha-1)}t\mathbf{B}(\sigma_1,k_1+1).$$

It follows that

(15) $t^{3-\alpha}(T_1y)'(t)$ is uniformly convergent as $t\to 0$.

From (14) and (15), we know that $\{t^{3-\alpha}(T_1y)'(t) : y\in\Omega\}$ is equi-continuous on $(0,1]$.

Thirdly, let $t_1,t_2\in(0,1]$ with $t_1<t_2$ and $y\in\Omega$. Then

$$|t_1^{4-\alpha}(T_1y)''(t_1)-t_2^{4-\alpha}(T_1y)''(t_2)|$$
$$\le\bigg|t_1^{4-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds$$
$$-t_2^{4-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-3}}{\Gamma(\alpha-2)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\bigg|$$
$$+|t_1-t_2|\frac{1}{\Gamma(\alpha-2)}\int_0^1|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds$$
$$\le\bigg|t_1^{4-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds$$
$$-t_2^{4-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-3}}{\Gamma(\alpha-2)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\bigg|$$
$$+M_r|t_1-t_2|\frac{1}{\Gamma(\alpha-2)}\mathbf{B}(\sigma_1+1,k_1+1).$$

We find that

$$\bigg|t_1^{4-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds$$
$$-t_2^{4-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-3}}{\Gamma(\alpha-2)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\bigg|$$

$$\leq |t_2^{4-\alpha} - t_1^{4-\alpha}| \int_0^{t_1} \frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\, ds$$

$$+ t_2^{4-\alpha} \int_{t_1}^{t_2} \frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\, ds$$

$$+ t_2^{4-\alpha} \int_0^{t_2} \frac{|(t_1-s)^{\alpha-3}-(t_2-s)^{\alpha-3}|}{\Gamma(\alpha-2)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\, ds$$

$$\leq M_r \bigg[|t_2^{4-\alpha} - t_1^{4-\alpha}| \int_0^{t_1} \frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)} s^{k_1}(1-s)^{\sigma_1}\, ds$$

$$+ t_2^{4-\alpha} \int_{t_1}^{t_2} \frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)} s^{k_1}(1-s)^{\sigma_1}\, ds$$

$$+ t_2^{4-\alpha} \int_0^{t_2} \frac{|(t_1-s)^{\alpha-3}-(t_2-s)^{\alpha-3}|}{\Gamma(\alpha-2)} s^{k_1}(1-s)^{\sigma_1}\, ds \bigg]$$

$$\leq M_r \bigg[|t_2^{4-\alpha} - t_1^{4-\alpha}| \int_0^{t_1} \frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)} s^{k_1}(t_1-s)^{\sigma_1}\, ds$$

$$+ t_2^{4-\alpha} \int_{t_1}^{t_2} \frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)} s^{k_1}(t_2-s)^{\sigma_1}\, ds$$

$$+ t_2^{4-\alpha} \int_0^{t_2} \frac{|(t_1-s)^{\alpha-3}-(t_2-s)^{\alpha-3}|}{\Gamma(\alpha-2)} s^{k_1}(1-s)^{\sigma_1}\, ds \bigg].$$

We can prove that $1 - x^\tau \leq (1-x)^\tau$ for all $x \in [0,1]$ and $\tau \in (0,1)$. Choose $\tau = \alpha - 3$ and $x = v/u$. Then $u^{\alpha-3} - v^{\alpha-3} \leq (u-v)^{\alpha-3}$ if $u \geq v$. Hence

$$\bigg| t_1^{4-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-3}}{\Gamma(\alpha-2)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\, ds - t_2^{4-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\Gamma(\alpha-2)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\, ds \bigg|$$

$$\leq M_r \bigg[|t_2^{4-\alpha} - t_1^{4-\alpha}| \frac{\mathbf{B}(\alpha+\sigma_1-2, k_1+1)}{\Gamma(\alpha-2)} + \int_{t_1/t_2}^{1} \frac{(1-w)^{\alpha+\sigma_1-3}}{\Gamma(\alpha-2)} w^{k_1}\, dw + t_2^{4-\alpha}(t_2-t_1)^{\alpha-3} \frac{1}{\Gamma(\alpha-2)} \mathbf{B}(\sigma_1+1, k_1+1) \bigg].$$

As $t_1 \to t_2$, the right-hand side tends to zero uniformly. So

$$|t_1^{4-\alpha}(T_1y)''(t_1) - t_2^{4-\alpha}(T_1y)''(t_2)| \to 0 \text{ uniformly as } t_1 \to t_2 \text{ on } (0,1]. \tag{16}$$

Fourthly, let $t_1, t_2 \in [e,d] \subset (0,1]$ with $0 < e \le t_1 < t_2 \le d \le 1$ and $y \in \Omega$. Then

$$\begin{aligned}
&|t_1^{2+m-\alpha} D_{0+}^m (T_1y)(t_1) - t_2^{2+m-\alpha} D_{0+}^m (T_1y)(t_2)| \\
&\quad\le \bigg| t_1^{2+m-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \\
&\qquad - t_2^{2+m-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \bigg| \\
&\qquad + |t_1-t_2| \frac{1}{\Gamma(\alpha-m)} \int_0^1 |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds \\
&\quad\le \bigg| t_1^{2+m-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \\
&\qquad - t_2^{2+m-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \bigg| \\
&\qquad + M_r|t_1-t_2| \frac{1}{\Gamma(\alpha-m)} \mathbf{B}(\sigma_1+1,k_1+1).
\end{aligned}$$

We find that

$$\begin{aligned}
&\bigg| t_1^{2+m-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \\
&\qquad - t_2^{2+m-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds \bigg| \\
&\quad\le |t_2^{2+m-\alpha} - t_1^{+m-\alpha}| \int_0^{t_2} \frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds \\
&\qquad + t_1^{2+m-\alpha} \int_{t_1}^{t_2} \frac{(t_1-s)^{\alpha-m-1}}{\Gamma(\alpha-m)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds \\
&\qquad + t_1^{2+m-\alpha} \int_0^{t_1} \frac{|(t_1-s)^{\alpha-m-1} - (t_2-s)^{\alpha-m-1}|}{\Gamma(\alpha-m)} \\
&\qquad\qquad\qquad \times |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le M_r\Bigg[|t_2^{2+m-\alpha}-t_1^{2+m-\alpha}|\int_0^{t_2}\frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(1-s)^{\sigma_1}\,ds\\
&\quad+t_1^{2+m-\alpha}\int_{t_1}^{t_2}\frac{(t_1-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(1-s)^{\sigma_1}\,ds\\
&\quad+t_1^{2+m-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-m-1}-(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(1-s)^{\sigma_1}\,ds\Bigg]\\
&\le M_r\Bigg[|t_2^{2+m-\alpha}-t_1^{2+m-\alpha}|t_2^{\alpha+\sigma_1+k_1-3}\int_0^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\\
&\quad+t_1^{2+m-\alpha}t_2^{\alpha+\sigma_1+k_1-m}\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\\
&\quad+t_1^{2+m-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(t_1-s)^{\sigma_1}\,ds\\
&\quad-t_1^{2+m-\alpha}\int_0^{t_1}\frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(t_1-s)^{\sigma_1}\,ds\Bigg]\\
&\le M_r\Bigg[|t_2^{2+m-\alpha}-t_1^{2+m-\alpha}|\max\{e^{\alpha+\sigma_1+k_1-m},d^{\alpha+\sigma_1+k_1-m}\}\\
&\qquad\qquad\times\frac{\mathbf{B}(\alpha+\sigma_1-m,k_1+1)}{\Gamma(\alpha-m)}\\
&\quad+\max\{e^{\alpha+\sigma_1+k_1-m},d^{\alpha+\sigma_1+k_1-m}\}\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\\
&\quad+t_1^{2+m-\alpha}t_1^{\alpha+\sigma_1+k_1-m}\int_0^{1}\frac{(1-w)^{\alpha-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\\
&\quad-t_1^{2+m-\alpha}\int_0^{t_1}\frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(t_2-s)^{\sigma_1}\,ds\Bigg]\\
&\le M_r\Bigg[|t_2^{2+m-\alpha}-t_1^{2+m-\alpha}|\max\{e^{\alpha+\sigma_1+k_1-m},d^{\alpha+\sigma_1+k_1-m}\}\\
&\qquad\qquad\times\frac{\mathbf{B}(\alpha+\sigma_1-m,k_1+1)}{\Gamma(\alpha-m)}\\
&\quad+\max\{e^{\alpha+\sigma_1+k_1-m},d^{\alpha+\sigma_1+k_1-m}\}\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\\
&\quad+t_1^{2+m-\alpha}t_1^{\alpha+\sigma_1+k_1-m}\int_0^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\\
&\quad-t_1^{2+m-\alpha}t_2^{\alpha+\sigma_1+k_1-m}\int_0^{t_1/t_2}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\Bigg]
\end{aligned}
$$

$$\leq M_r\bigg[|t_2^{2+m-\alpha}-t_1^{2+m-\alpha}|\max\{e^{\alpha+\sigma_1+k_1-m},d^{\alpha+\sigma_1+k_1-m}\}\times\frac{\mathbf{B}(\alpha+\sigma_1-m,k_1+1)}{\Gamma(\alpha-m)}$$

$$+\max\{e^{\alpha+\sigma_1+k_1-m},d^{\alpha+\sigma_1+k_1-m}\}\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw$$

$$+t_1^{2+m-\alpha}t_1^{\alpha+\sigma_1+k_1-m}\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw$$

$$+t_1^{2+m-\alpha}|t_1^{\alpha+\sigma_1+k_1-m}-t_2^{\alpha+\sigma_1+k_1-m}|\int_{0}^{t_1/t_2}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\bigg].$$

We consider two cases:

CASE 1: $\alpha+\sigma_1+k_1-m>0$. We have

$$\bigg|t_1^{2+m-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}f(s,y(s),y'(s),y''(s),D_{0+}^{n}y(s))\,ds$$

$$-t_2^{2+m-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}f(s,y(s),y'(s),y''(s),D_{0+}^{n}y(s))\,ds\bigg|$$

$$\leq M_r\bigg[|t_2^{2+m-\alpha}-t_1^{2+m-\alpha}|\max\{e^{\alpha+\sigma_1+k_1-m},f^{\alpha+\sigma_1+k_1-m}\}\times\frac{\mathbf{B}(\alpha+\sigma_1-m,k_1+1)}{\Gamma(\alpha-m)}$$

$$+\max\{e^{\alpha+\sigma_1+k_1-m},d^{\alpha+\sigma_1+k_1-m}\}\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw$$

$$+\int_{t_1/t_2}^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw$$

$$+|t_1^{\alpha+\sigma_1+k_1-m}-t_2^{\alpha+\sigma_1+k_1-m}|\int_{0}^{1}\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\bigg].$$

As $t_1\to t_2$, the right-hand side tends to zero uniformly. So

$$|t_1^{2+m-\alpha}D_{0+}^{m}(T_1y)(t_1)-t_2^{2+m-\alpha}D_{0+}^{m}(T_1y)(t_2)|\to 0 \quad \text{uniformly as } t_1\to t_2 \text{ on } [e,d]\subset(0,1]. \tag{17}$$

CASE 2: $\alpha+\sigma_1+k_1-m\le 0$. We have

$$\begin{aligned}&\bigg|t_1^{2+m-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\\&\quad-t_2^{2+m-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\bigg|\\&\le M_r\bigg[|t_2^{2+m-\alpha}-t_1^{2+m-\alpha}|\max\{e^{\alpha+\sigma_1+k_1-m},d^{\alpha+\sigma_1+k_1-m}\}\\&\qquad\qquad\times\frac{\mathbf{B}(\alpha+\sigma_1-m,k_1+1)}{\Gamma(\alpha-m)}\\&\quad+\max\{e^{\alpha+\sigma_1+k_1-m},f^{\alpha+\sigma_1+k_1-m}\}\int_{t_1/t_2}^1\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\\&\quad+\int_{t_1/t_2}^1\frac{(1-w)^{\alpha+\sigma_1-m-1}}{\Gamma(\alpha-m)}w^{k_1}\,dw\bigg].\end{aligned}$$

As $t_1\to t_2$, the right-hand side tends to zero uniformly. Hence (17) holds.

Furthermore,

$$\begin{aligned}&\bigg|t^{2+m-\alpha}D_{0+}^m(T_1y)(t)-\bigg(-\frac{1}{\Gamma(\alpha-m-1)}\int_0^1 f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\\&\qquad+\frac{1}{\Gamma(\alpha-m-1)}\int_0^1(1-s)f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))\,ds\bigg)\bigg|\\&\le t^{2+m-\alpha}\int_0^t\frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds\\&\quad+\frac{1}{\Gamma(\alpha-m)}t\int_0^1|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))|\,ds\\&\le M_rt^{2+m-\alpha}\int_0^t\frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(1-s)^{\sigma_1}\,ds+\frac{M_r}{\Gamma(\alpha-m)}t\int_0^1 s^{k_1}(1-s)^{\sigma_1}\,ds\\&\le M_rt^{2+\sigma_1+k_1}\int_0^1\frac{(1-w)^{\alpha+\sigma_1-m}}{\Gamma(\alpha-m)}w^{k_1}\,dw+\frac{M_r}{\Gamma(\alpha-m)}t\mathbf{B}(\sigma_1+1,k_1+1).\end{aligned}$$

It follows that

(18) $\qquad t^{2+m-\alpha}D_{0+}^m(T_1y)(t)$ is uniformly convergent as $t\to 0$.

From (17) and (18), we deduce that $\{t^{2+m-\alpha}D_{0+}^{m}(T_1y)(t) : y \in \Omega\}$ is equicontinuous on $(0,1]$.

Therefore, $T_1\Omega$ is relatively compact.

From the above discussion, T_1 is completely continuous. The proof is complete. ∎

LEMMA 2.4. *Suppose that*

(B2) $g(t,x,y,z,w)$ *is continuous on* $(0,1)\times\mathbb{R}^4$ *and for each* $r>0$ *there exist* $N_r>0$, $k_2>-1$, $\sigma_2\le 0$ *such that* $2+\sigma_2+k_2>0$ *and*
$$|g(t,t^{\alpha-2}x,t^{\alpha-3}y,t^{\alpha-4}z,t^{\alpha-m-2}w)|\le N_r t^{k_2}(1-t)^{\sigma_2}$$
for all $t\in(0,1)$ *and* $|x|,|y|,|z|,|w|\le r$.

Then $T_2 : X\to Y$ *is completely continuous.*

Proof. The proof is similar to that of Lemma 2.3 and is omitted. ∎

REMARK 2.6. Suppose that (B1) and (B2) hold. It follows from Lemmas 2.3 and 2.4 that $T : X\times Y\to X\times Y$ is completely continuous.

3. Main results. In this section, we prove our main results. We need assumptions imposed on f,g given in Section 1 and the following assumptions:

(B3) $\psi\in C^0(0,1)$ and there exist $c_0>-1$, $-1<d_0\le 0$ $(2+c_0+d_0>0)$ and $N_0>0$ such that $|\psi(t)|\le N_0t^{c_0}(1-t)^{d_0}$ for all $t\in(0,1)$.

(B4) $\phi\in C^0(0,1)$ and there exist $a_0>-1$, $-1<b_0\le 0$ $(2+a_0+b_0>0)$ and $M_0>0$ such that $|\phi(t)|\le M_0t^{a_0}(1-t)^{b_0}$ for all $t\in(0,1)$.

(B5) $f:(0,1)\times\mathbb{R}^4\to[0,\infty)$ is continuous and there exist $a_i>-1$, $b_i\le 0$ $(i=1,\dots,\omega)$ with $2+a_i+b_i>0$, $\tau_{i1},\tau_{i2},\tau_{i3},\tau_{i4}\ge 0$ $(i=1,\dots,\omega)$ and $A_i\ge 0$ $(i=1,\dots,\omega)$ such that
$$|f(t,t^{\beta-2}x,t^{\beta-3}y,t^{\beta-4}z,t^{\beta-n-2}w)-\psi(t)| \le \sum_{i=1}^{\omega}A_it^{a_i}(1-t)^{b_i}|x|^{\tau_{i1}}|y|^{\tau_{i2}}|z|^{\tau_{i3}}|w|^{\tau_{i4}}$$
for all $t\in(0,1)$ and $x,y,z,w\in\mathbb{R}$.

(B6) $g:(0,1)\times\mathbb{R}^4\to[0,\infty)$ is continuous and there exist $c_i>-1$, $d_i\le 0$ $(i=1,\dots,\omega)$ with $2+c_i+d_i>0$, $\delta_{i1},\delta_{i2},\delta_{i3},\delta_{i4}\ge 0$ $(i=1,\dots,\omega)$ and $B_i\ge 0$ $(i=1,\dots,\omega)$ such that
$$|g(t,t^{\alpha-2}x,t^{\alpha-3}y,t^{\alpha-4}x,t^{\alpha-m-2}y)-\phi(t)| \le \sum_{i=1}^{\omega}B_it^{c_i}(1-t)^{d_i}|x|^{\delta_{i1}}|y|^{\delta_{i2}}|z|^{\delta_{i3}}|w|^{\delta_{i4}}$$
for all $t\in(0,1)$ and $x,y,z,w\in\mathbb{R}$.

Set

$$\tau_i = \tau_{i1} + \tau_{i2} + \tau_{i3} + \tau_{i4}\ (i = 1, \dots, \omega), \quad \tau = \max\{\tau_i : i = 1, \dots, \omega\},$$
$$\delta_i = \delta_{i1} + \delta_{i2} + \delta_{i3} + \delta_{i4}\ (i = 1, \dots, \omega), \quad \delta = \max\{\delta_i : i = 1, \dots, \omega\},$$

$$\Psi(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi(s)\,ds - \left[\frac{t^{\beta-1}}{\Gamma(\beta)} - \frac{t^{\beta-2}}{\Gamma(\beta-1)}\right]\int_0^1 \psi(s)\,ds - \frac{t^{\beta-2}}{\Gamma(\beta-1)}\int_0^1 (1-s)\psi(s)\,ds,$$

$$\Phi(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)\,ds - \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\right]\int_0^1 \phi(s)\,ds - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\phi(s)\,ds,$$

$$P_i = \frac{\mathbf{B}(\alpha+b_i, a_i+1)}{\Gamma(\alpha)} + \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right]\mathbf{B}(b_i+1, a_i+1) + \frac{\mathbf{B}(b_i+2, a_i+1)}{\Gamma(\alpha-1)},$$

$$Q_i = \frac{\mathbf{B}(\alpha+b_i-1, a_i+1)}{\Gamma(\alpha-1)} + \left[\frac{1}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha-2)}\right]\mathbf{B}(b_i+1, a_i+1) + \frac{\mathbf{B}(b_i+2, a_i+1)}{\Gamma(\alpha-2)},$$

$$R_i = \frac{\mathbf{B}(\alpha+b_i-2, a_i+1)}{\Gamma(\alpha-2)} + \left[\frac{1}{\Gamma(\alpha-2)} + \frac{1}{\Gamma(\alpha-3)}\right]\mathbf{B}(b_i+1, a_i+1) + \frac{\mathbf{B}(b_i+2, a_i+1)}{\Gamma(\alpha-3)},$$

$$W_i = \frac{\mathbf{B}(\alpha+b_i-m, a_i+1)}{\Gamma(\alpha-m)} + \left[\frac{1}{\Gamma(\alpha-m)} + \frac{1}{\Gamma(\alpha-m-1)}\right]\mathbf{B}(b_i+1, a_i+1) + \frac{\mathbf{B}(b_i+2, a_i+1)}{\Gamma(\alpha-m-1)},$$

$$\Xi_i = \max\{P_i, Q_i, R_i, W_i\}, \quad i = 1, \dots, \omega,$$

and

$$\overline{P}_i = \frac{\mathbf{B}(\beta+d_i, c_i+1)}{\Gamma(\beta)} + \left[\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-1)}\right]\mathbf{B}(d_i+1, c_i+1) + \frac{\mathbf{B}(d_i+2, c_i+1)}{\Gamma(\beta-1)},$$

$$\overline{Q}_i = \frac{\mathbf{B}(\beta+d_i-1,c_i+1)}{\Gamma(\beta-1)} + \left[\frac{1}{\Gamma(\beta-1)} + \frac{1}{\Gamma(\beta-2)}\right]\mathbf{B}(d_i+1,c_i+1) + \frac{\mathbf{B}(d_i+2,c_i+1)}{\Gamma(\beta-2)},$$

$$\overline{R}_i = \frac{\mathbf{B}(\beta+d_i-2,c_i+1)}{\Gamma(\beta-2)} + \left[\frac{1}{\Gamma(\beta-2)} + \frac{1}{\Gamma(\beta-3)}\right]\mathbf{B}(d_i+1,c_i+1) + \frac{\mathbf{B}(d_i+2,c_i+1)}{\Gamma(\beta-3)},$$

$$\overline{W}_i = \frac{\mathbf{B}(\beta+d_i-n,c_i+1)}{\Gamma(\beta-n)} + \left[\frac{1}{\Gamma(\beta-n)} + \frac{1}{\Gamma(\beta-n-1)}\right]\mathbf{B}(d_i+1,c_i+1) + \frac{\mathbf{B}(d_i+2,c_i+1)}{\Gamma(\beta-n-1)},$$

$$\overline{\Xi}_i = \max\{\overline{P}_i,\overline{Q}_i,\overline{R}_i,\overline{W}_i\}, \quad i=1,\dots,\omega.$$

THEOREM 3.1. *Suppose that* (B3), (B4), (B5) *and* (B6) *hold. Then* BVP(1) *has at least one positive solution if*

(i) $\tau\delta < 1$, *or*
(ii) $\tau\delta = 1$ *with*

$$\sum_{i=1}^{\omega} B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta}\Big[\sum_{i=1}^{\omega} A_i\Xi_i\|\Psi\|^{\tau_i-\tau}\Big]^{1/\tau} < 1,$$

or
(iii) $\tau\delta > 1$ *with*

$$\frac{\left(\frac{\|\Phi\|}{\delta\tau-1}+\|\Phi\|\right)^{\delta}}{\left(\frac{\|\Phi\|}{\delta\tau-1}\right)^{1/\tau}}\sum_{i=1}^{\omega} B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta} + \frac{\|\Psi\|}{\left(\frac{\|\Phi\|}{\delta\tau-1}\right)^{1/\tau}} \le \frac{1}{[\sum_{i=1}^{\omega} A_i\Xi_i\|\Psi\|^{\tau_i-\tau}]^{1/\tau}}.$$

Proof. It is easy to show that (B5) and (B6) imply (B1) and (B2). Let the Banach space $X\times Y$ and the operator T on $X\times Y$ be as defined in Section 2. By Lemmas 2.3 and 2.4, $T: X\times Y \to X\times Y$ is well defined and completely continuous. By the assumptions on f,g in Section 1, together with Remark 2.5, we know that $(x,y)\in X\times Y$ is a nonnegative solution of BVP(1) if and only if $(x,y)\in X\times Y$ is a fixed point of T. If there exists $t_0\in(0,1)$ such that $[x(t_0)]^2+[y(t_0)]^2=0$, we have $x(t_0)=y(t_0)=0$. Then similarly to Remark 2.2, we get

$$[g(t,t^{\alpha-2}x,t^{\alpha-3}y,t^{\alpha-4}z,t^{\alpha-m-2}w)]^2 + [f(t,t^{\beta-2}x,t^{\beta-3}y,t^{\beta-4}z,t^{\beta-m-2}w)]^2 \equiv 0$$

on some subinterval $(0,c]$ with $c\in(0,1)$, contradicting the assumptions of Section 1. So (x,y) is a positive solution of BVP(1) if (x,y) is a fixed point of T.

It is easy to see from (B3), (B4) and Lemmas 2.1 and 2.2 that $\Psi \in Y$ and $\Phi \in X$. By direct computation, we have

$$\Psi'(t) = \int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}\psi(s)\,ds - \left[\frac{t^{\beta-2}}{\Gamma(\beta-1)} - \frac{t^{\beta-3}}{\Gamma(\beta-2)}\right]\int_0^1 \psi(s)\,ds - \frac{t^{\beta-3}}{\Gamma(\beta-2)}\int_0^1 (1-s)\psi(s)\,ds,$$

$$\Psi''(t) = \int_0^t \frac{(t-s)^{\beta-3}}{\Gamma(\beta-2)}\psi(s)\,ds - \left[\frac{t^{\beta-3}}{\Gamma(\beta-2)} - \frac{t^{\beta-4}}{\Gamma(\beta-3)}\right]\int_0^1 \psi(s)\,ds - \frac{t^{\beta-4}}{\Gamma(\beta-3)}\int_0^1 (1-s)\psi(s)\,ds,$$

$$D_{0+}^{n}\Psi(t) = \int_0^t \frac{(t-s)^{\beta-n-1}}{\Gamma(\beta-n)}\psi(s)\,ds - \left[\frac{t^{\beta-n-1}}{\Gamma(\beta-n)} - \frac{t^{\beta-n-2}}{\Gamma(\beta-n-1)}\right]\int_0^1 \psi(s)\,ds - \frac{t^{\beta-n-2}}{\Gamma(\beta-n-1)}\int_0^1 (1-s)\psi(s)\,ds,$$

$$\Phi'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\phi(s)\,ds - \left[\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{t^{\alpha-3}}{\Gamma(\alpha-2)}\right]\int_0^1 \phi(s)\,ds - \frac{t^{\alpha-3}}{\Gamma(\alpha-2)}\int_0^1 (1-s)\phi(s)\,ds,$$

$$\Phi''(t) = \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}\phi(s)\,ds - \left[\frac{t^{\alpha-3}}{\Gamma(\alpha-2)} - \frac{t^{\alpha-4}}{\Gamma(\alpha-3)}\right]\int_0^1 \phi(s)\,ds - \frac{t^{\alpha-4}}{\Gamma(\alpha-3)}\int_0^1 (1-s)\phi(s)\,ds,$$

$$D_{0+}^{m}\Phi(t) = \int_0^t \frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}\phi(s)\,ds - \left[\frac{t^{\alpha-m-1}}{\Gamma(\alpha-m)} - \frac{t^{\alpha-m-2}}{\Gamma(\alpha-m-1)}\right]\int_0^1 \phi(s)\,ds - \frac{t^{\alpha-m-2}}{\Gamma(\alpha-m-1)}\int_0^1 (1-s)\phi(s)\,ds.$$

For $r_1, r_2 > 0$, denote

$$\Omega_{r_1,r_2} = \{(x,y) \in X \times Y : \|y-\Psi\| \le r_2,\ \|x-\Phi\| \le r_1\}.$$

One sees, for $(x,y) \in \Omega_{r_1,r_2}$, that

$$\|x\| \le \|x-\Psi\| + \|\Psi\| \le r_1 + \|\Psi\|,$$
$$\|y\| \le \|y-\Phi\| + \|\Phi\| \le r_2 + \|\Phi\|.$$

Hence for $(x,y) \in \Omega_r$, we have

$$\begin{aligned}
&|f(t,y(t),y'(t),y''(t),D_{0+}^n y(t)) - \psi(t)| \\
&\le \sum_{i=1}^{\omega} A_i t^{a_i}(1-t)^{b_i} |t^{2-\beta}y(t)|^{\tau_{i1}} |t^{3-\beta}y'(t)|^{\tau_{i2}} |t^{4-\beta}y''(t)|^{\tau_{i3}} |t^{2+n-\beta} D_{0+}^n y(t)|^{\tau_{i4}} \\
&\le \sum_{i=1}^{\omega} A_i t^{a_i}(1-t)^{b_i} [r_2 + \|\Psi\|]^{\tau_{i1}+\tau_{i2}+\tau_{i3}+\tau_{i4}} \\
&= \sum_{i=1}^{\omega} A_i t^{a_i}(1-t)^{b_i} [r_2 + \|\Psi\|]^{\tau_i}
\end{aligned}$$

and

$$|g(t,x(t),x'(t),x''(t),D_{0+}^m x(t)) - \phi(t)| \le \sum_{i=1}^{\omega} B_i t^{c_i}(1-t)^{d_i} [r_1 + \|\Phi\|]^{\delta_i}.$$

Firstly, we have

$$\begin{aligned}
&t^{2-\alpha}|(T_1 y)(t) - \Phi(t)| \\
&\quad \le t^{2-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s)) - \phi(s)|\, ds \\
&\qquad + \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right] \int_0^1 |f(s,y(s),y'(s),y''(s),D_{0+}^n y(s)) - \phi(s)|\, ds \\
&\qquad + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s)) - \phi(s)|\, ds \\
&\quad \le t^{2-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} A_i s^{a_i}(1-s)^{b_i} [r_2 + \|\Psi\|]^{\tau_i}\, ds \\
&\qquad + \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right] \int_0^1 \sum_{i=1}^{m} A_i s^{a_i}(1-s)^{b_i} [r_2 + \|\Psi\|]^{\tau_i}\, ds \\
&\qquad + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s) \sum_{i=1}^{m} A_i s^{a_i}(1-s)^{b_i} [r_2 + \|\Psi\|]^{\tau_i}\, ds
\end{aligned}$$

$$\le t^{2-\alpha}\sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{a_i}(t-s)^{b_i}\,ds$$

$$+\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right]\sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}\int_0^1 s^{a_i}(1-s)^{b_i}\,ds$$

$$+\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}\int_0^1(1-s)s^{a_i}(1-s)^{b_i}\,ds$$

$$= t^{2-\alpha}\sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}t^{\alpha+a_i+b_i}\int_0^1\frac{(1-w)^{\alpha+b_i-1}}{\Gamma(\alpha)}w^{a_i}\,dw$$

$$+\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right]\sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}\mathbf{B}(b_i+1,a_i+1)$$

$$+\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}\mathbf{B}(b_i+2,a_i+1)$$

$$\le \sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}\frac{\mathbf{B}(\alpha+b_i,a_i+1)}{\Gamma(\alpha)}$$

$$+\left[\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right]\sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}\mathbf{B}(b_i+1,a_i+1)$$

$$+\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{m}A_i[r_2+\|\Psi\|]^{\tau_i}\mathbf{B}(b_i+2,a_i+1)$$

$$= \sum_{i=1}^{\omega}A_iP_i[r_2+\|\Psi\|]^{\tau_i}.$$

It follows that

$$\sup_{t\in(0,1]}t^{2-\alpha}|(T_1y)(t)-\Phi(t)|\le\sum_{i=1}^{\omega}A_iP_i[r_2+\|\Psi\|]^{\tau_i}. \tag{19}$$

Secondly,

$$t^{3-\alpha}|(T_1y)'(t)-\Phi'(t)|$$

$$\le\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,y(s),y'(s),y''(s),D^n_{0+}y(s))-\phi(s)|\,ds$$

$$+\left[\frac{1}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-2)}\right]\int_0^1|f(s,y(s),y'(s),y''(s),D^n_{0+}y(s))-\phi(s)|\,ds$$

$$+\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-s)|f(s,y(s),y'(s),y''(s),D_{0+}^n y(s))-\phi(s)|\,ds$$
$$\le \sum_{i=1}^{\omega} A_i Q_i[r_2+\|\Psi\|]^{\tau_i}.$$

It follows that

$$\sup_{t\in(0,1]} t^{3-\alpha}|(T_1y)'(t)-\Phi'(t)| \le \sum_{i=1}^{\omega} A_i Q_i[r_2+\|\Psi\|]^{\tau_i}. \tag{20}$$

Thirdly, we have similarly

$$\sup_{t\in(0,1]} t^{4-\alpha}|(T_1y)''(t)-\Phi''(t)| \le \sum_{i=1}^{\omega} A_i R_i[r_2+\|\Psi\|]^{\tau_i}. \tag{21}$$

Finally,

$$\sup_{t\in(0,1]} t^{2+m-\alpha}|D_{0+}^m(T_1y)(t)-D_{0+}^m\Phi(t)| \le \sum_{i=1}^{\omega} A_i W_i[r_2+\|\Psi\|]^{\tau_i}. \tag{22}$$

Thus (19)–(22) imply that

$$\|T_1y-\Phi\| \le \sum_{i=1}^{\omega} A_i\Xi_i[r_2+\|\Psi\|]^{\tau_i} \le [r_2+\|\Psi\|]^{\tau}\sum_{i=1}^{\omega} A_i\Xi_i\|\Psi\|^{\tau_i-\tau}. \tag{23}$$

Similarly we can prove that

$$\|T_2x-\Psi\| \le \sum_{i=1}^{\omega} B_i\overline{\Xi}_i[r_1+\|\Phi\|]^{\delta_i} \le [r_1+\|\Phi\|]^{\delta}\sum_{i=1}^{\omega} B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta}. \tag{24}$$

In order to apply Schauder's fixed point theorem, we will construct Ω_{r_1,r_2} such that $T(\Omega_{r_1,r_2})\subseteq\Omega_{r_1,r_2}$. From (23) and (24), our purpose is to choose r_1 and r_2 such that

$$[r_2+\|\Psi\|]^{\tau}\sum_{i=1}^{\omega} A_i\Xi_i\|\Psi\|^{\tau_i-\tau} \le r_1,$$
$$[r_1+\|\Phi\|]^{\delta}\sum_{i=1}^{\omega} B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta} \le r_2,$$

i.e.,

$$[r_1+\|\Phi\|]^{\delta}\sum_{i=1}^{\omega} B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta} \le r_2 \le \frac{r_1^{1/\tau}}{[\sum_{i=1}^{\omega} A_i\Xi_i\|\Psi\|^{\tau_i-\tau}]^{1/\tau}}-\|\Psi\|$$

or

$$[r_2+\|\Psi\|]^{\tau}\sum_{i=1}^{\omega} A_i\Xi_i\|\Psi\|^{\tau_i-\tau} \le r_1 \le \frac{r_2^{1/\delta}}{[\sum_{i=1}^{\omega} B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta}]^{1/\delta}}-\|\Phi\|.$$

We consider three cases:

Case (i): $\tau\delta < 1$. Since $\tau\delta < 1$ implies that there exists $r_1 > 0$ sufficiently large such that

$$[r_1 + \|\Phi\|]^\delta \sum_{i=1}^{\omega} B_i \overline{\Xi}_i \|\Phi\|^{\delta_i - \delta} + \|\Psi\| \leq \frac{r_1^{1/\tau}}{[\sum_{i=1}^{\omega} A_i \Xi_i \|\Psi\|^{\tau_i - \tau}]^{1/\tau}},$$

we can choose

$$r_2 = [r_1 + \|\Phi\|]^\delta \sum_{i=1}^{\omega} B_i \overline{\Xi}_i \|\Phi\|^{\delta_i - \delta}.$$

Then

$$[r_1 + \|\Phi\|]^\delta \sum_{i=1}^{\omega} B_i \overline{\Xi}_i \|\Phi\|^{\delta_i - \delta} = r_2 \leq \frac{r_1^{1/\tau}}{[\sum_{i=1}^{\omega} A_i \Xi_i \|\Psi\|^{\tau_i - \tau}]^{1/\tau}} - \|\Psi\|.$$

Set $\Omega_{r_1,r_2} = \{(x,y) \in X \times Y : \|y - \Psi\| \leq r_2,\ \|x - \Phi\| \leq r_1\}$. From the above discussion,

$$\|T_1 y - \Phi\| \leq [r_2 + \|\Psi\|]^\tau \sum_{i=1}^{\omega} A_i \Xi_i \|\Psi\|^{\tau_i - \tau} \leq r_1,$$

$$\|T_2 x - \Psi\| \leq [r_1 + \|\Phi\|]^\delta \sum_{i=1}^{\omega} B_i \overline{\Xi}_i \|\Phi\|^{\delta_i - \delta} \leq r_2.$$

Hence $T(x,y) \in \Omega_{r_1,r_2}$. By Schauder's fixed point theorem, T has at least one fixed point $(x,y) \in \Omega_{r_1,r_2}$. Then (x,y) is a solution of BVP(1).

Case (ii): $\tau\delta = 1$. Since

$$\sum_{i=1}^{\omega} B_i \overline{\Xi}_i \|\Phi\|^{\delta_i - \delta} < \frac{1}{[\sum_{i=1}^{\omega} A_i \Xi_i \|\Psi\|^{\tau_i - \tau}]^{1/\tau}},$$

there exists $r_1 > 0$ sufficiently large such that

$$[r_1 + \|\Phi\|]^\delta \sum_{i=1}^{\omega} B_i \overline{\Xi}_i \|\Phi\|^{\delta_i - \delta} + \|\Psi\| \leq \frac{r_1^{1/\tau}}{[\sum_{i=1}^{\omega} A_i \Xi_i \|\Psi\|^{\tau_i - \tau}]^{1/\tau}}.$$

Choose

$$r_2 = [r_1 + \|\Phi\|]^\delta \sum_{i=1}^{\omega} B_i \overline{\Xi}_i \|\Phi\|^{\delta_i - \delta}.$$

Then

$$[r_1 + \|\Phi\|]^\delta \sum_{i=1}^{\omega} B_i \overline{\Xi}_i \|\Phi\|^{\delta_i - \delta} = r_2 \leq \frac{r_1^{1/\tau}}{[\sum_{i=1}^{\omega} A_i \Xi_i \|\Psi\|^{\tau_i - \tau}]^{1/\tau}} - \|\Psi\|. \tag{25}$$

Set $\Omega_{r_1,r_2} = \{(x,y) \in X \times Y : \|y - \Psi\| \leq r_2, \|x - \Phi\| \leq r_1\}$. From the above discussion,

$$\|T_1y-\Phi\| \le [r_2+\|\Psi\|]^{\tau}\sum_{i=1}^{\omega}A_i\Xi_i\|\Psi\|^{\tau_i-\tau} \le r_1,$$

$$\|T_2x-\Psi\| \le [r_1+\|\Phi\|]^{\delta}\sum_{i=1}^{\omega}B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta} \le r_2.$$

Hence $T(x,y)\in\Omega_{r_1,r_2}$. By Schauder's fixed point theorem, T has at least one fixed point $(x,y)\in\Omega_{r_1,r_2}$. Then (x,y) is a solution of BVP(1).

Case (iii): $\tau\delta>1$. Choose

$$r_1=\frac{\|\Phi\|}{\delta\tau-1}.$$

It is easy to show from the inequality in Theorem 3.1(iii) that

$$\frac{(r_1+\|\Phi\|)^{\delta}}{r_1^{1/\tau}}\sum_{i=1}^{\omega}B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta}+\frac{\|\Psi\|}{r_1^{1/\tau}} \le \frac{1}{[\sum_{i=1}^{\omega}A_i\Xi_i\|\Psi\|^{\tau_i-\tau}]^{1/\tau}}.$$

Then

$$(r_1+\|\Phi\|)^{\delta}\sum_{i=1}^{\omega}B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta}+\|\Psi\| \le \frac{r_1^{1/\tau}}{[\sum_{i=1}^{\omega}A_i\Xi_i\|\Psi\|^{\tau_i-\tau}]^{1/\tau}}.$$

Choose

$$r_2=(r_1+\|\Phi\|)^{\delta}\sum_{i=1}^{\omega}B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta}.$$

We also have (25). Set $\Omega_{r_1,r_2}=\{(x,y)\in X\times Y:\|y-\Psi\|\le r_2,\ \|x-\Phi\|\le r_1\}$. From the above discussion,

$$\|T_1y-\Phi\| \le [r_2+\|\Psi\|]^{\tau}\sum_{i=1}^{\omega}A_i\Xi_i\|\Psi\|^{\tau_i-\tau} \le r_1,$$

$$\|T_2x-\Psi\| \le [r_1+\|\Phi\|]^{\delta}\sum_{i=1}^{\omega}B_i\overline{\Xi}_i\|\Phi\|^{\delta_i-\delta} \le r_2.$$

Hence $T(x,y)\in\Omega_{r_1,r_2}$. By Schauder's fixed point theorem, T has at least one fixed point $(x,y)\in\Omega_{r_1,r_2}$. Then (x,y) is a solution of BVP(1).

The proof of Theorem 3.1 is complete. ■

Theorem 3.2. *Suppose that both* (B3) *and* (B4) *hold, and there exists* $M>0$ *such that*

$$f(t,t^{\beta-2}x,t^{\beta-3}y,t^{\beta-4}z,t^{\beta-n-2}w)|\le M,$$
$$g(t,t^{\alpha-2}x,t^{\alpha-3}y,t^{\alpha-4}z,t^{\alpha-m-2}w)|\le M$$

for all $t\in(0,1)$ *and* $x,y,z,w\in\mathbb{R}$. *Then* BVP(1) *has at least one positive solution.*

Proof. In Theorem 3.1, choose $\psi(t) = \phi(t) = 0$ and $\omega = 1$, $\tau_{1j} = \delta_{1j} = 0$ $(j = 1, 2, 3, 4)$, $A_1 = B_1 = M$. ■

Now, we list the following assumptions:

(B7) $\psi \in L^1(0,1)$ and there exists $N_0 > 0$ such that $|\psi(t)| \le N_0$ for all $t \in (0,1)$.

(B8) $\phi \in L^1(0,1)$ and there exists $M_0 > 0$ such that $|\phi(t)| \le M_0$ for all $t \in (0,1)$.

(B9) $f : (0,1) \times \mathbb{R}^4 \to [0,\infty)$ is continuous and there exist $\tau_1, \tau_2, \tau_3, \tau_4 \ge 0$ and $A_i \ge 0$ $(i = 1, 2, 3, 4)$ such that

$$|f(t, t^{\beta-2}x, t^{\beta-3}y, t^{\beta-4}z, t^{\beta-n-2}w) - \psi(t)| \le A_1|x|^{\tau_1} + A_2|y|^{\tau_2} + A_3|z|^{\tau_3} + A_4|w|^{\tau_4}$$

for all $t \in (0,1)$ and $x, y, z, w \in \mathbb{R}$.

(B10) $g : (0,1) \times \mathbb{R}^4 \to [0,\infty)$ is continuous and there exist $\delta_1, \delta_2, \delta_3, \delta_4 \ge 0$ and $B_i \ge 0$ $(i = 1, 2, 3, 4)$ such that

$$|g(t, t^{\alpha-2}x, t^{\alpha-3}y, t^{\alpha-4}x, t^{\alpha-m-2}y) - \phi(t)| \le B_1|x|^{\delta_1} + B_2|y|^{\delta_2} + B_3|z|^{\delta_3} + B_4|w|^{\delta_4}$$

for all $t \in (0,1)$ and $x, y, z, w \in \mathbb{R}$.

Denote $\tau = \max\{\tau_i : i = 1, 2, 3, 4\}$ and $\delta = \max\{\delta_i : i = 1, 2, 3, 4\}$ and

$$\Psi(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi(s)\,ds - \left[\frac{t^{\beta-1}}{\Gamma(\beta)} - \frac{t^{\beta-2}}{\Gamma(\beta-1)}\right]\int_0^1 \psi(s)\,ds - \frac{t^{\beta-2}}{\Gamma(\beta-1)}\int_0^1 (1-s)\psi(s)\,ds,$$

$$\Phi(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)\,ds - \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\right]\int_0^1 \phi(s)\,ds - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\phi(s)\,ds,$$

$$P_i = \frac{1}{\Gamma(\alpha+1)} + \frac{\alpha}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} = \frac{3\alpha^2-\alpha+2}{2\Gamma(\alpha+1)},$$

$$Q_i = \frac{1}{\Gamma(\alpha)} + \frac{\alpha-1}{\Gamma(\alpha-1)} + \frac{1}{2\Gamma(\alpha-2)} = \frac{3\alpha^2-7\alpha+6}{2\Gamma(\alpha)},$$

$$R_i = \frac{1}{\Gamma(\alpha-1)} + \frac{\alpha-2}{\Gamma(\alpha-2)} + \frac{1}{2\Gamma(\alpha-3)} = \frac{3\alpha^2-13\alpha+16}{2\Gamma(\alpha-1)},$$

$$W_i = \frac{1}{\Gamma(\alpha-m+1)} + \frac{\alpha-m}{\Gamma(\alpha-m)} + \frac{1}{2\Gamma(\alpha-m-1)} = \frac{3\alpha^2-(6m+1)\alpha+3m^2+m+2}{2\Gamma(\alpha-m+1)},$$

$$\Xi = \max\left\{\frac{3\alpha^2-(6j+1)\alpha+3j^2+j+2}{2\Gamma(\alpha-j+1)} : j=0,1,2,m\right\},$$

and

$$\overline{P}_i = \frac{1}{\Gamma(\beta+1)} + \frac{\beta}{\Gamma(\beta)} + \frac{1}{2\Gamma(\beta-1)} = \frac{3\beta^2-\beta+2}{2\Gamma(\beta+1)},$$
$$\overline{Q}_i = \frac{1}{\Gamma(\beta)} + \frac{\beta-1}{\Gamma(\beta-1)} + \frac{1}{2\Gamma(\beta-2)} = \frac{3\beta^2-7\beta+6}{2\Gamma(\beta)},$$
$$\overline{R}_i = \frac{1}{\Gamma(\beta-1)} + \frac{\beta-2}{\Gamma(\beta-2)} + \frac{1}{2\Gamma(\beta-3)} = \frac{3\beta^2-13\beta+16}{2\Gamma(\beta-1)},$$
$$\overline{W}_i = \frac{1}{\Gamma(\beta-n+1)} + \frac{\beta-n}{\Gamma(\beta-n)} + \frac{1}{2\Gamma(\beta-n-1)} = \frac{3\beta^2-(6n+1)\beta+3n^2+n+2}{2\Gamma(\beta-n+1)}, \quad i=1,2,3,4,$$
$$\overline{\Xi} = \max\left\{\frac{3\beta^2-(6j+1)\beta+3j^2+j+2}{2\Gamma(\beta-j+1)} : j=0,1,\dots,m\right\}.$$

THEOREM 3.3. *Suppose that* (B7), (B8), (B9) *and* (B10) *hold. Then* BVP(1) *has at least one positive solution if*

(i) $\tau\delta<1$, *or*

(ii) $\tau\delta=1$ *with*

$$\overline{\Xi}\,\Xi^{1/\tau}\sum_{i=1}^{4} B_i\|\Phi\|^{\delta_i-\delta}\Big[\sum_{i=1}^{4} A_i\|\Psi\|^{\tau_i-\tau}\Big]^{1/\tau} < 1,$$

or

(iii) $\tau\delta>1$ *with*

$$\overline{\Xi}\Big(\frac{\delta\tau}{\delta\tau-1}\Big)^{\delta}\sum_{i=1}^{4} B_i\|\Phi\|^{\delta_i} + \|\Psi\| \le \frac{\big(\frac{\|\Phi\|}{\delta\tau-1}\big)^{1/\tau}}{[\Xi\sum_{i=1}^{4} A_i\|\Psi\|^{\tau_i-\tau}]^{1/\tau}}.$$

Proof. Choose $a_0=b_0=c_0=d_0=0$. Then (B7) and (B8) imply (B3) and (B4) respectively. Choose $\omega=4$, $a_i=b_i=c_i=d_i=0$ $(i=1,2,3,4)$, and

$$(\tau_{11},\tau_{12},\tau_{13},\tau_{14})=(\tau_1,0,0,0), \quad (\tau_{21},\tau_{22},\tau_{23},\tau_{24})=(0,\tau_2,0,0),$$
$$(\tau_{31},\tau_{32},\tau_{33},\tau_{34})=(0,0,\tau_3,0), \quad (\tau_{41},\tau_{42},\tau_{43},\tau_{44})=(0,0,0,\tau_4),$$
$$(\delta_{11},\delta_{12},\delta_{13},\delta_{14})=(\delta_1,0,0,0), \quad (\delta_{21},\delta_{22},\delta_{23},\delta_{24})=(0,\delta_2,0,0),$$
$$(\delta_{31},\delta_{32},\delta_{33},\delta_{34})=(0,0,\delta_3,0), \quad (\delta_{41},\delta_{42},\delta_{43},\delta_{44})=(0,0,0,\delta_4).$$

Then (B9) and (B10) imply (B5) and B6) respectively. By Theorem 3.1, we get (i), (ii) and (iii).

REMARK 3.1. From (B5), (B6), we know that our assumptions on f, g are different from (BA1) and (BA2). When one considers BVP(I), BVP(II) and BVP(III), similar results can be obtained. We omit the details. But we point out that our assumptions on f and g are weaker than (BA1) and (BA2) since $\rho\sigma = \max\{\rho_1, \rho_2\} \max\{\sigma_1, \sigma_2\} < 1$ is weaker than $\max\{\rho_1, \rho_2\} < 1$ and $\max\{\sigma_1, \sigma_2\} < 1$.

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