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Real Jacobian mates

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Abstract. Let p be a real polynomial in two variables. We say that a polynomial q is a real Jacobian mate of p if the Jacobian determinant of the mapping $(p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ vanishes nowhere. We present a class of polynomials that do not have real Jacobian mates.

1. Introduction. This paper is inspired by [2] where Braun and dos Santos Filho proved that every polynomial mapping $(p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ which is a local diffeomorphism with $\deg p \le 3$ is a global diffeomorphism.

A pair of polynomials $p, q \in \mathbb{R}[x, y]$ such that the Jacobian determinant $\mathrm{Jac}(p, q) = \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}$ vanishes nowhere, or equivalently the mapping $(p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a local diffeomorphism, will be called *real Jacobian mates.* The statement that $p = x(1 + xy)$ does not have a real Jacobian mate is crucial for [2]. Theorem 2.1 below provides a new proof of this fact. In Theorem 4.1, a wide class of polynomials that do not have real Jacobian mates is characterized. In particular, every polynomial whose Newton polygon has an edge as described in Corollary 4.2 belongs to this class. This gives a new proof of [3, Theorem 5.5] that polynomials of degree 4 with at least one disconnected level set do not have real Jacobian mates (see Example 4.4 below for details).

2. Glacial tongues

Theorem 2.1. *Let p be a real polynomial in two variables and let $B \subset A$ be subsets of the real plane such that:*

(i) *the set B is bounded,*
(ii) *for every $t \in \mathbb{R}$ the set $p^{-1}(t) \cap A$ is either empty, or contained in B, or homeomorphic to a segment and has endpoints in B,*
(iii) *the border of A contains a half-line.*

Then for every $q \in \mathbb{R}[x, y]$ there exists $v \in \mathbb{R}^2$ such that $\mathrm{Jac}(p, q)(v) = 0$.

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Proof. Suppose that there exists a polynomial $q \in \mathbb{R}[x, y]$ such that the mapping $\Phi = (p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a local diffeomorphism.

Take any $t \in \mathbb{R}$ such that the set $A_t = p^{-1}(t) \cap A$ is nonempty. If $A_t \subset B$ then $\Phi(A_t) \subset \Phi(B)$. If A_t is homeomorphic to a segment with endpoints in B, then the restriction of Φ to A_t is a locally injective continuous mapping from A_t, which is homeomorphic to a segment, to the vertical line $\{t\} \times \mathbb{R}$, homeomorphic to $\mathbb{R}$. By the extreme value theorem and the mean value theorem, such a mapping is either increasing or decreasing. Hence, $\Phi(A_t)$ is a vertical segment with endpoints in $\Phi(B)$.

Since $\Phi(B)$ is bounded, so is $\Phi(A)$.

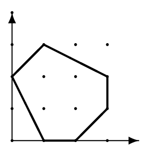
Let L be a half-line contained in the border of A. As Φ is bounded on A, it is also bounded on L. Consequently, the polynomials p and q restricted to L are constant (because they behave on L like polynomials in one variable). Hence Φ restricted to L is constant, which contradicts the assumption that Φ is a local diffeomorphism. ■

Every set A satisfying the assumptions of Theorem 2.1 will be called a *glacial tongue with a straight border*.

EXAMPLE 2.2. Let $p = x(1 + xy)$. In [2, Lemma 4.1 and Remark 1] it is established that $A = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, -1/x < y \leq -1\}$ is a glacial tongue with a straight border for p. Hence, p does not have a real Jacobian mate.

3. Newton polygon and branches at infinity. Let $p = \sum a_{i,j}x^i y^j$ be a nonzero polynomial. By definition, the *Newton polygon* $\Delta(p)$ is the convex hull of the set $\{(i, j) \in \mathbb{Z}^2 : a_{i,j} \neq 0\}$. An edge S of $\Delta(p)$ will be called an *outer edge* if it has a normal vector $\vec{v} = (v_1, v_2)$ pointing outwards from $\Delta(p)$ such that $v_1 > 0$ or $v_2 > 0$ (if $\Delta(p)$ reduces to a segment, then by convention all normal vectors point outwards). If $v_1 > 0$, then S will be called a *right outer edge*. With every right outer edge S, we associate a rational number $\theta(S) = v_2/v_1$, called the *slope* of S.

EXAMPLE 3.1. The Newton polygon of $p = x+x^2+x^3y+y^2+x^3y^2+xy^3$ has 4 outer edges. Three of them are right outer edges with slopes -1, 0, and 2.



The objective of this section is to describe branches at infinity of the curve $p(x, y) = 0$ and associate with each branch a certain outer edge of the Newton polygon of p.

Let $V = \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}$. Assume that V is unbounded and consider the standard one-point algebraic compactification $\widehat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$ of the real plane (see [1, Definition 3.6.12]). Then ∞ belongs to the Zariski closure of V in $\widehat{\mathbb{R}}^2$. By [4, Lemma 3.3] in a suitable neighborhood of ∞, the curve $V \cup \{\infty\}$ is the union of finitely many branches which intersect only at ∞. Each branch is homeomorphic to an open interval under an analytic homeomorphism $\gamma : (-\epsilon, \epsilon) \to V \cup \{\infty\}$, $\gamma(0) = \infty$.

It follows that after passing to x and y coordinates in $\mathbb{R}^2$ and substituting $s = 1/t$ in γ, we obtain the following characterization of branches at infinity.

Lemma 3.2. *Assume that* $V = \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}$ *is an unbounded polynomial curve. Then, in the complement of some compact set* $K \subset \mathbb{R}^2$, V *is the union of finitely many pairwise disjoint "branches at infinity". Each branch at infinity is homeomorphic to a union of open intervals* $(-\infty, -R) \cup (R, \infty)$ *under a homeomorphism* $(x, y) = (\tilde{x}(t), \tilde{y}(t))$ *given by Laurent power series*

$$\tilde{x}(t) = a_k t^k + a_{k-1} t^{k-1} + a_{k-2} t^{k-2} + \cdots, \tag{3.1}$$

$$\tilde{y}(t) = b_l t^l + b_{l-1} t^{l-1} + b_{l-2} t^{l-2} + \cdots, \tag{3.2}$$

convergent for $|t| > R$.

Lemma 3.3. *Keep the assumptions and notation of Lemma 3.2. If* $a_k, b_k \neq 0$, *then* (k, l) *is a normal vector to an outer edge of the Newton polygon of* p.

Proof. Let $d = \max\{ki + lj : (i, j) \in \Delta(p)\}$. Writing $p = \sum_{ki+lj \le d} c_{i,j} x^i y^j$, substituting $(x, y) = (\tilde{x}(t), \tilde{y}(t))$ and collecting the terms of the highest degree we obtain

$$0 = p(\tilde{x}(t), \tilde{y}(t)) = \Big(\sum_{ki+lj=d} c_{i,j} a_k^i b_l^j\Big) t^d + \text{terms of lower degrees.}$$

A necessary condition for this identity to hold is the vanishing of the sum in parentheses, hence there are at least two nonzero $c_{i,j}$ such that $ki + lj = d$. Thus, the Newton polygon and the line $\{(i, j) \in \mathbb{R}^2 : ki + lj = d\}$ intersect along an edge.

Since $(x, y) = (\tilde{x}(t), \tilde{y}(t))$ is a Laurent parametrization of a branch at infinity, we have $\|(\tilde{x}(t), \tilde{y}(t))\| \to \infty$ as $t \to \infty$, which proves that $k > 0$ or $l > 0$, and $\Delta(p) \cap \{(i, j) \in \mathbb{R}^2 : ki + lj = d\}$ is an outer edge. ∎

It follows from Lemma 3.3 that every branch at infinity of the curve $p = 0$, which is not contained in coordinate axes, is associated with one of the outer edges of the Newton polygon of p. In the next lemma, we will show that the slope of the associated edge characterizes the asymptotic behavior of the branch at infinity.

For real valued functions g, h defined in (R,∞), we will write $g(x) \sim h(x)$ if there exist constants $c, C, r > 0$ such that $c|h(x)| \le |g(x)| \le C|h(x)|$ for all $x > r$.

Lemma 3.4. *Let $p(x,y)$ be a nonzero real polynomial such that for every x_0 the set $X_{x_0} = \{(x,y) \in \mathbb{R}^2 : x > x_0,\ y > 0,\ p(x,y) = 0\}$ is nonempty. Then for sufficiently large x_0 there exists a finite collection of continuous semialgebraic functions $f_i : (x_0,\infty) \to \mathbb{R}$, $i = 1,\dots,s$, such that*

(i) $0 < f_1(x) < \cdots < f_s(x)$ *for* $x > x_0$,
(ii) X_{x_0} *is the union of the graphs* $\{(x,y) \in \mathbb{R}^2 : y = f_i(x),\ x > x_0\}$, $i = 1,\dots,s$,
(iii) *for every f_i there exists a right outer edge S_i of the Newton polygon of $p(x,y)$ such that $f_i(x) \sim x^{\theta(S_i)}$.*

Proof. Parts (i) and (ii) follow from the cylindrical decomposition theorem for semialgebraic sets (see for example [1, Theorem 2.2.1]).

To prove (iii) observe that the graph of f_i is unbounded and homeomorphic to an open interval. Thus, increasing x_0 if necessary, we may assume that this graph is a half-branch at infinity. By Lemma 3.2, there exists a homeomorphism of (R,∞) and the graph given by Laurent power series (3.1), (3.2) with $a_k, b_l \neq 0$. The condition $\tilde{x}(t) \to \infty$ for $t \to \infty$ gives $k > 0$. By $\tilde{x}(t) \sim t^k$, $\tilde{y}(t) \sim t^l$ and the identity $f_i(\tilde{x}(t)) = \tilde{y}(t)$, we get $f_i(x) \sim x^{l/k}$. Finally, by Lemma 3.3, there exists a right outer edge S_i of the Newton polygon of p such that $l/k = \theta(S_i)$. ∎

4. Main result

Theorem 4.1. *Assume that the Newton polygon of $p \in \mathbb{R}[x,y]$ has a right outer edge S of negative slope with endpoint $(0,1)$, and the curve $p = 0$ has a real branch at infinity associated with the edge S. Then p has a glacial tongue with a straight border.*

Proof. By the assumptions, there exists a half-branch at infinity of the curve $p = 0$ with a Laurent parametrization $(x,y) = (\tilde{x}(t), \tilde{y}(t))$, where $\tilde{x}(t) \sim t^k$, $\tilde{y}(t) \sim t^l$, $k > 0$, $l < 0$ and (k,l) is a normal vector to S. Reversing signs of variables if necessary, we may assume that $\tilde{x}(t), \tilde{y}(t) > 0$ for sufficiently large t.

Then, in the notation of Lemma 3.4, this half-branch at infinity is a graph $y = f(x)$, where f is one of the functions f_i, $i = 1,\dots,s$. Comparing the asymptotic behavior of these functions, we see that $\theta(S_1) \le \cdots \le \theta(S_s)$. From the assumptions on S, it has the smallest slope among all right outer edges of the Newton polygon $\Delta(p)$, hence $S = S_1$ and we may assume that $f = f_1$. One has $p(x, f(x)) = p(x,0) = 0$ and $p(x,y) \neq 0$ for $x > x_0$, $0 < y < f(x)$.

Let $W_{x_0} = \{(x,y) \in \mathbb{R}^2 : x > x_0,\, 0 < y < f(x)\}$. The polynomial p vanishes nowhere on W_{x_0}, hence, without loss of generality, we may assume that p is positive on this set.

CLAIM 1. *For every $t \neq 0$ the set $p^{-1}(t) \cap W_{x_0}$ is bounded.*

Proof of Claim 1. If not, then by the Curve Selection Lemma there exists a half-branch at infinity of the curve $p(x,y) = t$ contained in W_{x_0}. Let $y = g(x)$ be the graph of this half-branch. By Lemma 3.4, $g(x) \sim x^{\theta(T)}$, where T is one of the right outer edges of the Newton polygon $\Delta(p-t)$. From $0 < g(x) < f(x)$, we get $\theta(T) \leq \theta(S)$. This is impossible because all right outer edges of $\Delta(p-t)$ have slopes greater than the slope of S.

CLAIM 2. *For x_0 sufficiently large, W_{x_0} does not contain any critical points of p.*

Proof of Claim 2. If the intersection of W_{x_0} with the set of critical points is bounded, then it is enough to enlarge x_0. If this intersection is unbounded, then by the Curve Selection Lemma it contains an unbounded semialgebraic arc $\Gamma \subset W_{x_0}$. It follows that p restricted to Γ is constant and nonzero, contrary to Claim 1.

We denote $V = W_{x_0}$ as in Claim 2. We can assume, enlarging x_0 if necessary, that p is positive on $\{x_0\} \times (0, f(x_0))$. We denote $V_t = p^{-1}(t) \cap V$ for $t \in \mathbb{R}$. Since V_t is a one-dimensional smooth semialgebraic manifold, it has finitely many connected components, and each connected component is homeomorphic to a circle or to an open interval.

CLAIM 3. *No connected component of V_t is homeomorphic to a circle.*

Proof of Claim 3. Suppose there is such a component. Then by Jordan's Theorem it divides the set V into two open regions. One of these regions is bounded. Since p is constant on the boundary of this region, it attains an extreme value at some point inside. This is impossible because p has no critical points in V.

Let $h(y) = p(x_0, y)$ be the restriction of p to $\{x_0\} \times \mathbb{R}$. The function h vanishes at the endpoints of the closed interval $[0, f(x_0)]$ and is positive inside. It is easy to find $t_0 > 0$ and two points $a < b$ inside $[0, f(x_0)]$ such that:

- $h'(y) \neq 0$ for $y \in (0, a] \cup [b, f(x_0))$,
- h increases from 0 to t_0 in $[0, a]$,
- $h(y) > t_0$ for $a < y < b$,
- h decreases from t_0 to 0 in $[b, f(x_0)]$.

CLAIM 4. *For every t such that $0 < t \leq t_0$ the set V_t is connected and homeomorphic to an open interval. The topological closure of V_t intersects the vertical segment $\{x_0\} \times (0, f(x_0))$ in two points.*

Proof of Claim 4. By the discussion preceding Claim 4, the polynomial p attains value t precisely at two points of the boundary of V. These are the points (x_0, y_1), where $0 < y_1 \leq a$ and (x_0, y_2), where $b \leq y_2 < f(x_0)$. Moreover $\partial p/\partial y$ does not vanish at these points.

By Claims 2 and 3, the set V_t is a one-dimensional smooth manifold having a finite number of connected components; each component is semi-algebraic and homeomorphic to an open interval. Thus, the closure of V_t is a graph with vertices (x_0, y_1), (x_0, y_2) and edges which are connected components of V_t.

By the Implicit Function Theorem, in a small neighborhood of (x_0, y_i) for $i = 1, 2$, the closure of V_t has the topological type of an interval $[0, 1)$, which shows that there is exactly one edge that connects (x_0, y_1) and (x_0, y_2).

By Claim 4, the closure of V_{t_0} is a line with endpoints (x_0, a) and (x_0, b). Joining these points with a vertical segment, we get a nonsmooth oval. By Jordan's Theorem, this oval divides the plane into two open regions. Let B_0 be the bounded region, let $B = B_0 \cup (\{x_0\} \times (0, f(x_0)))$, and let $A = V \cup (\{x_0\} \times (0, f(x_0)))$.

If $t \leq 0$, then $A_t = p^{-1}(t) \cap A$ is empty. If $0 < t \leq t_0$, then A_t is homeomorphic to a segment with endpoints in $\{x_0\} \times (0, f(x_0))$. If $t > t_0$, then either A_t is empty, or the closure of every connected component of A_t intersects the border of A along $x_0 \times (a, b)$. In this case, $A_t \subset B$. ∎

COROLLARY 4.2. *Assume that the Newton polygon of a polynomial $p \in \mathbb{R}[x, y]$ has a right outer edge that begins at $(0, 1)$, has a negative slope, and its only lattice points are the endpoints. Then p does not have a real Jacobian mate.*

Proof. Let S be the edge satisfying the assumptions of the corollary. Its endpoints are the lattice points $(0, 1)$ and (a, b) with $a \geq 1$, $b \geq 2$, and it has slope $\theta(S) = -a/(b-1)$. Moreover, a or $b - 1$ is odd since otherwise $(a/2, (b+1)/2)$ would be a lattice point on S.

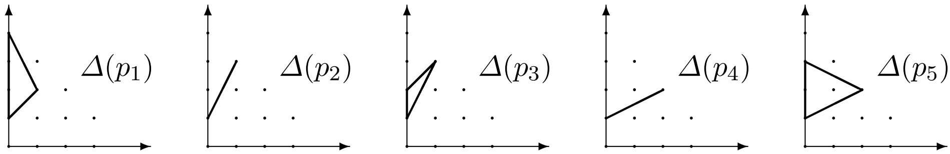
It is enough to prove that the curve $p = 0$ has a branch at infinity associated with S and to apply Theorems 4.1 and 2.1.

The polynomial p has two nonzero terms Ay and $Bx^a y^b$ corresponding to the endpoints of S. Using the conditions on a and b and reversing the sign of x or y if necessary, we may assume that A and B have opposite signs. For $(x(t), y(t)) = (ct^{b-1}, t^{-a})$, where c is a positive constant, we get $p(x(t), y(t)) = (Bc^a + A)t^{-a} +$ terms of lower degrees. Hence, the sign of p on the curve $(x(t), y(t))$ for large t depends only on the sign of $Bc^a + A$. The curve $(x(t), y(t))$ for large t is the graph of $h_c(x) = c^{a/(b-1)} x^{\theta(S)}$. By an appropriate choice of c, we get functions $g_1 = h_{c_1}$ and $g_2 = h_{c_2}$ such that p has opposite signs on their graphs, $g_1(x) \sim g_2(x) \sim x^{\theta(S)}$ and $0 < g_1 < g_2$. Then, by Lemma 3.4, there is a half-branch at infinity of $p = 0$ which is the graph of a function f such that $g_1(x) < f(x) < g_2(x)$ for large x. From the above inequalities we get $f(x) \sim x^{\theta(S)}$, which ends the proof. ∎

REMARK 4.3. Using toric modifications of the real plane, one can give a shorter proof of Corollary 4.2.

EXAMPLE 4.4. The polynomials $p_1 = y + xy^2 + y^4$, $p_2 = y + xy^3$, $p_3 = y + y^2 + xy^3$, $p_4 = y + x^2y^2$, $p_5 = y + ay^2 + y^3 + x^2y^2$, where $a^2 < 3$, all satisfy the assumptions of Corollary 4.2.

The Newton polygons of these polynomials are drawn below.



The polynomials in the above example are taken from [3]. Theorem 1.3 in that paper states that these polynomials are canonical forms up to affine substitution of polynomials of degree 4 without critical points and with at least one disconnected level set. Theorem 5.5 of [3] says that none of these polynomials has a real Jacobian mate. The method of its proof uses integration based on Green's formula and requires an analysis of each case separately.

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