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Diophantine triples of Fibonacci numbers

by

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1. Introduction. The sequence $\{F_n\}_{n\geq 1}$ of Fibonacci numbers is given by

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 1.$$

A *Diophantine m-tuple* is a set $\{a_1, \ldots, a_m\}$ of positive integers such that $a_i a_j + 1$ is a perfect square for all $i \neq j$ in $\{1, \ldots, m\}$. In 1977, Hoggatt and Bergum [4] proved that $\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\}$ is a Diophantine quadruple. In 1999, Dujella [2] proved that if d is a positive integer such that $\{F_{2n}, F_{2n+2}, F_{2n+4}, d\}$ is a Diophantine quadruple, then $d = 4F_{2n+1}F_{2n+2}F_{2n+3}$. Here, we take this one step further, by fixing a positive integer n and looking at positive integers k such that $\{F_{2n}, F_{2n+2}, F_k\}$ is a Diophantine triple. Our result is the following.

THEOREM 1. *If $\{F_{2n}, F_{2n+2}, F_k\}$ is a Diophantine triple, then $k \in \{2n+4, 2n-2\}$, except when $n = 2$, in which case we have the additional solution $k = 1$.*

Note that the exception $k = 1$ in case $n = 2$ is not truly an exception: it appears merely due to the fact that $F_1 = F_2$.

2. Intersection of two sequences. For any fixed positive integer n, assume that there exist positive integers k, x, y such that

$$F_{2n}F_k + 1 = x^2, \quad F_{2n+2}F_k + 1 = y^2. \tag{1}$$

Eliminating F_k, we deduce the norm form equation

$$F_{2n} \cdot y^2 - F_{2n+2} \cdot x^2 = F_{2n} - F_{2n+2}. \tag{2}$$

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Since $F_{2n} < F_{2n+2} < 4F_{2n}$, by [5, Theorem 8], we have

$$y\sqrt{F_{2n}} + x\sqrt{F_{2n+2}} = (\pm\sqrt{F_{2n}} + \sqrt{F_{2n+2}})(F_{2n+1} + \sqrt{F_{2n}F_{2n+2}})^j, \quad j \geq 0.$$

Define

$$V_j + U_j\sqrt{F_{2n}F_{2n+2}} := (F_{2n+1} + \sqrt{F_{2n}F_{2n+2}})^j.$$

Then we obtain

$$x = x_j = V_j \pm F_{2n}U_j. \tag{3}$$

Substituting (3) into the first equation of (1), we get

$$F_k = \pm 2V_jU_j + (F_{2n} + F_{2n+2})U_j^2. \tag{4}$$

This is the main equation to be solved. Let

$$C_j^{(\pm)} := \pm 2V_jU_j + (F_{2n} + F_{2n+2})U_j^2 \quad \text{for } j = 1, 2, \ldots. \tag{5}$$

Therefore, we have to solve the equation

$$C_j^{(\pm)} = F_k \tag{6}$$

for some positive integers j and k. Notice that the above equation has solutions

$$C_1^{(+)} = F_{2n+4} \quad \text{and} \quad C_1^{(-)} = F_{2n-2}, \tag{7}$$

which are the ones appearing in the statement of Theorem 1. We need to show that there are no other solutions (except when $n = 2$, for which $C_1^{(-)} = F_{2\cdot 2-2} = F_2 = F_1$). So, we shall assume that $j \geq 2$ in order to get a contradiction.

Let

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \bar{\alpha} = \frac{1-\sqrt{5}}{2}.$$

We use the well-known Binet formula

$$F_k = \frac{\alpha^k - \bar{\alpha}^k}{\sqrt{5}} \quad \text{for all } k \geq 1. \tag{8}$$

We set

$$\beta_n := F_{2n+1} + \sqrt{F_{2n+1}^2 - 1},$$

and also

$$V_j := \frac{\beta_n^j + \beta_n^{-j}}{2}, \qquad U_j = \frac{\beta_n^j - \beta_n^{-j}}{2\sqrt{F_{2n+1}^2 - 1}}. \tag{9}$$

Technically, V_j and U_j depend on n, but we shall assume that n is fixed throughout the argument. Define

$$\gamma_n^{(\pm)} := \pm\frac{1}{2\sqrt{F_{2n+1}^2 - 1}} + \frac{F_{2n} + F_{2n+2}}{4(F_{2n+1}^2 - 1)}. \tag{10}$$

Formula (5) leads to

$$(11)\qquad C_j^{(\pm)} = \pm\frac{\beta_n^{2j}-\beta_n^{-2j}}{2\sqrt{F_{2n+1}^2-1}} + (F_{2n}+F_{2n+2})\cdot\frac{\beta_n^{2j}-2+\beta_n^{-2j}}{4(F_{2n+1}^2-1)}$$
$$= \beta_n^{2j}\gamma_n^{(\pm)} - \frac{F_{2n}+F_{2n+2}}{2(F_{2n+1}^2-1)} + \beta_n^{-2j}\gamma_n^{(\mp)}.$$

Therefore, by (8) and (11), equation (6) becomes

$$(12)\qquad \beta_n^{2j}\gamma_n^{(\pm)} - \frac{F_{2n}+F_{2n+2}}{2(F_{2n+1}^2-1)} + \beta_n^{-2j}\gamma_n^{(\mp)} = \frac{\alpha^k-\bar{\alpha}^k}{\sqrt{5}}.$$

3. A linear form in three logarithms

Lemma 1. *We have*

(i) $1.46\alpha^{-2n} < \gamma_n^{(+)} < 1.66\alpha^{-2n}$;
(ii) $0.08\alpha^{-2n} < \gamma_n^{(-)} < 0.13\alpha^{-2n}$.

Proof. From the definition of $\gamma_n^{(\pm)}$, we deduce

$$(13)\qquad 2\sqrt{\gamma_n^{(\pm)}} = \frac{1}{\sqrt{F_{2n}}} \pm \frac{1}{\sqrt{F_{2n+2}}}$$
$$= 5^{1/4}\alpha^{-n}\left(\frac{1}{\sqrt{1-1/\alpha^{4n}}} \pm \frac{1}{\alpha\sqrt{1-1/\alpha^{4n+4}}}\right).$$

The Taylor series of $(1-x)^{-1/2}$ is

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots,$$

which implies

$$1 + \frac{1}{2}x < \frac{1}{\sqrt{1-x}} < 1 + \frac{x}{2(1-x)} \qquad \text{for } x \in (0,1).$$

Therefore,

$$(14)\qquad 1 \pm \frac{1}{\alpha} < \frac{1}{\sqrt{1-1/\alpha^{4n}}} \pm \frac{1}{\alpha\sqrt{1-1/\alpha^{4n+4}}} < 1.1 \pm \frac{1}{\alpha}.$$

Using (13) and (14), we get

$$1 \pm \frac{1}{\alpha} < \frac{2\sqrt{\gamma_n^{(\pm)}}}{5^{1/4}\alpha^{-n}} < 1.1 \pm \frac{1}{\alpha}.$$

Straightforward calculations give the results (i) and (ii) in the lemma. ■

Now, we define the following linear form in three logarithms:

$$(15)\qquad \Lambda := 2j\log\beta_n - k\log\alpha + \log(\sqrt{5}\cdot\gamma_n^{(\pm)}).$$

The next result determines an upper bound for Λ.

LEMMA 2. *If $j \geq 2$, then $0 < \Lambda < 66\beta_n^{-2j}$.*

Proof. Using equation (12), we have

$$\beta_n^{2j}\gamma_n^{(\pm)} - \frac{\alpha^k}{\sqrt{5}} = \frac{F_{2n}+F_{2n+2}}{2(F_{2n+1}^2-1)} - \beta_n^{-2j}\gamma_n^{(\mp)} - \frac{\bar{\alpha}^k}{\sqrt{5}}.$$

Suppose first that $\beta_n^{2j}\gamma_n^{(\pm)} \leq \alpha^k/\sqrt{5}$. Then

$$\frac{\sqrt{5}}{\alpha^k} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(\pm)}} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(-)}}$$

and

$$\begin{aligned}\frac{1}{F_{2n+2}} < \frac{1}{2F_{2n}} + \frac{1}{2F_{2n+2}} &= \frac{F_{2n}+F_{2n+2}}{2(F_{2n+1}^2-1)}\\ &< \beta_n^{-2j}\gamma_n^{(\mp)} + \frac{\bar{\alpha}^k}{\sqrt{5}} \leq \beta_n^{-2j}\gamma_n^{(+)} + \frac{1}{\sqrt{5}\cdot\alpha^k}\end{aligned}$$

imply

$$\frac{1}{F_{2n+2}} < \beta_n^{-2j}\left(\gamma_n^{(+)} + \frac{1}{5\gamma_n^{(-)}}\right). \tag{16}$$

Inequality (16) and Lemma 1 give

$$4^j F_{2n}^j F_{2n+2}^j < \beta_n^{2j} < F_{2n+2}\left(\gamma_n^{(+)} + \frac{1}{5\gamma_n^{(-)}}\right) < F_{2n+2}(1.66\alpha^{-2n} + 2.5\alpha^{2n}),$$

so

$$4^j F_{2n}^j F_{2n+2}^{j-1} < 1.66\alpha^{-2n} + 2.5\alpha^{2n}. \tag{17}$$

Inequality (17) implies easily that $j < 2$, which contradicts the assumption. So, we have $\beta_n^{2j}\gamma_n^{(\pm)} > \alpha^k/\sqrt{5}$. Therefore, $\Lambda > 0$. Moreover, as

$$|\alpha^k 5^{-1/2}\beta_n^{-2j}\gamma_n^{(\pm)-1} - 1| < \frac{F_{2n}+F_{2n+2}}{2(F_{2n+1}^2-1)}\cdot\frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}} < \frac{1}{F_{2n}}\cdot\frac{1}{\beta_n^{2j}\gamma_n^{(-)}} < 33\beta_n^{-2j}$$

and the rightmost quantity above is $< 1/2$, we deduce that $\Lambda < 66\beta_n^{-2j}$. Here, we have used the fact that

$$|\Lambda| < 2|e^\Lambda - 1| \quad \text{whenever} \quad |e^\Lambda - 1| < 1/2. \tag{18}$$ ■

For any non-zero algebraic number γ of degree d over $\mathbb{Q}$ whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^d (X - \gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{d}\Big(\log a + \sum_{j=1}^d \log\max(1, |\gamma^{(j)}|)\Big)$$

its absolute logarithmic height. We need the following result due to Matveev [8].

Lemma 3. *Let Λ be a linear form in logarithms of multiplicatively independent totally real algebraic numbers $\alpha_1, \ldots, \alpha_N$ with rational integer coefficients $b_1, \ldots, b_N$ $(b_N \neq 0)$. Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \leq j \leq N$. Define the numbers D, A_j $(1 \leq j \leq N)$ and E by $D := [\mathbb{Q}(\alpha_1, \ldots, \alpha_N) : \mathbb{Q}]$, $A_j := \max\{Dh(\alpha_j), |\log \alpha_j|\}$, $E := \max\{1, \max\{|b_j| A_j / A_N\,; 1 \leq j \leq N\}\}$. Then*

$$\log |\Lambda| > -C(N) C_0 W_0 D^2 \Omega,$$

where

$$\begin{aligned} C(N) &:= \frac{8}{(N-1)!}(N+2)(2N+3)(4e(N+1))^{N+1}, \\ C_0 &:= \log(e^{4.4N+7} N^{5.5} D^2 \log(eD)), \\ W_0 &:= \log(1.5eED \log(eD)), \qquad \Omega = A_1 \cdots A_N. \end{aligned}$$

In order to apply Lemma 3 to the linear form in three logarithms

$$\Lambda = 2j \log \beta_n - k \log \alpha + \log(\sqrt{5} \cdot \gamma_n^{(\pm)}),$$

we take

$$N = 3, \qquad D = 4, \qquad b_1 = 2j, \qquad b_2 = -k, \qquad b_3 = 1,$$

and

$$\alpha_1 = \beta_n, \qquad \alpha_2 = \alpha, \qquad \alpha_3 = \sqrt{5} \cdot \gamma_n^{(\pm)}.$$

We need to justify that $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent. But note that $\alpha_2 \in \mathbb{Q}(\sqrt{5})$ and $\alpha_1, \alpha_3^2 \in \mathbb{Q}(\sqrt{F_{2n}F_{2n+2}})$. Let us show that $F_{2n}F_{2n+2}$ is neither a square nor 5 times a square. Indeed, otherwise, since $\gcd(F_{2n}, F_{2n+2}) = F_{\gcd(2n,2n+2)} = F_2 = 1$, one of F_{2n} or F_{2n+2} would be a square. It is well-known that the only squares in the Fibonacci sequence are 1 and 144, leading to $n = 1, 5, 6$, but none of F_2F_4, $F_{10}F_{12}$, $F_{12}F_{14}$ is a square or 5 times a square. Thus, if we write $F_{2n}F_{2n+2} = du^2$ for an integer u and a square-free integer d, then $d > 1$ and $d \neq 5$. So, if $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively dependent, then α_1 and α_3^2 are multiplicatively dependent (because no power of α_2 of a non-zero integer exponent is in $\mathbb{Q}(\sqrt{d})$). Since α_1 is a unit in $\mathbb{Q}(\sqrt{d})$, we deduce that $\alpha_3^2 = 5(\gamma_n^{(\pm)})^2$ is a unit, which is false since the norm of $5(\gamma_n^{(\pm)})^2$ is

$$25(\gamma_n^{(+)}\gamma_n^{(-)})^2 = \frac{25F_{2n+1}^4}{256F_{2n}^4F_{2n+2}^4},$$

and the above fraction is not an integer for any $n \geq 1$.

One can see that

$$h(\alpha_1) = h(\beta_n) = \frac{1}{2}\log \beta_n \quad \text{and} \quad h(\alpha_2) = h(\alpha) = \frac{1}{2}\log \alpha.$$

As $\gamma_n^{(+)}, \gamma_n^{(-)}$ are conjugate and are roots of the quadratic polynomial

$$16F_{2n}^2F_{2n+2}^2X^2 - 8(F_{2n}^2F_{2n+2} + F_{2n}F_{2n+2}^2)X + (F_{2n+2} - F_{2n})^2,$$

and furthermore

$$|\gamma_n^{(\pm)}| \le |\gamma_n^{(+)}| = \frac{1}{4}\left(\frac{1}{\sqrt{F_{2n}}} + \frac{1}{\sqrt{F_{2n+2}}}\right)^2 < 1,$$

we get

$$h(\gamma_n^{(\pm)}) \le \frac{1}{2}\log(16F_{2n}^2F_{2n+2}^2) = \log(4F_{2n}F_{2n+2}) < \log(4/5) + (4n+2)\log\alpha,$$

where we have used the fact that $F_\ell < \alpha^\ell/\sqrt{5}$ for $\ell \in \{2n, 2n+2\}$. This implies

$$\begin{aligned} h(\alpha_3) = h(\sqrt{5}\cdot\gamma_n^{(\pm)}) &\le h(\sqrt{5}) + h(\gamma_n^{(\pm)}) \\ &< \frac{1}{2}\log 5 + \log(4/5) + (4n+2)\log\alpha \\ &= \log(4/\sqrt{5}) + (4n+2)\log\alpha < 4(n+1)\log\alpha, \end{aligned}$$

where we have used the inequality $4/\sqrt{5} < \alpha^2$. We set

$$A_1 = 2\log\beta_n, \qquad A_2 = 2\log\alpha, \qquad A_3 = 16(n+1)\log\alpha.$$

Relabeling the three numbers for the purpose of computing E (so making the substitution $\alpha_2 \leftrightarrow \alpha_3$), we see that we can take

$$E = \max\left\{\frac{2j\log\beta_n}{\log\alpha}, 8(n+1), k\right\} \le 4j(n+1).$$

For the last inequality above we have used, on the one hand, the fact that $\alpha^{\ell-2} \le F_\ell \le \alpha^{\ell-1}$ for all $\ell \ge 1$ to deduce that

$$\beta_n < 2F_{2n+1} < 2\alpha^{2n} < \alpha^{2(n+1)},$$

because $\alpha^2 > 2$, and on the other hand the fact that for $n \ge 2$ we have

$$\begin{aligned} \alpha^{k-1} < 2\alpha^{k-2} \le 2F_k &\le 4U_jV_j + 2(F_{2n} + F_{2n+2})U_j^2 \\ &= (V_j + U_j\sqrt{F_{2n}F_{2n+2}})^2 = (F_{2n+1} + \sqrt{F_{2n}F_{2n+2}})^{2j} \\ &< (2\alpha^{2n})^{2j} < \alpha^{4j(n+1)}, \end{aligned}$$

while for $n = 1$ we have $k \le 6$ by a result of Robbins [9].

By Lemmas 2 and 3 we get

$$\begin{aligned} C(3) &= \frac{8}{2!}(3+2)(6+3)(4^2e)^4 < 6.45\cdot 10^8, \\ C_0 &= \log(e^{4.4\cdot 3+7}3^{5.5}4^2\log(4e)) < 30, \\ W_0 &= \log(1.5\cdot 4eE\log(4e)) < \log(156j(n+1)), \\ \Omega &= (2\log\beta_n)(2\log\alpha)(16(n+1)\log\alpha), \end{aligned}$$

so

$$2j\log\beta_n - \log 66 < -\log|\Lambda| < (6.45\cdot 10^8)\cdot 30\cdot 4^2\log(156j(n+1))(2\log\beta_n)(2\log\alpha)(16(n+1)\log\alpha),$$

which leads to

$$j < 2.3\cdot 10^{12}(n+1)\log(156j(n+1)).$$

We record what we have obtained:

PROPOSITION 1. *If equation* (4) *has a positive integer solution* (j,k) *with* $j>1$, *then*

$$j < 2.3\cdot 10^{12}(n+1)\log(156j(n+1)). \tag{19}$$

4. A linear form in two logarithms. From (7), when $j=1$, one can see that equation (4) has the solutions

$$k = \begin{cases} 2n-2 & \text{for } C = C_1^{(-)}, \\ 2n+4 & \text{for } C = C_1^{(+)}. \end{cases} \tag{20}$$

This leads us to define

$$\Lambda_0 = 2\log\beta_n - ((2n+1)\pm 3)\log\alpha + \log(\sqrt{5}\cdot\gamma_n^{(\pm)}). \tag{21}$$

LEMMA 4. *We have* $|\Lambda_0| < 66\beta_n^{-2}$.

Proof. For $n=1$, this can be checked directly. Assume that $n\ge 2$. Substituting (20) into (12), we have

$$\beta_n^2\gamma_n^{(\pm)} - \frac{\alpha^{(2n+1)\pm 3}}{\sqrt{5}} = \frac{F_{2n}+F_{2n+2}}{2(F_{2n+1}^2-1)} - \beta_n^{-2}\gamma_n^{(\mp)} - \frac{\alpha^{-(2n+1)\mp 3}}{\sqrt{5}}.$$

If $\beta_n^2\gamma_n^{(\pm)} \le \alpha^{(2n+1)\pm 3}/\sqrt{5}$, then $\alpha^{-(2n+1)\mp 3}/\sqrt{5} \le 1/(5\beta_n^2\gamma^{(\pm)})$ and hence

$$|\alpha^{((2n+1)\pm 3)}5^{-1/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1| < \frac{\beta_n^{-2}\gamma_n^{(\mp)} + \alpha^{-(2n+1)\mp 3}/\sqrt{5}}{\beta_n^2\gamma_n^{(\pm)}} < \frac{\gamma_n^{(\mp)} + 1/(5\gamma_n^{(\pm)})}{\beta_n^4\gamma_n^{(\pm)}} < \frac{20.75+31.25\alpha^{4n}}{\beta_n^4}.$$

This inequality together with $\beta_n > 2+\sqrt{3}$ and $\beta_n > \alpha^{2n}$ gives

$$|\alpha^{((2n+1)\pm 3)}5^{-1/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1| < 32.8\beta_n^{-2}.$$

On the other hand, if $\beta_n^2\gamma_n^{(\pm)} > \alpha^{(2n+1)\pm 3}/\sqrt{5}$, then

$$|\alpha^{((2n+1)\pm 3)}5^{-1/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1| < \frac{1/(2F_{2n}) + 1/(2F_{2n+2})}{\beta_n^2\gamma_n^{(\pm)}} < \frac{1}{F_{2n}\beta_n^2\gamma_n^{(\pm)}} < 8.7\beta_n^{-2}.$$

In both cases,

$$|\alpha^{((2n+1)\pm 3)}5^{-1/2}\beta_n^{-2}/\gamma_n^{(\pm)} - 1| < 32.8\beta_n^{-2}. \tag{22}$$

Since $n \geq 2$, we have $\beta_n \geq 5 + \sqrt{24}$, so $\beta_n^2 > 66$, and inequality (22) implies $|\Lambda_0| < 66\beta_n^{-2}$ via (18). ∎

Let $K := (2j-1)(2n+1) - k \pm 3$ and

$$\Lambda_1 := K \log\alpha - (j-1)\log(5/4). \tag{23}$$

Lemma 5. *We have* $|\Lambda_1| < (6j+192)\alpha^{-4n-2}$.

Proof. We know that

$$\begin{aligned}\beta_n &= F_{2n+1} + \sqrt{F_{2n+1}^2 - 1} = 2F_{2n+1} - \frac{1}{F_{2n+1} + \sqrt{F_{2n+1}^2 - 1}} \\ &= 2F_{2n+1}\left(1 - \frac{1}{2F_{2n+1}(F_{2n+1} + \sqrt{F_{2n+1}^2 - 1})}\right)\end{aligned} \tag{24}$$

and

$$2F_{2n+1} = \frac{2}{\sqrt{5}}(\alpha^{2n+1} - \bar{\alpha}^{2n+1}) = \frac{2}{\sqrt{5}}\alpha^{2n+1}\left(1 + \frac{1}{\alpha^{4n+2}}\right).$$

We define

$$\delta_n = \left(1 - \frac{1}{2F_{2n+1}(F_{2n+1} + \sqrt{F_{2n+1}^2 - 1})}\right)\left(1 + \frac{1}{\alpha^{4n+2}}\right).$$

From the above, we deduce that

$$\log\beta_n = \log(2/\sqrt{5}) + (2n+1)\log\alpha + \log\delta_n.$$

Using (15) and (21), we have

$$\begin{aligned}\Lambda - \Lambda_0 &= (2j-2)\log\beta_n - (k - (2n+1) \mp 3)\log\alpha \\ &= (2j-2)\log\frac{2}{\sqrt{5}} + (2j-2)(2n+1)\log\alpha \\ &\quad + (2j-2)\log\delta_n - (k - (2n+1) \mp 3)\log\alpha \\ &= (2j-2)\log\delta_n + K\log\alpha - (j-1)\log(5/4).\end{aligned}$$

The above calculation and the definition of Λ_1 give

$$\Lambda_1 = \Lambda - \Lambda_0 - (2j-2)\log\delta_n.$$

One can see that Lemmas 2, 4 and the inequalities

$$\begin{aligned}|\log\delta_n| &\leq \left|\log\left(1 - \frac{1}{2F_{2n+1}(F_{2n+1} + \sqrt{F_{2n+1}^2 - 1})}\right)\right| + \left|\log\left(1 + \frac{1}{\alpha^{4n+2}}\right)\right| \\ &< \frac{1.2}{2F_{2n+1}(F_{2n+1} + \sqrt{F_{2n+1}^2 - 1})} + \frac{1}{\alpha^{4n+2}} < \frac{3}{\alpha^{4n+2}}\end{aligned}$$

imply

$$|\Lambda_1| \le |\Lambda| + |\Lambda_0| + |(2j-2)\log\delta_n| < \frac{132}{\beta_n^2} + \frac{6(j-1)}{\alpha^{4n+2}}. \tag{25}$$

Clearly,

$$\begin{aligned}\beta_n &= F_{2n+1}\left(1+\sqrt{1-\frac{1}{F_{2n+1}^2}}\right) \ge F_{2n+1}(1+\sqrt{3}/2)\\ &> \alpha^{2n+1}(1+\sqrt{3}/2)/\sqrt{5},\end{aligned}$$

so

$$\beta_n^2 > \alpha^{4n+2}\frac{(1+\sqrt{3}/2)^2}{5} > \frac{2\alpha^{4n+2}}{3}. \tag{26}$$

From (25) and (26), we get the desired conclusion. ∎

At this point, we recall the following result of Laurent [7].

LEMMA 6. *Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $b_1, b_2 \in \mathbb{Z}$ not both 0 and*

$$\Lambda = b_2\log\gamma_2 - b_1\log\gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1,\gamma_2):\mathbb{Q}]$. Let

$$h_i \ge \max\left\{h(\gamma_i), \frac{|\log\gamma_i|}{D}, \frac{1}{D}\right\} \quad \text{for } i=1,2, \qquad b' \ge \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

Then

$$\log|\Lambda| \ge -17.9D^4\left(\max\left\{\log b' + 0.38, \frac{30}{D}, \frac{1}{2}\right\}\right)^2 h_1h_2.$$

To apply Lemma 6 to Λ_1, we set

$$D=2, \qquad \gamma_1 = 5/4, \qquad \gamma_2 = \alpha, \qquad b_1 = j-1, \qquad b_2 = K.$$

The conditions of the lemma are fulfilled for our choices of parameters. Furthermore, we can take $h_1 = \log 5$, $h_2 = 1/2$. By Lemma 5, we have

$$\begin{aligned}K &< \frac{(j-1)\log(5/4) + (6j+192)\alpha^{-4n-2}}{\log\alpha}\\ &< 0.47(j-1) + 0.7j + 22.3 < 1.2j + 22.\end{aligned}$$

So, we can take

$$b' = 1.38j + 6.9 > (j-1) + \frac{K}{2\log 5} = \frac{|b_1|}{D\,h_2} + \frac{|b_2|}{D\,h_1}.$$

Therefore, Lemma 6 yields

$$\log|\Lambda_1| > -17.9\cdot 8\log 5\cdot(\max\{\log(1.38j+6.9)+0.38, 15\})^2. \tag{27}$$

From Lemma 5, we get

$$\log|\Lambda_1| < -(4n+2)\log\alpha + \log(6j+192). \tag{28}$$

Combining the two bounds (27) and (28) on $\log|\Lambda_1|$, we have

$$n < 120(\max\{\log(1.38j + 6.9) + 0.38, 15\})^2 + 0.6\log(6j + 192). \tag{29}$$

If

$$\log(1.38j + 6.9) + 0.38 \le 15,$$

then

$$j < 1619963 \quad \text{and} \quad n < 120 \cdot 15^2 + 0.6\log(6 \cdot 1619963 + 192) < 27010.$$

Otherwise,

$$n < 120(\log(1.38j + 6.9) + 0.38)^2 + 0.6\log(6j + 192). \tag{30}$$

Substituting inequality (30) into Proposition 1, we have

$$\begin{aligned} j < {} & 2.3 \cdot 10^{12}(120(\log(1.38j + 6.9) + 0.38)^2 + 0.6\log(6j + 192) + 1) \\ & \times \log\big(156j\big(120(\log(1.38j+6.9)+0.38)^2+0.6\log(6j + 192)+1\big)\big). \end{aligned} \tag{31}$$

A straightforward calculation gives $j < 4 \cdot 10^{19}$, which together with (30) implies $n < 252158$. Therefore, we get the following result.

Lemma 7. *If equation* (4) *has a positive integer solution* (j, k) *with* $j > 1$, *then*

$$j < 4 \cdot 10^{19} \quad \textit{and} \quad n < 252158.$$

5. Better bounds on j and n. From Lemma 5, we have

$$|K \log\alpha - (j - 1)\log(5/4)| < (6j + 192)\alpha^{-4n-2}.$$

Hence,

$$\left|\frac{\log(5/4)}{\log\alpha} - \frac{K}{j-1}\right| < \frac{6j + 192}{(j-1)\alpha^{4n+2}\log\alpha}. \tag{32}$$

Assume first that

$$\frac{6j + 192}{(j-1)\alpha^{4n+2}\log\alpha} < \frac{1}{2(j-1)^2}. \tag{33}$$

Then

$$\left|\frac{\log(5/4)}{\log\alpha} - \frac{K}{j-1}\right| < \frac{1}{2(j-1)^2}.$$

This implies, by a criterion of Legendre, that $K/(j-1)$ is a convergent in the simple continued fraction expansion of $\log(5/4)/\log\alpha$. We know that

$$\begin{aligned} \frac{\log(5/4)}{\log\alpha} = {} & [0, 2, 6, 2, 1, 1, 3, 7, 1, 3, 1, 1, 22, 2, 1, 1, 4, 3, 1, 2, 1, 1, 1, 1, 4, \\ & 1, 12, 6, 1, 1, 4, 1, 8, 2, 1, 49, 1, 10, 6, 1, 1, 3, 1, 1, 1, 5, 22, 1, \ldots]. \end{aligned}$$

The denominator of the 46th convergent

$$\frac{25158053660121411107}{54253653513327093513}$$

is greater than the upper bound $4 \cdot 10^{19}$ on j. The 45th convergent

$$\frac{4460457560349832575}{9619031832089360168}$$

provides the lower bound

$$\left|\frac{\log(5/4)}{\log\alpha} - \frac{K}{j-1}\right| > 1.9 \cdot 10^{-39}. \tag{34}$$

From (32) and (34), we get

$$1.9 \cdot 10^{-39} < \frac{6j+192}{(j-1)\alpha^{4n+2}\log\alpha} < 204\alpha^{-4n-2}(\log\alpha)^{-1},$$

which implies that $n < 49$. It is known (see [6]) that if p_r/q_r is the rth such convergent of $\log(5/4)/\log\alpha$, then

$$\left|\frac{\log(5/4)}{\log\alpha} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1}+2)q_r^2},$$

where a_{r+1} is the $(r+1)$st partial quotient of $\log(5/4)/(\log\alpha)$. We thus have

$$\min\left\{\frac{1}{(a_{r+1}+2)(j-1)^2}\right\} < \frac{6j+192}{(j-1)\alpha^{4n+2}\log\alpha} \qquad \text{for } 2 \le r \le 45. \tag{35}$$

Since $\max\{a_{r+1} : 2 \le r \le 45\} = a_{36} = 49$, from (35) we get

$$\alpha^{4n+2} < 51(j-1)(6j+192)(\log\alpha)^{-1}.$$

All this was when inequality (33) holds. On the other hand, if (33) does not hold, then

$$\alpha^{4n+2} \le 2(j-1)(6j+192)(\log\alpha)^{-1}.$$

Both possibilities give

$$\alpha^{4n+2} < 51(j-1)(6j+192)(\log\alpha)^{-1} < 636j(j+32) < 20988j^2.$$

Therefore, we deduce the following result.

Proposition 2. *If equation* (4) *has a positive integer solution* (j,k) *with* $j > 1$, *then*

$$n < 1.04\log j + 4.7. \tag{36}$$

This bound is better than that in (30). Combining Propositions 1 and 2, we get

$$j < 2.3 \cdot 10^{12}(1.04\log j + 5.7)\log(156j(1.04\log j + 5.7)),$$

which implies that $j < 5 \cdot 10^{15}$. Using Proposition 2, we get the following result.

Lemma 8. *If equation* (4) *has a positive integer solution* (j,k) *with* $j>1$, *then*

$$j < 5\cdot 10^{15} \quad \textit{and} \quad n < 43.$$

6. The Baker–Davenport reduction method. In order to deal with the remaining cases, for $1 \le n \le 42$ we used a Diophantine approximation algorithm called the Baker–Davenport reduction method. The following lemma is a slight modification of the original version of the Baker–Davenport reduction method (see [3, Lemma 5a]).

Lemma 9. *Assume that* κ *and* μ *are real numbers and* M *is a positive integer. Let* P/Q *be the convergent of the continued fraction expansion of* κ *such that* $Q > 6M$ *and let*

$$\eta = \|\mu Q\| - M\cdot\|\kappa Q\|,$$

where $\|\cdot\|$ *denotes the distance from the nearest integer. If* $\eta > 0$, *then there is no solution of the inequality*

$$0 < j\kappa - k + \mu < AB^{-j}$$

in integers j *and* k *with*

$$\frac{\log(AQ/\eta)}{\log B} \le j \le M.$$

As

$$0 < 2j\log\beta_n - k\log\alpha + \log(\sqrt{5}\cdot\gamma_n^{(\pm)}) < 66\beta_n^{-2j},$$

we apply Lemma 9 with

$$\kappa = \frac{2\log\beta_n}{\log\alpha}, \quad \mu = \frac{\log(\sqrt{5}\cdot\gamma_n^{(\pm)})}{\log\alpha}, \quad A = \frac{66}{\log\alpha}, \quad B = \beta_n^2, \quad M = 5\cdot 10^{15}.$$

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions. In one minute all the computations were done. In all cases, we obtained $j \le 17$. From Proposition 2, we deduce that $n < 8$. We set $M = 17$ to check again in the range $1 \le n \le 7$. The second run of the reduction method yields $j \le 6$ and then $n \le 6$. So we have the following result.

Lemma 10. *If equation* (4) *has a positive integer solution* (j,k) *with* $j > 1$, *then*

$$j \le 6 \quad \textit{and} \quad n \le 6.$$

We are now ready to prove Theorem 1.

7. The proof of Theorem 1. For $1 \le n \le 6$, $2 \le j \le 6$, we compute all $C_j^{(\pm)}$. None of them is a Fibonacci number. This means that equation (4) has no positive integer solution (j,k) with $j \ge 2$. When $j = 1$, we have

$$C_1^{(+)} = 2V_1U_1 + (F_{2n} + F_{2n+2})U_1^2 = 2F_{2n+1} + F_{2n} + F_{2n+2} = F_{2n+4}$$

for $n \ge 1$, and

$$C_1^{(-)} = -2V_1U_1 + (F_{2n} + F_{2n+2})U_1^2 = -2F_{2n+1} + F_{2n} + F_{2n+2} = F_{2n-2}$$

for $n = 1, 2$. Since $F_1 = F_2 = 1$, the additional solutions come from the triple $\{F_1, F_4, F_6\} = \{F_2, F_4, F_6\} = \{1, 3, 8\}$.

8. Final remark. We close by offering the following conjectures.

CONJECTURE 1. *There are no four positive integers* p, q, m, n *such that* $\{F_p, F_q, F_m, F_n\}$ *is a Diophantine quadruple.*

CONJECTURE 2. *If there exist three positive integers* $p < q < r$ *such that* $\{F_p, F_q, F_r\}$ *is a Diophantine triple, then* $(p,q,r) = (2n, 2n+2, 2n+4)$ *for some* $n \ge 1$ *with the additional exception* $(p,q,r) = (1,4,6)$.

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