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Congruences modulo powers of 3 for generalized Frobenius partitions with six colors

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1. Introduction. This paper is concerned with congruences modulo powers of 3 for the number of 6-colored generalized Frobenius partitions. In particular, we confirm a conjecture on a congruence modulo 243 that that appeared in [19].

The notion of *generalized Frobenius partition* of n was introduced by Andrews in his 1984 AMS Memoir [1]; it is a two-row array of the form

$$\begin{pmatrix} a_1 & \cdots & a_r \\ b_1 & \cdots & b_r \end{pmatrix}$$

of non-negative integers a_i, b_i (which we shall call 'parts') with

$$n = r + \sum_{i=1}^{r}(a_i + b_i), \tag{1.1}$$

where each row is arranged in non-increasing order. Andrews [1] also discussed a variant of generalized Frobenius partition which is called a k-colored generalized Frobenius partition. A *k-colored generalized Frobenius partition* is an array of the above form where the integer entries are taken from k distinct copies of non-negative integers distinguished by color, the rows are ordered first according to size and then according to color with no two consecutive like entries in any row. For any positive integer k, let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n. Andrews [1] also proved that for $n \geq 0$,

$$c\phi_2(5n+3) \equiv 0 \pmod{5}. \tag{1.2}$$

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Since then, a number of congruences for $c\phi_k(n)$ have been proved, typically for a small number of colors k. See, for example, Baruah and Sarmah [2, 3], Eichhorn and Sellers [5], Garvan and Sellers [6], Hirschhorn [7], Hirschhorn and Sellers [9], Kolitsch [10, 11], Lin [12], Lovejoy [13], Ono [14], Paule and Radu [15], Sellers [16, 17, 18], Xia [19, 20] and Zhang and Wang [21].

Recently, Baruah and Sarmah [3] established 2- and 3-dissection formulas of the generating function for $c\phi_6(n)$ which imply that for $n \geq 0$,

$$c\phi_6(2n+1) \equiv 0 \pmod{4}, \tag{1.3}$$

$$c\phi_6(3n+1) \equiv 0 \pmod{9}, \tag{1.4}$$

$$c\phi_6(3n+2) \equiv 0 \pmod{9}. \tag{1.5}$$

Baruah and Sarmah [3] also conjectured that for $n \geq 0$,

$$c\phi_6(3n+2) \equiv 0 \pmod{27}. \tag{1.6}$$

Recently, Xia [19] proved (1.6) by employing the generating function for $c\phi_6(3n+2)$ due to Baruah and Sarmah [3]. Moreover, Xia [19] also conjectured that for $n \geq 0$,

$$c\phi_6(9n+7) \equiv 0 \pmod{27}, \tag{1.7}$$

$$c\phi_6(27n+16) \equiv 0 \pmod{243}. \tag{1.8}$$

Very recently, Hirschhorn [7] established the generating functions for $c\phi_6(3n)$, $c\phi_6(3n+1)$ and $c\phi_6(3n+2)$ and proved (1.6) and (1.7).

In this paper, utilizing the generating function for $c\phi_6(3n+1)$ given by Hirschhorn [7], we prove several congruences modulo powers of 3 for $c\phi_6(n)$. In particular, we establish (1.8). The main results of this paper can be stated as follows.

THEOREM 1.1. *For $n \geq 0$,*

$$c\phi_6(27n+7) \equiv 3c\phi_6(3n+1) \pmod{3^4}, \tag{1.9}$$

$$c\phi_6(81n+61) \equiv 0 \pmod{3^4}, \tag{1.10}$$

$$c\phi_6(27n+16) \equiv 0 \pmod{3^5}, \tag{1.11}$$

$$c\phi_6(81n+61) \equiv 3c\phi_6(9n+7) \pmod{3^5}, \tag{1.12}$$

$$c\phi_6(729n+547) \equiv 0 \pmod{3^5}, \tag{1.13}$$

$$c\phi_6(243n+142) \equiv 0 \pmod{3^6}, \tag{1.14}$$

$$c\phi_6(729n+547) \equiv 3c\phi_6(81n+61) \pmod{3^6}, \tag{1.15}$$

$$c\phi_6(6561n+4921) \equiv 0 \pmod{3^6}, \tag{1.16}$$

$$c\phi_6(2187n+1276) \equiv 0 \pmod{3^7}, \tag{1.17}$$

$$c\phi_6(6561n+4921) \equiv 3c\phi_6(729n+547) \pmod{3^7}, \tag{1.18}$$

(1.19) $$c\phi_6(19683n + 11482) \equiv 0 \pmod{3^7},$$

(1.20) $$c\phi_6(59049n + 44287) \equiv 0 \pmod{3^7}.$$

To end this section, we present the following conjecture:

CONJECTURE 1.2. *For any integer $j \geq 8$, there exist positive integers a and k such that for all $n \geq 0$,*

(1.21) $$c\phi_6(3^k n + a) \equiv 0 \pmod{3^j}.$$

2. Preliminaries. Recall that the *Ramanujan theta function* $f(a, b)$ is defined by

(2.1) $$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

where $|ab| < 1$. The *Jacobi triple product identity* can be restated as

(2.2) $$f(a,b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty,$$

where

(2.3) $$(a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

One special case of (2.1) is defined by

(2.4) $$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

In this paper, for any positive integer n, we use f_n to denote $f(-q^n)$, that is,

$$f_n = (q^n; q^n)_\infty := \prod_{k=1}^{\infty} (1 - q^{nk}).$$

From Corollaries (i) and (ii) on p. 49 of Berndt's book [4], we have

(2.5) $$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_{18}^5}{f_9^2 f_{36}^2} + 2q \frac{f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}},$$

(2.6) $$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9}.$$

Replacing q by $-q$ in (2.5) and (2.6) and using the fact that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we get

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{2.7}$$

$$\frac{f_1 f_4}{f_2} = \frac{f_3 f_{12} f_{18}^5}{f_6^2 f_9^2 f_{36}^2} - q\frac{f_9 f_{36}}{f_{18}}. \tag{2.8}$$

Replacing q by q^2 in (2.7) yields

$$\frac{f_2^2}{f_4} = \frac{f_{18}^2}{f_{36}} - 2q^2\frac{f_6 f_{36}^2}{f_{12} f_{18}}. \tag{2.9}$$

The following identities were proved by Hirschhorn, Garvan and Borwein [8]:

$$a(q) = a(q^3) + 6q\frac{f_9^3}{f_3}, \tag{2.10}$$

$$f_1^3 = f_3 a(q^3) - 3q f_9^3, \tag{2.11}$$

where

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}. \tag{2.12}$$

Hirschhorn et al. [8] proved that

$$a(q) = 1 + 6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\right), \tag{2.13}$$

which implies

$$a(q) \equiv 1 \pmod 3 \quad \text{and} \quad a^3(q) \equiv 1 \pmod 9. \tag{2.14}$$

By the binomial theorem, for any positive integer k,

$$f_1^{3^k} \equiv f_3^{3^{k-1}} \pmod{3^k}. \tag{2.15}$$

3. Proof of Theorem 1.1. Recently, Hirschhorn [7] proved (there is a typo in his paper):

$$\begin{aligned} \sum_{n=0}^{\infty} c\phi_6(3n+1)q^n &= 9\left\{\frac{f_2^5 f_3^6}{f_1^{22} f_4^2}\left(2a^5(q)\frac{f_3^3}{f_1} + 189qa^2(q)\frac{f_3^{12}}{f_1^4}\right)\right. \\ &\quad + \frac{f_3^9 f_4 f_6^2}{f_1^{23} f_2 f_{12}}\left(2a^6(q) + 378qa^3(q)\frac{f_3^9}{f_1^3} + 1458q^2\frac{f_3^{18}}{f_1^6}\right) \\ &\quad \left. - \frac{f_3^9 f_{12}^2}{f_1^{23} f_6}\left(36qa^5(q)\frac{f_3^3}{f_1} + 1944q^2a^2(q)\frac{f_3^{12}}{f_1^4}\right)\right\}. \end{aligned} \tag{3.1}$$

Identity (3.1) yields

$$\sum_{n=0}^{\infty} c\phi_6(3n+1)q^n \equiv 18\frac{f_2^5 f_3^9}{f_1^{23} f_4^2}a^5(q) + 18\frac{f_3^9 f_4 f_6^2}{f_1^{23} f_2 f_{12}}a^6(q) + 1701q\frac{f_2^5 f_3^{18}}{f_1^{26} f_4^2}a^2(q) + 1215q\frac{f_3^{18} f_4 f_6^2}{f_1^{26} f_2 f_{12}}a^3(q) + 1863q\frac{f_3^{12} f_{12}^2}{f_1^{24} f_6}a^5(q) \pmod{3^7}. \tag{3.2}$$

By (2.15),

$$\frac{f_2^5 f_3^9}{f_1^{23} f_4^2} = \frac{f_2^5}{f_1^2 f_4^2}\frac{f_1^{222} f_3^9}{f_1^{243}} \equiv \frac{f_2^5}{f_1^2 f_4^2}\frac{f_1^{222}}{f_3^{72}} \pmod{3^5}, \tag{3.3}$$

$$\frac{f_3^9 f_4 f_6^2}{f_1^{23} f_2 f_{12}} = \frac{f_1 f_4}{f_2}\frac{f_1^{219} f_3^9 f_6^2}{f_1^{243} f_{12}} \equiv \frac{f_1 f_4}{f_2}\frac{f_1^{219} f_6^2}{f_3^{72} f_{12}} \pmod{3^5}, \tag{3.4}$$

$$\frac{f_2^5 f_3^{18}}{f_1^{26} f_4^2} = \frac{f_2^5}{f_1^2 f_4^2}\frac{f_1^3 f_3^{18}}{f_1^{27}} \equiv \frac{f_2^5}{f_1^2 f_4^2}f_1^3 f_3^9 \pmod{9}, \tag{3.5}$$

$$\frac{f_3^{18} f_4 f_6^2}{f_1^{26} f_2 f_{12}} = \frac{f_1 f_4}{f_2}\frac{f_3^{18} f_6^2}{f_1^{27} f_{12}} \equiv \frac{f_1 f_4}{f_2}\frac{f_3^9 f_6^2}{f_{12}} \pmod{9}, \tag{3.6}$$

$$\frac{f_3^{12} f_{12}^2}{f_1^{24} f_6} = \frac{f_1^3 f_3^{12} f_{12}^2}{f_1^{27} f_6} \equiv \frac{f_1^3 f_3^3 f_{12}^2}{f_6} \pmod{27}. \tag{3.7}$$

Combining (3.2)–(3.7) yields

$$\sum_{n=0}^{\infty} c\phi_6(3n+1)q^n \equiv 18\frac{f_2^5}{f_1^2 f_4^2}\frac{f_1^{222}}{f_3^{72}}a^5(q) + 18\frac{f_1 f_4}{f_2}\frac{f_1^{219} f_6^2}{f_3^{72} f_{12}}a^6(q) + 1701q\frac{f_2^5}{f_1^2 f_4^2}f_1^3 f_3^9 a^2(q) + 1215q\frac{f_1 f_4}{f_2}\frac{f_3^9 f_6^2}{f_{12}}a^3(q) + 1863q\frac{f_1^3 f_3^3 f_{12}^2}{f_6}a^5(q) \pmod{3^7}. \tag{3.8}$$

By (2.11), it is easy to check that

$$f_1^{222} \equiv f_3^{74}a^{74}(q^3) + 21q f_3^{73} f_9^3 a^{73}(q^3) + 9q^2 f_3^{72} f_9^6 a^{72}(q^3) + 81q^3 f_3^{71} f_9^9 a^{71}(q^3) \pmod{3^5}, \tag{3.9}$$

$$f_1^{219} \equiv f_3^{73}a^{73}(q^3) + 24q f_3^{72} f_9^3 a^{72}(q^3) + 81q^2 f_3^{71} f_9^6 a^{71}(q^3) + 81q^3 f_3^{70} f_9^9 a^{70}(q^3) \pmod{3^5}. \tag{3.10}$$

Substituting (2.5), (2.8), (2.10), (2.11), (3.9) and (3.10) into (3.8) and extracting the terms of the form q^{3n+2}, then dividing by q^2 and replacing q^3

by q, we get

$$\sum_{n=0}^{\infty} c\phi_6(9n+7)q^n \equiv 28\cdot 3^3\frac{f_2^2f_3^4f_{12}}{f_4f_6}a^{78}(q)+4\cdot 3^5\frac{f_3^4f_6^5}{f_{12}^2}a^{77}(q) + 3^6q\frac{f_2^2f_3^{13}f_{12}}{f_1^3f_4f_6}a^{75}(q) \pmod{3^7}. \tag{3.11}$$

In view of (2.14) and (2.15), we can write (3.11) as

$$\sum_{n=0}^{\infty} c\phi_6(9n+7)q^n \equiv 28\cdot 3^3\frac{f_2^2f_3^4f_{12}}{f_4f_6}a^{78}(q)+4\cdot 3^5\frac{f_3^4f_6^5}{f_{12}^2}a^{77}(q) + 3^6q\frac{f_2^2f_9^4f_{12}}{f_4f_6} \pmod{3^7}. \tag{3.12}$$

Based on (2.10),

$$a^{77}(q)\equiv a^{77}(q^3)+3q\frac{f_9^3}{f_3}a^{76}(q^3) \pmod{9} \tag{3.13}$$

and

$$a^{78}(q)\equiv a^{78}(q^3)+63q\frac{f_9^3}{f_3}a^{77}(q^3)+54q^2\frac{f_9^6}{f_3^2}a^{76}(q^3)+27q^3\frac{f_9^9}{f_3^3}a^{75}(q^3) \pmod{81}. \tag{3.14}$$

If we substitute (2.9), (3.13) and (3.14) into (3.12) and extract the terms of the form q^{3n+j} ($j\in\{0,1\}$), then divide by q^j and replace q^3 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_6(27n+7)q^n &\equiv 28\cdot 3^3\frac{f_1^4f_4}{f_2}\frac{f_6^2}{f_{12}}a^{78}(q)+4\cdot 3^5\frac{f_1^4f_2^5}{f_4^2}a^{77}(q)+3^6q\frac{f_3^4f_{12}^2}{f_6}\\ &\quad+3^6q\frac{f_1f_4}{f_2}\frac{f_6^2f_3^9}{f_{12}}a^{75}(q)+4\cdot 3^5q\frac{f_1^3f_3^3f_{12}^2}{f_6}a^{77}(q) \pmod{3^7}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_6(27n+16)q^n &\equiv 7\cdot 3^5\frac{f_1^3f_4f_6^2f_3^3}{f_2f_{12}}a^{77}(q)+3^6\frac{f_3^4f_4f_6^2}{f_2f_{12}}+3^6\frac{f_1^3f_2^5f_3^3}{f_4^2}a^{76}(q)\\ &\quad+2\cdot 3^6q\frac{f_1^2f_3^6f_{12}^2}{f_6}a^{76}(q) \pmod{3^7}.\end{aligned} \tag{3.16}$$

Congruence (1.11) follows from (3.16).

In view of (2.14), (2.15) and (3.15),

$$\sum_{n=0}^{\infty} c\phi_6(27n+7)q^n \tag{3.17}$$
$$\equiv 28\cdot 3^3\frac{f_1f_4}{f_2}f_1^3\frac{f_6^2}{f_{12}}a^{78}(q)+4\cdot 3^5\frac{f_2^5}{f_1^2f_4^2}f_1^6a^{77}(q)+3^6q\frac{f_3^4f_{12}^2}{f_6}$$
$$+3^6q\frac{f_1f_4}{f_2}\frac{f_6^2f_{27}}{f_{12}}+4\cdot 3^5qf_1^3\frac{f_3^3f_{12}^2}{f_6}a^{77}(q) \pmod{3^7}.$$

By (2.14), (2.15) and (3.2),

$$\sum_{n=0}^{\infty} c\phi_6(3n+1)q^n \equiv 9\frac{f_2^5f_3^9}{f_1^{23}f_4^2} \pmod{27}. \tag{3.18}$$

Congruence (1.9) follows from (2.14), (2.15), (3.17) and (3.18).

Substituting (2.5), (2.8), (2.11), (3.13) and (3.14) into (3.17) and extracting the terms of the form q^{3n+2}, then dividing by q^2 and replacing q^3 by q yields

$$\sum_{n=0}^{\infty} c\phi_6(81n+61)q^n \equiv 2\cdot 3^6\frac{f_2^2f_3f_9f_{12}}{f_4f_6}+3^6\frac{f_3^4f_6^5}{f_{12}^2}a^{77}(q) \tag{3.19}$$
$$+16\cdot 3^4\frac{f_2^2f_3^4f_{12}}{f_4f_6}a^{78}(q) \pmod{3^7},$$

which yields (1.10). Congruence (1.12) follows from (3.12) and (3.19).

If we substitute (2.9), (3.13) and (3.14) into (3.19) and extract the terms of the form q^{3n+j} ($j\in\{0,1\}$), then divide by q^j and replace q^3 by q, we obtain

$$\sum_{n=0}^{\infty} c\phi_6(243n+61)q^n \tag{3.20}$$
$$\equiv 2\cdot 3^6\frac{f_1f_3f_4f_6^2}{f_2f_{12}}+3^6\frac{f_1^4f_2^5}{f_4^2}a^{77}(q)+16\cdot 3^4\frac{f_1^4f_4f_6^2}{f_2f_{12}}a^{78}(q)$$
$$+3^6q\frac{f_1^3f_3^3f_{12}^2}{f_6}a^{77}(q) \pmod{3^7}$$

and

$$\sum_{n=0}^{\infty} c\phi_6(243n+142)q^n \equiv 3^6\frac{f_1^3f_3^3f_4f_6^2}{f_2f_{12}} \pmod{3^7}. \tag{3.21}$$

Congruence (1.14) follows from (3.21).

By (2.14) and (2.15),

$$\frac{f_1f_3f_4f_6^2}{f_2f_{12}} \equiv \frac{f_1^4f_2^5}{f_4^2}a^{77}(q) \pmod{3}. \tag{3.22}$$

Thanks to (2.14), (2.15), (3.20) and (3.22),

$$\sum_{n=0}^{\infty} c\phi_6(243n+61)q^n \equiv 16\cdot 3^4 f_1^3 \frac{f_1f_4f_6^2}{f_2f_{12}} a^{78}(q) + 3^6 q\frac{f_3f_9f_{12}^2}{f_6} \pmod{3^7}. \tag{3.23}$$

Substituting (2.8), (2.11) and (3.14) into (3.23) and extracting the terms of the form q^{3n+2}, then dividing by q^2 and replacing q^3 by q yields

$$\sum_{n=0}^{\infty} c\phi_6(729n+547)q^n \equiv 4\cdot 3^5 \frac{f_2^2f_3^4f_{12}}{f_4f_6} a^{78}(q) \pmod{3^7}. \tag{3.24}$$

Congruence (1.13) follows from (3.24); and (1.15) follows from (3.19) and (3.24).

In view of (2.14) and (3.24),

$$\sum_{n=0}^{\infty} c\phi_6(729n+547)q^n \equiv 4\cdot 3^5 \frac{f_2^2f_3^4f_{12}}{f_4f_6} \pmod{3^7}. \tag{3.25}$$

If we substitute (2.9) into (3.25), we get

$$\sum_{n=0}^{\infty} c\phi_6(729n+547)q^n \equiv 4\cdot 3^5 \frac{f_3^4f_{12}f_{18}^2}{f_6f_{36}} + 3^5q^2\frac{f_3^4f_{36}^2}{f_{18}} \pmod{3^7}, \tag{3.26}$$

which yields (1.17). Congruence (3.26) also implies that

$$\sum_{n=0}^{\infty} c\phi_6(2187n+547)q^n \equiv 4\cdot 3^5 \frac{f_1^4f_4f_6^2}{f_2f_{12}} \pmod{3^7}. \tag{3.27}$$

Substituting (2.8) and (2.11) into (3.27) and extracting the terms of the form q^{3n+2}, then dividing by q^2 and replacing q^3 by q yields

$$\sum_{n=0}^{\infty} c\phi_6(6561n+4921)q^n \equiv 3^6 \frac{f_2^2f_3^4f_{12}}{f_4f_6} \pmod{3^7}. \tag{3.28}$$

Congruence (1.16) follows from (3.28) and congruence (1.18) follows from (3.25) and (3.28). Replacing n by $3n+1$ in (1.18) and utilizing (1.17), we get (1.19). Replacing n by $9n+6$ in (1.18) and employing (1.16), we obtain (1.20). This completes the proof of Theorem 1.1. ∎

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