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An asymptotic formula related to the sums of divisors

by

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1. Introduction. Let $d(n)$ be the number of divisors of n, and k a positive integer. For $X > 1$, consider the sums of divisors of the form

$$T(k,s;X) := \sum_{1\le m_1,\dots,m_s\le X} d(m_1^k + \dots + m_s^k). \tag{1.1}$$

The earliest result about the asymptotics of such sums was given by Gafurov [2, 3] who studied the divisors of the quadratic form with $s = 2$ in (1.1) and obtained

$$T(2,2;X) = A_1X^2\log X + A_2X^2 + O(X^{5/3}\log^9 X),$$

where A_1 and A_2 are certain constants. The above error term was improved to $O(X^{3/2+\varepsilon})$ by Yu [13]. In 2000, C. Calderón and M. J. de Velasco [1] investigated (1.1) with $k = 2$, $s = 3$ and established the asymptotic formula

$$T(2,3;X) = \frac{8\zeta(3)}{5\zeta(4)}X^3\log X + O(X^3), \tag{1.2}$$

where $\zeta(s)$ is the Riemann zeta-function. Let γ be the Euler constant, and write

$$C_{i,k,s} = \sum_{q=1}^{\infty} \frac{(-2\log q + 2\gamma)^{i-1}}{q^{s+1}} \sum_{\substack{1\le a<q\\ (a,q)=1}} \Big(\sum_{r=1}^{q} e\Big(\frac{ar^k}{q}\Big)\Big)^s, \quad i = 1,2, \tag{1.3}$$

and

$$I_{j,k,s} = \int_{-\infty}^{\infty} \Big(\int_0^1 e(v^k\lambda)\,dv\Big)^s \Big(\int_0^s e(-v\lambda)(\log v)^{j-1}\,dv\Big)\,d\lambda, \quad j = 1,2. \tag{1.4}$$

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In 2012, Guo and Zhai [4] improved (1.2) to

$$T(2,3;X) = \frac{8\zeta(3)}{5\zeta(4)}X^3\log X + (C_{1,3}I_{2,3} + C_{2,3}I_{1,3})X^3 + O(X^{8/3+\varepsilon}),$$

where

$$C_{i,s} = C_{i,2,s} \quad \text{and} \quad I_{j,s} = I_{j,2,s}, \qquad i,j = 1,2. \tag{1.5}$$

The error term above was refined to $O(X^2\log^7 X)$ by Zhao [14] in 2014. Recently, Hu [7] investigated $T(2,4;X)$ and obtained the asymptotic formula

$$T(2,4;X) = 2C_{1,4}I_{1,4}X^4\log X + (C_{1,4}I_{2,4}+C_{2,4}I_{1,4})X^4 + O(X^{7/2+\varepsilon}), \tag{1.6}$$

where $C_{i,s}$, $I_{j,s}$ $(i,j=1,2)$ are defined in (1.5).

In this paper, we consider $T(k,s;X)$ for general $k \geq 2$, and establish asymptotic formulas for (1.1). We state our main results separately for $k=2$ and $k \geq 3$.

THEOREM 1.1. *Let $T(k,s;X)$ be as defined in* (1.1), *and let $C_{i,s}$, $I_{j,s}$ $(i,j=1,2)$ be as defined in* (1.5). *Then for $k=2$ and $s \geq 3$,*

$$T(2,s;X) = 2C_{1,s}I_{1,s}X^s\log X + (C_{1,s}I_{2,s} + C_{2,s}I_{1,s})X^s + O_s(X^{(s+1)/2}\log^{s+4}X + X^{s-2}\log X),$$

where the singular series $C_{i,s}$ $(i=1,2)$ are absolutely convergent and satisfy $C_{i,s} \gg 1$.

Note that $2C_{1,3}I_{1,3} = 8\zeta(3)/(5\zeta(4))$ and the error term for $s=3$ in Theorem 1.1 is $O(X^2\log^7 X)$ which implies the result of Zhao [14]. For $s=4$, the error term in Theorem 1.1 is $O(X^{5/2}\log^8 X)$, which is better than the result in (1.6).

THEOREM 1.2. *Let $T(k,s;X)$ be as defined in* (1.1). *Then for $k \geq 3$ and $s > \min\{2^{k-1}, k^2+k-2\}$,*

$$T(k,s;X) = kC_{1,k,s}I_{1,k,s}X^s\log X + (C_{1,k,s}I_{2,k,s}+C_{2,k,s}I_{1,k,s})X^s + O(X^{s-\theta+\varepsilon})$$

for every $\varepsilon > 0$, where

$$\theta = \begin{cases} k\left(\dfrac{1}{2^{k-1}} - \dfrac{1}{s}\right) & \text{when } 3 \leq k \leq 6 \text{ and } s > 2^{k-1}, \\[2ex] \dfrac{s-k^2-k+2}{2sk-2s} & \text{when } k \geq 7 \text{ and } s > k^2+k-2, \end{cases}$$

and $C_{i,k,s}$, $I_{j,k,s}$ $(i,j=1,2)$ are defined in (1.3) *and* (1.4), *respectively. Moreover, the singular series $C_{i,k,s}$ $(i=1,2)$ are absolutely convergent and satisfy $C_{i,k,s} \gg 1$.*

To prove our theorems we use the circle method. For Theorem 1.1, instead of using the classical circle method, we apply the Hardy–Littlewood–Kloosterman circle method which avoids decomposing the unit interval into the major arcs and the minor arcs. Also, we use Voronoi's summation formula to deal with the exponential sum related to $d(n)$. Moreover, to obtain a relatively good error term, we employ the estimates of the quadratic Gauss sum in [8].

In fact, the proof of Theorem 1.1 depends on the properties of the quadratic Gauss sum. As these properties are not available for general Gauss sums, the same method cannot be applied to prove Theorem 1.2. For Theorem 1.2, we use the classical circle method. We will treat the integral over the major arcs much as in [4]. For the minor arcs, we will use Weyl's inequality and Hua's lemma (see [11]) for $3 \le k \le 6$; and for $k \ge 7$, we will apply Wooley's new estimates of Weyl type for exponential sums together with the mean value estimates of [12].

Notation. Throughout the paper, X is a large positive integer and γ is the Euler constant. As usual, $e(x) = e^{2\pi i x}$ and $(a, b) = \gcd(a, b)$. The symbol $[X]$ denotes the integer part of X, and for an odd prime p, $\left(\frac{a}{p}\right)$ denotes the Legendre symbol. The letter ε denotes positive constants which are arbitrarily small, but may vary from statement to statement.

2. Proof of Theorem 1.1. To apply Voronoi's summation formula, we introduce a smooth weight. Let $\phi_0 \in C^2[0, \infty)$ be a function compactly supported on $[0, s+1]$ which is identically 1 on $[0, s]$, and let $\phi_1 \in C^2[0, \infty)$ be supported on $[1/2, s)$ with $\phi_1(x) = 1$ for $x \in (1, s-1)$. For $X \ge 2$, we define

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x < 1, \\ 1 & \text{if } 1 \le x \le sX^2, \\ \phi_0(x/X^2) & \text{if } x > sX^2. \end{cases}$$

Then ϕ is a smooth function supported on $[1/2, (s+1)X^2]$, and

$$T(2, s; X) = \sum_{\substack{1 \le m_1, \ldots, m_s \le X \\ m_1^2 + \cdots + m_s^2 = n}} d(n)\phi(n).$$

In order to apply the circle method, we introduce the exponential sums

$$f_k(\alpha) = \sum_{1 \le m \le X} e(m^k \alpha), \qquad h(\alpha) = \sum_n d(n)\phi(n)e(n\alpha). \tag{2.1}$$

Then we have

$$T(2, s; X) = \int_0^1 f_2^s(\alpha)h(-\alpha)\, d\alpha = \int_{-1/(\tau+1)}^{\tau/(\tau+1)} f_2^s(\alpha)h(-\alpha)\, d\alpha,$$

where τ is a large integer to be chosen later. We will evaluate $T(2, s; X)$ by dissecting the interval $(-1/(\tau+1), \tau/(\tau+1)]$ with Farey's points of order τ

(see [9, Chapter 11]). Let $a'/q' < a/q < a''/q''$ be adjacent points with q' and q'' satisfying

$$\tau \le q+q', q+q'' \le q+\tau, \quad aq' \equiv 1 \pmod q, \quad aq'' \equiv -1 \pmod q.$$

Then

$$\left(\frac{-1}{\tau+1}, \frac{\tau}{\tau+1}\right] = \bigcup_{q\le\tau} \bigcup_{\substack{0\le a<q\\(a,q)=1}} \left(\frac{a}{q} - \frac{1}{q(q+q')}, \frac{a}{q} + \frac{1}{q(q+q'')}\right].$$

Hence

$$T(2,s;X) = \sum_{q\le\tau} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathcal{M}(q,a)} f_2^s\left(\frac{a}{q}+\lambda\right) h\left(-\frac{a}{q}-\lambda\right) d\lambda,$$

where

$$\mathcal{M}(q,a) = \left[-\frac{1}{q(q+q')}, \frac{1}{q(q+q')}\right].$$

Following the argument in [6, Section 3] (see also [14, (5.1)–(5.2)]), we get

$$T(2,s;X) = \sum_{q\le\tau} \int_{|\lambda|\le 1/(q\tau)} \sum_{|v|\le\tau} \sigma(v;\lambda,q) \sum_{\substack{a=1\\(a,q)=1}}^{q} f_2^s\left(\frac{a}{q}+\lambda\right) h\left(-\frac{a}{q}-\lambda\right) e\left(-\frac{\bar{a}v}{q}\right) d\lambda, \tag{2.2}$$

where

$$\sigma(v;\lambda,q) \ll \frac{1}{1+|v|}. \tag{2.3}$$

To prove Theorem 1.1, we use two propositions which will be proved at the end of this section. Write

$$S_k(q,a,b) = \sum_{h=1}^{q} e\left(\frac{ah^k+bh}{q}\right), \qquad S_k(q,a) = S_k(q,a,0), \tag{2.4}$$

and define

$$\begin{gathered} \mathcal{F}(q;b_1,\dots,b_s,m) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \prod_{j=1}^{s} S_2(q,a,b_j) e\left(\frac{\bar{a}m}{q}\right), \\ \mathcal{F}(q) = \mathcal{F}(q;\underbrace{0,\dots,0}_{s},0). \end{gathered} \tag{2.5}$$

Proposition 2.1. *Let q be a positive integer and $q=q_1q_2$ with $(q_1,q_2)=1$, q_1 square-free and q_2 square-full. Then*

$$\mathcal{F}(q;b_1,\dots,b_s,m) \ll_s q_1^{(s+1)/2} q_2^{(s+2)/2}.$$

PROPOSITION 2.2. *Let X be a sufficiently large real number and $\tau=[5X]$. Suppose that $\alpha = a/q + \lambda$ with $(a,q) = 1$, $q \le \tau$ and $|\lambda| \le 1/(q\tau)$. Then for any $v \in \mathbb{Z}$,*

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} f_2^s\Big(\frac{a}{q}+\lambda\Big)h\Big(-\frac{a}{q}-\lambda\Big)e\Big(-\frac{\overline{a}v}{q}\Big) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S_2^s(q,a)}{q^{s+1}}\mathcal{J}(\lambda,q)e\Big(-\frac{\overline{a}v}{q}\Big) + \Delta(q,\lambda),$$

where

$$\mathcal{J}(\lambda,q) = \nu^s(\lambda)\vartheta(-\lambda,q) \tag{2.6}$$

with

$$\begin{aligned}\nu(\lambda) &= \int_0^X e(x^2\lambda)\,dx,\\ \vartheta(\lambda,q) &= \int(\log x + 2\gamma - 2\log q)e(x\lambda)\phi(x)\,dx,\end{aligned} \tag{2.7}$$

and $\Delta(q,\lambda)$ satisfies

$$\sum_{q\le\tau}\int_{|\lambda|\le 1/(q\tau)} |\Delta(q,\lambda)|\,d\lambda \ll_s X^{(s+1)/2}\log^{s+3} X. \tag{2.8}$$

Proof of Theorem 1.1. Let $X \ge 2$, $\tau = [5X]$ and $\alpha = a/q + \lambda$ be a real number with $(a,q) = 1$, $q \le \tau$ and $|\lambda| \le 1/(q\tau)$. It follows from Proposition 2.2 and (2.2) that

$$\begin{aligned}T(2,s;X) = &\sum_{q\le\tau}\int_{|\lambda|\le1/(q\tau)}\sum_{|v|\le\tau}\sigma(v;\lambda,q)\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{S_2^s(q,a)}{q^{s+1}}\mathcal{J}(\lambda,q)e\Big(-\frac{\overline{a}v}{q}\Big)\,d\lambda\\ &+\sum_{q\le\tau}\int_{|\lambda|\le1/(q\tau)}\sum_{|v|\le\tau}\sigma(v;\lambda,q)\Delta(q,\lambda)\,d\lambda.\end{aligned}$$

Thus by (2.3) and (2.8), we have

$$T(2,s;X) = \sum_{q\le\tau}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\lambda\in\mathcal{M}(q,a)}\frac{S_2^s(q,a)}{q^{s+1}}\mathcal{J}(\lambda,q)\,d\lambda + O_s(X^{(s+1)/2}\log^{s+4} X). \tag{2.9}$$

Applying the technique used in [6, Section 3], we get

$$\sum_{q\le\tau}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\lambda\in\mathcal{M}(q,a)\setminus[-\frac{1}{2q\tau},\frac{1}{2q\tau}]}\frac{S_2^s(q,a)}{q^{s+1}}\mathcal{J}(\lambda,q)\,d\lambda$$

$$=\sum_{q\le\tau}\int_{\frac{1}{2q\tau}\le|\lambda|\le\frac{1}{q\tau}}\sum_{|v|\le\tau}\sigma(v;\lambda,q)\frac{\mathcal{F}(q;\overbrace{0,\dots,0}^{s},-v)}{q^{s+1}}\mathcal{J}(\lambda,q)\,d\lambda,$$

where $\mathcal{F}(q;0,\dots,0,m)$ is defined in (2.5). By Proposition 2.1 and (2.3), the expression on the right is

$$\ll_s\sum_{|v|\le\tau}\frac{1}{1+|v|}\sum_{q\le\tau}\int_{\frac{1}{2q\tau}\le|\lambda|\le\frac{1}{q\tau}}\frac{q_1^{(s+1)/2}q_2^{(s+2)/2}}{q^{s+1}}|\mathcal{J}(\lambda,q)|\,d\lambda. \tag{2.10}$$

For $\nu(\lambda)$ and $\vartheta(\lambda,q)$ defined in (2.7), integration by parts shows that

$$\nu(\lambda)\ll\frac{X}{\sqrt{1+X^2|\lambda|}}\quad\text{and}\quad\vartheta(\lambda,q)\ll\frac{X^2(\log q+\log X)}{1+X^2|\lambda|}. \tag{2.11}$$

This together with (2.6) gives

$$\mathcal{J}(\lambda,q)\ll X^{s+2}(\log X)(1+X^2|\lambda|)^{-(s+2)/2}. \tag{2.12}$$

Inserting (2.12) into (2.10), we get

$$\begin{aligned}\sum_{q\le\tau}\sum_{\substack{a=1\\(a,q)=1}}^{q}&\int_{\lambda\in\mathcal{M}(q,a)\setminus[-\frac{1}{2q\tau},\frac{1}{2q\tau}]}\frac{S_2^s(q,a)}{q^{s+1}}\mathcal{J}(\lambda,q)\,d\lambda\\&\ll_s\sum_{|v|\le\tau}\frac{1}{1+|v|}\sum_{q\le\tau}\int_{\frac{1}{2q\tau}\le|\lambda|\le\frac{1}{q\tau}}\frac{q_1^{(s+1)/2}q_2^{(s+2)/2}X^{s+2}(\log X)}{q^{s+1}(1+X^2|\lambda|)^{(s+2)/2}}\,d\lambda\\&\ll_s X^{(s+1)/2}\log^3 X.\end{aligned} \tag{2.13}$$

Let $\mathcal{F}(q)$ be defined in (2.5). Then by Proposition 2.1 and (2.12),

$$\begin{aligned}\sum_{q\le\tau}\sum_{\substack{a=1\\(a,q)=1}}^{q}&\int_{|\lambda|>1/(2q\tau)}\frac{S_2^s(q,a)}{q^{s+1}}\mathcal{J}(\lambda,q)\,d\lambda\\&\ll\sum_{q\le\tau}\frac{|\mathcal{F}(q)|}{q^{s+1}}\int_{|\lambda|>1/(2q\tau)}|\mathcal{J}(\lambda,q)|\,d\lambda\\&\ll_s X^{(s+1)/2}\log^2 X.\end{aligned} \tag{2.14}$$

Back to (2.9), we deduce from (2.13) and (2.14) that

$$
\begin{aligned}
(2.15)\qquad & T(2,s;X)\\
&=\sum_{q\le\tau}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int\limits_{|\lambda|\le 1/(2q\tau)}\frac{S_2^s(q,a)}{q^{s+1}}\mathcal{J}(\lambda,q)\,d\lambda+O_s(X^{(s+1)/2}\log^{s+4}X)\\
&=\sum_{q\le\tau}\frac{\mathcal{F}(q)}{q^{s+1}}\int\limits_{|\lambda|\le 1/(2q\tau)}\mathcal{J}(\lambda,q)\,d\lambda+O_s(X^{(s+1)/2}\log^{s+4}X)\\
&=\sum_{q\le\tau}\frac{\mathcal{F}(q)}{q^{s+1}}\int\limits_{-\infty}^{\infty}\mathcal{J}(\lambda,q)\,d\lambda+O_s(X^{(s+1)/2}\log^{s+4}X).
\end{aligned}
$$

Define

$$
\widetilde{\mathcal{J}}(\lambda,q)=\Big(\int\limits_0^X e(x^2\lambda)\,dx\Big)^s\int(\log x+2\gamma-2\log q)e(-x\lambda)\phi_0(x/X^2)\,dx.
$$

By changing variables, $\widetilde{\mathcal{J}}(\lambda,q)$ can be written as

$$
\begin{aligned}
(2.16)\qquad \widetilde{\mathcal{J}}(\lambda,q)&=2X^{s+2}(\log X)\mathcal{G}_1(X^2\lambda)\\
&\quad+X^{s+2}\big(\mathcal{G}_2(X^2\lambda)+(2\gamma-2\log q)\mathcal{G}_1(X^2\lambda)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{G}_1(\lambda)&=\Big(\int\limits_0^1 e(x^2\lambda)\,dx\Big)^s\int e(-x\lambda)\phi_0(x)\,dx,\\
\mathcal{G}_2(\lambda)&=\Big(\int\limits_0^1 e(x^2\lambda)\,dx\Big)^s\int(\log x)e(-x\lambda)\phi_0(x)\,dx.
\end{aligned}
$$

Note that $|\mathcal{J}(\lambda,q)-\widetilde{\mathcal{J}}(\lambda,q)|\ll X^s(\log X)(1+X^2|\lambda|)^{-s/2}$. By Proposition 2.1,

$$
(2.17)\qquad \sum_{q\le\tau}\frac{\mathcal{F}(q)}{q^{s+1}}\int\limits_{-\infty}^{\infty}(\mathcal{J}(\lambda,q)-\widetilde{\mathcal{J}}(\lambda,q))\,d\lambda\ll_s X^{s-2}\log X.
$$

Therefore by (2.15)–(2.17),

$$
\begin{aligned}
T(2,s;X)&=\sum_{q\le\tau}\frac{\mathcal{F}(q)}{q^{s+1}}\int\limits_{-\infty}^{\infty}\widetilde{\mathcal{J}}(\lambda,q))d\lambda+O_s(X^{(s+1)/2}\log^{s+4}X+X^{s-2}\log X)\\
&=2\sum_{q\le\tau}\frac{\mathcal{F}(q)}{q^{s+1}}X^s(\log X)I_{1,s}+\sum_{q\le\tau}\frac{\mathcal{F}(q)}{q^{s+1}}X^sI_{2,s}\\
&\quad+\sum_{q\le\tau}\frac{\mathcal{F}(q)(2\gamma-2\log q)}{q^{s+1}}X^sI_{1,s}+O_s(X^{(s+1)/2}\log^{s+4}X+X^{s-2}\log X),
\end{aligned}
$$

where

$$I_{1,s}=\int_{-\infty}^{\infty}\mathcal{G}_1(\lambda)\,d\lambda+O(1)\quad\text{and}\quad I_{2,s}=\int_{-\infty}^{\infty}\mathcal{G}_2(\lambda)\,d\lambda+O(1)$$

with $I_{j,s}$ $(j=1,2)$ defined in (1.5). Moreover, it is easy to see that

$$\sum_{q\le\tau}\frac{\mathcal{F}(q)}{q^{s+1}}=C_{1,s}+O_s(X^{(1-s)/2}\log X),$$
$$\sum_{q\le\tau}\frac{\mathcal{F}(q)(2\gamma-2\log q)}{q^{s+1}}=C_{2,s}+O_s(X^{(1-s)/2}\log^2 X),$$

where $C_{i,s}$ $(i=1,2)$ are defined in (1.5). Therefore, for $s\ge 3$, we obtain

$$T(2,s;X)=2C_{1,s}I_{1,s}X^s\log X+(C_{1,s}I_{2,s}+C_{2,s}I_{1,s})X^s + O_s(X^{(s+1)/2}\log^{s+4}X+X^{s-2}\log X).$$

This finishes the proof of Theorem 1.1. ∎

Now we turn to the proofs of Propositions 2.1 and 2.2. To prove Proposition 2.1, we need two lemmas which are related to the Gauss sum. The following lemma can be found in [8].

LEMMA 2.3.

(1) *If* $(2,a)=1$, *then* $|S_2(2^r,a,b)|\le 2^{1+r/2}$.
(2) *If* $(2a,q)=1$, *then*

$$S_2(q,a,b)=e\Big(-\frac{\overline{4a}b^2}{q}\Big)\Big(\frac{a}{q}\Big)S_2(q,1);$$

moreover, $|S_2(q,1)|=q^{1/2}$.

LEMMA 2.4. *Let* s *be a positive integer. Then*

$$\Big|\sum_{\substack{a=1\\(a,p)=1}}^{p}\Big(\frac{a}{p}\Big)^s e\Big(\frac{a}{p}\Big)\Big|\le p^{1/2}.$$

Proof. If s is odd and $(a,p)=1$, then $\big(\frac{a}{p}\big)^s=\big(\frac{a}{p}\big)$. Hence the desired inequality follows from [14, Lemma 3.4]. If s is even and $(a,p)=1$, then the sum on the left hand side is equal to $\mu(p)$, and the desired result follows, because $|\mu(p)|=1$. ∎

Proof of Proposition 2.1. Following the proof of [14, Lemma 3.5], we find that $\mathcal{F}(q;b_1,\dots,b_s,m)$ is multiplicative in q, that is, for $q=q_1q_2$ with $(q_1,q_2)=1$,

$$\mathcal{F}(q_1q_2;b_1,\dots,b_s,m)=\mathcal{F}(q_1;b_1,\dots,b_s,m)\mathcal{F}(q_2;b_1,\dots,b_s,m). \tag{2.18}$$

To prove Proposition 2.1, we only need to deal with $\mathcal{F}(p^r; b_1, \ldots, b_s, m)$ $(r \geq 1)$. By Lemma 2.3, we get

$$(2.19)\qquad \begin{aligned} |\mathcal{F}(2^r; b_1, \ldots, b_s, m)| &\leq 2^{s-1+(s+2)r/2}, \\ \mathcal{F}(p^r; b_1, \ldots, b_s, m)| &\leq p^{(s+2)r/2} \quad (p > 2). \end{aligned}$$

In particular, if $r = 1$ and $p > 2$, we change variables and deduce from Lemma 2.3(2) that

$$\begin{aligned} \mathcal{F}(p; b_1, \ldots, b_s, m) &= S_2^s(p,1) \sum_{\substack{a=1 \\ (a,p)=1}}^{p} e\left(\frac{-\overline{4a}(b_1^2 + \cdots + b_s^2 - 4m)}{p}\right)\left(\frac{a}{p}\right)^s \\ &= S_2^s(p,1)\left(\frac{-1}{p}\right)^s \left(\frac{b_1^2 + \cdots + b_s^2 - 4m}{p}\right)^s \sum_{\substack{c=1 \\ (c,p)=1}}^{p} e\left(\frac{c}{p}\right)\left(\frac{c}{p}\right)^s. \end{aligned}$$

Then by Lemmas 2.4 and 2.3(2), we get

$$(2.20)\qquad \mathcal{F}(p; b_1, \ldots, b_s, m) \leq p^{(s+1)/2}.$$

Assume that $q = q_1 q_2$ with $(q_1, q_2) = 1$, q_1 square-free and q_2 square-full. By (2.18)–(2.20), we get

$$\mathcal{F}(q_1; b_1, \ldots, b_s, m) \ll_s q_1^{(s+1)/2} \quad \text{and} \quad \mathcal{F}(q_2; b_1, \ldots, b_s, m) \ll q_2^{(s+2)/2}.$$

This completes the proof of Proposition 2.1. ∎

Now we turn to the proof of Proposition 2.2. The following lemma provides an asymptotic formula for $f_2(\alpha)$ which is [11, Theorem 4.1] with $k = 2$ (see also [14, Lemma 4.1]).

Lemma 2.5. *Let X be a sufficiently large real number and $\tau = [5X]$. Suppose that $\alpha = a/q + \lambda$ with $(a,q) = 1$, $q \leq \tau$ and $|\lambda| \leq 1/(q\tau)$. Then*

$$f_2(\alpha) = \frac{S_2(q,a)}{q} \int_0^X e(x^2\lambda)\,dx + \sum_{-3q/2 < b \leq 3q/2} S_2(q,a,b)\mathcal{D}(b,q,\lambda),$$

where $S_k(q,a,b)$, $S_k(q,a)$ are defined in (2.4), *and*

$$(2.21)\qquad \sum_{-3q/2 < b \leq 3q/2} |\mathcal{D}(b,q,\lambda)| \ll \log(q+2).$$

To estimate $h(\alpha)$, we employ Voronoi's summation formula to get the following result:

Lemma 2.6. *Let α be defined as in Proposition* 2.2. *Then*

$$h(\alpha) = q^{-1} \int (\log x + 2\gamma - 2\log q) e(x\lambda)\phi(x)\,dx + \sum_{|n| \neq 0} e\left(-\frac{\overline{a}n}{q}\right)\mathcal{H}(n,q,\lambda),$$

where

$$\sum_{|n|\neq 0} |\mathcal{H}(n,q,\lambda)| = O_s(q\log^2(q+2) + |\lambda|^2 q^{3/2} X^{7/2}). \tag{2.22}$$

Proof. The proof is similar to that of [14, Lemma 4.2]. The only difference is that we replace the smooth weight function $\omega(x)$ in [14] by $\phi(x)$ here, which results in the dependence on s of the O-term in (2.22). We omit the details. ∎

Proof of Proposition 2.2. Let $\mathcal{F}(q,b_1,\dots,b_s,m)$ be as in (2.5). Applying Lemmas 2.5 and 2.6, we get

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} f_2^s(\alpha)h(-\alpha)e\Big(-\frac{\overline{a}v}{q}\Big) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S_2^s(q,a)}{q^{s+1}}\mathcal{J}(\lambda,q)e\Big(-\frac{\overline{a}v}{q}\Big) + \varDelta(q,\lambda),$$

where $\mathcal{J}(\lambda,q)$ is defined in (2.6) and

$$\varDelta(q,\lambda) = \sum_{i=0}^{s} R_i + \sum_{k=0}^{s-1} E_k$$

with

$$R_i = \frac{s!}{i!(s-i)!}\,\frac{\nu^i(\lambda)}{q^i} \sum_{-3q/2<b_1,\dots,b_{s-i}\le 3q/2}\ \sum_{|n|\neq 0}\Big(\prod_{j=1}^{s-i}\mathcal{D}(b_j,q,\lambda)\Big)\mathcal{H}(n,q,-\lambda) \times \mathcal{F}(q;b_1,\dots,b_{s-i},\underbrace{0,\dots,0}_{i},n-v)$$

and

$$E_k = \frac{s!}{k!(s-k)!}\,\frac{\vartheta(-\lambda,q)\nu^k(\lambda)}{q^{k+1}} \sum_{-3q/2<b_1,\dots,b_{s-k}\le 3q/2}\Big(\prod_{j=1}^{s-k}\mathcal{D}(b_j,q,\lambda)\Big) \times \mathcal{F}(q;b_1,\dots,b_{s-k},\underbrace{0,\dots,0}_{k},-v).$$

In the following, we will only deal with integrals involving R_0, R_s, E_0 and E_{s-1}. The treatment of other terms related to R_s $(1\le i\le s-1)$ and E_k $(1\le k\le s-2)$ is similar. By Proposition 2.1 together with (2.11), (2.21) and (2.22), we get

$$R_0 \ll_s q_1^{(s+3)/2} q_2^{(s+4)/2}\log^{s+2} X + |\lambda|^2 q_1^{(s+4)/2} q_2^{(s+5)/2} X^{7/2}\log^s X,$$

$$R_s \ll_s q_1^{(3-s)/2} q_2^{(4-s)/2}\frac{X^s\log^2 X}{(1+X^2|\lambda|)^{s/2}} + |\lambda|^2 q_1^{(4-s)/2} q_2^{(5-s)/2}\frac{X^{7/2+s}}{(1+X^2|\lambda|)^{s/2}},$$

$$E_0 \ll_s q_1^{(s-1)/2} q_2^{s/2} \frac{X^2 \log^{s+1} X}{1+X^2|\lambda|},$$

$$E_{s-1} \ll_s q_1^{(1-s)/2} q_2^{(2-s)/2} \frac{X^{s+1} \log^2 X}{(1+X^2|\lambda|)^{(s+1)/2}}.$$

Thus

$$\sum_{q\le\tau} \int\limits_{|\lambda|\le 1/(q\tau)} |R_0|\, d\lambda \ll_s X^{(s+1)/2} \log^{s+3} X,$$

$$\sum_{q\le\tau} \int\limits_{|\lambda|\le 1/(q\tau)} (|R_s| + |E_1| + |E_{s-1}|)\, d\lambda \ll_s X^{(s+1)/2} \log^{s+2} X.$$

Similarly, we have

$$\sum_{q\le\tau} \int\limits_{|\lambda|\le 1/(q\tau)} \Big(\sum_{i=1}^{s-1} |R_i| + \sum_{k=1}^{s-2} |E_k|\Big)\, d\lambda \ll_s X^{(s+1)/2} \log^{s+3} X.$$

This finishes the proof of Proposition 2.2. ∎

3. Proof of Theorem 1.2. To prove Theorem 1.2 we use the classical circle method. Let $f_k(\alpha)$ be as in (2.1), and write

$$g(\alpha) = \sum_{1\le n\le sX^k} d(n) e(n\alpha)$$

with $X \ge 2$ and $k \ge 3$. Then for any $Q > 0$, (1.1) can be written as

$$T(k,s;X) = \int_0^1 f_k^s(\alpha) g(-\alpha)\, d\alpha = \int_{1/Q}^{1+1/Q} f_k^s(\alpha) g(-\alpha)\, d\alpha.$$

By Dirichlet's lemma on rational approximations, each $\alpha \in [1/Q, 1+1/Q]$ can be written in the form

$$\alpha = \frac{a}{q} + \lambda, \qquad |\lambda| \le \frac{1}{qQ},$$

for some integers a, q with $1 \le a \le q \le Q$ and $(a,q) = 1$. We set

$$P = \tfrac{1}{8} X^{k/s} \quad \text{and} \quad Q = 2X^{k(s-1)/s}. \tag{3.1}$$

Define the *major arcs* $\mathfrak{M}$ and the *minor arcs* $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \mathfrak{M}(P,Q) = \bigcup_{q\le P} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} \mathfrak{M}(q,a), \qquad \mathfrak{m} = \Big[\frac{1}{Q}, 1 + \frac{1}{Q}\Big] \setminus \mathfrak{M}, \tag{3.2}$$

where $\mathfrak{M}(q,a) = [a/q - 1/(qQ), a/q + 1/(qQ)]$. Then

$$T(k,s;X) = \Big\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \Big\} f_k^s(\alpha) g(-\alpha)\, d\alpha.$$

For the integral over the major arcs, we have the following:

Proposition 3.1. *Let the major arcs* $\mathfrak{M}(P,Q)$ *be as defined in* (3.2) *with* P, Q *defined in* (3.1). *Then for* $k \ge 3$ *and* $s > k$,

$$\int_{\mathfrak{M}} f_k^s(\alpha) g(-\alpha)\, d\alpha = kC_{1,k,s} I_{1,k,s} X^s \log X + (C_{1,k,s} I_{2,k,s} + C_{2,k,s} I_{1,k,s}) X^s + O(X^{s-\delta+\varepsilon}),$$

where $\delta = 1 - k/s$ *and* $C_{i,k,s}$, $I_{j,k,s}$ $(i,j = 1,2)$ *are defined in* (1.3) *and* (1.4), *respectively. Moreover,* $C_{i,k,s}$ *are absolutely convergent and satisfy* $C_{i,k,s} \gg 1$.

For the minor arcs, we have

Proposition 3.2. *Let the minor arcs* $\mathfrak{m}$ *be as defined in* (3.2). *Then for* $k \ge 3$ *and* $s > \min\{2^{k-1}, k^2 + k - 2\}$, *and every* $\varepsilon > 0$,

$$\int_{\mathfrak{m}} f_k^s(\alpha) g(-\alpha)\, d\alpha = O(X^{s-\theta+\varepsilon}),$$

where

$$\theta = \begin{cases} k\left(\dfrac{1}{2^{k-1}} - \dfrac{1}{s}\right) & \text{when } 3 \le k \le 7 \text{ and } s > 2^{k-1}, \\[2ex] \dfrac{s - k^2 - k + 2}{2sk - 2s} & \text{when } k \ge 8 \text{ and } s > k^2 + k - 2. \end{cases}$$

Proof of Theorem 1.2. Theorem 1.2 is an immediate consequence of Propositions 3.1 and 3.2. ∎

Our task now is to prove Propositions 3.1 and 3.2. To prove Proposition 3.1, we need some lemmas. The following lemma can be found in [11].

Lemma 3.3. *Let* $S_k(a,b,q)$ *and* $S_k(q,a)$ *be as defined in* (2.4). *If* $(a,b,q) = 1$, *then*

$$S_k(a,b,q) = O(q^{1-1/k+\varepsilon}).$$

Moreover, if $(a,q) = 1$, *then*

$$S_k(q,a) = O(q^{1-1/k}).$$

Lemmas 3.4 and 3.5 give estimates of the exponential sums over the major arcs.

Lemma 3.4. *Let the major arcs* $\mathfrak{M}(P,Q)$ *be as defined in* (3.2) *with* P, Q *defined in* (3.1). *Then for* $\alpha = a/q + \lambda \in \mathfrak{M}$,

$$f_k\left(\frac{a}{q} + \lambda\right) = \frac{S_k(q,a)}{q} X \int_0^1 e(v^k X^k \lambda)\, dv + O(q^{1-1/k+\varepsilon}).$$

Lemma 3.5. *Let the major arcs* $\mathfrak{M}(P,Q)$ *be as defined in* (3.2) *with* P, Q *defined in* (3.1). *Then for* $\alpha = a/q + \lambda \in \mathfrak{M}$,

$$g\left(-\frac{a}{q} - \lambda\right) = J + O(q^{1/2} X^{k+\varepsilon} Q^{-1} + q^{2/3} X^{k/3}),$$

where

$$J = \frac{kX^k \log X}{q} \int_0^s e(-X^k v\lambda)\, dv + \frac{X^k}{q} \int_0^s e(-X^k v\lambda) \log v\, dv + \frac{-2\log q + 2\gamma}{q} X^k \int_0^s e(-X^k v z)\, dv.$$

The proofs of Lemmas 3.4 and 3.5 are similar to those of [4, Lemmas 4.1 and 5.1]. In fact, the proofs in [4] require the conditions $PQ \le X^k$ and $Q > X^{k-1+\varepsilon}$. One can verify that P, Q defined in (3.1) meet this requirement. To prove Lemma 3.4, we follow the argument of [4, Section 4] step by step, replacing [4, Lemma 3.4] by Lemma 3.3. For Lemma 3.5, we employ the estimate in [5] of the sum of the divisors over an arithmetic progression (see also [4, Sections 6 and 7.1]), and the proof is almost the same as in [4] except for replacing $S_2(-\alpha; 3X^2)$ in [4, Section 7.2] by $g(-\alpha)$ here. We omit the details. ∎

LEMMA 3.6 (see [10, Lemma 8.10]). *Suppose that for some $k \ge 1$ and $\Delta > 0$, we have $|f^{(k)}(x)| \ge \Delta$ for any $x \in [a, b]$. Then*

$$\int_a^b e(f(x))\, dx = O(\Delta^{-1/k}).$$

Proof of Proposition 3.1. Let $\alpha \in \mathfrak{M}(P, Q)$ with P, Q defined in (3.1). By (3.2), we have

$$\int_{\mathfrak{M}} f_k^s(\alpha) g(-\alpha)\, d\alpha = \sum_{1 \le q \le P} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{-1/(qQ)}^{1/(qQ)} f_k^s\Big(\frac{a}{q} + \lambda\Big) g\Big(-\frac{a}{q} - \lambda\Big)\, d\lambda. \tag{3.3}$$

Using Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned} f_k^s(\alpha) g(-\alpha) = {} & \frac{kS_k^s(q,a) X^{s+k} \log X}{q^{s+1}} \Big(\int_0^1 e(v^k X^k \lambda)\, dv\Big)^s \Big(\int_0^s e(-X^k v\lambda)\, dv\Big) \\ & + \frac{S_k^s(q,a) X^{s+k}}{q^{s+1}} \Big(\int_0^1 e(v^k X^k \lambda)\, dv\Big)^s \Big(\int_0^s e(-X^k v\lambda) \log v\, dv\Big) \\ & + \frac{S_k^s(q,a) X^{s+k} (-2\log q + 2\gamma)}{q^{s+1}} \Big(\int_0^1 e(v^k X^k \lambda)\, dv\Big)^s \\ & \times \Big(\int_0^s e(-X^k v\lambda)\, dv\Big) \\ & + O(X^{s+k-1+\varepsilon} q^{-s/k} + X^{k/3+s+\varepsilon} q^{2/3-s/k} + X^{s+k+\varepsilon} q^{1/2-s/k} Q^{-1}). \end{aligned}$$

Define

$$H_j(z) = \Big(\int_0^1 e(v^k z)\,dv\Big)^s \Big(\int_0^s e(-vz)(\log v)^{j-1}\,dv\Big), \qquad j = 1, 2.$$

Changing the variable $z = X^k \lambda$, we get

$$\begin{aligned}
(3.4)\qquad & \int_{-1/(qQ)}^{1/(qQ)} f_k^s(\alpha) g(-\alpha)\,d\alpha \\
&= \frac{k S_k^s(q,a) X^s \log X}{q^{s+1}} \int_{-X^k/(qQ)}^{X^k/(qQ)} H_1(z)\,dz \\
&\quad + \frac{S_k^s(q,a) X^s}{q^{s+1}} \int_{-X^k/(qQ)}^{X^k/(qQ)} H_2(z)\,dz \\
&\quad + \frac{S_k^s(q,a) X^s(-2\log q + 2\gamma)}{q^{s+1}} \int_{-X^k/(qQ)}^{X^k/(qQ)} H_1(z)\,dz \\
&\quad + O(X^{s+k-1+\varepsilon} q^{-s/k-1} Q^{-1} + X^{k/3+s+\varepsilon} q^{-1/3-s/k} Q^{-1} \\
&\qquad\qquad + X^{s+k+\varepsilon} q^{-1/2-s/k} Q^{-2}).
\end{aligned}$$

We first deal with the integrals $\int_{-X^k/(qQ)}^{X^k/(qQ)} H_i(z)\,dz$ ($i = 1, 2$) in (3.4). By Lemma 3.6, we have

$$H_1(z) = O\bigg(\frac{1}{|z|^{s/k+1}}\bigg) \quad \text{and} \quad H_2(z) = O\bigg(\frac{\log|z|}{|z|^{s/k+1}}\bigg).$$

Then for any $U > 2$,

$$\int_{|z|>U} H_1(z)\,dz = O(U^{-s/k}) \quad \text{and} \quad \int_{|z|>U} H_1(z)\,dz = O(U^{-s/k}\log U).$$

Thus the infinite integrals $\int_{-\infty}^{\infty} H_j(z)\,dz$ ($j = 1, 2$) converge for $s > k$. Taking $U = X^k/(qQ)$, one can easily check that $U > 2$ for $q \le P$ and P, Q defined in (3.1). Then we obtain

$$\int_{-X^k/(qQ)}^{X^k/(qQ)} H_1(z)\,dz = \int_{-\infty}^{\infty} H_1(z)\,dz + O\bigg(\Big(\frac{qQ}{X^k}\Big)^{s/k}\bigg),$$

$$\int_{-X^k/(qQ)}^{X^k/(qQ)} H_2(z)\,dz = \int_{-\infty}^{\infty} H_2(z)\,dz + O\bigg(\Big(\frac{qQ}{X^k}\Big)^{s/k} \log X\bigg).$$

Inserting the above two estimates into (3.4), by Lemma 3.3, we get

$$(3.5)\qquad \int_{-1/(qQ)}^{1/(qQ)} f_k^s(\alpha)g(-\alpha)\,d\alpha = \frac{kS_k^s(q,a)X^s\log X}{q^{s+1}}\int_{-\infty}^{\infty} H_1(z)\,dz + \frac{S_k^s(q,a)X^s}{q^{s+1}}\int_{-\infty}^{\infty} H_2(z)\,dz + \frac{S_k^s(q,a)X^s(-2\log q+2\gamma)}{q^{s+1}}\int_{-\infty}^{\infty} H_1(z)\,dz + O(Q^{s/k}q^{-1}X^{\varepsilon} + X^{s+k-1+\varepsilon}q^{-s/k-1}Q^{-1} + X^{k/3+s+\varepsilon}q^{-1/3-s/k}Q^{-1} + X^{s+k+\varepsilon}q^{-1/2-s/k}Q^{-2}) =: T_1+T_2+T_3+O\text{-term}.$$

For $s>k$, the contribution of the O-term to (3.3) is

$$(3.6)\qquad \ll \sum_{1\le q\le P}\sum_{\substack{1\le a\le q\\(a,q)=1}} (Q^{s/k}q^{-1}X^{\varepsilon} + X^{s+k-1+\varepsilon}q^{-s/k-1}Q^{-1} + X^{k/3+s+\varepsilon}q^{-1/3-s/k}Q^{-1} + X^{s+k+\varepsilon}q^{-1/2-s/k}Q^{-2})$$
$$\ll PQ^{s/k}X^{\varepsilon} + Q^{-1}X^{s+k-1+\varepsilon} + X^{k/3+s+\varepsilon}Q^{-1}\max\{1,P^{5/3-s/k}\} + X^{s+k+\varepsilon}Q^{-2}\min\{1,P^{3/2-s/k}\}$$
$$\ll X^{s-(1-k/s)+\varepsilon}.$$

Then back to (3.3), for $s>k$, we obtain

$$\int_{\mathfrak{M}} f_k^s(\alpha)g(-\alpha)\,d\alpha = \sum_{1\le q\le P}\sum_{\substack{1\le a\le q\\(a,q)=1}}(T_1+T_2+T_3) + O(X^{s-(1-k/s)+\varepsilon})$$
$$= \sum_{q=1}^{\infty}\sum_{\substack{1\le a\le q\\(a,q)=1}}(T_1+T_2+T_3) + O(X^{s-(1-k/s)+\varepsilon})$$
$$= kC_{1,k,s}I_{1,k,s}X^s\log X + (C_{1,k,s}I_{2,k,s}+C_{2,k,s}I_{1,k,s})X^s + O(X^{s-(1-k/s)+\varepsilon}),$$

where $C_{i,k,s}$, $I_{j,k,s}$ $(i,j=1,2)$ are defined in (1.3) and (1.4). This proves Proposition 3.1. ■

Now we turn to the proof of Proposition 3.2. We distinguish two cases: $3\le k\le 6$ and $k\ge 7$. For $3\le k\le 6$, we need the following two lemmas, which are Lemmas 2.4 and 2.5 in [11].

LEMMA 3.7. *Suppose that* $(a,q)=1$ *and* $|\alpha - a/q| \le q^{-2}$. *Then*

$$f_k(\alpha) \ll X^{1+\varepsilon}(q^{-1}+X^{-1}+qX^{-k})^{1/2^{k-1}}.$$

LEMMA 3.8. *Fix* $k \ge 1$. *Then*

$$\int_0^1 |f_k(\alpha)|^{2^k}\,d\alpha \ll X^{2^k-k+\varepsilon}.$$

Proof of Proposition 3.2 for $3 \le k \le 6$. By Cauchy's inequality,

$$\begin{aligned}\int_{\mathfrak{m}} f_k^s(\alpha)g(-\alpha)\,d\alpha &\ll \max_{\alpha\in\mathfrak{m}} |f_k(\alpha)|^{s-2^{k-1}}\Big(\int_{\mathfrak{m}} |f_k(\alpha)|^{2^k}\,d\alpha\Big)^{1/2}\Big(\int_{\mathfrak{m}} |g(-\alpha)|^2\,d\alpha\Big)^{1/2}\\ &\ll \max_{\alpha\in\mathfrak{m}} |f_k(\alpha)|^{s-2^{k-1}}\Big(\int_{\mathfrak{m}} |f_k(\alpha)|^{2^k}\,d\alpha\Big)^{1/2}\Big(\sum_{1\le m\le sX^k} d(n)\Big)^{1/2}.\end{aligned}$$

Note that

$$\sum_{1\le m\le t} d(n) \ll t\log t. \tag{3.7}$$

By Lemma 3.7, for $\alpha \in \mathfrak{m}$ we obtain

$$f_k(\alpha) \ll X^{1+\varepsilon}(P^{-1}+X^{-1}+QX^{-k})^{1/2^{k-1}}. \tag{3.8}$$

Applying Lemma 3.8, one can deduce that

$$\int_{\mathfrak{m}} |f_k(\alpha)|^{2^k}\,d\alpha \ll \int_0^1 |f_k(\alpha)|^{2^k}\,d\alpha \ll X^{2^k-k+\varepsilon}.$$

This together with (3.7) and (3.8) gives, for $s > 2^{k-1}$,

$$\begin{aligned}\int_{\mathfrak{m}} f_k^s(\alpha)g(-\alpha)\,d\alpha &\ll X^{s+\varepsilon}(P^{-1}+X^{-1}+QX^{-k})^{s/2^{k-1}-1}\\ &\ll X^{s-k(1/2^{k-1}-1/s)+\varepsilon}.\end{aligned}$$

This finishes the proof of Proposition 3.2 for $3 \le k \le 6$. ∎

For $k \ge 7$, we use the following lemmas which are Theorem 1.5 and Corollary 10.2 in [12].

LEMMA 3.9. *Let* $k \ge 2$ *be an integer. Suppose that* $(a,q)=1$, $|\alpha - a/q| \le q^{-2}$ *and* $q \le X^k$. *Then*

$$f_k(\alpha) \ll X^{1+\varepsilon}(q^{-1}+X^{-1}+qX^{-k})^{\frac{1}{2k(k-1)}}.$$

LEMMA 3.10. *For* $s \ge k^2+k-2$,

$$\int_0^1 |f_k(\alpha)|^{2s}\,d\alpha \ll X^{2s-k+\varepsilon}.$$

Proof of Proposition 3.2 for $k \geq 7$. By Cauchy's inequality,

$$\int_{\mathfrak{m}} f_k^s(\alpha) g(-\alpha)\, d\alpha \ll \max_{\alpha\in\mathfrak{m}} |f_k(\alpha)|^{s-(k^2+k-2)} \Big(\int_{\mathfrak{m}} |f_k(\alpha)|^{2k^2+2k-4}\, d\alpha\Big)^{1/2} \times \Big(\int_{\mathfrak{m}} |g(-\alpha)|^2\, d\alpha\Big)^{1/2}$$
$$\ll \max_{\alpha\in\mathfrak{m}} |f_k(\alpha)|^{s-(k^2+k-2)} \Big(\int_{\mathfrak{m}} |f_k(\alpha)|^{2k^2+2k-4}\, d\alpha\Big)^{1/2} \times \Big(\sum_{1\leq m\leq sX^k} d(n)\Big)^{1/2}.$$

Then it follows from Lemmas 3.9 and 3.10 that

$$f_k(\alpha) \ll X^{1+\varepsilon}(P^{-1} + X^{-1} + QX^{-k})^{\frac{1}{2k(k-1)}} \tag{3.9}$$

for $\alpha \in \mathfrak{m}$, and

$$\int_{\mathfrak{m}} |f_k(\alpha)|^{2k^2+2k-4}\, d\alpha \ll \int_0^1 |f_k(\alpha)|^{2k^2+2k-4}\, d\alpha \ll X^{2k^2+k-4+\varepsilon}. \tag{3.10}$$

By (3.7), (3.9) and (3.10) we obtain, for $s > k^2 + k - 2$,

$$\int_{\mathfrak{m}} f_k^s(\alpha) g(-\alpha) d\alpha \ll X^{s+\varepsilon}(P^{-1} + X^{-1} + QX^{-k})^{\frac{s-(k^2+k-2)}{2k(k-1)}}$$
$$\ll X^{s-\frac{s-(k^2+k-2)}{2s(k-1)}+\varepsilon}.$$

This completes the proof of Proposition 3.2. ■

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