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On the structure of long unsplittable minimal zero-sum sequences

by

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1. Introduction. Throughout the paper, let G be an additively written abelian group and $\mathcal{F}(G)$ be the free abelian (multiplicative) monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write a sequence $S \in \mathcal{F}(G)$ in the form

$$S = g_1 \cdot \ldots \cdot g_s = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $s \in \mathbb{N}_0$ (the set of non-negative integers), $g_1, \ldots, g_s \in G$ and $\mathsf{v}_g(S) \in \mathbb{N}_0$. We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S, and $|S| = s = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S. The unit element in $\mathcal{F}(G)$ is the *empty sequence*. Denote by $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$ the *support* of S, and by $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) : g \in G\}$ the *height* of S.

A sequence S_1 is called a *subsequence* of S if $S_1 \mid S$ in $\mathcal{F}(G)$ (i.e. $\mathsf{v}_g(S_1) \le \mathsf{v}_g(S)$ for all $g \in G$), and a *proper subsequence* of S if S_1 is a non-empty subsequence of S with $S_1 \neq S$. If S_1 is a subsequence of S, we use $S(S_1)^{-1}$ to denote the subsequence obtained by deleting the terms of S_1 from S (equivalently, $S = (S(S_1)^{-1}) \cdot S_1$).

For a sequence S defined above, we denote by

$$\sigma(S) = \sum_{i=1}^{s} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$$

the *sum* of S, and by

$$\sum(S) = \Big\{ \sum_{i \in I} g_i : \emptyset \neq I \subset [1, s] \Big\}$$

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the set of *subsums* of S, where for real numbers a and b, $[a,b] = \{x \in \mathbb{Z} : a \le x \le b\}$. Sometimes we also use $\sum_0(S) = \sum(S) \cup \{0\}$ for convenience.

A sequence S is called

- *zero-sum* if $\sigma(S) = 0$,
- *minimal zero-sum* if $\sigma(S) = 0$ and $\sigma(T) \neq 0$ for every proper subsequence $T \mid S$,
- *zero-sum free* if $0 \notin \sum(S)$.

In zero-sum theory, one of the main objects of study is the (minimal) zero-sum sequences. On the one hand, researchers investigate several important invariants such as the Davenport constant $D(G)$ (the maximal possible length of a minimal zero-sum sequence in G) and the EGZ constant $E(G)$ (the minimal positive integer t such that every sequence S of length t must have a zero-sum subsequence of length $|G|$). On the other hand, people are interested in determining the structure of minimal zero-sum sequences, which is the main goal of this paper.

When G is not cyclic, the best known result regarding the structure of minimal zero-sum sequences is due to C. Reiher [8], who determined the structure of such sequences of maximal length $2p-1$ in $\mathbb{Z}_p \oplus \mathbb{Z}_p$, where p is a prime.

When G is cyclic, there are several important results on this topic. To state them, we need the following notion introduced by Chapman, Freeze and Smith [1].

Definition 1.1. Let G be a cyclic group of order n, and S a sequence over G of the form

$$S = (n_1 g) \cdot \ldots \cdot (n_s g), \qquad \text{where } s \in \mathbb{N}_0 \text{ and } n_1, \ldots, n_s \in [1,n].$$

We denote the *g-norm* of S by

$$\|S\|_g = \frac{n_1 + \cdots + n_s}{n},$$

and the *index* of S by

$$\mathrm{Ind}(S) = \min\{\|S\|_g : g \text{ is any generator of } G\}.$$

Clearly, S is zero-sum if and only if $\mathrm{Ind}(S) \in \mathbb{N}_0$. If $\mathrm{Ind}(S) = 1$, then $S = (n_1 g) \cdot \ldots \cdot (n_s g)$ for some generator g and $n_1, \ldots, n_s \in [1,n]$ with $\sum_{i=1}^{s} n_i = n$, so clearly S is a minimal zero-sum sequence and its structure is clear (as $n_1, \ldots, n_s$ form a partition of n). Several authors have attempted to show that the converse is true under some conditions (i.e. with some restriction, a minimal zero-sum sequence has index 1). Gao [2] showed that any minimal zero-sum sequence of length roughly greater than $2n/3$ has index 1. Later, Yuan [12] and Savchev and Chen [9] independently extended the above result to the case of minimal zero-sum sequences of length greater than $n/2+1$. It was conjectured that if $\gcd(n,6)=1$, then every minimal zero-sum sequence of length 4 has index 1; some relevant results can be found in [5, 6, 10] for example.

In contrast to the above results, it has been known that for each k in $[5, n/2+1]$, there is a minimal zero-sum sequence S of length k with $\mathrm{Ind}(S) \geq 2$, and that the same is true for $k=4$ and $\gcd(n,6) \neq 1$. When $\mathrm{Ind}(S) \geq 2$, the structure of S is not yet understood (if we only know the index of S). In order to characterize explicitly the structure of such a sequence, we need to overcome many difficulties. Next we introduce a concept which will be helpful in that investigation.

Definition 1.2. Let S be a minimal zero-sum sequence over G. An element g in S is called *splittable* if there exist $x, y \in G$ such that $x+y=g$ and $Sg^{-1}xy$ is a minimal zero-sum sequence as well; otherwise, g is called *unsplittable.* The sequence S is called *splittable* if at least one element of S is splittable; otherwise, it is *unsplittable.*

By applying the splitting operation repeatedly to any minimal zero-sum sequence S, we eventually obtain an unsplittable minimal zero-sum sequence S' and the index will not decrease (i.e. $\mathrm{Ind}(S) \leq \mathrm{Ind}(S')$). In this way, we can try to understand the structure of minimal zero-sum sequences through characterizing that of unsplittable minimal zero-sum sequences. Recently several new results in this direction have been obtained. Xia and Yuan [11] obtained the structure of unsplittable minimal zero-sum sequences of length $\lfloor n/2 \rfloor + 1$, where for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the maximal integer less than or equal to x. Peng and Sun [7] took one step forward to describe unsplittable minimal zero-sum sequences of length $(n-1)/2$ when $n > 155$ is a prime. Most recently, Yuan and Li [13] determined the structure of unsplittable minimal zero-sum sequences of length $\geq \lfloor n/3 \rfloor + 8$ when $n > 20585$ is an odd positive integer with least prime divisor greater than 13.

In the present paper, we continue these investigations; we improve the last mentioned result significantly by removing the constraint on $|G| = n$, and by sharpening the lower bound for $|S|$ from $\lfloor n/3 \rfloor + 8$ to $\lfloor n/3 \rfloor + 3$. Our main result is as follows, and it is best possible.

Theorem 1.3. *Let $n \geq 9$ be an odd integer and G an abelian group of order n. If there exists an unsplittable minimal zero-sum sequence S over G of length $|S| \geq \lfloor n/3 \rfloor + 3$, then G is cyclic, and either $S = g^n$, or*

$$S = g^{(n-r)/2-1-tr} \cdot \left(\frac{n+r}{2} g\right)^{2(t+1)} \cdot \left(\left(\frac{n-r}{2}+1\right) g\right),$$

where g is a generator of G and $r, t \in \mathbb{N}_0$ with r odd and $3 \leq r \leq (n-r)/2 - 1 - tr$. Moreover, $\mathrm{Ind}(S) = 2$ *in the latter case.*

If $\mathrm{supp}(S)$ contains only one element g, then since $|G| = n$ is odd and $|S| \geq \lfloor n/3 \rfloor + 3$, we conclude that G is generated by g and $S = g^n$. Hence, in what follows, we may always assume that S is a minimal zero-sum sequence consisting of at least two distinct elements, and show that S must have the second form described in the theorem.

The paper is organized as follows. In Section 2, some basic lemmas are given. We divide the proof of the main theorem into two steps. The first step is handled in Section 3, where it is shown that if S is not of the form described in the main theorem, then it must contain a suitable subsequence with large subsum set. Then in Section 4, we make the second step by proving that the existence of the above mentioned subsequence leads to a contradiction. In the final section, we make some comments regarding our main result and present an example to show that it is best possible.

2. Some basic lemmas. In this section, some basic lemmas are given; they will be frequently used in this paper.

Lemma 2.1 ([3]). *Let G be an abelian group of order $n \geq 3$. If S is a zero-sum free sequence over G of length*

$$|S| \geq \frac{6n + 28}{19},$$

then S contains an element $g \in G$ with multiplicity

$$\mathsf{v}_g(S) \geq \frac{6|S| - n + 1}{17}.$$

Lemma 2.2 ([4, Theorem 5.3.1]). *Let S be a zero-sum free sequence over an abelian group, and let $S_1, \ldots, S_k$ be disjoint subsequences of S. Then $|\sum(S)| \geq \sum_{i=1}^{k} |\sum(S_i)|$.*

Lemma 2.3 ([12]). *Let S be a minimal zero-sum sequence over an abelian group G. If $a, ta \in \mathrm{supp}(S)$ with $t \in [2, \mathrm{ord}(a) - 1]$ and ta is unsplittable in S, then $t \geq \mathsf{v}_a(S) + 2$. In particular, if S is unsplittable, then for any $a \in G \setminus \{0\}$, a and $2a$ cannot occur in S simultaneously.*

The above lemma is slightly different from the one in [12], but the same proof works.

Lemma 2.4 ([11, Lemma 2.14]). *Let S be a minimal zero-sum sequence over an abelian group G. Then $a \in \mathrm{supp}(S)$ is unsplittable in S if and only if $\sum(Sa^{-1}) = G \setminus \{0\}$. Moreover, S is unsplittable if and only if $\sum(Sa^{-1}) = G \setminus \{0\}$ for every $a \in \mathrm{supp}(S)$. In particular, if S is unsplittable, then $G = \langle \mathrm{supp}(S) \rangle$.*

Lemma 2.5 ([11, Lemma 2.15]). *Let S be a minimal zero-sum sequence over an abelian group, and suppose S consists of two distinct elements. Then S is splittable.*

We remark that in [11], the above two lemmas were stated only for cyclic groups, but the same proofs remain valid for all abelian groups. In particular, in Lemma 2.4, if S is unsplittable, then $G \subset \sum(S) \subset \langle \mathrm{supp}(S)\rangle$, so $G = \langle \mathrm{supp}(S)\rangle$. The following lemma is an easy observation.

Lemma 2.6. *Let S be a minimal zero-sum or zero-sum free sequence over an abelian group, and T a proper subsequence of S. Let a be a term of ST^{-1}. Then $\sigma(T) + a \notin \sum(T)$.*

Lemma 2.7 (Replacement Lemma). *Let S be a minimal zero-sum sequence over an abelian group G, and T a proper subsequence of S. Let T' be another sequence over G such that $\sum(T') = \sum(T)$ and $\sigma(T') = \sigma(T)$. Then:*

- *$S' = (ST^{-1}) \cdot T'$ is a minimal zero-sum sequence.*
- *If $a \in \mathrm{supp}(ST^{-1})$ is unsplittable in S, then it is so in S'. In particular, if S is unsplittable and $\mathrm{supp}(T') \subset \mathrm{supp}(ST^{-1})$, then S' is unsplittable.*

Proof. Since $\sigma(S') = \sigma(ST^{-1}) + \sigma(T') = \sigma(ST^{-1}) + \sigma(T) = \sigma(S)$, S' is a zero-sum sequence. Let $a \in \mathrm{supp}(ST^{-1})$. Then by the assumption,

$$\begin{aligned}\textstyle\sum(S'a^{-1}) &= \textstyle\sum(ST^{-1}a^{-1}) \cup \sum(T') \cup (\sum(ST^{-1}a^{-1}) + \sum(T'))\\ &= \textstyle\sum(ST^{-1}a^{-1}) \cup \sum(T) \cup (\sum(ST^{-1}a^{-1}) + \sum(T))\\ &= \textstyle\sum(Sa^{-1}) \subset G \setminus \{0\}.\end{aligned}$$

Thus $0 \notin \sum(S'a^{-1})$ and $S'a^{-1}$ is zero-sum free, which implies that S' is minimal zero-sum.

Moreover, if $a \in \mathrm{supp}(ST^{-1})$ is unsplittable in S, then $\sum(S'a^{-1}) = \sum(Sa^{-1}) = G \setminus \{0\}$ by Lemma 2.4. Hence a is unsplittable in S'. ∎

Remark. From the proof, we can see that replacing T by T' in any sequence S containing T does not change the subsum set of S and the sum of S either, which will be very useful in what follows.

Lemma 2.8. *Let $S = a^l bc$ be a minimal zero-sum sequence over an abelian group G of odd order, and $b, c \neq a$. Then b and c are splittable.*

Proof. Assume that (say) b is unsplittable. By Lemma 2.4, $\sum(a^l c) = G\setminus\{0\}$, so $G \subseteq \langle a\rangle \cup (c + \langle a\rangle)$. Since $|G|$ is odd, if $G \neq \langle a\rangle$, then G contains at least three cosets of $\langle a\rangle$, giving a contradiction. Thus, G is cyclic generated by a. Let $b = ka$ and $c = ta$ with $k, t \in [2, n-1]$. Since S is a minimal zero-sum sequence, $t + l \le n - 1$. Since b is unsplittable, $(n-1)a \in \sum(a^l c)$, which implies that $t + l = n - 1$, so $k + l + t < 2n - 1$, forcing $k + l + t = n$. Thus $b = (n - l - t)a = a$, a contradiction. ∎

LEMMA 2.9. *Let $S = a^\alpha b^\beta$ with $\alpha, \beta \in \mathbb{N}$ be a zero-sum free sequence over an abelian group G, and let t be the minimal positive integer such that $tb \in \{-\alpha a, (-\alpha+1)a, \ldots, \alpha a\}$.*

(1) *If $\beta < t$, then $\sum_0(S) = \bigcup_{i=0}^{\beta}(ib + \{0, a, \ldots, \alpha a\})$, which is a disjoint union. In this case, $|\sum(S)| = (\alpha+1)(\beta+1) - 1$.*
(2) *If $\beta \geq t-1$, let $tb = ka$ with $k \in [-\alpha, \alpha]$ and $\beta - (t-1) = tq + r$ with $r \in [0, t-1]$. Then $k > 0$ if $\beta \geq t$, and*

$$\begin{aligned}\textstyle\sum_0(S) = &\bigcup_{i=0}^{r-1}(ib + \{0, a, \ldots, (\alpha+kq+k)a\})\\ &\cup \bigcup_{i=r}^{t-1}(ib + \{0, a, \ldots, (\alpha+kq)a\}),\end{aligned}$$

which is a disjoint union. In this case, $|\sum(S)| = t(\alpha+1) + (\beta - t + 1)k - 1$.

Proof. First, since $|G|b = 0 \in \{-\alpha a, (-\alpha+1)a, \ldots, \alpha a\}$, such a t exists.

(1) Clearly, $\sum_0(S) = \bigcup_{i=0}^{\beta}(ib + \{0, a, \ldots, \alpha a\})$. To show the union is disjoint, assume $(ib + \{0, a, \ldots, \alpha a\}) \cap (jb + \{0, a, \ldots, \alpha a\}) \neq \emptyset$ for some distinct $i, j \in [0, \beta]$. Then $ib + p_0 a = jb + p_1 a$ for some $p_0, p_1 \in [0, \alpha]$. We may assume $i > j$. Then $(i-j)b = (p_1 - p_0)a \in \{-\alpha a, (-\alpha+1)a, \ldots, \alpha a\}$, contradicting the definition of t.

(2) If $t = 1$, then $b = ka$, $q = \beta$ and $r = 0$. Since S is zero-sum free, $0 < k \leq \alpha$. It is easy to show that $\sum_0(S) = \{0, a, \ldots, (\alpha + k\beta)a\}$, as claimed.

Next, suppose that $t \geq 2$. It is easy to see that $0 < k \leq \alpha$ if $\beta \geq t$, for otherwise $a^{-k}b^t$ is a zero-sum subsequence of S.

We now use induction on β. The base step when $\beta = t-1$ has been proved in (1). Next, suppose that the result holds for $\beta = m$, and let $\beta = m+1$. Write $m - (t-1) = tq + r$. Clearly, $(m+1) - (t-1) = tq + (r+1)$ when $r < t-1$, and $(m+1) - (t-1) = t(q+1) + 0$ otherwise.

As $S = a^\alpha b^{m+1}$ is zero-sum free, so is $a^\alpha b^m$. Thus by the induction hypothesis,

$$\begin{aligned}\textstyle\sum_0(a^\alpha b^m) = &\bigcup_{i=0}^{r-1}(ib + \{0, a, \ldots, (\alpha+kq+k)a\})\\ &\cup \bigcup_{i=r}^{t-1}(ib + \{0, a, \ldots, (\alpha+kq)a\}),\end{aligned}$$

where the union is disjoint. Note that

$$\textstyle\sum_0(a^\alpha b^{m+1}) = \sum_0(a^\alpha b^m) \cup (b + \sum_0(a^\alpha b^m)),$$

and the latter summand is

$$\begin{aligned}
&b+\textstyle\sum_0(a^\alpha b^m)\\
&\quad=\bigcup_{i=1}^{r}(ib+\{0,a,\ldots,(\alpha+kq+k)a\})\cup\bigcup_{i=r+1}^{t}(ib+\{0,a,\ldots,(\alpha+kq)a\})\\
&\quad=\bigcup_{i=1}^{r}(ib+\{0,a,\ldots,(\alpha+kq+k)a\})\cup\bigcup_{i=r+1}^{t-1}(ib+\{0,a,\ldots,(\alpha+kq)a\})\\
&\qquad\cup\{ka,\ldots,(\alpha+kq+k)a\}.
\end{aligned}$$

If $r=0$, then

$$\begin{aligned}
\textstyle\sum_0(a^\alpha b^{m+1})&=\bigcup_{i=0}^{t-1}(ib+\{0,a,\ldots,(\alpha+kq)a\})\cup\{ka,\ldots,(\alpha+kq+k)a\}\\
&=\bigcup_{i=1}^{t-1}(ib+\{0,a,\ldots,(\alpha+kq)a\})\cup\{0,\ldots,(\alpha+kq+k)a\}.
\end{aligned}$$

It remains to show that $\{(\alpha+kq+1)a,\ldots,(\alpha+kq+k)a\}$ is disjoint from $\bigcup_{i=1}^{t-1}(ib+\{0,a,\ldots,(\alpha+kq)a\})$. If not, then there are $i\in[1,t-1]$, $p_0\in[0,\alpha+kq]$ and $p_1\in[0,k-1]$ such that $ib+p_0a=(\alpha+kq+k-p_1)a$, that is, $ib=(\alpha+kq+k-p_1-p_0)a$. By the definition of t, $\alpha+kq+k-p_1-p_0>\alpha$, and thus $0\le p_0+p_1<kq+k\le\alpha+kq$. Hence, $(\alpha+kq+k)a=ib+(p_0+p_1)a\in\sum_0(a^\alpha b^m)$. Since S is zero-sum free, $\sigma(S)=\sigma(a^\alpha b^{m+1})=(\alpha+kq+k)a\notin\sum_0(a^\alpha b^m)$ by Lemma 2.6. So we have a contradiction, and thus the union is disjoint.

If $r>0$, then

$$\begin{aligned}
&\textstyle\sum_0(a^\alpha b^{m+1})\\
&\quad=\bigcup_{i=0}^{r-1}(ib+\{0,a,\ldots,(\alpha+kq+k)a\})\cup\bigcup_{i=r}^{t-1}(ib+\{0,a,\ldots,(\alpha+kq)a\})\\
&\qquad\cup(rb+\{0,\ldots,(\alpha+kq+k)a\})\\
&\quad=\bigcup_{i=0}^{r}(ib+\{0,a,\ldots,(\alpha+kq+k)a\})\cup\bigcup_{i=r+1}^{t-1}(ib+\{0,a,\ldots,(\alpha+kq)a\}).
\end{aligned}$$

The same argument as above shows that the union is disjoint.

In both cases, the value of $|\sum(S)|$ can be obtained by a direct calculation. ∎

3. The first step. We first explain why S must be in the form described in the main theorem.

Proposition 3.1. *Let G be an abelian group of odd order n, and $S=a^lU$ be an unsplittable minimal zero-sum sequence where U is the subsequence of S consisting of all terms different from a. Suppose that U is not empty, and for any $x,y\in G$ (possibly $x=y$) with $xy\,|\,U$, there is some $\gamma\in[1,l]$ such that $x+y=\gamma a$. Then G is cyclic and generated by a, and there exist $r,t\in\mathbb{N}_0$ with r odd and $3\le r\le l$ such that $l=(n-r)/2-1-tr$ and*

$$U=\Big(\frac{n+r}{2}a\Big)^{2(t+1)}\cdot\Big(\Big(\frac{n-r}{2}+1\Big)a\Big).$$

Proof. Since S is an unsplittable minimal zero-sum sequence, by Lemma 2.5, U must contain at least two distinct elements and by Lemma 2.8, $|U|\ge 3$.

We first prove $G=\langle a\rangle$. Take any three terms x,y,z in U (this is possible because $|U|\ge 3$). Then by the assumption on U the sums $x+y$, $y+z$ and $x+z$ are in $\langle a\rangle$, so $2x,2y,2z\in\langle a\rangle$. Since $|G|$ is odd, we obtain $x,y,z\in\langle a\rangle$, which implies $\operatorname{supp}(S)\subset\langle a\rangle$. Now by Lemma 2.4, $G=\langle\operatorname{supp}(S)\rangle=\langle a\rangle$.

We can now assume that $U=(t_1a)\cdots(t_ka)$ with $2\le t_1\le\cdots\le t_k<n$. Since S is minimal zero-sum, $t_k<n-l$. Since S is unsplittable, by Lemma 2.3, $t_1\ge l+2$. By the assumption on U, we have $n<t_i+t_j\le n+l$ for all distinct $i,j\in[1,k]$.

Consider a new sequence $S'=S(t_ka)^{-1}\cdot a((t_k-1)a)$, obtained from S by replacing t_ka with two terms a and $(t_k-1)a$. Since S is unsplittable, S' is not minimal zero-sum, so it can be decomposed into two disjoint zero-sum subsequences. Let V be the one containing $(t_k-1)a$. It is clear that V does not contain a: otherwise, we can replace a and $(t_k-1)a$ by t_ka and obtain a proper zero-sum subsequence of S, which contradicts S being minimal zero-sum.

Now, let $V=(t_{i_1}a)\cdots(t_{i_v}a)\cdot((t_k-1)a)$ where $1\le i_1<\cdots<i_v\le k-1$. We next prove two claims.

Claim 1. $t_2=t_k$.

Since S is unsplittable, U contains at least two distinct elements, so $t_2=t_k$ implies that U contains exactly two distinct elements.

Assume to the contrary that $t_2<t_k$. Note that $n<t_1+t_2\le t_{i_1}+t_k-1$, so $v\ge 2$. Let $\gamma\in[1,l]$ with $t_{i_1}+t_{i_2}=n+\gamma$. Then

$$\begin{aligned}\sigma(V)&=(t_{i_1}+t_{i_2})a+\sum_{j=3}^{v}(t_{i_j}a)+(t_k-1)a\\&=\gamma a+\sum_{j=3}^{v}(t_{i_j}a)+(t_k-1)a=(\gamma-1)a+\sum_{j=3}^{v}(t_{i_j}a)+t_ka.\end{aligned}$$

Thus $a^{\gamma-1} \cdot (t_{i_3}a) \cdots (t_{i_v}a) \cdot (t_k a)$ is a zero-sum sequence; moreover, it is a proper subsequence of S, giving a contradiction.

Claim 2. $t_1 + t_2 = n + 1$.

Assume that $t_1 + t_2 > n + 1$. Note that $n < t_1 + t_2 - 1 \leq t_{i_1} + t_k - 1$, hence $v \geq 2$. Let $\gamma \in [1, l]$ with $t_{i_1} + t_{i_2} = n + \gamma$. Then the same calculation as before shows that $\sigma(V) = (\gamma-1)a + \sum_{j=3}^{v}(t_{i_j}a) + t_k a$. Again, we can find a proper zero-sum subsequence of S, yielding a contradiction.

We can now prove the proposition. Let $r \in [1, l]$ with $2t_2 = n+r$. Clearly, r is odd and $r \geq 3$ as S is unsplittable. It follows from the above two claims that

$$U = \left(\frac{n+r}{2}a\right)^{k-1} \cdot \left(\left(\frac{n-r}{2}+1\right)a\right).$$

Note that

$$\sum\left(a^{l-1}\left(\frac{n+r}{2}a\right)^3\right) = \sum\left(a^{l-1+r}\left(\frac{n+r}{2}a\right)\right),$$
$$\sigma\left(a^{l-1}\left(\frac{n+r}{2}a\right)^3\right) = \sigma\left(a^{l-1+r}\left(\frac{n+r}{2}a\right)\right).$$

By Lemma 2.7, we can perform the replacement operation on S and obtain a longer sequence with r more copies of a and 2 fewer copies of $\frac{n+r}{2}a$. Repeat the same operation until the resulting sequence contains less than three copies of $\frac{n+r}{2}a$ (do nothing if S only contains less than three copies of $\frac{n+r}{2}a$). Let the resulting sequence be

$$a^{\alpha} \cdot \left(\frac{n+r}{2}a\right)^{\beta} \cdot \left(\left(\frac{n-r}{2}+1\right)a\right).$$

If $k-1$ is odd, then $\beta = 1$. Since $\left(\frac{n-r}{2}+1\right)a$ is unchanged through any replacement operation, $\left(\frac{n-r}{2}+1\right)a$ remains unsplittable by Lemma 2.7. But this is impossible by Lemma 2.8. Therefore, $k-1$ must be even, say $k-1 = 2(t+1)$. In this case, $\beta = 2$ and $\alpha = l + tr$. Thus $\alpha + 2\frac{n+r}{2} + \left(\frac{n-r}{2}+1\right) \equiv 0 \pmod{n}$, so $\alpha = \frac{n-r}{2} - 1$, that is, $l = \frac{n-r}{2} - 1 - tr$, as desired. ∎

Next we give two lemmas which describe the structure of some simple sequences with small subsum sets.

Lemma 3.2. *Let G be an abelian group of odd order, S an unsplittable minimal zero-sum sequence over G, and $T = a^l bc$ a proper subsequence of S such that $l \geq 1$ and a, b, c are distinct. Suppose that $|\sum(T)| < 3|T|-4$. Then $b, c \in \langle a\rangle$ and $b+c \in \{a, \ldots, la\}$. Morevoer, if we set $b = ta$ and $c = ka$ with $1 \leq t \leq k < \mathrm{ord}(a)$, then $k \leq t + l$ and $\mathrm{ord}(a) < t + k \leq 2k \leq \mathrm{ord}(a) + l$.*

Proof. For convenience, we consider $\sum_0(T)$ instead of $\sum(T)$, so the given condition can be restated as $|\sum_0(T)| < 3|T| - 3 = 3l + 3$. It is clear that

$$\textstyle\sum_0(T) = \{0, b, c, b+c\} + \{0, a, 2a, \dots, la\} = A_0 \cup A_b \cup A_c \cup A_{b+c},$$

where $A_x = x + \{0, a, \dots, la\}$ for $x \in \{0, b, c, b+c\}$. Since S is an unsplittable minimal zero-sum sequence, $A_0 \cap A_b = \emptyset$, $A_0 \cap A_c = \emptyset$, $A_{b+c} \cap A_b = \emptyset$ and $A_{b+c} \cap A_c = \emptyset$.

If $A_b \cap A_c = \emptyset$ or $A_0 \cap A_{b+c} = \emptyset$, then $|\sum_0(T)| \geq 3l + 3$, a contradiction. Hence, $A_b \cap A_c \neq \emptyset$ and $A_0 \cap A_{b+c} \neq \emptyset$, implying that $b - c \in \{-la, (-l+1)a, \dots, la\}$ and $b + c \in \{a, \dots, la\}$. It follows that $b, c \in \langle a \rangle$. Moreover, if we set $b = ta$ and $c = ka$ with $1 \leq t \leq k < \operatorname{ord}(a)$, then $k \leq t + l$ and $\operatorname{ord}(a) < t + k < \operatorname{ord}(a) + l$. Thus

$$\textstyle\sum_0(T) = \{0, a, \dots, (l + t + k - \operatorname{ord}(a))a\} \cup \{ta, \dots, (k + l)a\},$$

and so $|\sum_0(T)| = l + t + k - \operatorname{ord}(a) + 1 + k + l - t + 1 = 2l + 2k + 2 - \operatorname{ord}(a)$. This together with $|\sum_0(T)| < 3l + 3$ implies that $2k \leq \operatorname{ord}(a) + l$. ∎

LEMMA 3.3. *Let G be an abelian group of odd order, S an unsplittable minimal zero-sum sequence over G, and $T = a^l b^2 c$ a proper subsequence of S such that $l \geq 3$ and a, b, c are distinct. Suppose that $2b = \gamma a$ for some $\gamma \in [3, l]$ and $|\sum(T)| < 3|T| - 4$. Then $b, c \in \langle a \rangle$ and $b + c \in \{a, \dots, la\}$. If we set $b = ta$ and $c = ka$ with $1 \leq t, k < \operatorname{ord}(a)$, then $t - l \leq k \leq t + l$ and $\operatorname{ord}(a) < t + k \leq 2\max\{t, k\} \leq \operatorname{ord}(a) + l$.*

Proof. As before, we consider $\sum_0(T)$ instead of $\sum(T)$, so the given condition can be restated as $|\sum_0(T)| < 3|T| - 3 = 3l + 6$. With the same notation as in the proof of Lemma 3.2, we obtain

$$\textstyle\sum_0(T) = A_0 \cup A_b \cup A_{2b} \cup A_c \cup A_{b+c} \cup A_{2b+c}.$$

Since S is an unsplittable minimal zero-sum sequence, we have $A_x \cap A_y = \emptyset$ if $x - y \in \{\pm b, \pm c\}$. Since $2b = \gamma a$ for some $\gamma \in [3, l]$, we have $A_0 \cup A_{2b} = \{0, a, \dots, (l + \gamma)a\}$.

We first show that $A_{2b+c} \cap A_0 = \emptyset$: if not, then $2b + c = \beta a$ with $\beta \in [1, l]$, hence $c = (\beta - \gamma)a \in \{-la, \dots, la\}$, a contradiction.

Next, we show that $A_{2b+c} \cap A_b \neq \emptyset$: if not, then $\sum_0(T)$ includes three pairwise disjoint parts: $A_0 \cup A_{2b}$, A_b and A_{2b+c}, and thus $|\sum_0(T)| \geq l + \gamma + 1 + 2(l + 1) \geq 3l + 6$, a contradiction.

Thus, we may assume that $2b + c \in b + \{-la, \dots, la\}$, so $b + c \in \{-la, \dots, la\}$. Since S is minimal zero-sum, $b + c \notin \{-la, \dots, -a, 0\}$. Hence $b + c \in \{a, \dots, la\}$.

Since $2b \in \langle a \rangle$ and $|G|$ is odd, we have $b \in \langle a \rangle$. Since $b + c \in \langle a \rangle$ from the above paragraph, we infer that $c \in \langle a \rangle$. We have thus proven the first part of the lemma.

Let $b = ta$ and $c = ka$ with $1 \le t, k < \mathrm{ord}(a)$. Since S is unsplittable minimal zero-sum, $l + 2 \le t, k < \mathrm{ord}(a) - l$. We have shown that $2t = \mathrm{ord}(a) + \gamma$ and $t + k = \mathrm{ord}(a) + \alpha$ with $\alpha, \gamma \in [1, l]$. So, $t - k = 2t - (t + k) = \gamma - \alpha \in [-l, l]$, that is, $t - l \le k \le t + l$.

It remains to show that $\mathrm{ord}(a) < 2\max\{t, k\} \le \mathrm{ord}(a) + l$. By the above, it suffices to consider the case $t < k$ and to prove that $2k < \mathrm{ord}(a) + l$. Now

$$\begin{aligned} A_0 \cup A_{2b} \cup A_{b+c} &= \{0, a, \ldots, (l + \alpha)a\}, \\ A_b \cup A_c \cup A_{2b+c} &= \{ta, (t + 1)a, \ldots, (t + \alpha + l)a\}. \end{aligned}$$

Note that $l + \alpha = l + (k + t - \mathrm{ord}(a)) = (l + k - \mathrm{ord}(a)) + t < t$. Hence $|\sum_0(T)| = 2(\alpha + l + 1) = 2k + 2t + 2l - 2\mathrm{ord}(a) + 2$. This together with $|\sum_0(T)| < 3l + 6$ implies that $2k \le (\mathrm{ord}(a) + l) + (\mathrm{ord}(a) + 3 - 2t) = (\mathrm{ord}(a) + l) - (\gamma - 3) \le \mathrm{ord}(a) + l$. ∎

The next two theorems form the first step of the proof of our main theorem.

THEOREM 3.4. *Let G be an abelian group of odd order n, and S an unsplittable minimal zero-sum sequence over G with $\mathsf{h}(S) \ge 2$ and $|\mathrm{supp}(S)| \ge 2$. Then one of the following holds:*

(i) *G is a cyclic group and*

$$S = g^{(n-r)/2-1-tr} \cdot \left(\frac{n+r}{2} g\right)^{2(t+1)} \cdot \left(\left(\frac{n-r}{2} + 1\right) g\right),$$

where g is a generator of G and $r, t \in \mathbb{N}_0$ with r odd and $3 \le r \le (n - r)/2 - 1 - tr$.

(ii) *There are distinct $a, b \in \mathrm{supp}(S)$ with $\mathsf{v}_a(S) \ge 2$ and a subsequence $T \mid S$ such that $\mathsf{v}_a(T) = \mathsf{v}_a(S) - 1$, $\mathsf{v}_b(T) = \mathsf{v}_b(S) - 1$ and $|\sum(T)| \ge 3|T| - 4$. Moreover, if there are distinct elements in S with multiplicities greater than 1, we may choose b such that $\mathsf{v}_b(S) \ge 2$.*

Proof. Since $\mathsf{h}(S) \ge 2$, we can choose $a \in \mathrm{supp}(S)$ such that $\mathsf{v}_a(S) \ge 2$, and let $l_a = \mathsf{v}_a(S) - 1$. Once we have determined the desired $b \in \mathrm{supp}(S)$, we always set $l_b = \mathsf{v}_b(S) - 1$ and $U = Sa^{-\mathsf{v}_a(S)} b^{-\mathsf{v}_b(S)}$. By Lemma 2.5, $\mathrm{supp}(S)$ contains at least three distinct elements, and thus U is always non-empty.

First, assume $Sa^{-\mathsf{v}_a(S)}$ is square free. By Lemma 2.8, $|Sa^{-\mathsf{v}_a(S)}| \ge 3$. If there are distinct $x, y \in \mathrm{supp}(S) \setminus \{a\}$ such that $|\sum(a^{l_a} xy)| \ge 3(l_a + 2) - 4$, then we may choose any $b \in \mathrm{supp}(S) \setminus \{a, x, y\}$, set $T = a^{l_a} xy$, and (ii) holds. If $|\sum(a^{l_a} xy)| < 3(l_a + 2) - 4$ for any distinct $x, y \in \mathrm{supp}(S) \setminus \{a\}$, then by Lemma 3.2, $x + y \in \{a, \ldots, l_a a\}$ for any such x, y. Hence, Proposition 3.1 yields (i).

Next assume $Sa^{-\mathsf{v}_a(S)}$ is not square free. Choose $b \in \mathrm{supp}(S) \setminus \{a\}$ with $\mathsf{v}_b(S) \ge 2$. Let t_b be the minimal positive integer such that $t_b b \in \{-l_a a, \ldots, l_a a\}$. Since S is unsplittable minimal zero-sum, by Lemma 2.3

we have $b \notin \{-l_a a, \dots, l_a a\}$, so $t_b \geq 2$. Let $t_b b = \gamma a$ for some $\gamma \in [-l_a, l_a]$. We consider several cases.

Case 1: $l_b = 1$. Suppose there is some $x \in \operatorname{supp}(S) \setminus \{a, b\}$ such that $|\sum(a^{l_a} bx)| \geq 3(l_a + 2) - 4$. Then as before, $T = a^{l_a} bx$ is as desired.

Suppose $|\sum(a^{l_a} bx)| < 3l_a + 2$ for any $x \in \operatorname{supp}(S) \setminus \{a, b\}$. Then $x \in \langle a \rangle$ for any $x \in \operatorname{supp}(S) \setminus \{a, b\}$ and $b \in \langle a \rangle$ by Lemma 3.2. Hence $G = \langle a \rangle$. Write $b = ta$ and $S = a^{l_a+1}(ta)^2 \cdot (k_1 a) \cdots (k_s a)$ with $t \in [1, n-1]$ and $1 \leq k_1 \leq \cdots \leq k_s < n$. Applying Lemma 3.2 to $a^{l_a}(ta)(k_s a)$, we obtain $2\max\{t, k_s\} \leq l_a + n$. Hence any two terms in $t, t, k_1, \dots, k_s$ have sum $\leq 2\max\{t, k_s\} \leq l_a + n$.

If $t < k_1$, we can apply Lemma 3.2 to $a^{l_a}(ta)(k_1 a)$ and obtain $t + k_1 > n$ and $k_1 \leq t + l_a$. We assert that $2t > n$: otherwise, $2t \leq n < t + k_1 \leq 2t + l_a$, and thus $a^{l_a} b^2$ contains a zero-sum subsequence, which is impossible. Hence any two terms in $t, t, k_1, \dots, k_s$ must have sum $\geq 2t > n$.

If $k_1 < t < k_2$, again we obtain $t + k_1 > n$. Hence, any two terms in $t, t, k_1, \dots, k_s$ have sum $\geq k_1 + t > n$.

If $k_2 < t$, apply Lemma 3.2 to $a^{l_a}(ta)(k_1 a)$ and $a^{l_a}(ta)(k_2 a)$, which gives $t + k_1 > n$ and $t \leq k_2 + l_a$. We assert that $k_1 + k_2 > n$: otherwise, $k_1 + k_2 \leq n < t + k_1 \leq k_1 + k_2 + l_a$, which is impossible. Hence any two terms in $t, t, k_1, \dots, k_s$ must have sum $\geq k_1 + k_2 > n$.

We have shown that any two terms in $b^2 U$ have sum in $\{a, \dots, l_a a\}$. It follows from Proposition 3.1 that (i) holds.

From now on, we always assume that $l_b \geq 2$. Also, we may assume that $l_a \geq 2$, for otherwise, by switching the roles of a and b, we are back in Case 1.

Case 2: $2 \leq l_b < t_b$. By Lemma 2.9, $|\sum(a^{l_a} b^{l_b})| = (l_a + 1)(l_b + 1) - 1 = l_a l_b + l_a + l_b \geq 3(l_a + l_b) - 4$.

Case 3: $l_b \geq t_b = 2$. Since $l_b \geq t_b$, by Lemma 2.9 we have $2b = \gamma a$ with $\gamma \in [1, l_a]$. Since $|G|$ is odd and S is unsplittable, γ is odd and $\gamma \geq 3$ by Lemma 2.3. Note that

$$\sum(a^{l_a} b^3) = \sum(a^{l_a+\gamma} b) \quad \text{and} \quad \sigma(a^{l_a} b^3) = \sigma(a^{l_a+\gamma} b).$$

We can replace $a^{l_a} b^3$ by $a^{l_a+\gamma} b$ and obtain a longer sequence with γ more a's and two fewer b's. Repeat this until the sequence contains only three or two copies of b depending on whether l_b is even or odd (do nothing if $l_b \leq 2$). Note that at least one a, at least one b and all elements of U are not involved in the replacing operation, hence the new sequence is also an unsplittable minimal zero-sum sequence.

Subcase 3.1: *l_b is odd*. The resulting sequence is $S' = a^{r'+1} b^2 U$ where $r' = l_a + \gamma(l_b - 1)/2$ and U is defined at the beginning of the proof. Applying

Case 1 to S', we see that either $|\sum(a^{r'}bx)| \geq 3|a^{r'}bx| - 4$ for some x in U, or any two terms in b^2U have sum in $\{a, \dots, r'a\}$.

In the former case, $\sum(a^{l_a}b^{l_b}) = \sum(a^{r'}b)$, and thus

$$|\sum(a^{l_a}b^{l_b}x)| = |\sum(a^{r'}bx)| \geq 3|a^{r'}bx| - 4 \geq 3|a^{l_a}b^{l_b}x| - 4.$$

This is just (ii) with $T = a^{l_a}b^{l_b}x$.

In the latter case, applying Proposition 3.1 to S' shows that U contains only one term $a - b$. Thus S has the desired form, and (i) follows.

Subcase 3.2: l_b *is even*. The resulting sequence is $S' = a^{r'+1}b^3U$ where $r' = l_a + \gamma(l_b - 2)/2$ and U is defined at the beginning of the proof.

Suppose there is some x in U such that $|\sum(a^{r'}b^2x)| \geq 3|a^{r'}b^2x| - 4$. Then $|\sum(a^{l_a}b^{l_b}x)| = |\sum(a^{r'}b^2x)| \geq 3|a^{r'}b^2x| - 4 \geq 3|a^{l_a}b^{l_b}x| - 4$, and thus (ii) holds with $T = a^{l_a}b^{l_b}x$.

Suppose $|\sum(a^{r'}b^2x)| < 3|a^{r'}b^2x| - 4$ for any $x \,|\, U$. Then $G = \langle a\rangle$ by Lemma 3.3. Write $b = ta$ and $S' = a^{r'+1}(ta)^3 \cdot (k_1a) \cdots (k_sa)$ with $2t = n + \gamma$ and $1 \leq k_1 \leq \cdots \leq k_s < n$. By choosing $x = k_sa$, we see that any two terms in $t, t, t, k_1, \dots, k_s$ have sum $\leq 2\max\{t, k_s\} \leq n + r'$.

If $t < k_2$, by considering $x = k_1a$, any two terms in $t, t, t, k_1, \dots, k_s$ have sum $\geq \min\{2t, t + k_1\} > n$.

If $t > k_2$, we can choose $x = k_1a$ and then $x = k_2a$. We assert that $k_1 + k_2 > n$, for otherwise $k_1 + k_2 \leq n < k_1 + t \leq k_1 + k_2 + r'$, which is impossible. Hence any two terms in $t, t, t, k_1, \dots, k_s$ have sum $\geq k_1 + k_2 > n$.

Now any two terms in b^3U have sum in $\{a, \dots, r'a\}$. Proposition 3.1 implies that S' contains an even number of copies of $b = ta$, a contradiction.

Case 4: $l_b \geq t_b \geq 3$. Since S is minimal zero-sum, $\gamma > 0$. By Lemma 2.3, $\gamma \neq 1$. If $\gamma = 2$, then $t_bb = 2a$. Hence we can exchange the roles of a and b and apply the result of Case 3. If $\gamma \geq 3$, then by Lemma 2.9,

$$\begin{aligned}
|\sum(a^{l_a}b^{l_b})| &= t_b(l_a + 1) + \gamma(l_b - t_b + 1) - 1 \\
&= (l_a + 1)(l_b + 1) - (l_a + 1 - \gamma)(l_b + 1 - t_b) - 1 \\
&\geq (l_a + 1)(l_b + 1) - (l_a + 1 - 3)(l_b + 1 - 3) - 1 \\
&= 3(l_a + l_b) - 4.
\end{aligned}$$

Hence we can take $T = a^{l_a}b^{l_b}$. ∎

Using a similar (but longer) argument, we can obtain the following stronger result. A detailed proof is in the Appendix.

Theorem 3.5. *Let G be an abelian group of order n with* $\gcd(n, 6) = 1$, *and S an unsplittable minimal zero-sum sequence over G with* $\mathsf{h}(S) \geq 2$ *and* $|\mathrm{supp}(S)| \geq 2$. *Then one of the following holds:*

(i) *G is a cyclic group and*

$$S = g^{(n-r)/2-1-tr} \cdot \left(\frac{n+r}{2}g\right)^{2(t+1)} \cdot \left(\left(\frac{n-r}{2}+1\right)g\right),$$

where g is a generator of G and $r,t \in \mathbb{N}_0$ with r odd and $3 \le r \le (n-r)/2 - 1 - tr$.

(ii) *There are distinct $a, b \in \operatorname{supp}(S)$ with $\mathsf{v}_a(S) \ge 2$ and a subsequence $T \mid S$ such that $\mathsf{v}_a(T) = \mathsf{v}_a(S) - 1$, $\mathsf{v}_b(T) = \mathsf{v}_b(S) - 1$ and $|\sum(T)| \ge 3|T| - 3$. Moreover, if there are distinct elements in S with multiplicities greater than 1, then we may choose b such that $\mathsf{v}_b(S) \ge 2$.*

4. The second step. To prove our main theorem, it suffices to show that no subsequence T (as described in statement (ii) of either Theorem 3.4 or Theorem 3.5) exists under the assumption that $|S| \ge \lfloor n/3 \rfloor + 3$. We first state two lemmas.

Lemma 4.1 ([9]). *Let $S = Tg$ be a zero-sum free sequence over an abelian group with T non-empty. Suppose that $|\sum(S)| = |\sum(T)| + 1$. Then*

(1) *$\sum(T) = \{g, 2g, \ldots, sg\} \cup H$, where $s \in [1, \operatorname{ord}(g) - 2]$ and H is the union of several (possibly none) $\langle g \rangle$ cosets.*
(2) *$\sigma(T) = sg$.*

Lemma 4.2. *Let $S = Tg$ be a zero-sum free sequence over an abelian group of odd order, where T is not empty. Suppose that $|\sum(S)| = |\sum(T)| + 2$. Then one of the following holds:*

(1) *$\sum(T) = \{g, 2g, \ldots, sg\} \cup \{kg, \ldots, (k+t)g\} \cup H$, where $t, s \ge 0$, $s + 2 \le k \le k + t \le \operatorname{ord}(g) - 2$ and H is the union of several (possibly none) $\langle g \rangle$ cosets. In this case $\sigma(T) \in \{sg, (k+t)g\}$.*
(2) *$\sum(T) = \{g, 2g, \ldots, sg\} \cup (c + \{0, g, \ldots, sg\}) \cup H$, where $0 \le s \le \operatorname{ord}(g) - 2$, $c \notin \langle g \rangle$ and H is the union of several (possibly none) $\langle g \rangle$ cosets. In this case $\sigma(T) = c + sg$.*

Proof. For convenience, we consider $\sum_0(T)$ instead of $\sum(T)$. Now $|\sum_0(S)| = |\sum(S)| + 1 = |\sum(T)| + 2 + 1 = |\sum_0(T)| + 2$ and $\sum_0(S) = \sum_0(T) + \{0, g\}$. It is easy to see that $\sum_0(T)$ is the union of two arithmetical progressions with the same difference g and several (possibly none) $\langle g \rangle$ cosets. Since S is zero-sum free, $-g \notin \sum_0(T)$, and thus $\langle g \rangle \not\subset \sum_0(T)$. Since $0 \in \sum_0(T)$ and $-g \notin \sum_0(T)$, 0 occurs as the first term in one of the two arithmetical progressions, that is, one of the progressions is $\{0, g, \ldots, sg\}$ with $0 \le s \le \operatorname{ord}(g) - 2$.

Suppose first that the other arithmetical progression is included in $\langle g \rangle$, say $\{kg, \ldots, (k+t)g\}$ with $0 \le k \le k+t \le \operatorname{ord}(g) - 1$. Since the two progressions cannot be joined to be an arithmetical progression, and $-g \notin \sum_0(T)$, we infer $s+2 \le k \le k+t \le \operatorname{ord}(g)-2$. By Lemma 2.6, $\sigma(T)+g \notin \sum_0(T)$, and thus $\sigma(T)+g \in \{sg+g,(k+t)g+g\}$, that is, $\sigma(T) \in \{sg,(k+t)g\}$. Statement (1) follows.

Now suppose that the other arithmetical progression is not in $\langle g\rangle$, say $c+\{0,\dots,tg\}$ with $0 \le t \le \operatorname{ord}(g)-1$ and $c \notin \langle g\rangle$. Since $c+\{0,\dots,tg\}$ is not a full coset, $t \le \operatorname{ord}(g)-2$. As above, it is easy to see that $\sigma(T) \in \{sg, c+tg\}$.

If $\sigma(T) = sg$, then $sg-c = \sigma(T)-c \in \sum_0(T)$. Since the group is of odd order, $sg-c \notin \langle g\rangle \cup (c+\langle g\rangle)$, and thus $sg-c \in H$. By the definition of H, $-c+\langle g\rangle \subset \sum_0(T)$. Thus $c+\langle g\rangle = \sigma(T)-(-c+\langle g\rangle) \subset \sum_0(T)$, a contradiction. Therefore $\sigma(T) = c+tg$.

Finally, we prove $s=t$. Since $tg = \sigma(T)-c \in \sum_0(T)$, we have $t \le s$. Since $c+(t-s)g = \sigma(T)-sg \in \sum_0(T)$, either $t-s \ge 0$ or $\operatorname{ord}(g)+t-s \le t$. The latter is impossible because $\operatorname{ord}(g)-s>0$. Thus $t=s$. ∎

We now finish the proof of the main theorem, which is restated as follows.

Theorem 4.3. *Let $n \ge 9$ be an odd integer, and G an abelian group of order n. Let S be an unsplittable minimal zero-sum sequence over G of length $|S| \ge \lfloor n/3\rfloor + 3$. Then G is cyclic, and either $S = g^n$ or*

$$S = g^{(n-r)/2-1-tr} \cdot \left(\frac{n+r}{2}g\right)^{2(t+1)} \cdot \left(\left(\frac{n-r}{2}+1\right)g\right),$$

where g is a generator of G and $r,t \in \mathbb{N}_0$ with r odd and $3 \le r \le (n-r)/2-1-tr$. Moreover, $\operatorname{Ind}(S)=2$ *in the latter case.*

Proof. If $\operatorname{supp}(S)$ contains only one element g, then G is generated by g and $S=g^n$. Therefore we only need to consider the case when $|\operatorname{supp}(S)| \ge 2$. By Lemma 2.1, $\mathsf{h}(S) \ge \mathsf{h}(Sx^{-1}) \ge (6(|S|-1)-n+1)/17 > 1$, where x is any term in S.

By Theorems 3.4 and 3.5, either S has the desired form or there exists a subsequence T with the properties stated in those theorems. We need to prove that the existence of such a T is impossible.

Assume that such a T exists. Then we can choose distinct $a,b \in \operatorname{supp}(S)$ with $\mathsf{v}_a(S) \ge 2$ and a subsequence $U \mid S$ such that $\mathsf{v}_a(U) = \mathsf{v}_a(S)-1$, $\mathsf{v}_b(U) = \mathsf{v}_b(S)-1$ and $|\sum(U)| \ge 3|U|-\delta$, where $\delta = 4$ if $3 \mid n$, and $\delta = 3$ otherwise. Moreover, we may choose U to be the one with greatest possible length and write $S = UVab$. We state some facts:

- $\{a,b\} \cap \operatorname{supp}(V) = \emptyset$.
- $|\sum(Ug)| \le |\sum(U)|+2$ for all $g \in \operatorname{supp}(V)$.
- $|U| \ge 2$.
- $a \in \sum(U) \setminus \{\sigma(U)\}$ and $b \in \sum(U) \setminus \{\sigma(U)\}$ if S contains at least two distinct elements with multiplicities greater than 1.

- $|V| \geq 2$ (for otherwise, $n-3 \geq |\sum(U)| \geq 3|U| - \delta \geq 3(|S|-2) - \delta \geq 3\lfloor n/3 \rfloor + 3 - \delta \geq n-2$ if $|V| = 0$, yielding a contradiction, and $n-6 = n-1-|\sum(abV) \setminus \{\sigma(abV)\}| \geq |\sum(U)| \geq 3|U| - \delta \geq 3(|S|-3) - \delta \geq 3\lfloor n/3 \rfloor - \delta \geq n-5$ if $|V| = 1$, yielding a contradiction again).

We divide the proof into two cases.

Case 1: *There is $g \in \mathrm{supp}(V)$ such that $|\sum(Ug)| = |\sum(U)| + 1$.* By Lemma 4.1, $\sum(U)$ has the form $\{g, 2g, \ldots, sg\} \cup H$, where $s \in [1, \mathrm{ord}(g)-2]$ and $H + \langle g \rangle = H$. Moreover, $\sigma(U) = sg$.

We first show that $a \in \langle g \rangle$: if not, $a + \langle g \rangle \subset \sum(U) \setminus \{\sigma(U)\}$, and thus $-a \in -a + \langle g \rangle = \sigma(U) - (a + \langle g \rangle) \subset \sum(U)$, which is impossible since S is minimal zero-sum.

Suppose $s = 1$. Since $a \in \sum(U) \setminus \{\sigma(U)\}$, we have $a \notin \langle g \rangle$, a contradiction. Hence we may assume $s \geq 2$.

Suppose there is $h \in \mathrm{supp}(V)$ with $h \neq g$. First note that $h + sg \notin \sum(U)$ by Lemma 2.6. If $h \notin \langle g \rangle$, we have $(h + \langle g \rangle) \cap \sum(U) = \emptyset$. Then $\{h+sg, h+(s-1)g, h+(s-2)g\} \subset \sum(Uh) \setminus \sum(U)$, a contradiction. If $h \in \langle g \rangle$, we have $\{h+sg, h+(s-1)g, h+(s-2)g\} \subset \sum(Uh) \setminus \sum(U)$ or $h = 2g$, which contradicts the definition of U or Lemma 2.3. Hence $\mathrm{supp}(V) = \{g\}$.

Now, $b = 0 - \sigma(U) - \sigma(V) - a \in \langle g \rangle$, $s + |V| \leq \mathrm{ord}(g) - 2$ and $\sum(UV) = \{g, 2g, \ldots, (s+|V|)g\} \cup H$. Since a is unsplittable in S, we have $G \setminus \{0\} = \sum(UVb) = \{g, 2g, \ldots, (s+|V|)g\} \cup \{b, \ldots, b+(s+|V|)g\} \cup H$, and thus $b + (s+|V|)g = -g$, which implies $a = 0 - \sigma(U) - \sigma(V) - b = g$, a contradiction.

Case 2: *For all $x \in \mathrm{supp}(V)$, $|\sum(Ux)| = |\sum(U)| + 2$.* Clearly, $|\sum(U)| = 3|U| - \delta$, for otherwise we can add any term of V to U and obtain a longer subsequence U. Choose $g \in \mathrm{supp}(V)$ such that $\mathsf{v}_g(V) = \mathsf{h}(V)$. By Lemma 4.2, $\sum(U)$ has two possible structures:

Subcase 2.1: *$\sum(U) = \{g, 2g, \ldots, sg\} \cup (c + \{0, g, \ldots, sg\}) \cup H$, where $0 \leq s \leq \mathrm{ord}(g) - 2$, $c \notin \langle g \rangle$ and H is the union of several (possibly none) $\langle g \rangle$ cosets.* Moreover, in this case $\sigma(U) = c + sg$.

We remark that if $h \in \mathrm{supp}(V) \cap \langle g \rangle$, then $h = g$, for otherwise

$$\{(s+1)g, sg+h, (s+1)g+h, c+(s+1)g, c+sg+h, c+(s+1)g+h\}$$

is a subset of $\sum(Ugh) \setminus \sum(U)$, which contradicts the choice of U.

We claim that $\mathsf{v}_x(S) \leq \mathrm{ord}(x) - 2$ for any $x \in \mathrm{supp}(S)$. Since S is minimal zero-sum, $\mathsf{v}_x(S) \leq \mathrm{ord}(x) - 1$. If $\mathsf{v}_x(S) = \mathrm{ord}(x) - 1$, then $|S| - (\mathrm{ord}(x) - 1) \geq \lfloor n/3 \rfloor - \mathrm{ord}(x) + 4 \geq n/\mathrm{ord}(x) + 1$ (here we have used the fact that n is odd). Thus $Sx^{-\mathsf{v}_x(S)}$ contains a proper subsequence T such that $\sigma(T) \in \langle x \rangle$, and so $\sigma(T)x^{\mathsf{v}_x(S)}$ contains a zero-sum subsequence, which implies that $Tx^{\mathsf{v}_x(S)}$ contains a zero-sum subsequence, a contradiction. This proves the claim.

We now tackle the simplest case $\mathrm{supp}(V)=\{g\}$. It is easy to see that

$$\sum(UV)=\{g,2g,\ldots,(s+|V|)g\}\cup\{c,c+g,\ldots,c+(s+|V|)g\}\cup H,$$

$s+|V|\le \mathrm{ord}(g)-1$, $\sigma(UV)=c+(s+|V|)g$ and $|\sum(UV)|=3|U|+2|V|-\delta=3(|S|-2)-|V|-\delta\ge 3\lfloor n/3\rfloor-|V|+3-\delta$.

If $H\ne G\setminus(\{0,c\}+\langle g\rangle)$, then $|\sum(UV)|\le n-|\langle g\rangle|-1$. Thus by the claim above, $n\ge 3\lfloor n/3\rfloor+|\langle g\rangle|-|V|+4-\delta\ge 3\lfloor n/3\rfloor+6-\delta$, which is impossible. Therefore we may assume $H=G\setminus(\{0,c\}+\langle g\rangle)$.

Since S is minimal zero-sum, $\{-a,-b,0\}\notin\sum(UV)$, and thus $G\setminus\sum(UV)$ contains at least three elements, which implies $s+|V|\le \mathrm{ord}(g)-2$. Since $\sigma(UV)+a\notin\sum(UV)$ and $\sigma(UV)+b\notin\sum(UV)$, we have $a,b\in\{0,-c\}+\langle g\rangle$. Since $a+b=-\sigma(UV)\in -c+\langle g\rangle$, a or b is in $\langle g\rangle$ and the other one is in $-c+\langle g\rangle$. We let $a\in\langle g\rangle$ and $b\in -c+\langle g\rangle$. Since S is unsplittable, $\sum(UVb)=G\setminus\{0\}$. Then $c+(s+|V|)g+b=-g$, otherwise $-g=(\mathrm{ord}(g)-1)g\notin\sum(UVb)$. Thus $a=-\sigma(UV)-b=-(c+(s+|V|)g+b)=g\in\mathrm{supp}(V)$, a contradiction.

Now, we only need to consider the case when V contains at least two distinct elements. By the remark above, there exists $s\in\mathrm{supp}(V)\setminus\langle g\rangle$. We divide the proof into three subcases according to the value of s.

Subcase 2.1.1: $s\ge 2$. First, suppose that there is $h\in\mathrm{supp}(V)$ such that $h\notin(-c+\langle g\rangle)\cup\langle g\rangle$, which implies $c+sg+h\notin(c+\langle g\rangle)\cup\langle g\rangle$. By Lemma 2.6, $c+sg+h=\sigma(U)+h\notin\sum(U)$. Thus $(c+h+\langle g\rangle)\cap\sum(U)=\emptyset$, which implies $\{c+h,c+g+h,c+2g+h\}\subset\sum(Uh)\setminus\sum(U)$, a contradiction.

Next, suppose that there is $h\in\mathrm{supp}(V)$ such that $h\in -c+\langle g\rangle$. Note that $-c+\langle g\rangle\subset\sum(U)$, for otherwise $(-c+\langle g\rangle)\cap\sum(U)=\emptyset$, and thus $\{h,h+g,h+2g\}\subset\sum(Uh)\setminus\sum(U)$, which is impossible. Now let t_1 be the minimal positive integer such that $-t_1c+\langle g\rangle\not\subset\sum(U)$, and t_2 be the minimal positive integer such that $-t_2c\in\langle g\rangle$. Since $-c+\langle g\rangle\subset\sum(U)$ and $\sum(U)$ does not contain the full coset $c+\langle g\rangle=-(t_2-1)c+\langle g\rangle$, such a t_1 exists and $2\le t_1\le t_2-1$. By the definition of t_1, we have $-t_1c+\langle g\rangle=-(t_1-1)c+\langle g\rangle+h\subset\sum(Uh)$.

If $t_1<t_2-1$, then $(-t_1c+\langle g\rangle)\cap\sum(U)=\emptyset$. Thus $-t_1c+\langle g\rangle\subset\sum(Uh)\setminus\sum(U)$, which implies $|\sum(Uh)|-|\sum(U)|\ge|\langle g\rangle|\ge 3$, a contradiction.

If $t_1=t_2-1$, then $c+\langle g\rangle=-(t_1c)+\langle g\rangle\subset\sum(Uh)$, and thus $\{c+(s+1)g,\ldots,c+(\mathrm{ord}(g)-1)g\}\subset\sum(Uh)\setminus\sum(U)$. Since $\sigma(Uh)=c+sg+h\in\langle g\rangle$ is another element in $\sum(Uh)\setminus\sum(U)$, we must have $s+1=\mathrm{ord}(g)-1$ to ensure that $|\sum(Uh)|\le|\sum(U)|+2$. Then $s=\mathrm{ord}(g)-2$ and $\sigma(Uh)=(\mathrm{ord}(g)-1)g=-g$, which implies $\sigma(Uhg)=-g+g=0$, a contradiction with S being minimal zero-sum.

Subcase 2.1.2: $s=1$. Note that $|\langle g\rangle|$ is not a multiple of 3, for otherwise $3\mid n$, $|\sum(U)|=3+|H|\equiv 0\pmod 3$ and thus $|\sum(U)|\ne 3|U|-4$. Hence $|\langle g\rangle|\ge 5$.

Choose $h \in \mathrm{supp}(V) \setminus \langle g \rangle$.

We claim that $(\{g, c, c+g\} + \langle g, h\rangle) \cap \sum(U) = \{g, c, c+g\}$ and H is the union of some $\langle g, h\rangle$ cosets. Indeed, let $d + \langle g \rangle \subset \sum(U)$; then $d + h + \langle g \rangle \subset \sum(U)$, as otherwise $|\sum(Uh) \setminus \sum(U)| \geq |(d+h+\langle g \rangle) \setminus \sum(U)| \geq |\langle g \rangle| - 2 \geq 3$, a contradiction. This proves the claim.

By the claim, $|\langle g, h\rangle|$ is not a multiple of 3, since otherwise $3 \mid n$, but $|\sum(U)| \neq 3|U| - 4$. Hence the order of $h + \langle g \rangle$ in $G/\langle g \rangle$ is at least 5.

Suppose that $h \notin c + \langle g \rangle$. Then $\{h, g+h\} \subset \sum(Uh) \setminus \sum(U)$. By Lemma 2.6, $c + g + h = \sigma(U) + h$ is another element in $\sum(Uh) \setminus \sum(U)$. Thus $\sum(Uh) \setminus \sum(U)$ contains at least three elements, a contradiction.

Suppose now $h \in c + \langle g \rangle$. Since $\{c+h, c+h+g\} \subset \sum(Uh) \setminus \sum(U)$, we must have $\{h, g+h\} \subset \sum(U)$ to ensure that $|\sum(Uh) \setminus \sum(U)| \leq 2$. Thus $h = c$, so $\mathrm{supp}(V) = \{g, c\}$.

If $\mathsf{v}_c(V) \geq 2$, then $\mathsf{v}_g(V) = \mathsf{h}(V) \geq 2$. We have

$$\{2g, 3g, c+2g, c+3g, 2c, 2c+g, 2c+2g, 2c+3g, 3c, 3c+g, 3c+2g, 3c+3g\} \subset \textstyle\sum(Ug^2c^2) \setminus \sum(U),$$

and thus $|\sum(Ug^2c^2) \setminus \sum(U)| \geq 12$, a contradiction.

If $\mathsf{v}_c(V) = 1$, then there are at least two $\langle g \rangle$ cosets outside $\sum(UV)$, and thus $n \geq |\sum(UV)| + 2\mathrm{ord}(g) + 1 \geq |\sum(U)| + |\sum(V)| + 2(\mathsf{v}_g(S) + 2) + 1 \geq 3|U| - \delta + |V| + 2(|V| - 1 + 2) + 1 \geq 3\lfloor n/3 \rfloor + 6 - \delta$, which is impossible.

Subcase 2.1.3: $s = 0$. Choose $h \in \mathrm{supp}(V) \setminus \langle g \rangle$. By the same argument as in Subcase 2.1.2, we have $\mathrm{ord}(g) = |\langle g \rangle| \geq 5$, the order of $h + \langle g \rangle$ in $G/\langle g \rangle$ is at least 5, $(\{0, c\} + \langle g, h\rangle) \cap \sum(U) = \{c\}$ and H is the union of some $\langle g, h\rangle$ cosets.

If $h \notin \{c-g, c, c+g, -c+g\}$, then it is easy to check that $\{g, h, h+g, c+g, c+h, c+h+g\} \subset \sum(Uhg) \setminus \sum(U)$, a contradiction. If $h = c$, then $\sum(Uh) \setminus \sum(U) = \{2c\}$, which contradicts the assumption of Case 2. Therefore we may assume that $\mathrm{supp}(V) \subset \{c-g, c+g, -c+g, g\}$ by the arbitrariness of h.

We claim that only one of $c-g$, $c+g$, $-c+g$ occurs in V, with multiplicity 1.

Indeed, if $c-g$ occurs in V with multiplicity ≥ 2, then $\mathsf{v}_g(V) = \mathsf{h}(V) \geq 2$ and

$$\textstyle\sum(U(c-g)^2g^2) \setminus \sum(U) \supset \{g, 2g, c-g, c+g, c+2g, 2c-2g, 2c-g, 2c, 2c+g, 3c-2g, 3c-g, 3c\},$$

which implies $|\sum(U(c-g)^2g^2) \setminus \sum(U)| \geq 12$, a contradiction.

If $c+g$ or $-c+g$ occurs in V with multiplicity ≥ 2, a similar calculation leads to a contradiction.

If $c-g$ and $-c+g$ occur in V simultaneously, then $(c-g)+(-c+g)=0$, a contradiction.

If $c-g$ and $c+g$ occur in V simultaneously, then

$$\{c-g, c+g, 2c-g, 2c, 2c+g, 3c\} \subset \sum(U(c-g)(c+g)) \setminus \sum(U),$$

a contradiction.

If $-c+g$ and $c+g$ occur in V simultaneously, then

$$\{-c+g, g, 2g, c+g, c+2g, 2c+g\} \subset \sum(U(-c+g)(c+g)) \setminus \sum(U)$$

also yields a contradiction, proving the claim.

Now we have $V = hg^{|V|-1}$ with $h \in \{c-g, c+g, -c+g\}$. However, the same calculation as used at the end of Subcase 2.1.2 leads to a contradiction.

Subcase 2.2: $\sum(U) = \{g, 2g, \dots, sg\} \cup \{kg, \dots, (k+t)g\} \cup H$, *where* $s \ge 0$, $s+2 \le k \le k+t \le \operatorname{ord}(g)-2$ *and* H *is the union of several* (*possibly none*) $\langle g \rangle$ *cosets.* Moreover, in this case $\sigma(U) \in \{sg, (k+t)g\} \subset \langle g \rangle$.

First we show that $\operatorname{supp}(abV) \subset \langle g \rangle$ and $s+t \ge 1$. If $a \notin \langle g \rangle$, then $a + \langle g \rangle \subset \sum(U)$ because $a \in \sum(U) \setminus \{\sigma(U)\}$. Hence $-a \in (-a + \langle g \rangle) = \sigma(U) - (a + \langle g \rangle) \subset \sum(U)$, a contradiction. Therefore $a \in \langle g \rangle$.

If $s+t=0$, then $\sum(U) \cap \langle g \rangle = \{kg\} = \{\sigma(U)\}$. Since $a \in \sum(U) \setminus \{\sigma(U)\}$, we have $a \notin \langle g \rangle$, a contradiction. Therefore $s+t \ge 1$.

If there is $x \in \operatorname{supp}(V) \setminus \langle g \rangle$, then $(x + \langle g \rangle) \cap \sum(U) = \emptyset$ because $\sigma(U)+x \notin \sum(U)$. Hence $\{x, \dots, x+sg, x+kg, \dots, x+(k+t)g\} \subset \sum(Ux) \setminus \sum(U)$, which implies $|\sum(Ux) \setminus \sum(U)| \ge s+1+t+1 \ge 3$, a contradiction. Therefore $\operatorname{supp}(V) \subset \langle g \rangle$. Finally, $b = -\sigma(U) - \sigma(V) - a \in \langle g \rangle$.

Next we show that if $\operatorname{supp}(V) \ne \{g\}$, then $k \ge s+3$. Choose $h \in \operatorname{supp}(V) \setminus \{g\}$, write $h = lg$ with $l \in [2, \operatorname{ord}(g)-2]$, and suppose that $k = s+2$. Since S is unsplittable, $l \ge 3$. Clearly $\sum(Ug) = \{g, 2g, \dots, (k+t+1)g\} \cup H$.

If $l \ge 4$, then

$$\begin{aligned}&\textstyle\sum(Ugh) \setminus \sum(Ug) \supset \\ &\qquad \{(k+t+1)g+h, (k+t)g+h, (k+t-1)g+h, (k+t-2)g+h\},\end{aligned}$$

which implies $|\sum(Ugh)| \ge |\sum(Ug)| + 4 = |\sum(U)| + 6$, a contradiction. Therefore $l = 3$ and $h = 3g$.

By the arbitrariness of h, we have $\operatorname{supp}(V) = \{g, 3g\}$. By Lemma 2.3, $\mathsf{h}(V) = \mathsf{v}_g(V) = 1$ and thus $V = g \cdot (3g)$. Then $\sum(UV) = \{g, 2g, \dots, (k+t+4)g\} \cup H$. Since $\sum(UVa) = G \setminus \{0\}$, $(k+t+4)g + a = (\operatorname{ord}(g)-1)g = -g$. Similarly, $(k+t+4)g+b = (\operatorname{ord}(g)-1)g = -g$. Hence $a = b$, a contradiction.

We divide the proof of this subcase into two situations according to the value of $\sigma(U)$.

Subcase 2.2.1: $\sigma(U) = sg$. In this subcase, $s \ge 1$.

We show that $kg + (k+t)g = sg$. Let $x \in \{kg, \dots, (k+t)g\}$. Then $sg - x = \sigma(U) - x \in \sum(U)$. It is easy to see that $\sigma(U) - x \notin \{g, 2g, \dots, sg\}$.

Thus $\sigma(U) - x \in \{kg, \dots, (k+t)g\}$. Therefore $sg - \{kg, \dots, (k+t)g\} \subset \{kg, \dots, (k+t)g\}$, which implies $sg - kg = (k+t)g$.

We first consider the case when $|\mathrm{supp}(V)| = 1$, that is, $V = g^{|V|}$. Clearly $\sum(UV) = \{g, \dots, (s+|V|)g\} \cup \{kg, \dots, (k+t+|V|)g\} \cup H$.

If $s+|V| \geq k-1$, then $\sum(UV) = \{g, \dots, (k+t+|V|)g\} \cup H$. Since S is unsplittable, $\sum(UVa) = G \setminus \{0\}$ and thus $(k+t+|V|)g + a = (\mathrm{ord}(g)-1)g$. Similarly, $(k+t+|V|)g + b = (\mathrm{ord}(g)-1)g$. Thus $a = b$, a contradiction.

If $s+|V| < k-1$, then by Lemma 2.6, $\sigma(UV) + a \notin \sum(UV)$ and thus $\sigma(UV) + a = (s+|V|)g + a \in \{(s+|V|+1)g, \dots, (k-1)g\} \cup \{(k+t+|V|+1)g, \dots, (\mathrm{ord}(g)-1)g\}$ (the last subset may be empty). Since S is unsplittable, $\sum(UVa) = G \setminus \{0\}$. We have $(s+|V|)g + a = (k-1)g$ or $(\mathrm{ord}(g)-1)g$, for otherwise $(s+|V|)g + a + g \notin \sum(UVa)$. Similarly $(s+|V|)g + b = (k-1)g$ or $(\mathrm{ord}(g)-1)g$. Since $a \neq b$, one of them, say a, satisfies $(s+|V|)g + a = (\mathrm{ord}(g)-1)g$. Hence $b = -(\sigma(UV)+a) = g \in \mathrm{supp}(V)$, a contradiction.

Next we suppose that there is $h \in \mathrm{supp}(V) \setminus \{g\}$. Write $h = lg$ with $l \in [2, \mathrm{ord}(g)-2]$. Note that $l \geq 3$ as S is unsplittable. Since $\sigma(U) + h \notin \sum(U)$, $sg + h = (s+l)g \in \{(s+1)g, \dots, (k-1)g\} \cup \{(k+t+1)g, \dots, (\mathrm{ord}(g)-1)g\}$, that is, $s+l \in [s+1, k-1] \cup [k+t+1, \mathrm{ord}(g)-1]$.

If $s = 1$ and $t = 0$, then from the beginning of this Subcase 2.2.1, we have $2kg = g$. Since $a \in \sum(U) \setminus \{\sigma(U)\}$ and $a \in \langle g \rangle$, we get $a = kg$. Hence $2a = g$, a contradiction. Therefore $s = 1$ and $t = 0$ cannot occur simultaneously.

If $s+l \in [s+1, k-1] \cup [k+t+3, \mathrm{ord}(g)-1]$, then $\{(s+l)g, (s+l-1)g\} \subset \sum(Uh) \setminus \sum(U)$ when $s = 1$, and $\{(s+l)g, (s+l-1)g, (s+l-2)g\} \subset \sum(Uh) \setminus \sum(U)$ when $s \geq 2$. By the choice of U, only the former case can occur, that is, $s = 1$. Again $|\sum(Uh) \setminus \sum(U)| = 2$ yields $\{kg+h, \dots, (k+t)g+h\} \subset \sum(U)$. Hence $\{kg+h, \dots, (k+t)g+h\} \subset \{g, \dots, sg\}$, which implies $t+1 \leq s = 1$. Hence $t = 0$, a contradiction.

If $s+l = k+t+2$, then $\{(s+l)g, (s+l-1)g\} \subset \sum(Uh) \setminus \sum(U)$. Since $|\sum(Uh) \setminus \sum(U)| = 2$, we have $\{h, g+h, \dots, sg+h\} \setminus \sum(U) = \{(s+l)g, (s+l-1)g\}$ and $\{kg+h, \dots, (k+t)g+h\} \subset \sum(U)$. Thus $\{h, g+h, \dots, sg+h\} \subset \{kg, \dots, (k+t+2)g\}$ and $\{kg+h, \dots, (k+t)g+h\} \subset \{g, \dots, sg\}$, which implies $s+1 \leq t+3$ and $t+1 \leq s$, that is, $s-1 \leq t+1 \leq s$. Since $\{kg+h, \dots, (k+t)g+h\} \subset \{g, \dots, sg\}$, $(k+t)g+h$ equals sg or $(s-1)g$. Together with the assumption $sg+h = (k+t+2)g$ of this paragraph, we have $2h = 2g$ or $2h = g$, a contradiction.

If $s+l = k+t+1$ and $t = 0$, then recall that $s = 1$ and $t = 0$ cannot occur simultaneously, hence $s > 1$. Then $\{(s+l)g, (s+l-2)g, (s+l-3)g\} \subset \sum(Uh) \setminus \sum(U)$ when $s \geq 3$, and $\{(s+l)g, (s+l-2)g\} \subset \sum(Uh) \setminus \sum(U)$ when $s = 2$. We only need to consider the latter case. Since $|\sum(Uh) \setminus \sum(U)| = 2$, we have $kg + h \in \sum(U)$. Hence $kg + h = g$ or $2g$. Together with $(s+l)g =$

$(k+t+1)g$, that is, $2g+h=kg+g$, we deduce that $2h=0$ or $2h=g$, a contradiction.

If $s+l=k+t+1$ and $t>0$, then since $(s+l)g\in\sum(Uh)\setminus\{\sigma(U)\}$, we get $|\{kg+h,\dots,(k+t)g+h\}\setminus\sum(U)|\le 1$. By the paragraph before Subcase 2.2.1, $k\ge s+3$, that is, there are at least two holes between sg and kg. Thus $\{kg+h,\dots,(k+t)g+h\}\subset\{(k-1)g,\dots,(k+t+1)g\}$ or $\{kg+h,\dots,(k+t)g+h\}\subset\{g,\dots,(s+1)g\}$. The former inclusion implies $h=-g,0,g$, which is impossible. So we only consider the latter, which implies $t+1\le s+1$. Since $|\{h,g+h,\dots,sg+h\}\setminus\sum(U)|\le 2$, we have $\{h,g+h,\dots,sg+h\}\subset\{(k-1)g,\dots,(k+t+1)g\}$, which implies $s+1\le t+2$. Thus $s\le t+1\le s+1$. Recall that $\{kg+h,\dots,(k+t)g+h\}\subset\{g,\dots,(s+1)g\}$. Hence $(k+t)g+h=sg$ or $(k+t)g+h=(s+1)g$. Together with $sg+h=(k+t+1)g$, we have $2h=g$ or $2h=2g$, a contradiction.

Subcase 2.2.2: $\sigma(U)=(k+t)g$. We show that $s=t\ge 1$. Indeed, $\sigma(U)-\{kg,\dots,(k+t-1)g\}\subset\sum(U)\setminus\{(k+t)g\}$ and $\sigma(U)-\{g,\dots,sg\}\subset\sum(U)\setminus\{(k+t)g\}$, hence $\sigma(U)-\{kg,\dots,(k+t-1)g\}\subset\{g,\dots,sg\}$ and $\sigma(U)-\{g,\dots,sg\}\subset\{kg,\dots,(k+t-1)g\}$, which implies $s=t$. Since $s+t\ge 1$, we get $s=t\ge 1$.

We first consider the case when $|\mathrm{supp}(V)|=1$, that is, $V=g^{|V|}$. Clearly, $\sum(UV)=\{g,\dots,(t+|V|)g\}\cup\{kg,\dots,(k+t+|V|)g\}\cup H$.

If $t+|V|\ge k-1$, then the proof is the same as the corresponding case in Subcase 2.2.1.

If $t+|V|<k-1$ and $k+t+|V|<\mathrm{ord}(g)-1$, then since $(k+t+|V|)g+a\notin\sum(UV)$ and $\sum(UVa)=G\setminus\{0\}$, we have $(k+t+|V|)g+a=(k-1)g$ or $(\mathrm{ord}(g)-1)g$, for otherwise $(k+t+|V|)g+a+g\notin\sum(UVa)$. In the same way we get $(k+t+|V|)g+b=(k-1)g$ or $(\mathrm{ord}(g)-1)g$. Thus a or b, say a, satisfies $(k+t+|V|)g+a=(\mathrm{ord}(g)-1)g$. Hence $b=-(\sigma(UV)+a)=g\in\mathrm{supp}(V)$, a contradiction.

If $t+|V|<k-1$ and $k+t+|V|=\mathrm{ord}(g)-1$, then note that in this case $\mathsf{v}_g(S)\ge|V|\ge 2$, thus b occurs in U. Now $\sigma(UVa)=-g+a\notin\sum(UV)$ and $a\in\sum(UV)$, thus $a=kg$. Similarly, $b=kg$. Hence $a=b$, a contradiction.

Next we suppose that there is $h\in\mathrm{supp}(V)\setminus\{g\}$. Write $h=lg$ with $l\in[2,\mathrm{ord}(g)-2]$. Note that $l\ge 3$ as S is unsplittable. Since $\sigma(U)+h\notin\sum(U)$, we have $(k+t)g+h=(k+t+l)g\in\{(t+1)g,\dots,(k-1)g\}\cup\{(k+t+1)g,\dots,(\mathrm{ord}(g)-1)g\}$, that is, $k+t+l\in[\mathrm{ord}(g)+t+1,\mathrm{ord}(g)+k-1]\cup[k+t+1,\mathrm{ord}(g)-1]$.

If $k+t+l\in[k+t+1,\mathrm{ord}(g)-1]\cup[\mathrm{ord}(g)+t+3,\mathrm{ord}(g)+k-1]$, then $\{(k+t+l)g,(k+t+l-1)g\}\subset\sum(Uh)\setminus\sum(U)$ when $t=1$, and $\{(k+t+l)g,(k+t+l-1)g,(k+t+l-2)g\}\subset\sum(Uh)\setminus\sum(U)$ when $t\ge 2$. By the choice of U, only the former case can occur, that is, $t=1$. Hence

$\{h, g+h\} \subset \sum(U)$, which implies $h = kg$. Now $a \in \sum(U) \setminus \{(k+1)g\}$, thus $a = g$ or $a = kg = h$, a contradiction.

If $k + t + l = \mathrm{ord}(g) + t + 2$, then $\{(k + t + l)g, (k + t + l - 1)g\} \subset \sum(Uh) \setminus \sum(U)$, and thus $\{h, h + g, \ldots, h + tg\} \subset \sum(U)$. Then $\{h, h + g, \ldots, h + tg\} \subset \{kg, (k + 1)g, \ldots, (k + t)g\}$, that is, $h = kg$. Together with $(k + t)g + h = (t + 2)g$, we obtain $2h = 2g$, that is, $h = g$, a contradiction.

If $k + t + l = \mathrm{ord}(g) + t + 1$, then $k + l = \mathrm{ord}(g) + 1$. Since $(k + t)g + h \in \sum(Uh) \setminus \sum(U)$, we have $|\{h, h + g, \ldots, h + tg\} \setminus \sum(U)| \leq 1$. By the paragraph before Subcase 2.2.1, $k \geq s + 3$. Thus $\{h, h + g, \ldots, tg + h\} \subset \{(k-1)g, \ldots, (k+t+1)g\}$, which implies $h = (k-1)g$, $h = kg$ or $h = (k+1)g$. Together with $(k + t)g + h = (t + 1)g$, we get $2h = 0$, $2h = g$ or $2h = 2g$, a contradiction.

This completes the main part of the proof.

Next we calculate the index of S. Since the only unsplittable minimal zero-sum sequence with index 1 has the form g^n (for some generator g), we have $\mathrm{Ind}(S) \geq 2$ in the latter form. Note that $\|S\|_h = 2$ with $2h = g$. Hence, $\mathrm{Ind}(S) \leq 2$ and thus $\mathrm{Ind}(S) = 2$. ∎

5. Discussion. In this section, we make some remarks on our main result, and explain why the lower bound for the length of the sequence S in Theorem 1.3 is best possible.

1. Let $D(G)$ be the maximal length that a minimal zero-sum sequence over G can attain. Clearly, a minimal zero-sum sequence of length $D(G)$ must be unsplittable by definition. By our main theorem, among all abelian groups of odd orders, only cyclic groups can have unsplittable minimal zero-sum sequences of length $\geq \lfloor |G|/3 \rfloor + 3$. Hence, $D(G) \leq \lfloor |G|/3 \rfloor + 2$ for any non-cyclic abelian group G of odd order.

2. In the main theorem, the restriction that r is odd and $3 \leq r \leq (n - r)/2 - 1 - tr$ does not ensure that $|S| \geq \lfloor n/3 \rfloor + 3$. For example, we may take $n = (2m + 1)^2 + 2$, $r = 2m + 1$ and $t = m - 1$ for some $m \geq 1$. It is clear that $r \geq 3$ is odd and $(n - r)/2 - 1 - tr = r$. But $|S| = (n - r)/2 - 1 - tr + 2(t + 1) + 1 = 2m + 2 = \sqrt{n - 2} + 1$, which is much smaller than $\lfloor n/3 \rfloor + 3$ when n is large.

However, this restriction guarantees that S is unsplittable. Let $m = (n - r)/2 - 1 - tr$. If $t \geq 1$, we can replace $a^{m-1}b^3$ by $a^{m-1+r}b$. This operation does not affect unsplittability: the original sequence is unsplittable if and only if so is the sequence after replacement. Hence, we need only consider the case $t = 0$, and a brute force calculation gives the conclusion.

3. The following example shows that the lower bound $\lfloor n/3 \rfloor + 3$ for $|S|$ is best possible to ensure that S has the structure described in the main theorem.

EXAMPLE 5.1. Let G be a cyclic group of odd order $n \geq 31$, and set $S = g^{m-3} \cdot ((n-m+1)g)^3 \cdot (mg)^2$ where g is a generator of G and $m = \lfloor n/3 \rfloor$. Then S is an unsplittable minimal zero-sum sequence of length $\lfloor n/3 \rfloor + 2$ and $\mathrm{Ind}(S) = 3$.

6. Appendix: The proof of Theorem 3.5. In this appendix, it is always assumed that G is an abelian group of order n with $\gcd(n, 6) = 1$.

LEMMA 6.1. *Let S be an unsplittable minimal zero-sum sequence over G, and $T = a^l bc$ a proper subsequence of S with $1 \leq l < \mathsf{v}_a(S)$ and a, b, c pairwise distinct. Suppose that $|\sum(T)| < 3|T| - 3$. Then $b, c \in \langle a \rangle$. If we let $b = ta$ and $c = ka$ with $1 \leq t \leq k < \mathrm{ord}(a)$, then $k \leq t + l$ and $\mathrm{ord}(a) < t + k \leq 2k \leq \mathrm{ord}(a) + l + 1$.*

Proof. For convenience, we consider $\sum_0(T)$ instead of $\sum(T)$, so the condition is $|\sum_0(T)| < 3|T| - 2 = 3l + 4$. It is clear that

$$\textstyle\sum_0(T) = \{0, b, c, b + c\} + \{0, a, 2a, \ldots, la\} = A_0 \cup A_b \cup A_c \cup A_{b+c},$$

where $A_x = x + \{0, a, \ldots, la\}$ for $x \in \{0, b, c, b+c\}$. Since S is an unsplittable minimal zero-sum sequence, we know that $A_0 \cap A_b = \emptyset$, $A_0 \cap A_c = \emptyset$, $A_{b+c} \cap A_b = \emptyset$ and $A_{b+c} \cap A_c = \emptyset$.

First suppose that $A_b \cap A_c = \emptyset$. By Lemma 2.6, $b + c + la \notin A_0 \cup A_b \cup A_c$. Now $\sum_0(T) \supset A_0 \cup A_b \cup A_c \cup \{b+c+la\}$. Thus $|\sum_0(T)| \geq 3(l+1)+1 = 3l+4$, a contradiction.

Next suppose that $A_0 \cap A_{b+c} = \emptyset$. Thus $\sum_0(T)$ consists of three pairwise disjoint parts: A_0, A_{b+c}, $A_b \cup A_c$. Hence, $|A_b \cup A_c| = |\sum_0(T)| - |A_0| - |A_{b+c}| \leq 3l + 3 - (l + 1) - (l + 1) = l + 1$, which implies $A_b = A_c$. Since $b \neq c$, A_b is a coset of $\langle a \rangle$. Hence, $\mathrm{ord}(a) \leq l + 1 \leq \mathsf{v}_a(S)$, a contradiction.

Finally, suppose that $A_b \cap A_c \neq \emptyset$ and $A_0 \cap A_{b+c} \neq \emptyset$, which implies $b - c \in \{-la, (-l + 1)a, \ldots, la\}$ and $b + c \in \{a, \ldots, la\}$. It follows that $b, c \in \langle a \rangle$. Moreover, if we let $b = ta$ and $c = ka$ with $1 \leq t \leq k < \mathrm{ord}(a)$, then $k \leq t + l$ and $\mathrm{ord}(a) < t + k < \mathrm{ord}(a) + l$. We now have

$$\textstyle\sum_0(T) = \{0, a, \ldots (l + t + k - \mathrm{ord}(a))a\} \cup \{ta, \ldots, (k + l)a\},$$

and thus $|\sum_0(T)| = l+t+k-\mathrm{ord}(a)+1+k+l-t+1 = 2l+2k+2-\mathrm{ord}(a)$. Since $|\sum_0(T)| < 3l + 4$, it follows that $2k \leq \mathrm{ord}(a) + l + 1$. ∎

LEMMA 6.2. *Let S be an unsplittable minimal zero-sum sequence over G, and $T = a^l b^2 c$ a proper subsequence of S with $2 \leq l < \mathsf{v}_a(S)$, $\mathsf{v}_b(S) \geq 3$ and a, b, c pairwise distinct. Suppose that $3b \neq 2a$ and $|\sum(T)| < 3|T| - 3$. Then $b, c \in \langle a \rangle$. If we let $b = ta$ and $c = ka$ with $1 \leq t, k < \mathrm{ord}(a)$, then $t - l \leq k \leq t + l$ and $\mathrm{ord}(a) < t + k, 2t, 2\max\{t, k\} \leq \mathrm{ord}(a) + l + 1$.*

Proof. As before, we consider $\sum_0(T)$ instead of $\sum(T)$, so the condition is $|\sum_0(T)| < 3|T| - 2 = 3l + 7$. With the same notation we have

$$\textstyle\sum_0(T) = A_0 \cup A_b \cup A_{2b} \cup A_c \cup A_{b+c} \cup A_{2b+c}.$$

Since S is an unsplittable minimal zero-sum sequence, we know $A_x \cap A_y = \emptyset$ if $x - y \in \{\pm b, \pm c\}$.

We now prove that $b, c \in \langle a \rangle$.

If $b \notin \langle a \rangle$ and $c \in \langle a \rangle$, then the six sets are pairwise disjoint, and thus $|\sum_0(T)| = 6(l + 1) > 3l + 7$, a contradiction.

If $b \in \langle a \rangle$ and $c \notin \langle a \rangle$, then $\sum_0(T)$ contains five pairwise disjoint parts: A_0, A_b, A_c, A_{c+b} and $\{2b+c+la\}$, and thus $|\sum_0(T)| \geq 4(l+1)+1 \geq 3l+7$, a contradiction.

If $b \notin \langle a \rangle$, $c \notin \langle a \rangle$ and $b - c \notin \langle a \rangle$, then $\{0, b, 2b, c, b + c\}$ intersects at least four cosets of $\langle a \rangle$. Note that $2b + c + la$ is another element in $\sum_0(T)$. Hence $|\sum_0(T)| \geq 3l + 7$, a contradiction.

If $b \notin \langle a \rangle$, $c \notin \langle a \rangle$ and $b - c \in \langle a \rangle$, then $|\sum_0(T)| \geq |A_0| + |A_b \cup A_c| + |A_{2b}| + |A_{2b+c}|$. Since $\mathrm{ord}(a) \geq \mathsf{v}_a(S) + 1 \geq l + 2$ and $b \neq c$, we have $|A_b \cup A_c| \geq l + 2$. Hence $|\sum_0(T)| \geq 4l + 5 \geq 3l + 7$, a contradiction. Having considered all possible cases, we conclude that $b, c \in \langle a \rangle$.

From now on, let $b = ta$ and $c = ka$ with $1 \leq t, k < \mathrm{ord}(a)$. Since S is unsplittable, we have $t, k \in [\mathsf{v}_a(S)+2, \mathrm{ord}(a)-l-1] \subset [l+3, \mathrm{ord}(a)-l-1]$.

We divide the remainder of the proof into several cases according to whether $b + c, 2b \in \{a, 2a, \ldots, la\}$ or not.

Case 1: $b + c \notin \{a, 2a, \ldots, (l + 1)a\}$ *and* $2b \notin \{a, 2a, \ldots, (l + 1)a\}$. In this case, $A_x \cap A_y = \emptyset$ if $x - y \in \{\pm(b + c), \pm 2b\}$.

Note that $\sum_0(T)$ contains three pairwise disjoint parts: $A_0 \cup A_{2b+c}$, $A_b \cup A_c$ and A_{b+c}. It is clear that $|A_0 \cup A_{2b+c}| \geq l+2$ and $|A_b \cup A_c| \geq l+2$. Since $|\sum_0(T)| \leq 3l + 6$ and $|A_{b+c}| = l + 1$, it follows that $|A_0 \cup A_{2b+c}| \leq l + 3$, $|A_b \cup A_c| \leq l + 3$ and $|A_0 \cup A_{2b+c}| + |A_b \cup A_c| \leq 2l + 5$. Thus $2b + c = \beta a$ and $b - c = \theta a$ with $(\beta, \theta) = (1, \pm 1), (1, \pm 2), (2, \pm 1)$. Hence $3b = (\beta + \theta)a = -a, 0, a, 2a, 3a$, which are all impossible. Therefore, this case cannot occur.

Case 2: $b+c \notin \{a, 2a, \ldots, (l+1)a\}$ *and* $2b = \gamma a \in \{3a, \ldots, (l+1)a\}$. In this case, $A_x \cap A_y = \emptyset$ if $x - y = \pm(b + c)$.

We show that $A_b \cap A_c \neq \emptyset$ (or equivalently $A_{2b} \cap A_{b+c} \neq \emptyset$): otherwise, $\sum_0(T)$ contains four pairwise disjoint parts: $A_0 \cup A_{2b}$, A_b, A_{b+c} and $\{2b + c + la\}$, hence, $|\sum_0(T)| \geq (l + \gamma + 1) + (l + 1) + (l + 1) + 1 \geq 3l + 7$, a contradiction.

Thus, we can assume that $k - t \in [-l, l]$. Then $b + c = 2b + (c - b) = (\gamma + k - t)a$. Recall that $b + c \notin \{-la, (-l + a)a, \ldots, (l + 1)a\}$. It follows that

$k-t>0$ and $\gamma+k-t>l+1$, that is, $k>t$ and $t+k-\mathrm{ord}(a)\ge l+2$. Now

$$A_0\cup A_{2b}\cup A_{b+c}=\{0,a,\dots,(k+t-\mathrm{ord}(a)+l)a\},$$
$$A_b\cup A_c\cup A_{2b+c}=\{ta,(t+1)a,\dots,(t+k+t-\mathrm{ord}(a)+l)a\}.$$

Note that $k+t-\mathrm{ord}(a)+l=t+(k+l-\mathrm{ord}(a))<t$, so $|\sum_0(T)|=2(k+t-\mathrm{ord}(a)+l+1)\ge 2(l+2+l+1)=4l+6>3l+7$, a contradiction.

Case 3: *$b+c=\alpha a\in\{a,2a,\dots,(l+1)a\}$ and $2b\notin\{a,2a,\dots,(l+1)a\}$.* In this case, $A_x\cap A_y=\emptyset$ if $x-y=\pm 2b$.

We first show that $A_{2b+c}\cap A_0=\emptyset$. Indeed, otherwise $2b+c=\beta a$ with $\beta\in[1,l]$; thus $b=(2b+c)-(b+c)=(\beta-\alpha)a\in\{-la,\dots,la\}$, a contradiction. Hence, we can infer that $A_0\cup A_{b+c}$ is disjoint from $A_b\cup A_{2b+c}$.

Next we prove that $A_b\cap A_c\ne\emptyset$ (or equivalently $A_{2b}\cap A_{b+c}\ne\emptyset$). Indeed, otherwise $\sum_0(T)$ consists of three pairwise disjoint parts: $A_0\cup A_{b+c}$, $A_b\cup A_{2b+c}$ and $A_c\cup A_{2b}$. It follows that $l+1\le|A_c\cup A_{2b}|=|\sum_0(T)|-|A_0\cup A_{b+c}|-|A_b\cup A_{2b+c}|\le 3l+6-2(l+\alpha+1)=l+4-2\alpha$, and thus $\alpha=1$ and $|A_c\cup A_{2b}|=l+1,l+2$. The equatlity $\alpha=1$ gives $b+c=a$, while $|A_c\cup A_{2b}|=l+1,l+2$ gives $2b-c=-a,0,a$. Hence $3b=(2b-c)+(b+c)=0,a,2a$, impossible.

From the above we may assume $t-k\in[-l,l]$. Then $2b=(b+c)+(b-c)=(\alpha+t-k)a$. Recall that $2b\notin\{-la,\dots,(l+1)a\}$. It follows that $t-k>0$ and $\alpha+t-k>l+1$, that is, $t>k$ and $2t-\mathrm{ord}(a)\ge l+2$. Now

$$A_0\cup A_{b+c}\cup A_{2b}=\{0,a,\dots,(2t-\mathrm{ord}(a)+l)a\},$$
$$A_c\cup A_b\cup A_{c+2b}=\{ka,\dots,(k+2t-\mathrm{ord}(a)+l)a\}.$$

If $2t-\mathrm{ord}(a)+l<k$, then $|\sum_0(T)|=2(2t-\mathrm{ord}(a)+l+1)\ge 2(l+2+l+1)>3l+7$, a contradiction. If $2t-\mathrm{ord}(a)+l\ge k$, then

$$\textstyle\sum_0(T)=\{0,a,\dots,(k+2t-\mathrm{ord}(a)+l)a\}.$$

Now $3l+6\ge|\sum_0(T)|=k+2t-\mathrm{ord}(a)+l+1\ge l+3+l+2+l+1=3l+6$. Hence $k=l+3$ and $2t-\mathrm{ord}(a)=l+2$. From $\mathsf{v}_a(S)>l$ and $\mathsf{v}_b(S)\ge 3$, we know that there are at least one a and one b outside T, so $k+2t-\mathrm{ord}(a)+l\le\mathrm{ord}(a)-2$. Hence $3l\le\mathrm{ord}(a)-7$. Now $\mathrm{ord}(a)+1\le t+k=(\mathrm{ord}(a)+l+2)/2+l+3=(\mathrm{ord}(a)+3l+8)/2\le(2\mathrm{ord}(a)+1)/2$, a contradiction.

Case 4: *$b+c=\alpha a\in\{a,2a,\dots,(l+1)a\}$ and $2b=\gamma a\in\{3a,\dots,(l+1)a\}$.* In this case, $t-k=(2t-\mathrm{ord}(a))-(t+k-\mathrm{ord}(a))=\gamma-\alpha\in[-l,l]$. We only need to consider the case $t<k$ and to prove that $2k\le\mathrm{ord}(a)+l+1$. Now we have

$$A_0\cup A_{2b}\cup A_{b+c}=\{0,a,\dots,(t+k-\mathrm{ord}(a)+l)a\},$$
$$A_b\cup A_c\cup A_{2b+c}=\{ta,\dots,(2t+k-\mathrm{ord}(a)+l)a\}.$$

Note that $t + k - \mathrm{ord}(a) + l = t + (k + l - \mathrm{ord}(a)) < t$. Hence $|\sum_0(T)| = 2(t + k - \mathrm{ord}(a) + l + 1)$. Together with $|\sum_0(T)| \le 3l + 6$, we get $2k \le \mathrm{ord}(a) + l + 1 - (2t - \mathrm{ord}(a) - 3) \le \mathrm{ord}(a) + l + 1$. ∎

We are now in a position to prove Theorem 3.5.

Theorem 6.3. *Let S be an unsplittable minimal zero-sum sequence over G with $\mathsf{h}(S) \ge 2$ and $|\mathrm{supp}(S)| \ge 2$. Then one of the following holds:*

(i) *G is a cyclic group and*

$$S = g^{(n-r)/2-1-tr} \cdot \left(\frac{n+r}{2}g\right)^{2(t+1)} \cdot \left(\left(\frac{n-r}{2}+1\right)g\right),$$

where g is a generator of G and $r, t \in \mathbb{N}_0$ with r odd and $3 \le r \le (n-r)/2 - 1 - tr$.

(ii) *There are distinct $a, b \in \mathrm{supp}(S)$ with $\mathsf{v}_a(S) \ge 2$ and a subsequence $T \mid S$ such that $\mathsf{v}_a(T) = \mathsf{v}_a(S) - 1$, $\mathsf{v}_b(T) = \mathsf{v}_b(S) - 1$ and $|\sum(T)| \ge 3|T| - 3$. Moreover, if there are distinct elements in S with multiplicities greater than 1, we may choose b such that $\mathsf{v}_b(S) \ge 2$.*

Proof. We choose $a \in \mathrm{supp}(S)$ such that $\mathsf{v}_a(S) \ge 2$ and let $l_a = \mathsf{v}_a(S) - 1$. Once we have determined the desired $b \in \mathrm{supp}(S)$, we always set $l_b = \mathsf{v}_b(S) - 1$ and $U = Sa^{-\mathsf{v}_a(S)}b^{-\mathsf{v}_b(S)}$. By Lemma 2.5, $\mathrm{supp}(S)$ contains at least three distinct elements, and thus U is always non-empty.

When $Sa^{-\mathsf{v}_a(S)}$ is square free, the proof is exactly the same as that of Theorem 3.4, except that here we use Lemma 6.1 instead of Lemma 3.2.

Now suppose $Sa^{-\mathsf{v}_a(S)}$ is not square free. Choose $b \in \mathrm{supp}(S) \setminus \{a\}$ with $\mathsf{v}_b(S) \ge 2$. Let t_b be the minimal positive integer such that $t_b b$ is in $\{-l_a a, \dots, l_a a\}$. Since S is unsplittable minimal zero-sum, we have $b \notin \{-l_a a, \dots, l_a a\}$, and thus $t_b \ge 2$. Let $t_b b = \gamma a$ for some $\gamma \in [-l_a, l_a]$.

The proof for $l_b = 1$ is the same as that in Theorem 3.4. Therefore, we assume $l_b \ge 2$. Moreover, we may assume $l_a \ge 2$, for otherwise we can exchange the roles of a and b. We divide the remainder of the proof into several cases.

Case 1: $t_b = 2$. Since $l_b \ge t_b = 2$, by Lemma 2.9 we get $\gamma > 0$. Since S is unsplittable and $a \ne b$, we have $\gamma \ge 3$. Also, $3b \ne 2a$: otherwise, $b + (\gamma - 2)a = b + 2b - 3b = 0$. Now we can replace $a^{l_a}b^3$ by $a^{l_a+\gamma}b$ if $l_b \ge 3$ and obtain a longer unsplittable minimal zero-sum sequence. Repeat this process until the resulting sequence contains only three or two copies of b, depending on whether l_b is even or odd (do nothing if $l_b \le 2$).

Subcase 1.1: *l_b is odd.* The resulting sequence is $S' = a^{r'+1}b^2U$ where $r' = l_a + \gamma(l_b - 1)/2$ and U is defined at the beginning of the proof. Now we go back to the case when $l_b = 1$. By Lemma 6.1 either $|\sum(a^{r'}bx)| \ge 3|a^{r'}bx| - 3$ for some x in U, or any two terms in b^2U have sum in $\{a, \dots, (r' + 1)a\}$.

In the former case, $\sum(a^{l_a}b^{l_b}) = \sum(a^{r'}b)$, and thus

$$|\textstyle\sum(a^{l_a}b^{l_b}x)| = |\sum(a^{r'}bx)| \geq 3|a^{r'}bx| - 3 \geq 3|a^{l_a}b^{l_b}x| - 3.$$

This is exactly (ii) with $T = a^{l_a}b^{l_b}x$.

In the latter case, applying Proposition 3.1 to S', we conclude that U contains only one term $a - b$. Thus S has the desired form, and (i) follows.

SUBCASE 1.2: *l_b is even.* The resulting sequence is $S' = a^{r'+1}b^3U$ where $r' = l_a + \gamma(l_b - 2)/2$ and U is defined at the beginning of the proof.

Suppose there is $x \in U$ such that $|\sum(a^{r'}b^2x)| \geq 3|a^{r'}b^2x| - 3$. Then

$$|\textstyle\sum(a^{l_a}b^{l_b}x)| = |\sum(a^{r'}b^2x)| \geq 3|a^{r'}b^2x| - 3 \geq 3|a^{l_a}b^{l_b}x| - 3,$$

and thus (ii) holds with $T = a^{l_a}b^{l_b}x$.

Suppose that $|\sum(a^{r'}b^2x)| < 3|a^{r'}b^2x| - 3$ for any $x \mid U$. Then $G = \langle a\rangle$ by Lemma 6.2. Write $b = ta$ and $S' = a^{r'+1}(ta)^3 \cdot (k_1a) \cdots (k_sa)$ with $2t = n + \gamma$ and $1 \leq k_1 \leq \cdots \leq k_s < n$. By choosing $x = k_sa$, any two terms in $t, t, t, k_1, \ldots, k_s$ have sum $\leq 2\max\{t, k_s\} \leq n + r' + 1$.

If $t < k_2$, by considering $x = k_1a$, any two terms in $t, t, t, k_1, \ldots, k_s$ have sum $\geq \min\{2t, t + k_1\} > n$.

If $t > k_2$, we can choose $x = k_1a$ and then $x = k_2a$. We assert that $k_1 + k_2 > n$: otherwise, $k_1 + k_2 \leq n < k_1 + t \leq k_1 + k_2 + r'$, which is impossible. Hence any two terms in $t, t, t, k_1, \ldots, k_s$ have sum $\geq k_1 + k_2 > n$.

Thus any two terms in b^3U must have sum in $\{a, \ldots, (r' + 1)a\}$. Proposition 3.1 implies that S' contains an even number of copies of $b = ta$, a contradiction. This completes the proof of Subcase 1.2 and thus that of Case 1.

From now on, we may always assume that $t_b \geq 3$. Moreover, we may also assume that $2a \notin \{b, 2b, \ldots, l_bb\}$.

CASE 2: $l_b = 2$. First, if $3b \neq 2a$, then the situation is exactly as in Subcase 1.2, because there is no condition on $2b$ in Lemma 6.2.

Next suppose that $3b = 2a$ and $l_a = 2$. It is clear that $3a \neq 2b$. So we can exchange the roles of a and b, and use the case discussed in the last paragraph.

Finally, let $3b = 2a$ and $l_a \geq 3$. It is clear that

$$\textstyle\sum(a^3b^2) = \{b, 2b, 3b, 4b, 5b, a, a + b, a + 2b, a + 3b, a + 4b, a + 5b\},$$

which implies $\sum(a^3b^2) = \sum(ab^5)$. So we can replace a^3b^2 by ab^5 and obtain a longer unsplittable minimal zero-sum sequence $S' = b^6a^{l_a-1}U$. Recall that $2a = 3b$. Hence, applying the previous cases to S', we see that either $|\sum(b^5a^{l_a-2}x)| \geq 3|b^5a^{l_a-2}x| - 3$ for some $x \mid U$, or U consists of only one term $b - a$. In the former case,

$$|\textstyle\sum(a^{l_a}b^2x)| = |\sum(b^5a^{l_a-2}x)| \geq 3|b^5a^{l_a-2}x| - 3 \geq 3|a^{l_a}b^2x| - 3,$$

so (ii) holds for $T = a^{l_a}b^2x$. In the latter case, $S = b^3a^{l_a+1}(b-a)$. Note that $2a = 3b$ and $a + (b-a) = b$. Proposition 3.1 implies that (i) holds.

From now on, we assume that $l_b \geq 3$ and $l_a \geq 3$ as well.

Case 3: $3 \leq l_b < t_b$. By Lemma 2.9, $|\sum(a^{l_a}b^{l_b})| = (l_a+1)(l_b+1) - 1 \geq 3(l_a+l_b) - 3$, and thus (ii) holds with $T = a^{l_a}b^{l_b}$.

Case 4: $3 = t_b \leq l_b$. Since S is minimal zero-sum, $\gamma > 0$. By Lemma 2.3, $\gamma \neq 1$. Since $2a \notin \{b, \dots, l_b b\}$, we have $\gamma \neq 2$. Also, $\gamma \neq 3$ as $a \neq b$. Apply Lemma 2.9 to obtain

$$\begin{aligned}
|\textstyle\sum(a^{l_a}b^{l_b})| &= t_b(l_a+1) + \gamma(l_b - t_b + 1) - 1 \\
&= (l_a+1)(l_b+1) - (l_a+1-\gamma)(l_b+1-t_b) - 1 \\
&\geq (l_a+1)(l_b+1) - (l_a+1-4)(l_b+1-3) - 1 \\
&= 3(l_a+l_b) + l_b - 6 \geq 3(l_a+l_b) - 3.
\end{aligned}$$

Hence, we can take $T = a^{l_a}b^{l_b}$.

Case 5: $4 \leq t_b \leq l_b$. The same argument as in Case 4 shows that $\gamma \geq 3$ and

$$\begin{aligned}
|\textstyle\sum(a^{l_a}b^{l_b})| &= (l_a+1)(l_b+1) - (l_a+1-\gamma)(l_b+1-t_b) - 1 \\
&\geq (l_a+1)(l_b+1) - (l_a+1-3)(l_b+1-4) - 1 \\
&= 3(l_a+l_b) + l_a - 6 \geq 3(l_a+l_b) - 3. \quad \blacksquare
\end{aligned}$$

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