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On uniform approximation to real numbers

by

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1. Introduction. Throughout the present paper, the *height* $H(P)$ of a complex polynomial $P(X)$ is the maximum of the moduli of its coefficients, and the *height* $H(\alpha)$ of an algebraic number α is the height of its minimal polynomial over $\mathbb{Z}$. For an integer $n \geq 1$, the exponents of Diophantine approximation w_n, w_n^*, $\widehat{w}_n$, and $\widehat{w}_n^*$ measure the quality of approximation to real numbers by algebraic numbers of degree at most n. They are defined as follows.

Let ξ be a real number. We denote by $w_n(\xi)$ the supremum of the real numbers w for which

$$0 < |P(\xi)| \leq H(P)^{-w}$$

has infinitely many solutions in polynomials P in $\mathbb{Z}[X]$ of degree at most n, and by $\widehat{w}_n(\xi)$ the supremum of the real numbers w for which the system

$$0 < |P(\xi)| \leq H^{-w}, \quad H(P) \leq H$$

has a solution P in $\mathbb{Z}[X]$ of degree at most n, for all large values of H.

Likewise, we denote by $w_n^*(\xi)$ the supremum of the real numbers w for which

$$0 < |\xi - \alpha| \leq H(\alpha)^{-w-1}$$

has infinitely many solutions in algebraic numbers α of degree at most n, and by $\widehat{w}_n^*(\xi)$ the supremum of the real numbers w for which the system

$$0 < |\xi - \alpha| \leq H(\alpha)^{-1}H^{-w}, \quad H(\alpha) \leq H$$

is satisfied by an algebraic number α of degree at most n, for all large values of H.

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It is easy to check that every real number ξ satisfies

$$w_1(\xi) = w_1^*(\xi) \quad \text{and} \quad \widehat{w}_1(\xi) = \widehat{w}_1^*(\xi).$$

Furthermore, if n is a positive integer and ξ a real number which is not algebraic of degree at most n, then Dirichlet's Theorem implies that

$$w_n(\xi) \geq \widehat{w}_n(\xi) \geq n. \tag{1.1}$$

By combining (1.1) with the Schmidt Subspace Theorem, we can deduce that, for all positive integers d, n, every real algebraic number ξ of degree d satisfies

$$w_n(\xi) = \widehat{w}_n(\xi) = w_n^*(\xi) = \widehat{w}_n^*(\xi) = \min\{n, d-1\};$$

see [6, Theorem 2.4]. Thus, we may restrict our attention to transcendental real numbers and, in what follows, ξ will always denote such a number. Furthermore, in the sense of Lebesgue measure, almost all real numbers ξ satisfy

$$w_n(\xi) = \widehat{w}_n(\xi) = w_n^*(\xi) = \widehat{w}_n^*(\xi) = n \quad \text{for } n \geq 1.$$

The survey [5] gathers the known results on the exponents $w_n^*, \widehat{w}_n^*, w_n, \widehat{w}_n$, along with some open questions; see also [4, 22].

A central open problem, often referred to as the *Wirsing conjecture* [23, 4], asks whether every transcendental real number ξ satisfies $w_n^*(\xi) \geq n$ for every integer $n \geq 2$. It has been solved by Davenport and Schmidt [7] for $n = 2$ (see also [13]), but remains wide open for $n \geq 3$. In this direction, Bernik and Tsishchanka [3] established that

$$w_n^*(\xi) \geq \frac{n + \sqrt{n^2 + 16n - 8}}{4} \tag{1.2}$$

for every integer $n \geq 3$ and every transcendental real number ξ. The lower bound (1.2) was subsequently slightly refined by Tsishchanka [21]; see [4] for additional references.

Among the known relations between the exponents $w_n^*, \widehat{w}_n^*, w_n, \widehat{w}_n$, let us mention that Schmidt and Summerer [19, (15.4′)] used their deep, new theory of parametric geometry of numbers to show that

$$w_n(\xi) \geq (n-1)\frac{\widehat{w}_n(\xi)^2 - \widehat{w}_n(\xi)}{1 + (n-2)\widehat{w}_n(\xi)} \tag{1.3}$$

for $n \geq 2$ and every transcendental real number ξ. This extends an earlier result of Jarník [10] which deals with the case $n = 2$. For $n = 3$ Schmidt and Summerer [20] established the better bound

$$w_3(\xi) \geq \frac{\widehat{w}_3(\xi) \cdot (\sqrt{4\widehat{w}_3(\xi) - 3} - 1)}{2}. \tag{1.4}$$

In 1969, Davenport and Schmidt [8] proved that every transcendental real number ξ satisfies

$$1 \le \widehat{w}_n^*(\xi) \le \widehat{w}_n(\xi) \le 2n - 1, \tag{1.5}$$

for every integer $n \ge 1$ (the case $n = 1$ is due to Khintchine [11]). The stronger inequality

$$\widehat{w}_2(\xi) \le \frac{3+\sqrt{5}}{2} \tag{1.6}$$

was proved by Arbour and Roy [2]; it can also be obtained by a direct combination of another result of [8] with a transference theorem of Jarník [9], which remained forgotten until 2004. The first inequality in (1.5) is sharp for every $n \ge 1$; see [6, Proposition 2.1]. Inequality (1.6) is also sharp: Roy [14, 15] proved the existence of transcendental real numbers ξ for which $\widehat{w}_2(\xi) = (3+\sqrt{5})/2$ and called them *extremal numbers*. We also point out the relations

$$w_n^*(\xi) \le w_n(\xi) \le w_n^*(\xi)+n-1, \quad \widehat{w}_n^*(\xi) \le \widehat{w}_n(\xi) \le \widehat{w}_n^*(\xi)+n-1, \tag{1.7}$$

valid for every integer $n \ge 1$ and every transcendental real number ξ; see [4, Lemma A.8] or [5, Theorem 2.3.1].

In view of the lower bound

$$w_n^*(\xi) \ge \frac{\widehat{w}_n(\xi)}{\widehat{w}_n(\xi) - n + 1}, \tag{1.8}$$

established in [6] and valid for every integer $n \ge 2$ and every real transcendental number ξ, any counterexample ξ to the Wirsing conjecture must satisfy $\widehat{w}_n(\xi) > n$ for some integer $n \ge 3$. It is unclear whether transcendental real numbers with the latter property do exist. The main purpose of the present paper is to obtain new upper bounds for $\widehat{w}_n(\xi)$ and, in particular, to improve the last inequality of (1.5) for every integer $n \ge 3$.

2. Main results. Our main result is the following improvement of the upper bound (1.5) of Davenport and Schmidt [8].

THEOREM 2.1. *Let $n \ge 2$ be an integer and ξ a real transcendental number. Then*

$$\widehat{w}_n(\xi) \le n - 1/2 + \sqrt{n^2 - 2n + 5/4}. \tag{2.1}$$

For $n = 3$ we have the stronger estimate

$$\widehat{w}_3(\xi) \le 3 + \sqrt{2} = 4.4142\ldots. \tag{2.2}$$

For $n = 2$, Theorem 2.1 provides an alternative proof of (1.6). This inequality is best possible, as already mentioned in the Introduction.

For $n \ge 3$, Theorem 2.1 gives the first improvement on (1.5). This is, admittedly, a small improvement, since for $n \ge 4$ the right hand side of (2.1) can be written as $2n - 3/2 + \varepsilon_n$, where ε_n is positive and $\lim_{n\to\infty} \varepsilon_n = 0$. There is no reason to believe that our bound is best possible for $n \geq 3$.

Theorem 2.1 follows from the next two statements combined with the lower bounds (1.3) and (1.4) of $w_n(\xi)$ in terms of $\widehat{w}_n(\xi)$ obtained by Schmidt and Summerer [19, 20].

THEOREM 2.2. *Let $m \geq n \geq 2$ be integers and ξ a transcendental real number. Then (at least) one of the two assertions*

$$w_{n-1}(\xi) = w_n(\xi) = w_{n+1}(\xi) = \cdots = w_m(\xi) \tag{2.3}$$

or

$$\widehat{w}_n(\xi) \leq m + (n-1)\frac{\widehat{w}_n(\xi)}{w_m(\xi)} \tag{2.4}$$

holds. In other words, the inequality $w_{n-1}(\xi) < w_m(\xi)$ implies (2.4).

We remark that $w_m(\xi)$ may be infinite in Theorem 2.2, and this is also the case in Theorems 2.3 and 2.4. By (1.5), inequality (2.4) always holds for $m \geq 2n-1$, thus Theorem 2.2 is of interest only for $n \leq m \leq 2n-2$.

For the proof of our main result (Theorem 2.1) we only need the case $m = n$ of Theorem 2.2. We believe that, in this case, inequality (2.4) holds even when $w_{n-1}(\xi) = w_n(\xi)$. Actually, this is true if $\widehat{w}_n(\xi) = \widehat{w}_n^*(\xi)$; see Theorem 2.4.

THEOREM 2.3. *Let m, n be positive integers and ξ be a transcendental real number. Then*

$$\min\{w_m(\xi), \widehat{w}_n(\xi)\} \leq m + n - 1.$$

Taking $m = n$ in Theorem 2.3 gives (1.5), but our proof differs from that of Davenport and Schmidt. The choice $m = 1$ in Theorem 2.3 yields the main claim of [17, Theorem 5.1], which asserts that every real number ξ with $w_1(\xi) \geq n$ satisfies $\widehat{w}_j(\xi) = j$ for $1 \leq j \leq n$. Theorem 2.3 provides new information for $2 \leq m \leq n-1$.

A slight modification of the proof of Theorem 2.2 gives the next result.

THEOREM 2.4. *Let m, n be positive integers and ξ a transcendental real number. Assume that either $m \geq n$ or*

$$w_m(\xi) > \min\{n + m - 1, w_n^*(\xi)\}. \tag{2.5}$$

Then

$$\widehat{w}_n^*(\xi) \leq \min\left\{m + (n-1)\frac{\widehat{w}_n^*(\xi)}{w_m(\xi)}, w_m(\xi)\right\}. \tag{2.6}$$

In particular, for any integer $n \geq 1$ and any transcendental real number ξ,

$$\widehat{w}_n^*(\xi) \leq n + (n-1)\frac{\widehat{w}_n^*(\xi)}{w_n(\xi)}.$$

By (1.5), inequality (2.6) always holds for $m \geq 2n-1$, thus Theorem 2.4 is of interest only for $1 \leq m \leq 2n-2$.

Let $m \geq 2$ be an integer. According to LeVeque [12], a real number ξ is a *U_m-number* if $w_m(\xi)$ is infinite and $w_{m-1}(\xi)$ is finite. Furthermore, the U_1-numbers are precisely the *Liouville numbers*, that is, the real numbers for which the inequalities $0 < |\xi - p/q| < q^{-w}$ have infinitely many rational solutions p/q for every real number w. A *T-number* is a real number ξ such that $w_n(\xi)$ is finite for every integer n and $\limsup_{n\to\infty} w_n(\xi)/n = \infty$. LeVeque [12] proved the existence of U_m-numbers for every positive integer m. Schmidt [18] was the first to confirm that T-numbers do exist. Additional results on U_m- and T-numbers and on Mahler's classification of real numbers are given in [4]. The next statement is an easy consequence of our theorems.

COROLLARY 2.5. *Let m be a positive integer. Every U_m-number ξ satisfies $\widehat{w}_m(\xi) = m$ and the inequalities $\widehat{w}_n^*(\xi) \leq m$ and $\widehat{w}_n(\xi) \leq m+n-1$ for every integer $n \geq 1$. Moreover, every T-number ξ satisfies $\liminf_{n\to\infty} \widehat{w}_n(\xi)/n = 1$.*

Proof. Let m be a positive integer and ξ a U_m-number. We have already mentioned that $\widehat{w}_1(\xi) = 1$. For $m \geq 2$, we have $w_{m-1}(\xi) < w_m(\xi)$ and we derive from Theorem 2.2 that $\widehat{w}_m(\xi) = m$. The bound for $\widehat{w}_n^*(\xi)$ follows from (2.6) as we check the conditions are satisfied in both cases $m \geq n$ and $n < m$ from the inequalities $\widehat{w}_n^*(\xi) \leq \widehat{w}_m^*(\xi) \leq \widehat{w}_m(\xi)$. The upper bound $\widehat{w}_n(\xi) \leq m+n-1$ is then a consequence of (1.7).

Let ξ be a T-number. Then, for any positive real number C, there are arbitrarily large integers n such that $w_n(\xi) > w_{n-1}(\xi)$ and $w_n(\xi) \geq Cn$. For such an n, inserting these relations in (2.4) with $m = n$ and using (1.5), we obtain

$$\widehat{w}_n(\xi) \leq n + \frac{(n-1)(2n-1)}{Cn} < n\left(1 + \frac{2}{C}\right).$$

It is then sufficient to let C tend to infinity. ∎

Roy [15] proved that every extremal number ξ satisfies

$$w_2(\xi) = \sqrt{5} + 2 = 4.2361\ldots = (\widehat{w}_2(\xi) - 1)\widehat{w}_2(\xi), \tag{2.7}$$

thus providing a non-trivial example that equality can hold in (1.3). Approximation to extremal numbers by algebraic numbers of bounded degree was studied in [1, 16]. We deduce from Theorems 2.4 and 2.3 some additional information.

COROLLARY 2.6. *Every extremal number ξ satisfies*

$$\widehat{w}_3^*(\xi) \leq 3\frac{2+\sqrt{5}}{1+\sqrt{5}} = 3.9270\ldots \quad \textit{and} \quad \widehat{w}_3(\xi) \leq 4.$$

Proof. Let $m = 2$, $n = 3$ and ξ be an extremal number. By (2.7) we have $w_2(\xi) = 2 + \sqrt{5} > 4 = m + n - 1$ and the first claim follows from (2.6). Theorem 2.3 implies the second assertion. ∎

We conclude this section with a new relation between the exponents $\widehat{w}_n$ and w_n^*.

THEOREM 2.7. *For every positive integer n and every transcendental real number ξ, we have*

$$\widehat{w}_n(\xi) \leq \frac{2(w_n^*(\xi) + n) - 1}{3}$$

and, if $w_n(\xi) \leq 2n - 1$,

$$\widehat{w}_n^*(\xi) \geq \frac{2w_n^*(\xi)^2 - w_n^*(\xi) - 2n + 1}{2w_n^*(\xi)^2 - nw_n^*(\xi) - n}. \tag{2.8}$$

It follows from the first assertion of Theorem 2.7 that any counterexample ξ to the Wirsing conjecture, that is, any transcendental real number ξ with $w_n^*(\xi) < n$ for some integer $n \geq 3$, must satisfy $\widehat{w}_n(\xi) < (4n - 1)/3$.

It follows from the second assertion of Theorem 2.7 that if $w_n^*(\xi)$ is close to $n/2$ for some integer n and some real transcendental number ξ, then $\widehat{w}_n^*(\xi)$ is also close to $n/2$. Note that (1.7) implies that (2.8) holds for any couterexample ξ to the Wirsing conjecture.

Theorem 2.7 can be combined with (1.8) to get a lower bound for $w_n^*(\xi)$ which is slightly smaller than the one obtained by Bernik and Tsishchanka [3]. However, if we insert (1.3) in the proof of Theorem 2.7, then we get

$$w_n^*(\xi) \geq \max\left\{ \widehat{w}_n(\xi), \frac{\widehat{w}_n(\xi)}{\widehat{w}_n(\xi) - n + 1}, \frac{n-1}{2} \cdot \frac{\widehat{w}_n(\xi)^2 - \widehat{w}_n(\xi)}{1 + (n-2)\widehat{w}_n(\xi)} + \widehat{w}_n(\xi) - n + \frac{1}{2} \right\}.$$

From this we derive a very slight improvement of (1.2), which, like (1.2), has the form $w_n^*(\xi) \geq n/2 + 2 - \varepsilon_n$, where ε_n is positive and tends to 0 when n tends to infinity. Note that the best known lower bound, established by Tsishchanka [21], has the form $w_n^*(\xi) \geq n/2 + 3 - \varepsilon_n'$, where ε_n' is positive and tends to 0 as $n \to \infty$.

3. Proofs. We first show how Theorem 2.1 follows from Theorems 2.2 and 2.3.

Proof of Theorem 2.1. We distinguish two cases.

If $w_{n-1}(\xi) = w_n(\xi)$, then Theorem 2.3 with $m = n-1$ implies that either

$$\widehat{w}_n(\xi) \leq w_n(\xi) = w_{n-1}(\xi) \leq n - 1 + n - 1 = 2n - 2$$

or

$$\widehat{w}_n(\xi) \leq 2n - 2.$$

It then suffices to observe that $2n-2$ is smaller than the bounds in (2.1) and (2.2).

If $w_{n-1}(\xi) < w_n(\xi)$, then we apply Theorem 2.2 with $m = n$ and we get

$$\widehat{w}_n(\xi) \le n + (n-1)\frac{\widehat{w}_n(\xi)}{w_n(\xi)},$$

thus,

$$\widehat{w}_n(\xi) \le \frac{n w_n(\xi)}{w_n(\xi) - n + 1}. \tag{3.1}$$

Rewriting inequality (1.3) as

$$\widehat{w}_n(\xi) \le \frac{1}{2}\Big(1 + \frac{n-2}{n-1} w_n(\xi)\Big) + \sqrt{\frac{1}{4}\Big(\frac{n-2}{n-1} w_n(\xi) + 1\Big)^2 + \frac{w_n(\xi)}{n-1}}, \tag{3.2}$$

we now have two upper bounds for $\widehat{w}_n(\xi)$, one given by a decreasing function and the other by an increasing function of $w_n(\xi)$. An easy calculation shows that the right hand sides of (3.1) and (3.2) are equal for

$$w_n(\xi) = \frac{1}{2}\Big(\frac{1 + 2n\sqrt{n^2 - 2n + 5/4}}{n-1} + 2n - 1\Big).$$

Inserting this value in (3.1) gives precisely the upper bound (2.1). For (2.2) we proceed similarly using (1.4) instead of (1.3). ∎

For the proofs of Theorems 2.2 and 2.3 we need the following slight variation of [8, Lemma 8].

The notation $a \gg_d b$ means that a exceeds b times a constant depending only on d. When $\gg$ is written without any subscript, it means that the constant is absolute.

LEMMA 3.1. *Let P, Q be coprime polynomials with integral coefficients and of degrees at most m and n, respectively. Let ξ be a real number such that $\xi P(\xi) Q(\xi) \neq 0$. Then at least one of the two estimates*

$$|P(\xi)| \gg_{m,n,\xi} H(P)^{-n+1} H(Q)^{-m}, \qquad |Q(\xi)| \gg_{m,n,\xi} H(P)^{-n} H(Q)^{-m+1}$$

holds. In particular,

$$\max\{|P(\xi)|, |Q(\xi)|\} \gg_{m,n,\xi} H(P)^{-n+1} H(Q)^{-m+1} \min\{H(P)^{-1}, H(Q)^{-1}\}.$$

Proof. We proceed as in the proof of [8, Lemma 8] and we consider the resultant $\mathrm{Res}(P,Q)$ of the polynomials P and Q, written as

$$P(T) = a_0 T^s + a_1 T^{s-1} + \cdots + a_s, \qquad a_0 \neq 0,\ s \le m,$$
$$Q(T) = b_0 T^t + b_1 T^{t-1} + \cdots + b_t, \qquad b_0 \neq 0,\ t \le n.$$

Clearly, $|\mathrm{Res}(P,Q)|$ is at least 1 since P and Q are coprime. Transform the corresponding $(s+t)\times(s+t)$-matrix by adding to the last column the sum, for $i = 1, \dots, s+t-1$, of the $(s+t-i)$th column multiplied by ξ^i, so that the last column reads

$$(\xi^{t-1}P(\xi), \xi^{t-2}P(\xi), \dots, P(\xi), \xi^{s-1}Q(\xi), \xi^{s-2}Q(\xi), \dots, Q(\xi)).$$

This transformation does not affect the value of $\mathrm{Res}(P,Q)$. Observe that by expanding the determinant of the new matrix, we can see that every product in the sum is in absolute value either $\ll_{s,t,\xi} |P(\xi)|H(P)^{t-1}H(Q)^s$ or $\ll_{s,t,\xi} |Q(\xi)|H(P)^t H(Q)^{s-1}$. Since there are only $(s+t)! \le (m+n)!$ such terms in the sum, we infer that

$$1 \le |\mathrm{Res}(P,Q)| \ll_{m,n,\xi} \max\{|P(\xi)|H(P)^{n-1}H(Q)^m, |Q(\xi)|H(P)^n H(Q)^{m-1}\}.$$

The lemma follows. ∎

Proof of Theorem 2.2. It is inspired by the proof of [6, Proposition 2.1]. Let $m \ge n \ge 2$ be integers. Let ξ be a transcendental real number. Assume first that $w_m(\xi) < \infty$. We will show that if (2.3) is not satisfied, that is, if we assume

$$w_{n-1}(\xi) < w_m(\xi), \tag{3.3}$$

then (2.4) must hold. Let $\epsilon > 0$ be a fixed small real number. By the definition of $w_m(\xi)$ there exist integer polynomials P of degree at most m and arbitrarily large height $H(P)$ such that

$$H(P)^{-w_m(\xi)-\epsilon} \le |P(\xi)| \le H(P)^{-w_m(\xi)+\epsilon}. \tag{3.4}$$

By an argument of Wirsing [23, Hilfssatz 4] (see also [4, p. 54]), we may assume that P is irreducible. We deduce from our assumption (3.3) that, if ϵ is small enough, then P has degree at least n. Moreover, by the definition of $\widehat{w}_n(\xi)$, if the height $H(P)$ is sufficiently large, then for all $X \ge H(P)$ the inequalities

$$0 < |Q(\xi)| \le X^{-\widehat{w}_n(\xi)+\epsilon} \tag{3.5}$$

are satisfied by an integer polynomial Q of degree at most n and height $H(Q) \le X$. Set $\tau(\xi,\epsilon) = (w_m(\xi)+2\epsilon)/(\widehat{w}_n(\xi)-\epsilon)$ and note that this quantity exceeds 1. Keep in mind that

$$\lim_{\epsilon\to 0} \tau(\xi,\epsilon) = \frac{w_m(\xi)}{\widehat{w}_n(\xi)}. \tag{3.6}$$

For any integer polynomial P satisfying (3.4), set $X = H(P)^{\tau(\xi,\epsilon)}$. Then (3.4) implies

$$|P(\xi)| \ge H(P)^{-w_m(\xi)-\epsilon} > H(P)^{-w_m(\xi)-2\epsilon} = X^{-\widehat{w}_n(\xi)+\epsilon}, \tag{3.7}$$

thus any polynomial Q satisfying (3.5) also satisfies $|Q(\xi)| < |P(\xi)|$. Since P is irreducible of degree at least n and Q has degree at most n, this implies that P and Q are coprime.

On the other hand, by (3.4), we have the estimate

$$|P(\xi)| \le H(P)^{-w_m(\xi)+\epsilon} = X^{(-w_m(\xi)+\epsilon)/\tau(\xi,\epsilon)}.$$

Thus, by (3.6), we get

$$|P(\xi)| \le X^{-\widehat{w}_n(\xi)+\epsilon'}, \tag{3.8}$$

for some ϵ' which depends on ϵ and tends to 0 as $\epsilon \to 0$. Since $|Q(\xi)| < |P(\xi)|$ we obviously obtain

$$\max\{|P(\xi)|, |Q(\xi)|\} \le X^{-\widehat{w}_n(\xi)+\epsilon'}. \tag{3.9}$$

We have constructed pairs (P, Q) of coprime integer polynomials of arbitrarily large height and satisfying (3.9).

We show that, provided $H(P)$ was chosen large enough, we have

$$H(Q) \ge H(P)^{1-\epsilon''}, \tag{3.10}$$

where ϵ'' depends on ϵ and tends to 0 as ϵ tends to 0. Observe that since $|Q(\xi)| < |P(\xi)|$ and by (3.4) we have

$$w_m(\xi) - \epsilon \le -\frac{\log|P(\xi)|}{\log H(P)} \le -\frac{\log|Q(\xi)|}{\log H(P)}.$$

On the other hand,

$$-\frac{\log|Q(\xi)|}{\log H(Q)} \le w_n(\xi) + \epsilon,$$

since Q has degree at most n and can be assumed of sufficiently large height $H(Q)$. Moreover the assumption $m \ge n$ implies $w_m(\xi) \ge w_n(\xi)$. Combination of these facts yields

$$\frac{\log H(Q)}{\log H(P)} = \left(-\frac{\log|Q(\xi)|}{\log H(P)}\right) \cdot \left(-\frac{\log|Q(\xi)|}{\log H(Q)}\right)^{-1} \ge \frac{w_m(\xi)-\epsilon}{w_n(\xi)+\epsilon} \ge \frac{w_n(\xi)-\epsilon}{w_n(\xi)+\epsilon},$$

and we indeed infer (3.10) as $\epsilon \to 0$.

Now observe that we can apply Lemma 3.1 to the coprime polynomials P and Q. It follows from (3.10) and $H(Q) \le X$ that

$$\min\{H(P)^{-1}, H(Q)^{-1}\} \ge X^{-1/(1-\epsilon'')}.$$

We then deduce from Lemma 3.1 that

$$\max\{|P(\xi)|, |Q(\xi)|\} \gg_{m,n,\xi} X^{-(n-1)/\tau(\xi,\epsilon)} X^{-m+1} X^{-1/(1-\epsilon'')}. \tag{3.11}$$

Combining (3.9) and (3.11) we deduce (2.4) as ϵ can be taken arbitrarily small. This completes the proof of the case $w_m(\xi) < \infty$.

If $w_m(\xi) = \infty$, we take a sequence $(P_j)_{j\ge1}$ of integer polynomials of degree at most m with increasing heights and such that the quantity $-\log|P_j(\xi)|/\log H(P_j)$ tends to infinity as $j \to \infty$. We then proceed exactly as above, by using this sequence of polynomials instead of the polynomials satisfying (3.4). We omit the details. ∎

Proof of Theorem 2.3. We assume $n \geq 2$ and $w_m(\xi) < \infty$, for similar reasons to those in the previous proof. Let $\epsilon > 0$ be a fixed small number. By the definition of $w_m(\xi)$, there exist integer polynomials P of degree at most m and arbitrarily large height $H(P)$ such that

$$|P(\xi)| \leq H(P)^{-w_m(\xi)+\epsilon/2}.$$

Again, by using an argument of Wirsing [23, Hilfssatz 4], we can assume that P is irreducible. Then, by [4, Lemma A.3], there exists a real number $K(n)$ in $(0,1)$ such that no integer polynomial Q of degree at most n and whose height satisfies $H(Q) \leq K(n)H(P)$ is a multiple of P. Set $X := H(P)K(n)/2$. If X is large enough, then P satisfies

$$|P(\xi)| \leq X^{-w_m(\xi)+\epsilon}. \tag{3.12}$$

On the other hand, by the definition of $\widehat{w}_n(\xi)$, we may consider only the polynomials P for which $H(P)$ is sufficiently large, so that the estimate

$$0 < |Q(\xi)| \leq X^{-\widehat{w}_n(\xi)+\epsilon} \tag{3.13}$$

holds for an integer polynomial Q of degree at most n and height $H(Q) \leq X$. Our choice of X ensures that Q is not a multiple of P. Since P is irreducible, P and Q are coprime. Thus we may apply Lemma 3.1, which yields

$$\max\{|P(\xi)|, |Q(\xi)|\} \gg_{m,n,\xi} X^{-m-n+1}. \tag{3.14}$$

Combining (3.12)–(3.14), we deduce that $\min\{w_m(\xi), \widehat{w}_n(\xi)\} \leq m+n-1$, as ϵ can be taken arbitrarily small. ∎

Proof of Theorem 2.4. Most estimates arise by a modification of the proof of Theorem 2.2. Define the irreducible polynomial P as in the proof of Theorem 2.2. In that proof a difficulty occurs since the polynomial Q which satisfies (3.5) is not a priori coprime with P. The assumption (3.3) was used to guarantee that Q is not a multiple of P.

Here, instead of (3.5), we use the fact that, for all $X \geq H(P)$, the inequalities

$$0 < |\xi - \beta| < H(\beta)^{-1} X^{-\widehat{w}_n^*(\xi)+\epsilon} \tag{3.15}$$

are satisfied by an algebraic number β of degree at most n and height at most X. Let Q be the minimal defining polynomial over $\mathbb{Z}$ of such a β. Then a standard argument yields

$$|Q(\xi)| \ll_{n,\xi} X^{-\widehat{w}_n^*(\xi)+\epsilon} \tag{3.16}$$

(see [4, Proposition 3.2]; actually, this proves the left inequalities of (1.7)). Next we define $\tau^*(\xi,\epsilon) := (w_m(\xi)+2\epsilon)/(\widehat{w}_n^*(\xi)-\epsilon)$ and set $X = H(P)^{\tau^*(\xi,\epsilon)}$.

Similarly to the proof of Theorem 2.2 we obtain the variant

$$\begin{aligned}(3.17)\qquad |P(\xi)| \geq H(P)^{-w_m(\xi)-\epsilon} &= H(P)^{\epsilon}H(P)^{-w_m(\xi)-2\epsilon} \\ &= H(P)^{\epsilon}X^{-\widehat{w}_n^*(\xi)+\epsilon}\end{aligned}$$

of (3.7). Observe that the combination of (3.16) and (3.17) implies that $|Q(\xi)| < |P(\xi)|$ and consequently $P \neq Q$, provided that $H(P)$ was chosen large enough. On the other hand, with essentially the argument used to get (3.8), we obtain

$$(3.18)\qquad |P(\xi)| \leq X^{-\widehat{w}_n^*(\xi)+\tilde{\epsilon}}$$

for some $\tilde{\epsilon}$ which depends on ϵ and tends to 0 as $\epsilon \to 0$. By (3.16) and $|Q(\xi)| < |P(\xi)|$ we infer

$$(3.19)\qquad \max\{|P(\xi)|, |Q(\xi)|\} \leq X^{-\widehat{w}_n^*(\xi)+\tilde{\epsilon}},$$

an inequality similar to (3.9).

Now if $m \geq n$, we proceed as in the proof of Theorem 2.2 observing that we may apply Lemma 3.1 here since $P \neq Q$ and both P and Q are irreducible. Indeed, (3.10) holds for exactly the same reason and we get (3.11) with τ replaced by τ^*. This yields the first inequality of (2.6), whereas the inequality $\widehat{w}_n^*(\xi) \leq w_m(\xi)$ is trivially implied by the assumption $m \geq n$.

If $m < n$ we treat the cases $H(P) \leq H(Q)$ and $H(P) > H(Q)$ separately. First consider the case $H(P) \leq H(Q)$ for infinitely many pairs (P, Q) as above. In this case we again prove (2.6). The first inequality of (2.6) is derived as for $m \geq n$, using $H(P) \leq H(Q) \leq X$ instead of (3.10). The other inequality $\widehat{w}_n^*(\xi) \leq w_m(\xi)$ remains to be shown. Assume on the contrary that $w_m(\xi)/\widehat{w}_n^*(\xi) < 1$. Then for sufficiently small ϵ we have $\tau^*(\xi, \epsilon) < 1$ and hence $H(Q) = H(\beta) \leq X < X^{1/\tau^*(\xi,\epsilon)} = H(P)$, a contradiction. This finishes the proof of the case $H(P) \leq H(Q)$.

Now assume $H(P) > H(Q)$ for infinitely many pairs (P, Q) as above. Note that (3.10) does not necessarily hold now as we needed $m \geq n$ for its deduction. It is sufficient to show that (2.5) is false, that is,

$$(3.20)\qquad w_m(\xi) \leq \min\{m + n - 1, w_n^*(\xi)\}.$$

Observe that (3.19) implies

$$(3.21)\qquad \max\{|P(\xi)|, |Q(\xi)|\} \leq H(P)^{-w_m(\xi)+\hat{\epsilon}}$$

for $\hat{\epsilon} = \tilde{\epsilon} \cdot w_m(\xi)/\widehat{w}_n^*(\xi)$, which again tends to 0 as $\epsilon \to 0$. On the other hand, Lemma 3.1 yields

$$(3.22)\qquad \max\{|P(\xi)|, |Q(\xi)|\} \gg_{m,n,\xi} H(P)^{-n}H(Q)^{-m+1} \geq H(P)^{-m-n+1}.$$

The combination of (3.21) and (3.22) gives $w_m(\xi) \leq m + n - 1$. It remains to show that $w_m(\xi) \leq w_n^*(\xi)$. Assume that $w_m(\xi) - w_n^*(\xi) = \rho > 0$. Then

(3.15) would imply, if ϵ were chosen small enough, that

$$|\xi-\beta| \le H(\beta)^{-1}\, X^{-\widehat{w}_n^*(\xi)+\epsilon} = H(Q)^{-1} H(P)^{-w_m(\xi)+\epsilon/\tau(\xi,\epsilon)} < H(Q)^{-w_m(\xi)-1+\epsilon/\tau(\xi,\epsilon)} \le H(Q)^{-w_n^*(\xi)-1-\rho/2},$$

contrary to the definition of $w_n^*(\xi)$ as $H(Q)\to\infty$. Hence (3.20) is established in this case and the proof is finished. ∎

Proof of Theorem 2.7. Let $n \ge 2$ be an integer and ξ be a real transcendental number.

We establish the first assertion. We follow the proof of Wirsing's theorem as given in [4] and keep the notation used therein. By the definition of $\widehat{w}_n$, observe that the inequality $|Q_k(\xi)| \ll H(P_k)^{-n}$ in [4, (3.16)] can be replaced by

$$|Q_k(\xi)| \ll H(P_k)^{-\widehat{w}_n(\xi)+\varepsilon}.$$

The lower bound for $w_n^*(\xi)$ on line -8 of [4, p. 57] then becomes

(3.23)
$$w_n^*(\xi) \ge \min\left\{\widehat{w}_n(\xi), w_n(\xi) - \frac{n-1}{2} + \frac{\widehat{w}_n(\xi)-n}{2}, \frac{w_n(\xi)+1}{2} + \widehat{w}_n(\xi) - n\right\}.$$

Since $w_n(\xi) \ge \widehat{w}_n(\xi)$, this gives

$$w_n^*(\xi) \ge \min\left\{\widehat{w}_n(\xi), \frac{3\widehat{w}_n(\xi)}{2} - n + \frac{1}{2}\right\} = \frac{3\widehat{w}_n(\xi)}{2} - n + \frac{1}{2},$$

by (1.5). Thus, we have established

$$\widehat{w}_n(\xi) \le \frac{2(w_n^*(\xi)+n)-1}{3},$$

as asserted.

Now, we prove (2.8). Inequality (1.8) can be rewritten as

$$\widehat{w}_n(\xi) \ge \frac{(n-1)w_n^*(\xi)}{w_n^*(\xi)-1}.$$

Assuming $w_n(\xi) \le 2n-1$, the smallest of the three terms in the curly brackets in (3.23) is the third one and we eventually get

$$w_n(\xi) \le \frac{2w_n^*(\xi)^2 - 2n - w_n^*(\xi) + 1}{w_n^*(\xi)-1}.$$

Combining this with the lower bound

$$\widehat{w}_n^*(\xi) \ge \frac{w_n(\xi)}{w_n(\xi)-n+1},$$

established in [6], we obtain (2.8). ∎

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References

[1] B. Adamczewski et Y. Bugeaud, *Mesures de transcendance et aspects quantitatifs de la méthode de Thue–Siegel–Roth–Schmidt*, Proc. London Math. Soc. 101 (2010), 1–26.

[2] B. Arbour and D. Roy, *A Gel'fond type criterion in degree two*, Acta Arith. 111 (2004), 97–103.

[3] V. I. Bernik and K. I. Tsishchanka, *Integral polynomials with an overfall of the coefficient values and Wirsing's theorem*, Dokl. Akad. Nauk Belarusi 37 (1993), no. 5, 9–11 (in Russian).

[4] Y. Bugeaud, *Approximation by Algebraic Numbers*, Cambridge Tracts in Math. 160, Cambridge Univ. Press, Cambridge, 2004.

[5] Y. Bugeaud, *Exponents of Diophantine approximation*, in: Dynamics and Analytic Number Theory (Durham, 2014), London Math. Soc. Lecture Note Ser. 437, D. Badziahin et al. (eds.), Cambridge Univ. Press, Cambridge, to appear.

[6] Y. Bugeaud and M. Laurent, *Exponents of Diophantine approximation and Sturmian continued fractions*, Ann. Inst. Fourier (Grenoble) 55 (2005), 773–804.

[7] H. Davenport and W. M. Schmidt, *Approximation to real numbers by quadratic irrationals*, Acta Arith. 13 (1967), 169–176.

[8] H. Davenport and W. M. Schmidt, *Approximation to real numbers by algebraic integers*, Acta Arith. 15 (1969), 393–416.

[9] V. Jarník, *Zum Khintchineschen "Übertragungssatz"*, Trav. Inst. Math. Tbilissi 3 (1938), 193–212.

[10] V. Jarník, *Contribution à la théorie des approximations diophantiennes linéaires et homogènes*, Czechoslovak Math. J. 4 (1954), 330–353 (in Russian).

[11] A. Khintchine, *Über eine Klasse linearer diophantischer Approximationen*, Rend. Circ. Mat. Palermo 50 (1926), 170–195.

[12] W. J. LeVeque, *On Mahler's U-numbers*, J. London Math. Soc. 28 (1953), 220–229.

[13] N. Moshchevitin, *A note on two linear forms*, Acta Arith. 162 (2014), 43–50.

[14] D. Roy, *Approximation simultanée d'un nombre et de son carré*, C. R. Math. Acad. Sci. Paris 336 (2003), 1–6.

[15] D. Roy, *Approximation to real numbers by cubic algebraic integers I*, Proc. London Math. Soc. 88 (2004), 42–62.

[16] D. Roy and D. Zelo, *Measures of algebraic approximation to Markoff extremal numbers*, J. London Math. Soc. 83 (2011), 407–430.

[17] J. Schleischitz, *On the spectrum of Diophantine approximation constants*, Mathematika 62 (2016), 79–100.

[18] W. M. Schmidt, *T-numbers do exist*, in: Symposia Mathematica, Vol. IV, Ist. Naz. Alta Mat., Roma, and Academic Press, London, 1970, 3–26.

[19] W. M. Schmidt and L. Summerer, *Diophantine approximation and parametric geometry of numbers*, Monatsh. Math. 169 (2013), 51–104.

[20] W. M. Schmidt and L. Summerer, *Simultaneous approximation to three numbers*, Moscow J. Combin. Number Theory 3 (2013), no. 1, 84–107.

[21] K. I. Tsishchanka, *On approximation of real numbers by algebraic numbers of bounded degree*, J. Number Theory 123 (2007), 290–314.
[22] M. Waldschmidt, *Recent advances in Diophantine approximation*, in: Number Theory, Analysis and Geometry, Springer, New York, 2012, 659–704.
[23] E. Wirsing, *Approximation mit algebraischen Zahlen beschränkten Grades*, J. Reine Angew. Math. 206 (1961), 67–77.

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