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SOME CONGRUENCES FOR SCHRÖDER TYPE POLYNOMIALS

BY

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Abstract. The nth Schröder number is given by $S_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}$. Motivated by these numbers, for any positive integer α we introduce the polynomials

$$S_n^{(\alpha)}(x)=\sum_{k=0}^n\binom{n}{k}^\alpha\binom{n+k}{k}^\alpha\frac{x^k}{(k+1)^\alpha}.$$

We prove that for any positive integers r, α, odd prime p and any integer m not divisible by p, and for $\varepsilon=\pm1$,

$$\sum_{k=1}^{p-1}\varepsilon^k(2k+1)S_k^{(2\alpha-1)}(m)^r\equiv 0 \pmod p,$$
$$\sum_{k=1}^{p-1}\varepsilon^k(2k+1)S_k^{(2\alpha)}(m)^r\equiv -2^r \pmod p.$$

1. Introduction. In combinatorics, the nth *Schröder number* is given by

$$S_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}.$$

It equals the number of paths from $(0,0)$ to (n,n), using only steps $(1,0)$, $(0,1)$ and $(1,1)$, that do not rise above the line $y=x$. For more information on the Schröder numbers, one may refer to [8]. Motivated by these numbers, for any positive integer α we introduce the polynomials

$$S_n^{(\alpha)}(x)=\sum_{k=0}^n\binom{n}{k}^\alpha\binom{n+k}{k}^\alpha\frac{x^k}{(k+1)^\alpha}.$$

Clearly, $S_n^{(1)}(1)=S_n$.

Some amazing arithmetic properties of the Schröder numbers have been studied by Z.-W. Sun. For example, Sun [9] proved that if p is an odd prime

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and m is an integer not divisible by p, then

$$\sum_{k=1}^{p-1} \frac{S_k}{m^k} \equiv \frac{m^2-6m+1}{2m}\left(1-\left(\frac{m^2-6m+1}{p}\right)\right) \pmod{p},$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Recently, Pan [7] introduced the generalized Apéry polynomials defined by

$$A_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k}^{\alpha} \binom{n+k}{k}^{\alpha} x^k,$$

where α is a positive integer. He proved that for any positive integers m and α, and for $\varepsilon = \pm 1$,

$$\sum_{k=0}^{n-1} \varepsilon^k (2k+1) A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n}, \tag{1.1}$$

and thus he confirmed the related conjectures by Sun [10, 11] and Guo and Zeng [6]. Chen and Guo [3] have generalized (1.1) by introducing multi-variable Schmidt polynomials. Note that the values $A_n^{(2)}(1)$ are called the *Apéry numbers*; they play an important role in Apéry's ingenious proof [2] of the irrationality of $\zeta(3)$.

Congruences for some similar numbers and polynomials have been widely studied by several authors: see, for example, Ahlgren and Ono [1], Gessel [4], Guo [5], Guo and Zeng [6] and Sun [10, 11].

The aim of this paper is to prove the following theorem:

Theorem 1.1. *Let r and α be any positive integers and p be an odd prime. Then, for $\varepsilon = \pm 1$,*

$$\sum_{k=0}^{p-1} \varepsilon^k (2k+1) S_k^{(\alpha)}(x)^r \equiv 1 - \left(1 + (-1)^{\alpha} x^{p-1}\right)^r \pmod{p}. \tag{1.2}$$

Letting $x = m$ be an integer not divisible by p in (1.2) and then noting that $m^{p-1} \equiv 1 \pmod{p}$ for $p \nmid m$ and $S_0^{(\alpha)}(m) = 1$, we get the following two congruences:

Corollary 1.2. *Let r and α be any positive integers, p be an odd prime, and m be an integer not divisible by p. Then, for $\varepsilon = \pm 1$,*

$$\sum_{k=1}^{p-1} \varepsilon^k (2k+1) S_k^{(2\alpha-1)}(m)^r \equiv 0 \pmod{p},$$

$$\sum_{k=1}^{p-1} \varepsilon^k (2k+1) S_k^{(2\alpha)}(m)^r \equiv -2^r \pmod{p}.$$

We shall prove Theorem 1.1 for $\alpha = 1$ in Section 2 and for $\alpha \geq 2$ in Section 3. The proof is based on some combinatorial identities and mathematical induction.

2. Proof of Theorem 1.1 for $\alpha = 1$. We first prove the case $\varepsilon = 1$. Note the following identity [5, (2.5)]:

$$\binom{k}{i}\binom{k+i}{i}\binom{k}{j}\binom{k+j}{j} = \sum_{r=0}^{i}\binom{i+j}{i}\binom{j}{i-r}\binom{j+r}{r}\binom{k}{j+r}\binom{k+j+r}{j+r} = \sum_{s=\max\{i,j\}}^{i+j}\binom{i+j}{s}\binom{s}{i}\binom{s}{j}\binom{k}{s}\binom{k+s}{s}. \tag{2.1}$$

Let $f(i,j,s) = \binom{i+j}{s}\binom{s}{i}\binom{s}{j}$. Using (2.1) $r-1$ times, we obtain

$$\begin{aligned}
&\binom{k}{i_1}\binom{k+i_1}{i_1}\binom{k}{i_2}\binom{k+i_2}{i_2}\cdots\binom{k}{i_r}\binom{k+i_r}{i_r} \\
&= \sum_{s_1} f(i_1,i_2,s_1)\binom{k}{s_1}\binom{k+s_1}{s_1}\binom{k}{i_3}\binom{k+i_3}{i_3}\cdots\binom{k}{i_r}\binom{k+i_r}{i_r} \\
&= \sum_{s_1,s_2} f(i_1,i_2,s_1)f(s_1,i_3,s_2) \\
&\qquad \times \binom{k}{s_2}\binom{k+s_2}{s_2}\binom{k}{i_4}\binom{k+i_4}{i_4}\cdots\binom{k}{i_r}\binom{k+i_r}{i_r} \\
&= \cdots \\
&= \sum_{s_1,\dots,s_{r-1}} f(i_1,i_2,s_1)\cdots f(s_{r-3},i_{r-1},s_{r-2})f(s_{r-2},i_r,s_{r-1}) \\
&\qquad \times \binom{k}{s_{r-1}}\binom{k+s_{r-1}}{s_{r-1}},
\end{aligned} \tag{2.2}$$

where we have the following relationships (replace i_1 by s_0):

$$\max\{s_{j-1}, i_{j+1}\} \leq s_j \leq s_{j-1} + i_{j+1} \quad \text{for } 1 \leq j \leq r-1. \tag{2.3}$$

It is clear that (2.3) implies

$$s_1 \leq \cdots \leq s_{r-1}, \tag{2.4}$$

and

$$i_1, \dots, i_{j+1} \leq s_j \quad \text{for } 1 \leq j \leq r-1. \tag{2.5}$$

By (2.2), we have

$$\begin{aligned}\sum_{k=0}^{p-1}&(2k+1)S_k^{(1)}(x)^r\\ &=\sum_{k=0}^{p-1}(2k+1)\sum_{0\le i_1,\ldots,i_r\le k}\binom{k}{i_1}\binom{k+i_1}{i_1}\cdots\binom{k}{i_r}\binom{k+i_r}{i_r}\\ &\qquad\times\frac{x^{i_1+\cdots+i_r}}{(i_1+1)\cdots(i_r+1)}\\ &=\sum_{k=0}^{p-1}(2k+1)\sum_{0\le i_1,\ldots,i_r\le k}\sum_{s_1,\ldots,s_{r-1}}f(i_1,i_2,s_1)\cdots f(s_{r-3},i_{r-1},s_{r-2})\\ &\qquad\times f(s_{r-2},i_r,s_{r-1})\binom{k}{s_{r-1}}\binom{k+s_{r-1}}{s_{r-1}}\frac{x^{i_1+\cdots+i_r}}{(i_1+1)\cdots(i_r+1)}.\end{aligned}$$

Exchanging the summation order, we get

$$\begin{aligned}\sum_{k=0}^{p-1}(2k+1)S_k^{(1)}(x)^r&=\sum_{0\le i_1,\ldots,i_r\le p-1}\sum_{s_1,\ldots,s_{r-1}}f(i_1,i_2,s_1)\cdots\\ &\qquad\times f(s_{r-3},i_{r-1},s_{r-2})f(s_{r-2},i_r,s_{r-1})\\ &\qquad\times\frac{x^{i_1+\cdots+i_r}}{(i_1+1)\cdots(i_r+1)}\sum_{k=\max\{i_1,\ldots,i_r\}}^{p-1}(2k+1)\binom{k}{s_{r-1}}\binom{k+s_{r-1}}{s_{r-1}}.\end{aligned}\tag{2.6}$$

Furthermore, noting that, by (2.5), $i_1,\ldots,i_r\le s_{r-1}$, and using the identity

$$\sum_{k=s}^{n-1}(2k+1)\binom{k}{s}\binom{k+s}{s}=n\binom{n+s}{s}\binom{n}{s+1},\tag{2.7}$$

which can be easily proved by induction on n, we may simplify (2.6) as

$$\begin{aligned}\sum_{k=0}^{p-1}&(2k+1)S_k^{(1)}(x)^r\\ &=p\sum_{0\le i_1,\ldots,i_r\le p-1}\sum_{s_1,\ldots,s_{r-1}}f(i_1,i_2,s_1)\cdots f(s_{r-3},i_{r-1},s_{r-2})\\ &\qquad\times f(s_{r-2},i_r,s_{r-1})\binom{p+s_{r-1}}{s_{r-1}}\binom{p}{s_{r-1}+1}\frac{x^{i_1+\cdots+i_r}}{(i_1+1)\cdots(i_r+1)}.\end{aligned}\tag{2.8}$$

We shall now prove Theorem 1.1 for $\alpha=1$ by induction on r.

When $r=1$, (2.8) reduces to

$$\sum_{k=0}^{p-1}(2k+1)S_k^{(1)}(x) = p\sum_{i=0}^{p-1}\binom{p+i}{i}\binom{p}{i+1}\frac{x^i}{i+1}. \tag{2.9}$$

Since the summands on the right-hand side of (2.9) are congruent to 0 mod p for $i \neq p-1$, we have

$$\sum_{k=0}^{p-1}(2k+1)S_k^{(1)}(x) \equiv x^{p-1} \pmod p, \tag{2.10}$$

where we have used the fact that $\binom{2p-1}{p-1} \equiv 1 \pmod p$.

When $r \geq 2$, by (2.5) we have

$$\frac{p}{(i_1+1)\cdots(i_r+1)} \equiv 0 \pmod p \quad \text{ for } s_{r-1} \leq p-2.$$

It follows from (2.8) that

$$\sum_{k=0}^{p-1}(2k+1)S_k^{(1)}(x)^r \equiv p\sum_{0\leq i_1,\ldots,i_r\leq p-1}\sum_{s_1,\ldots,s_{r-2}} \frac{f(i_1,i_2,s_1)\cdots f(s_{r-3},i_{r-1},s_{r-2})f(s_{r-2},i_r,p-1)}{(i_1+1)\cdots(i_r+1)} \times x^{i_1+\cdots+i_r} \pmod p. \tag{2.11}$$

Now we need the following lemma to continue the proof.

Lemma 2.1. *If* $(s_{r-2}, i_r) \notin \{(0,p-1),(p-1,0),(p-1,p-1)\}$, *then*

$$p\frac{f(i_1,i_2,s_1)\cdots f(s_{r-3},i_{r-1},s_{r-2})f(s_{r-2},i_r,p-1)}{(i_1+1)\cdots(i_r+1)} \equiv 0 \pmod p. \tag{2.12}$$

Before proving this lemma, let us draw some conclusions from it. If $(s_{r-2}, i_r) = (0, p-1)$, by (2.4)–(2.5) we have $s_1 = \cdots = s_{r-3} = i_1 = \cdots = i_{r-1} = 0$. Applying (2.12) to the right-hand side of (2.11) yields

$$\sum_{k=0}^{p-1}(2k+1)S_k^{(1)}(x)^r \equiv x^{p-1} + \left(p + x^{p-1}\binom{2p-2}{p-1}\right) \times \sum_{0\leq i_1,\ldots,i_{r-1}\leq p-1}\sum_{s_1,\ldots,s_{r-3}} \frac{f(i_1,i_2,s_1)\cdots f(s_{r-3},i_{r-1},p-1)}{(i_1+1)\cdots(i_{r-1}+1)} \times x^{i_1+\cdots+i_{r-1}} \pmod p.$$

Noting that

$$\frac{1}{p}\binom{2p-2}{p-1} \equiv -1 \pmod p$$

and using (2.11), we get the recurrence

$$(2.13)\qquad \sum_{k=0}^{p-1}(2k+1)S_k^{(1)}(x)^r \equiv x^{p-1}+(1-x^{p-1})\sum_{k=0}^{p-1}(2k+1)S_k^{(1)}(x)^{r-1} \pmod p.$$

Then the proof of (1.2) for $\alpha=1$ and $\varepsilon=1$ directly follows from (2.10) and (2.13) by induction on r.

If $\varepsilon=-1$, we need to apply the following identity instead of (2.7):

$$\sum_{k=s}^{n-1}(-1)^k(2k+1)\binom{k}{s}\binom{k+s}{s}=(-1)^{n-1}n\binom{n+s}{s}\binom{n-1}{s}.$$

The rest of the proof is similar to that for $\varepsilon=1$, and we omit the details.

Proof of Lemma 2.1. If $1\le s_{r-2}\le p-2$, then by (2.5) we see that $\binom{s_{r-2}+i_r}{p-1}/(i_1+1)\cdots(i_r+1)$ is a p-adic integer, so that

$$p\frac{f(s_{r-2},i_r,p-1)}{(i_1+1)\cdots(i_r+1)}\equiv 0 \pmod p,$$

which implies that LHS (2.12) $\equiv 0 \pmod p$.

If $s_{r-2}=0$, then from (2.3) and $s_{r-1}=p-1$ we have $i_r=p-1$.

If $s_{r-2}=p-1$ and $1\le i_r\le p-2$, then

$$\frac{f(p-1,i_r,p-1)}{i_r+1}\equiv 0 \pmod p$$

and

$$(2.14)\qquad p\frac{f(i_1,i_2,s_1)\cdots f(s_{r-4},i_{r-2},s_{r-3})f(s_{r-3},i_{r-1},p-1)}{(i_1+1)\cdots(i_{r-1}+1)}$$

is a p-adic integer, which can be proved as follows. If $s_{r-3}\le p-2$, then by (2.5), $p/(i_1+1)\cdots(i_{r-1}+1)$ is a p-adic integer, and hence so is (2.14). If $s_{r-3}=p-1$, then $f(p-1,i_{r-1},p-1)/(i_{r-1}+1)$ is a p-adic integer and we can prove that (2.14) is a p-adic integer by induction on r. In conclusion, for $s_{r-2}=p-1$ and $1\le i_r\le p-2$, we have LHS (2.12) $\equiv 0 \pmod p$. This completes the proof. ∎

3. Proof of Theorem 1.1 for $\alpha\ge 2$. Since the proof for $\varepsilon=-1$ is similar to that for $\varepsilon=1$, we shall only deal with the case $\varepsilon=1$. From (2.2), we have

$$(3.1)\qquad \binom{k}{i_1}^{\alpha}\binom{k+i_1}{i_1}^{\alpha}\cdots\binom{k}{i_r}^{\alpha}\binom{k+i_r}{i_r}^{\alpha} = \sum_{\substack{s_{1,1},\dots,s_{1,r-1}\\ \cdots\\ s_{\alpha,1},\dots,s_{\alpha,r-1}}} \prod_{j=1}^{\alpha} f(i_1,i_2,s_{j,1})\cdots\prod_{j=1}^{\alpha} f(s_{j,r-2},i_r,s_{j,r-1}) \times \prod_{j=1}^{\alpha}\binom{k}{s_{j,r-1}}\binom{k+s_{j,r-1}}{s_{j,r-1}},$$

where

$$\max\{s_{j,d-1}, i_{d+1}\} \le s_{j,d} \le s_{j,d-1}+i_{d+1}$$

for $1 \le j \le \alpha$ and $1 \le d \le r-1$ (replace i_1 by $s_{j,0}$). Applying (2.2) again, we obtain

$$(3.2)\qquad \prod_{j=1}^{\alpha}\binom{k}{s_{j,r-1}}\binom{k+s_{j,r-1}}{s_{j,r-1}} = \sum_{t_1,\dots,t_{\alpha-1}} f(s_{1,r-1},s_{2,r-1},t_1)f(t_1,s_{3,r-1},t_2)\cdots f(t_{\alpha-2},s_{\alpha,r-1},t_{\alpha-1}) \times \binom{k}{t_{\alpha-1}}\binom{k+t_{\alpha-1}}{t_{\alpha-1}}.$$

Substituting (3.2) into (3.1), then using (2.7) and the same idea as in the previous section, we get

$$(3.3)\qquad \sum_{k=0}^{p-1}(2k+1)S_k^{(\alpha)}(x)^r = p\sum_{0\le i_1,\dots,i_r\le p-1}\ \sum_{\substack{s_{1,1},\dots,s_{1,r-1}\\ \cdots\\ s_{\alpha,1},\dots,s_{\alpha,r-1}}}\ \sum_{t_1,\dots,t_{\alpha-1}} \prod_{j=1}^{\alpha} f(i_1,i_2,s_{j,1})\cdots\prod_{j=1}^{\alpha} f(s_{j,r-2},i_r,s_{j,r-1}) \times f(s_{1,r-1},s_{2,r-1},t_1)f(t_1,s_{3,r-1},t_2)\cdots f(t_{\alpha-2},s_{\alpha,r-1},t_{\alpha-1}) \times \binom{p+t_{\alpha-1}}{t_{\alpha-1}}\binom{p}{t_{\alpha-1}+1}\frac{x^{i_1+\cdots+i_r}}{(i_1+1)^{\alpha}\cdots(i_r+1)^{\alpha}}.$$

By (2.5), we obtain $i_1,\dots,i_r \le s_{j,r-1}$ for $1 \le j \le \alpha$. So we must have $s_{j,r-1} = p-1$ for $1 \le j \le \alpha$, unless

$$\frac{p}{(i_1+1)^{\alpha}\cdots(i_r+1)^{\alpha}} \equiv 0 \pmod p.$$

Furthermore, from (2.3), $t_1 \le \cdots \le t_{\alpha-1} \le p-1$ and $t_j \ge s_{j+1,r-1}$ for $1 \le j \le \alpha-1$, we conclude that $t_j = p-1$ for $1 \le j \le \alpha-1$, and so

$$f(s_{1,r-1}, s_{2,r-1}, t_1) f(t_1, s_{3,r-1}, t_2) \cdots f(t_{\alpha-2}, s_{\alpha,r-1}, t_{\alpha-1}) \times \binom{p+t_{\alpha-1}}{t_{\alpha-1}} \binom{p}{t_{\alpha-1}+1} = \binom{2p-2}{p-1}^{\alpha-1} \binom{2p-1}{p-1}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\sum_{k=0}^{p-1} (2k+1) S_k^{(\alpha)}(x)^r \equiv p \binom{2p-2}{p-1}^{\alpha-1} \sum_{0 \le i_1, \ldots, i_r \le p-1} \sum_{\substack{s_{1,1}, \ldots, s_{1,r-2} \\ \cdots \\ s_{\alpha,1}, \ldots, s_{\alpha,r-2}}} \frac{\prod_{j=1}^{\alpha} f(i_1, i_2, s_{j,1}) \cdots \prod_{j=1}^{\alpha} f(s_{j,r-2}, i_r, p-1)}{(i_1+1)^\alpha \cdots (i_r+1)^\alpha} x^{i_1+\cdots+i_r} \pmod{p}. \tag{3.5}$$

Note that $\binom{2p-2}{p-1} \equiv 0 \pmod{p}$. Now we will generalize Lemma 2.1 to continue the proof; the proof is similar to that of Lemma 2.1 and we omit it.

Lemma 3.1. *If* $(s_{1,r-2}, \ldots, s_{\alpha,r-2}, i_r)$ *is not in*

$$\{(0, \ldots, 0, p-1), (p-1, \ldots, p-1, 0), (p-1, \ldots, p-1, p-1)\},$$

then

$$\frac{\prod_{j=1}^{\alpha} f(i_1, i_2, s_{j,1}) \cdots \prod_{j=1}^{\alpha} f(s_{j,r-2}, i_r, p-1)}{(i_1+1)^\alpha \cdots (i_r+1)^\alpha} \times p \binom{2p-2}{p-1}^{\alpha-1} \equiv 0 \pmod{p}. \tag{3.6}$$

Applying (3.6) to the right-hand side of (3.5), we finally get

$$\sum_{k=0}^{p-1} (2k+1) S_k^{(\alpha)}(x)^r \equiv (-1)^{\alpha-1} x^{p-1} + (1 + (-1)^\alpha x^{p-1}) \sum_{k=0}^{p-1} (2k+1) S_k^{(\alpha)}(x)^{r-1} \pmod{p}. \tag{3.7}$$

Then the conclusion for $\alpha \ge 2$ directly follows from (3.7) by induction on r.

Now we only need to prove the case $r = 1$. Letting $r = 1$ in (3.5) and using the same idea as in the previous section, we can easily get

$$\sum_{k=0}^{p-1} (2k+1) S_k^{(\alpha)}(x) \equiv (-1)^{\alpha+1} x^{p-1} \pmod{p}.$$

This completes the proof of Theorem 1.1 for $\alpha \ge 2$.

REFERENCES

[1] S. Ahlgren and K. Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences*, J. Reine Angew. Math. 518 (2000), 187–212.
[2] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque 61 (1979), 11–13.
[3] Q.-F. Chen and V. J. W. Guo, *On the divisibility of sums involving powers of multi-variable Schmidt polynomials*, arXiv:1412.5734 (2014).
[4] I. Gessel, *Some congruences for Apéry numbers*, J. Number Theory 14 (1982), 362–368.
[5] V. J. W. Guo, *Some congruences involving powers of Legendre polynomials*, Integral Transforms Spec. Funct. 26 (2015), 660–666.
[6] V. J. W. Guo and J. Zeng, *Proof of some conjectures of Z.-W. Sun on congruences for Apéry polynomials*, J. Number Theory 132 (2012), 1731–1740.
[7] H. Pan, *On divisibility of sums of Apéry polynomials*, J. Number Theory 143 (2014), 214–223.
[8] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Univ. Press, Cambridge, 1999.
[9] Z.-W. Sun, *On Delannoy numbers and Schröder numbers*, J. Number Theory 131 (2011), 2387–2397.
[10] Z.-W. Sun, *On sums of Apéry polynomials and related congruences*, J. Number Theory 132 (2012), 2673–2699.
[11] Z.-W. Sun, *Congruences involving generalized trinomial coeffcients*, Sci. China Math. 57 (2014), 1375–1400.

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