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On Waring's problem for intermediate powers

by

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1. Introduction. Conforming to tradition, we denote by $G(k)$ the least number s such that every sufficiently large natural number is the sum of at most s positive integral kth powers. In this note we obtain new bounds for $G(k)$ by exploiting recent progress concerning Vinogradov's mean value theorem (see [8] and [1]).

THEOREM 1.1. *When* $7 \le k \le 16$, *one has* $G(k) \le H(k)$, *where* $H(k)$ *is defined by means of Table* 1.

Table 1. Upper bounds for $G(k)$ when $7 \le k \le 16$

k	7	8	9	10	11	12	13	14	15	16
$H(k)$	31	39	47	55	63	72	81	90	99	108

For comparison, Vaughan and Wooley [4, 5, 6] have obtained the bounds $G(7) \le 33$, $G(8) \le 42$, $G(9) \le 50$, $G(10) \le 59$, $G(11) \le 67$, $G(12) \le 76$, $G(13) \le 84$, $G(14) \le 92$, $G(15) \le 100$, $G(16) \le 109$, in work spanning the 1990s. We note in particular that our new bound $G(8) \le 39$ makes appreciable progress towards the conjectured conclusion $G(8) = 32$ that now seems only just beyond our grasp.

Our proof of Theorem 1.1 utilises a combination of the powerful estimates for mean values restricted to minor arcs recently made available in our work [8] concerning the asymptotic formula in Waring's problem, together with the progress on Vinogradov's mean value theorem due to Bourgain, Demeter and Guth [1]. In applications, this mean value estimate has the potential to deliver bounds considerably sharper than corresponding pointwise bounds. For intermediate values of k, these estimates combine with

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earlier mean value estimates for smooth Weyl sums due to Vaughan and the author [6] to deliver satisfactory estimates for mixed mean values involving both classical and smooth Weyl sums. This we describe in §3. The corresponding major arc estimates, which we handle in §4, are familiar territory for experts in the subject, and pose no new challenges. For larger values of k, the relative strength of minor arc estimates available for smooth Weyl sums proves superior to our use here of classical Weyl sums, and so no improvements are made available for $k \geq 17$.

Throughout, the letter ε will denote a positive number. We adopt the convention that whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. In addition, we use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on k and ε, as well as other ambient parameters apparent from the context. Finally, we write $e(z)$ for $e^{2\pi i z}$, and $[\theta]$ for the greatest integer not exceeding θ.

2. Preliminaries. Our proof of Theorem 1.1 proceeds by means of the circle method. We take the opportunity in this section of outlining our basic approach, introducing notation en route that underpins the discussion of subsequent sections. Throughout, we let k denote a fixed integer with $7 \leq k \leq 16$. We consider a positive number η sufficiently small in terms of k, and let n be a positive integer sufficiently large in terms of both k and η. Next, write $P = n^{1/k}$, and consider positive integers t and u to be fixed in due course. Define the set of smooth numbers $\mathcal{A}_\eta(P)$ by

$$\mathcal{A}_\eta(P) = \{n \in [1, P] \cap \mathbb{Z} : p \,|\, n \text{ and } p \text{ prime} \Rightarrow p \leq P^\eta\}.$$

We consider the number $R(n)$ of representations of n in the shape

$$n = x_1^k + \cdots + x_t^k + y_1^k + \cdots + y_u^k, \tag{2.1}$$

with $1 \leq x_i \leq P$ $(1 \leq i \leq t)$ and $y_j \in \mathcal{A}_\eta(P)$ $(1 \leq j \leq u)$. We seek to show that for appropriate choices of t and u, one has $R(n) \gg n^{(t+u)/k-1}$, whence in particular $R(n) \geq 1$. Hence, whenever n is a sufficiently large positive integer, it follows that n possesses a representation as the sum of at most $t + u$ positive integral kth powers, whence $G(k) \leq t + u$.

We define

$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k) \quad \text{and} \quad g(\alpha) = \sum_{x \in \mathcal{A}_\eta(P)} e(\alpha x^k).$$

When $\mathfrak{B} \subseteq [0, 1)$, we set

$$R(n; \mathfrak{B}) = \int_{\mathfrak{B}} f(\alpha)^t g(\alpha)^u e(-n\alpha)\, d\alpha. \tag{2.2}$$

Then it follows from (2.1) via orthogonality that $R(n) = R(n; [0, 1))$.

In order to make further progress, we must define a Hardy–Littlewood dissection of the unit interval. Let $\mathfrak{m}$ denote the set of real numbers $\alpha \in [0,1)$ satisfying the property that, whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q)=1$ and

$$|q\alpha - a| \le (2k)^{-1}P^{1-k}$$

then one has $q > P$. The set of major arcs $\mathfrak{M}$ corresponding to this set of minor arcs $\mathfrak{m}$ is then defined by setting $\mathfrak{M} = [0,1) \setminus \mathfrak{m}$. It is apparent that $\mathfrak{M}$ is the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \le (2k)^{-1}P^{1-k}\},$$

with $0 \le a \le q \le P$ and $(a,q)=1$.

In the next section, we establish under appropriate conditions on t and u that one has $R(n;\mathfrak{m}) = o(P^{t+u-k})$, whilst in §4 we confirm under the same conditions that $R(n;\mathfrak{M}) \gg P^{t+u-k}$. Since $[0,1) = \mathfrak{M} \cup \mathfrak{m}$, these conclusions combine to deliver the anticipated lower bound $R(n;[0,1)) \gg n^{(t+u)/k-1}$, achieving the goal advertised in the opening paragraph of this section.

3. The minor arc contribution. We now set about establishing that $R(n;\mathfrak{m}) = o(P^{t+u-k})$. This we achieve by combining two mean value estimates, the first of which concerns classical Weyl sums.

Lemma 3.1. *Whenever $w \ge k(k+1)$, one has*

$$\int_{\mathfrak{m}} |f(\alpha)|^w \, d\alpha \ll P^{w-k-1+\varepsilon}.$$

Proof. Denote by $J_{s,k}(X)$ the number of integral solutions of the system of equations

$$\sum_{i=1}^{s} (x_i^j - y_i^j) = 0 \qquad (1 \le j \le k),$$

with $1 \le x_i, y_i \le X$ $(1 \le i \le s)$. Then it follows from [8, Theorem 2.1] that

$$\int_{\mathfrak{m}} |f(\alpha)|^{2u} \, d\alpha \ll P^{\frac{1}{2}k(k-1)-1}(\log P)^{2u+1} J_{u,k}(P). \tag{3.1}$$

However, by reference to [1, Theorem 1.1], we find that whenever $2u \ge k(k+1)$, then $J_{u,k}(P) \ll P^{2u-k(k+1)/2+\varepsilon}$. The desired conclusion follows by substituting this estimate into (3.1). ∎

We also employ mean value estimates for smooth Weyl sums. We say that the positive real number $\lambda_{w,k}$ is *permissible* when, for each $\varepsilon > 0$, whenever η is a sufficiently small positive number, then

$$\int_0^1 |g(\alpha)|^{2w} \, d\alpha \ll P^{\lambda_{w,k}+\varepsilon}. \tag{3.2}$$

By reference to the tables of exponents in [6, §§9–18], we find that the exponents $\lambda_{w,k}$ and $\lambda_{w+1,k}$ recorded in Table 2 are permissible. We are at liberty in what follows to assume that η has been chosen small enough that the estimate (3.2) holds for all pairs (k,w) and $(k,w+1)$ occurring in Table 2.

Table 2. Choice of exponents for $7 \le k \le 16$

k	w	$\lambda_{w,k}$	$\lambda_{w+1,k}$	t	u	δ^{-1}	r	$[U]$
7	14	21.1139297	23.0528848	5	26	1267	17	47
8	18	28.0833353	30.0473193	5	34	1111	21	58
9	21	33.1033373	35.0727119	7	40	534	25	86
10	25	40.0895832	42.0677228	9	46	1792	30	128
11	27	43.1274069	45.1020502	13	50	2959	34	375
12	32	52.0919461	54.0752481	13	59	546	38	314
13	36	59.0849135	61.0698015	13	68	823	42	289
14	40	66.0795485	68.0657585	14	76	620	46	342
15	44	73.0747403	75.0620643	16	83	417	50	525
16	47	78.0829008	80.0711728	19	89	519	55	1780

We combine these mean value estimates via Hölder's inequality to obtain the bounds contained in the following lemma.

LEMMA 3.2. *Let k, t, u and δ be given as in Table 2. Then*

$$\int_{\mathfrak{m}} |f(\alpha)^t g(\alpha)^u|\, d\alpha \ll P^{t+u-k-\delta}.$$

Proof. Let w be as in Table 2. Then by Hölder's inequality, the integral in question is bounded above by

$$\Big(\int_{\mathfrak{m}} |f(\alpha)|^{k(k+1)}\, d\alpha\Big)^{\omega} \Big(\int_0^1 |g(\alpha)|^{2w}\, d\alpha\Big)^{\phi_1} \Big(\int_0^1 |g(\alpha)|^{2w+2}\, d\alpha\Big)^{\phi_2}, \tag{3.3}$$

where

$$\omega = \frac{t}{k(k+1)}, \qquad \phi_1 = (1-\omega)(w+1) - u/2, \qquad \phi_2 = u/2 - (1-\omega)w.$$

Here, in order to verify that this is indeed a valid application of Hölder's inequality, it may be useful to note that for each value of k in question, one has $w = \big[\frac{1}{2}u/(1-\omega)\big]$.

By applying Lemma 3.1 together with (3.2) within (3.3), we infer that

$$\begin{aligned} \int_{\mathfrak{m}} |f(\alpha)^t g(\alpha)^u|\, d\alpha &\ll P^{\varepsilon}(P^{k(k+1)-k-1})^{\omega}(P^{\lambda_{w,k}})^{\phi_1}(P^{\lambda_{w+1,k}})^{\phi_2} \\ &\ll P^{t+u-k+\Delta+\varepsilon}, \end{aligned} \tag{3.4}$$

where

$$\Delta = \phi_1 \Delta_w + \phi_2 \Delta_{w+1} - \omega,$$

in which

$$\Delta_v = \lambda_{v,k} - 2v + k \qquad (v = w, w+1).$$

By reference to Table 2, one verifies that whenever $\varepsilon > 0$ is sufficiently small, one has $\Delta + \varepsilon < -\delta$. The upper bound claimed in the statement of the lemma therefore follows for each k in question from (3.4). ∎

An application of the triangle inequality leads from (2.2) via Lemma 3.2 to the bound

$$R(n; \mathfrak{m}) = o(P^{t+u-k}), \tag{3.5}$$

heralded at the opening of this section.

4. The major arc contribution and the proof of Theorem 1.1. Our goal in this section is the proof of the lower bound $R(n; \mathfrak{M}) \gg P^{t+u-k}$. Experts will recognise the argument here to be routine, though not directly accessible from the literature. We consequently provide a reasonably complete proof. Our task is made easier by the presence of a relatively large number of classical Weyl sums in the integral (2.2). We require an auxiliary set of major arcs. Let $W = \log\log P$, and define $\mathfrak{N}$ to be the union of the intervals

$$\mathfrak{N}(q, a) = \{\alpha \in [0,1) : |\alpha - a/q| \le WP^{-k}\},$$

with $0 \le a \le q \le W$ and $(a, q) = 1$.

We recall from [2, Lemma 5.1] that whenever $k \ge 3$ and $s \ge k+2$, one has

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |f(\alpha)|^s \, d\alpha \ll W^{\varepsilon - 1/k} P^{s-k}. \tag{4.1}$$

Moreover, by reference to the tables of [6, §§9–18], in combination with the discussion concluding [6, §8] associated with the process D^s therein, one finds that, with r defined as in Table 2, one has

$$\int_0^1 |g(\alpha)|^{2r} \, d\alpha \ll P^{2r-k}. \tag{4.2}$$

An application of Hölder's inequality therefore leads from (2.2) to the bound

$$R(n; \mathfrak{M}\setminus\mathfrak{N}) \le \Big(\int_{\mathfrak{M}\setminus\mathfrak{N}} |f(\alpha)|^{k+4} \, d\alpha \Big)^{t/(k+4)} \Big(\int_0^1 |g(\alpha)|^U \, d\alpha \Big)^{1-t/(k+4)}, \tag{4.3}$$

where $U = u/(1 - t/(k+4))$. Observe here that for $7 \le k \le 16$, it follows from Table 2 that $t < k+4$. Also, a modicum of computation reveals that in each case, one has $U > 2r$. Indeed, there is ample room to spare in the latter inequality, as is evident from Table 2. By importing (4.1) and (4.2) into (4.3), we thus discern that

$$R(n;\mathfrak{M}\setminus\mathfrak{N}) \ll W^{-t/(k+4)^2}(P^4)^{t/(k+4)}(P^{U-k})^{1-t/(k+4)} \ll P^{t+u-k}(\log W)^{-1}.$$

By combining this estimate with (3.5), we may conclude thus far that

$$R(n) = R(n;\mathfrak{N}) + O(P^{t+u-k}(\log W)^{-1}). \tag{4.4}$$

The analysis of the contribution arising from the major arcs $\mathfrak{N}$ is routine. Define

$$S(q,a) = \sum_{r=1}^{q} e(ar^k/q) \quad \text{and} \quad v(\beta) = \int_0^P e(\beta\gamma^k)\,d\gamma.$$

Standard arguments (see [2, Lemma 5.4] and [7, Lemma 8.5]) show that there is a positive number ρ having the property that whenever $\alpha\in\mathfrak{N}(q,a)\subseteq\mathfrak{N}$, one has

$$g(\alpha) - \rho q^{-1}S(q,a)v(\alpha-a/q) \ll P(\log P)^{-1/2}.$$

Under the same conditions, the relation

$$f(\alpha) - q^{-1}S(q,a)v(\alpha-a/q) \ll \log P$$

is immediate from [3, Theorem 4.1]. Thus we find that when $\alpha\in\mathfrak{N}(q,a)\subseteq\mathfrak{N}$, one has

$$f(\alpha)^t g(\alpha)^u - \rho^u(q^{-1}S(q,a)v(\alpha-a/q))^{t+u} \ll P^{t+u}(\log P)^{-1/2}.$$

Integrating over $\mathfrak{N}$, we infer that

(4.5)

$$\int_{\mathfrak{N}} f(\alpha)^t g(\alpha)^u e(-n\alpha)\,d\alpha = \rho^u\mathfrak{S}(n;W)\mathfrak{J}(n;W) + O(P^{t+u-k}(\log P)^{-1/3}),$$

where

$$\mathfrak{S}(n;W) = \sum_{1\le q\le W}\sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S(q,a))^{t+u}e(-na/q),$$

$$\mathfrak{J}(n;W) = \int_{-WP^{-k}}^{WP^{-k}} v(\beta)^{t+u}e(-\beta n)\,d\beta.$$

A comparison with classical singular series and integrals conveys us from here, via [3, Chapter 4], for example, to the relations

$$\mathfrak{S}(n;W) = \mathfrak{S}(n) + o(1),$$

$$\mathfrak{J}(n;W) = \frac{\Gamma(1+1/k)^{t+u}}{\Gamma((t+u)/k)} n^{(t+u)/k-1} + o(n^{(t+u)/k-1}),$$

in which

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}(q^{-1}S(q,a))^{t+u}e(-na/q)$$

is the conventional singular series associated with Waring's problem for sums of $t+u$ integral kth powers.

Substituting these expressions into (4.5), and from there into (4.4), we conclude that

$$R(n)=\rho^u\mathfrak{S}(n)\frac{\Gamma(1+1/k)^{t+u}}{\Gamma((t+u)/k)}n^{(t+u)/k-1}+o(n^{(t+u)/k-1}).$$

Here, we have made use of the fact that since $t+u\geq 4k$ in each case under consideration, the standard theory of the singular series (see [3, Theorems 4.3 and 4.5]) suffices to confirm that $1\ll\mathfrak{S}(n)\ll 1$. In particular, one has $R(n)\gg n^{(t+u)/k-1}$. As discussed earlier, this establishes that $G(k)\leq t+u$, with t and u determined via Table 2, and thus the proof of Theorem 1.1 is complete.

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