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Chen's first inequality for Riemannian maps

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Abstract. We obtain a basic Chen inequality for Riemannian maps between Riemannian manifolds.

1. Introduction. In [C1] and [C2], B. Y. Chen established a sharp inequality for a submanifold in a real space form involving intrinsic invariants of submanifolds and squared mean curvature, the main extrinsic invariant. After that work many related results have been published by various authors for different submanifolds in different ambient spaces; much of those results have been included in the monograph [C6]. However, this subject is still a very active research area (see [FG], [G], [GKKT], [KTG], [LLY], [MR], [OD], [OM], [V], [ZZ], [ZZS], [Z]).

As indicated in [GRK], a major flaw in Riemannian geometry (as compared to other areas) is a shortage of suitable types of maps between Riemannian manifolds that will enable comparing their geometric properties. In this direction, Fischer [F] introduced Riemannian maps between Riemannian manifolds as a generalization of isometric immersions and Riemannian submersions. Isometric immersions and Riemannian submersions have been widely studied in differential geometry (see for example [C3] and [FIP]), but the theory of Riemannian maps is a new research field.

Let $F : (M_1, g_1) \to (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < \operatorname{rank} F < \min\{m, n\}$, where $\dim M_1 = m$ and $\dim M_2 = n$. Then we denote the kernel space of F_{*p} by $\ker F_{*p}$ at $p \in M_1$ and consider its orthogonal complement $\mathcal{H}_p = (\ker F_{*p})^\perp$ in T_pM_1. Thus

$$T_pM_1 = \ker F_{*p} \oplus \mathcal{H}_p.$$

We denote the range of F_{*p} by $\operatorname{range} F_{*p}$ and consider its orthogonal complement $(\operatorname{range} F_{*p})^\perp$ in $T_{F(p)}M_2$. Since $\operatorname{rank} F < \min\{m, n\}$, we always have

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$(\operatorname{range} F_*)^{\perp} \neq \{0\}$. Thus

$$T_{F(p)}M_2 = (\operatorname{range} F_{*p}) \oplus (\operatorname{range} F_{*p})^{\perp}.$$

Now, a smooth map $F : (M_1^m, g_1) \to (M_2^n, g_2)$ is called a *Riemannian map* at $p_1 \in M$ if the horizontal restriction $F^h_{*p_1} : (\ker F_{*p_1})^{\perp} \to \operatorname{range} F_{*p_1}$ is a linear isometry of inner product spaces. Thus, as stated by Fischer [F], a Riemannian map is a map which is as isometric as it can be.

Isometric immersions and Riemannian submersions are particular Riemannian maps with $\ker F_* = \{0\}$ and $(\operatorname{range} F_*)^{\perp} = \{0\}$. It is known that a Riemannian map is a subimmersion, which implies that the rank of the linear map $F_{*p} : T_pM_1 \to T_{F(p)}M_2$ is constant for p in each connected component of M_1 [AMR], [F]. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For Riemannian maps and their applications in spacetime geometry, see [GRK].

In this paper, we obtain a Chen inequality for Riemannian maps.

2. Preliminaries. In this section we recall some notions and results from [BW] and [N].

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $F : M \to N$ is a smooth map. Then the differential F_* of F can be viewed a section of the bundle $\operatorname{Hom}(TM, F^{-1}TN) \to M$, where $F^{-1}TN$ is the pullback bundle which has fibres $(F^{-1}TN)_p = T_{F(p)}N$, $p \in M$. $\operatorname{Hom}(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the *second fundamental form* of F is given by

$$(\nabla F_*)(X, Y) = \nabla^F_X F_*(Y) - F_*(\nabla^M_X Y) \tag{2.1}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \to (N, g_N)$ is said to be *harmonic* if $\operatorname{trace} \nabla\varphi_* = 0$. On the other hand, the *tension field* of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by $\tau(\varphi) = \operatorname{div} \varphi_* = \sum_{i=1}^m (\nabla\varphi_*)(e_i, e_i)$, where $\{e_1, \ldots, e_m\}$ is an orthonormal frame on M. It follows that φ is harmonic if and only if $\tau(\varphi) = 0$. In [S1] we showed that the second fundamental form $(\nabla F_*)(X, Y)$, $X, Y \in \Gamma((\ker F_*)^{\perp})$, of a Riemannian map has no components in $\operatorname{range} F_*$. More precisely we have the following.

LEMMA 2.1 ([S1]). *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). Then*

$$g_2((\nabla F_*)(X, Y), F_*(Z)) = 0, \qquad \forall X, Y, Z \in \Gamma((\ker F_*)^{\perp}).$$

As a consequence of Lemma 2.1, we have

$$(\nabla F_*)(X, Y) \in \Gamma((\operatorname{range} F_*)^{\perp}), \qquad \forall X, Y \in \Gamma((\ker F_*)^{\perp}). \tag{2.2}$$

In [S], Solórzano introduced the second fundamental form B for Riemannian maps as follows. Let $\varphi : (M, g) \to (\bar{M}, \bar{g})$ be a Riemannian map. The second fundamental form B of φ is a bilinear bundle map $B : \bigoplus^2(\ker(\varphi_*)^\perp) \to T\bar{M}$ over φ given by

$$B(u, x) = \bar{\nabla}_{\varphi_*(u)} x - \varphi_*(\nabla_u x) \tag{2.3}$$

for basic vector fields. Solórzano showed that B vanishes identically if and only if $\varphi(M)$ is a totally geodesic submanifold of $\bar{M}$.

From now on, for simplicity, we denote by ∇^2 both the Levi-Civita connection of (M_2, g_2) and its pullback along F. Then according to Nore [N], for any vector field X on M_1 and any section V of $(\operatorname{range} F_*)^\perp$, where $(\operatorname{range} F_*)^\perp$ is the subbundle of $F^{-1}(TM_2)$ with fibre $(F_*(T_pM))^\perp$ over p, we have $\nabla_X^{F\perp} V$ which is the orthogonal projection of $\nabla_X^2 V$ on $(F_*(TM))^\perp$. Nore showed that $\nabla^{F\perp}$ is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F\perp} g_2 = 0$. We now define $\mathcal{S}_V$ via

$$\nabla_X^F V = -\mathcal{S}_V F_* X + \nabla_X^{F\perp} V, \tag{2.4}$$

where $\mathcal{S}_V F_* X$ is the tangential component (a vector field along F) of $\nabla^2_{F_*X} V$. It is easy to see that $\mathcal{S}_V F_* X$ is bilinear in V and $F_* X$, and $\mathcal{S}_V F_* X$ at p depends only on V_p and $F_{*p} X_p$. By direct computation, we obtain

$$g_2(\mathcal{S}_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)) \tag{2.5}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\operatorname{range} F_*)^\perp)$. Since (∇F_*) is symmetric, it follows that $\mathcal{S}_V$ is a symmetric linear transformation of $\operatorname{range} F_*$, called the *shape operator* of a Riemannian map.

We also recall the following algebraic lemma which will be a key tool for our result.

LEMMA 2.2 ([C1]). *Let $n \geq 2$ and let $a_1, \ldots, a_n, b$ be real numbers such that*

$$\Big(\sum_{i=1}^n a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^n a_i^2 + b\Big). \tag{2.6}$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

3. Chen inequality for Riemannian maps. In this section we are going to obtain Chen's inequality. First let us recall the Gauss equation for Riemannian maps from [S3]:

$$\begin{aligned} g_N(R^N(F_*X, F_*Y)F_*Z, F_*T) &= g_M(R^M(X,Y)Z,T) \\ &\quad + g_N\big((\nabla F_*)(X,Z), (\nabla F_*)(Y,T)\big) \\ &\quad - g_N\big((\nabla F_*)(Y,Z), (\nabla F_*)(X,T)\big) \end{aligned} \tag{3.1}$$

for $X, Y, Z, T \in \Gamma((\ker F_*)^\perp)$, where R^M and R^N denote the curvature tensors of the metric connections ∇^M and ∇^N on M and N, respectively.

Now suppose that N is a space form $N(c)$. Since F is a Riemannian map, we have

$$\begin{aligned}(3.2)\quad g_M(R^M(X,Y)Z,T) &= c\big(g_M(Y,,Z)g_M(X,T) - g_M(X,Z)g_M(Y,T)\big)\\ &\quad - g_N\big((\nabla F_*)(X,Z),(\nabla F_*)(Y,T)\big)\\ &\quad + g_N\big((\nabla F_*)(Y,Z),(\nabla F_*)(X,T)\big).\end{aligned}$$

THEOREM 3.1. *Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a space form $(N(c), g_N)$ with* $\operatorname{rank} F = r \geq 3$. *Then for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$(3.3)\qquad K(\pi) \geq \rho_{\mathcal{H}} - \frac{r-2}{2}\Big((r+1)c + \frac{1}{r-1}\|\tau^{\mathcal{H}}\|^2\Big),$$

where $\rho_{\mathcal{H}}$ is the scalar curvature defined on $\mathcal{H} = (\ker F_*)^\perp$, and $\tau^{\mathcal{H}}$ is defined by

$$\tau^{\mathcal{H}} = \sum_{i=1}^{r} g_N\big((\nabla F_*)(e_i,e_i),(\nabla F_*)(e_i,e_i)\big).$$

Equality holds if and only if there exists an orthonormal basis $\{e_1,\dots,e_r\}$ of $(\ker F_{*p})^\perp$ and an orthonormal basis $\{V_{r+1},\dots,V_{r+d}\}$ of $(\operatorname{range} F_{*p})^\perp$ such that the shape operator takes the form

$$\mathcal{S}_{r+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0\\ 0 & b & 0 & \cdots & 0\\ 0 & 0 & \mu & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \qquad a+b=\mu,$$

and

$$\mathcal{S}_{\alpha} = \begin{pmatrix} B^\alpha_{11} & B^\alpha_{12} & 0 & \cdots & 0\\ B^\alpha_{12} & -B^\alpha_{11} & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \alpha = r+2,\dots,r+d.$$

Proof. Taking $X = T = e_i$ and $Y = Z = e_j$ in (3.2) we get

$$(3.4)\qquad r(r-1)c = 2\rho_{\mathcal{H}} + \sum_{i,j=1}^{r} g_N\big((\nabla F_*)(e_i,e_j),(\nabla F_*)(e_i,e_j)\big) - \|\tau^{\mathcal{H}}\|^2.$$

Set

$$\varepsilon = -r(r-1)c + 2\rho_{\mathcal{H}} - \frac{r-2}{r-1}\|\tau^{\mathcal{H}}\|^2. \tag{3.5}$$

Then

$$\|\tau^{\mathcal{H}}\|^2 = (r-1)\sum_{i,j=1}^{r}\big(g_N\big((\nabla F_*)(e_i,e_j),(\nabla F_*)(e_i,e_j)\big)+\varepsilon\big). \tag{3.6}$$

If we use the notation introduced in (2.3), we get

$$\|\tau^{\mathcal{H}}\|^2 = (r-1)\Big(\sum_{i,j=1}^{r} g_N(B(e_i,e_j),B(e_i,e_j))+\varepsilon\Big). \tag{3.7}$$

Now for $p \in M$, consider a plane $\pi \subset T_pM$ spanned by $\{e_1, e_2\}$. From Lemma 2.1 we know that $\tau^{\mathcal{H}} \in (\operatorname{range} F_*)^{\perp}$. Take an orthonormal frame $\{V_{r+1}, \ldots, V_d\}$ of $(\operatorname{range} F_*)^{\perp}$ such that V_{r+1} is parallel to $\tau^{\mathcal{H}}$. Also for convenience, set $B_{ij}^n = g_N((\nabla F_*)(e_i,e_j),V_n)$. Then

$$\Big(\sum_{i=1}^{r} B_{ii}^{r+1}\Big)^2 = (r-1)\Big(\sum_{i,j=1}^{r}\sum_{\alpha=r+1}^{r+d}(B_{ij}^{\alpha})^2+\varepsilon\Big) \tag{3.8}$$

or

$$\Big(\sum_{i=1}^{r} B_{ii}^{r+1}\Big)^2 = (r-1)\Big\{\sum_{i=1}^{r}(B_{ii}^{r+1})^2+\sum_{i\neq j=1}^{r}(B_{ij}^{r+1})^2 + \sum_{\alpha=r+2}^{r+d}\sum_{i,j=1}^{r}(B_{ij}^{\alpha})^2+\varepsilon\Big\}. \tag{3.9}$$

Applying Lemma 2.2 we get

$$2B_{11}^{r+1}B_{22}^{r+1} \geq \sum_{i\neq j}^{r}(B_{ij}^{r+1})^2+\sum_{\alpha=r+2}^{r+d}\sum_{i,j=1}^{r}(B_{ij}^{\alpha})^2+\varepsilon. \tag{3.10}$$

Thus we obtain

$$\begin{aligned} 2B_{11}^{r+1}B_{22}^{r+1} \geq{}& 2(B_{12}^{r+1})^2+\sum_{i\neq j>2}^{r}(B_{ij}^{r+1})^2+2\sum_{j>2}^{r}((B_{1j}^{r+1})^2+(B_{2j}^{r+1})^2)\\ &+2\sum_{\alpha=r+2}^{r+d}(B_{12}^{\alpha})^2+\sum_{\alpha=r+2}^{r+d}\sum_{i,j>2}^{r}(B_{ij}^{\alpha})^2+2\sum_{\alpha=r+2}^{r+d}\sum_{j>2}^{r}((B_{1j}^{\alpha})^2+(B_{2j}^{\alpha})^2)\\ &+\sum_{\alpha=r+2}^{r+d}((B_{11}^{\alpha})^2+(B_{22}^{\alpha})^2)+\varepsilon. \end{aligned} \tag{3.11}$$

Hence

$$(3.12)\qquad 2B_{11}^{r+1}B_{22}^{r+1}-2(B_{12}^{r+1})^2-2\sum_{\alpha=r+2}^{r+d}(B_{12}^{\alpha})^2+2\sum_{\alpha=r+2}^{r+d}B_{11}^{\alpha}B_{22}^{\alpha}$$
$$\geq \sum_{i\neq j>2}^{r}(B_{ij}^{r+1})^2+2\sum_{j>2}^{r}((B_{1j}^{r+1})^2+(B_{2j}^{r+1})^2)+\sum_{\alpha=r+2}^{r+d}\sum_{i,j>2}^{r}(B_{ij}^{\alpha})^2$$
$$+2\sum_{\alpha=r+2}^{r+d}\sum_{j>2}^{r}((B_{1j}^{\alpha})^2+(B_{2j}^{\alpha})^2)+\sum_{\alpha=r+2}^{r+d}(B_{11}^{\alpha}+B_{22}^{\alpha})^2+\varepsilon.$$

Taking $X=T=e_1$ and $Y=Z=e_2$ in (3.1) and using it in (3.11), we get

$$(3.13)\qquad K(\pi)\geq \sum_{\alpha=r+1}^{r+d}\sum_{j>2}^{r}((B_{1j}^{\alpha})^2+(B_{2j}^{\alpha})^2)+\frac{1}{2}\sum_{i\neq j>2}^{r}(B_{ij}^{r+1})^2$$
$$+\frac{1}{2}\sum_{\alpha=r+2}^{r+d}\sum_{i,j>2}(B_{ij}^{\alpha})^2+\frac{1}{2}\sum_{\alpha=r+2}^{r+d}(B_{11}^{\alpha}+B_{22}^{\alpha})^2+c+\frac{\varepsilon}{2}\geq c+\frac{\varepsilon}{2}.$$

Thus we arrive at (3.3). If equality holds in (3.3) at a point p, then the inequality (3.13) becomes an equality. In this case, from (3.13) we have

$$\begin{cases} B_{1j}^{r+1}=B_{2j}^{r+1}=B_{ij}^{r+1}=0, \quad i\neq j>2,\\ B_{ij}^{\alpha}=0, \quad \forall i\neq j,\ i,j=3,\dots,r,\ \alpha=r+2,\dots,r+d,\\ B_{11}^{\alpha}+B_{22}^{\alpha}=0, \quad \forall \alpha=r+2,\dots,r+d,\\ B_{11}^{r+2}+B_{22}^{r+2}=\dots=B_{11}^{r+d}+B_{22}^{r+d}=0.\end{cases}$$

Now, we choose e_1, e_2 such that $B_{12}^{r+1}=0$ and we denote $a=B_{11}^{\alpha}, b=B_{22}^{\alpha}$, $\mu=B_{33}^{r+1}=\dots=B_{33}^{\alpha}$. Thus by choosing a suitable orthonormal basis the shape operators $\mathcal{S}_V$ take the desired forms. ∎

From Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let F be a harmonic Riemannian map from a Riemannian manifold (M, g_M) to the Euclidean space $\mathbb{E}^n$ with* $\operatorname{rank} F=r\geq 3$. *Then for each point $p\in M$ and each plane section $\pi\subset T_pM$, we have*

$$K(\pi)\geq \rho_{\mathcal{H}},$$

where $\rho_{\mathcal{H}}$ is the scalar curvature defined on $\mathcal{H}=(\ker F_*)^{\perp}$.

Remark 3.3. In [C1], Chen obtained the following result. Let M be an n-dimensional ($n\geq 2$) submanifold of a Riemannian manifold $\bar{M}(c)$ of constant sectional curvature c. Then

$$(3.14)\qquad \inf K\geq \frac{1}{2}\Big\{\rho-\frac{n^2(n-2)}{(n-1)}\|H\|^2-(n+1)(n-2)c\Big\}.$$

Equality holds if and only if there exists an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\dots,e_m\}$ of $T^\perp M$ such that the shape operator takes the following form:

$$A_{n+1}=\begin{pmatrix} a & 0 & 0 & \cdots & 0\\ 0 & b & 0 & \cdots & 0\\ 0 & 0 & \mu & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix},\ a+b=\mu,\qquad A_r=\begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \cdots & 0\\ h^r_{12} & -h^r_{11} & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$r=n+2,\dots,m$, where h^r_{ij} are the components of the second fundamental form of the submanifold.

Let $\mathbb{C}^{m+1}$ denote the complex Euclidean $(m+1)$-space and let $S^{2m+1}=\{z=(z_1,\dots,z_{m+1})\in\mathbb{C}^{m+1}\mid\langle z,z\rangle=1\}$ be the unit hypersphere of $\mathbb{C}^{m+1}$. Then consider the Hopf fibration $\pi: S^{2m+1}\to\mathbb{CP}^m(4c)$. It is well known that this map is a Riemannian submersion with totally geodesic fibres. Let N be an n-dimensional submanifold of $\mathbb{CP}^m(4c)$. set $\pi^{-1}(N)=\tilde N$. Then $\bar\pi:\tilde N\to N$ is also a Riemannian submersion with totally geodesic fibres, where $\bar\pi$ is the restriction π_N. For a horizontal 2-plane $P_x\subset T_x\tilde N$, we denote by $\tilde P_x$ the $\dim(S^{2m+1})-\dim(\mathbb{CP}^m(4c))+2$-subspace spanned by P_x and the vertical space $\mathcal{V}_x$. Let $x\in\tilde N$ and let e_1,e_2 be orthonormal vectors at $\pi(x)\in N$. Denote by $\tilde e_1,\tilde e_2$ the horizontal lifts of e_1, e_2 at $x\in\tilde N$. Then $\tilde P_x$ is spanned by $\tilde e_1,\tilde e_2$ and $\mathcal{V}_x$. In [ACM], Alegre, Chen and Munteanu proved the following result: Let $\pi: S^{2m+1}\to\mathbb{CP}^m(4)$ be the Hopf fibration and let N be an n-dimensional submanifold of $\mathbb{CP}^m(4)$. Then

$$\rho_{\tilde N}(x)-\inf_{\tilde P_x}\rho_{\tilde N}\tilde P_x\le\frac{n^2(n-2)}{2(n-1)}\|H\|^2+\|\mathcal{P}\|^2+\frac12(n+1)(n-2)c,$$

where $\tilde P_x$ runs over $(m+3)$-subspaces associated with all horizontal 2-planes P_x at $x\in\tilde N$, $\mathcal{P}$ is the projection from $\mathbb{CP}^m$ to TN, and $\|H\|^2$ is the squared mean curvature of N in $\mathbb{CP}^m$. Equality holds if and only if there exists an orthonormal basis $\{e_1,\dots,e_m\}$ such that

(a) the shape operator A of N in $\mathbb{CP}^m(4)$ satisfies

$$A_s=\begin{pmatrix} B_s & 0\\ 0 & \mu_s I\end{pmatrix},\qquad s=n+1,\dots,m,$$

where I is the identity $(n-2)\times(n-2)$ matrix and B_s are symmetric 2×2 submatrices satisfying $\mu_s=\operatorname{trace}B_s$, $s=n+1,\dots,2m$, and

(b) $\mathcal{P}e_1=\mathcal{P}e_2=0$.

From the above remarks, one can see that if $r=\dim(M)$, then a Riemannian map becomes an isometric immersion and Theorem 3.1 gives the immersion case. Since the base space for the Hopf map is a complex manifold, the two inequalities seem different due to extra terms, but still they relate similar notions.

In [C4] and [C5], Chen obtained another inequality for Riemannian submersions and found an interesting result about non-existence of certain immersions defined on the same total space. Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibres and $\phi : M \to \bar{M}$ an isometric immersion into a Riemannian manifold $\bar{M}$. It was shown in [C4] that

$$\breve{A}_\pi \leq \frac{n^2}{4}\|H\|^2 + b(n-b)\max \bar{K}$$

where $\breve{A}_\pi$ is the submersion invariant defined by $\breve{A}_\pi = \sum_{i=1}^{b}\sum_{s=b+1}^{n}\|A_{e_i}e_s\|^2$, and $\max \bar{K}(p)$ denotes the maximum value of the sectional curvature function of $\bar{M}^m$ restricted to plane sections in T_pM. By using this inequality, Chen proved that π cannot be isometrically immersed in any Riemannian manifold of non-positive sectional curvature as a minimal submanifold.

We now give an example of Riemannian maps satisfying (3.3); we first recall the notion of totally umbilical Riemannian maps.

Lemma 3.4 ([S2]). *Let F be a Riemannian map between Riemannian manifolds (M, g) and (N, g_N). Then F is an umbilical Riemannian map if and only if*

$$(\nabla F_*)(X, Y) = g_M(X, Y)H_2 \tag{3.15}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$, where H_2 is a nowhere zero vector field on $(\operatorname{range} F_*)^\perp$.

Corollary 3.5. *For every umbilical Riemannian map F from a Riemannian manifold (M, g_M) to a space form $(N(c), g_N)$ with $\operatorname{rank} F = r \geq 3$, equality holds in (3.3).*

We also have the following result.

Proposition 3.6 ([S2]). *Let F_1 be a Riemannian submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) and F_2 a totally umbilical isometric immersion from (N, g_N) into a Riemannian manifold $(\bar{N}, g_{\bar{N}})$. Then $F_2 \circ F_1$ is an umbilical Riemannian map from (M, g_M) to $(\bar{N}, g_{\bar{N}})$.*

Considering Proposition 3.6, we have the following example.

Example 3.7. We consider the Hopf fibration $\pi : S^7 \to S^4$. This map is a Riemannian submersion with totally geodesic fibres and it has fibres S^3. We also consider the isometric immersion $i : S^4 \to \mathbb{E}^5$ as a hypersurface of $\mathbb{E}^5$. Then i is a totally umbilical isometric immersion. Thus $i \circ \pi$ is a totally umbilical Riemannian map and therefore it satisfies (3.3).

Concluding remarks. In [C6], there are many different versions of Chen's inequality for various ambient manifolds and applications of Chen's inequality in different manifolds. Still, many problems for Chen-like inequalities for Riemannian maps remain to be explored.

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