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On the Stability of f-Maximal Spacelike Hypersurfaces in Weighted Generalized Robertson–Walker Spacetimes

by

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Summary. Our purpose in this paper is to study the stability of f-maximal spacelike hypersurfaces immersed in a weighted generalized Robertson–Walker spacetime $-I\times_\rho M^n_f$, where M^n_f is a weighted Riemannian manifold endowed with a weight function f. In this setting, we obtain sufficient conditions to guarantee that an f-maximal hypersurface be L_f-stable, where L_f stands for the weighted Jacobi operator.

1. Introduction. Let $(\overline{M}^{n+1},\langle\ ,\ \rangle)$ be an orientable $(n+1)$-dimensional Lorentzian manifold endowed with a timelike vector field V and let $f:\overline{M}^{n+1}\to\mathbb{R}$ be a smooth function. The *weighted Lorentzian manifold* $\overline{M}^{n+1}_f$ associated with $\overline{M}^{n+1}$ and f is the triple $(\overline{M}^{n+1},\langle\ ,\ \rangle,e^{-f}d\overline{M})$, where $d\overline{M}$ denotes the standard volume element of $\overline{M}^{n+1}$ induced by the metric $\langle\ ,\ \rangle$. We will refer to the function f as being the *weight function* associated to $\overline{M}^{n+1}_f$. In this setting, an important tensor is the Bakry–Émery Ricci tensor $\overline{\mathrm{Ric}}_f$, a natural generalization of the Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}^{n+1}_f$ defined by

$$\overline{\mathrm{Ric}}_f=\overline{\mathrm{Ric}}+\overline{\mathrm{Hess}}\,f, \tag{1.1}$$

where $\overline{\mathrm{Hess}}\,f$ is the Hessian of f on $\overline{M}^{n+1}_f$.

Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many other subjects in differential geometry, weighted

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manifolds proved to be important nontrivial generalizations of Riemannian manifolds and they are objects of extensive ongoing investigation. For a brief overview of results in this area, we refer the articles of Morgan [M] and Wei–Wylie [WW].

In this context, we consider a (connected) spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}_f$ immersed in a weighted Lorentzian manifold $\overline{M}^{n+1}_f$, that is, the metric induced on Σ^n via ψ is a Riemannian metric. As usual, we also denote by $\langle\ ,\ \rangle$ the metric of Σ^n induced via ψ. Since V is a globally defined timelike vector field on $\overline{M}^{n+1}_f$, there exists a unique unitary timelike normal vector field N globally defined on Σ^n which is in the same time-orientation of V, that is, $\langle V, N\rangle < 0$. We will refer to this normal timelike vector field N as being the *future-pointing Gauss map* of Σ^n. Throughout this work, N will always denote the future-pointing Gauss map of a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}_f$. On the other hand, we note that the weight function f induces a weighted measure $e^{-f} d\sigma$ on Σ^n, where $d\sigma$ denotes the standard volume element of Σ^n with respect to the induced metric from the ambient space $\overline{M}^{n+1}_f$. So, we have an induced weighted Riemannian manifold $(\Sigma^n, \langle\ ,\ \rangle, e^{-f} d\sigma)$.

The f-divergence operator on Σ^n is defined by

$$\mathrm{Div}_f(X) = e^f\, \mathrm{Div}(e^{-f} X), \tag{1.2}$$

where X is a tangent vector field on Σ^n and Div denotes the standard divergence operator of Σ^n. From (1.2) we can define the f-Laplacian of Σ^n by

$$\Delta_f u = \mathrm{Div}_f(\nabla u) = \Delta u - \langle \nabla f, \nabla u\rangle, \tag{1.3}$$

where u is a smooth function on Σ^n, Δ denotes the Laplacian induced by Div and ∇ stands for the Levi-Civita connection of Σ^n induced from the Levi-Civita connection $\overline{\nabla}$ on the ambient space $\overline{M}^{n+1}_f$.

Following Gromov [G], the weighted mean curvature, or simply *f-mean curvature*, H_f of Σ^n is defined by

$$nH_f = nH - \langle \overline{\nabla} f, N\rangle, \tag{1.4}$$

where $H = -\frac{1}{n}\mathrm{tr}(A)$ denotes the standard mean curvature of Σ^n and $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, given by $AX = -\overline{\nabla}_X N$, is the shape operator of Σ^n with respect to its future-pointing Gauss map N. So, a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}_f$ immersed in a weighted Lorentzian manifold $\overline{M}^{n+1}_f$ is called an *f-maximal* hypersurface if its f-mean curvature H_f vanishes identically or, equivalently, if its mean curvature H satisfies

$$nH = \langle \overline{\nabla} f, N\rangle.$$

Stability questions concerning constant mean curvature compact hypersurfaces in Riemannian space forms began with Barbosa–do Carmo [BC], and Barbosa–do Carmo–Eschenburg [BCE]. In the former paper, the authors introduced the notion of stability and proved that spheres are the only stable critical points for the area functional, for volume preserving variations. In the setting of spacelike hypersurfaces in Lorentz manifold, Barbosa–Oliker [BO] proved that constant mean curvature spacelike hypersurfaces are critical points for volume preserving variations. Moreover, by computing the second variation formula they showed that constant mean curvature embedded spheres in the de Sitter space $\mathbb{S}_1^{n+1}$ maximize the area functional for such variations. Later on, Barros–Brasil–Caminha [BBC] classified strongly stable spacelike hypersurfaces with constant mean curvature immersed into so-called *generalized Robertson–Walker spacetimes* $-I\times_\rho M^n$, that is, Lorentzian warped products with 1-dimensional negative definite base $I\subset\mathbb{R}$, Riemannian fiber M^n and warping function $\rho : I\to\mathbb{R}$. Assuming a certain convexity condition on the warping function, they showed that a closed strongly stable spacelike hypersurface immersed with constant mean curvature in $-I\times_\rho M^n$ is either maximal or a spacelike slice $\{t_0\}\times M^n$.

Proceeding in this direction, our aim is to investigate the L_f-stability of f-maximal spacelike hypersurfaces immersed in a weighted generalized Robertson–Walker spacetime $-I\times_\rho M_f^n$, where L_f stands for the weighted Jacobi operator defined by

$$L_f = \Delta_f - (|A|^2 + \overline{\mathrm{Ric}}_f(N,N)). \tag{1.5}$$

Here, motivated by a splitting theorem due to Case [C], we will suppose that the weight function f does not depend on the parameter $t\in I$.

This manuscript is organized as follows. In Section 2 we compute the first and second variation formulas for a spacelike hypersurface in a weighted Lorentzian manifold (see Lemmas 2.1 and 2.2). Next, in Section 3 we establish an L_f-stability criterion (see Lemma 3.2) and, finally, we apply it to determine when a f-maximal spacelike hypersurface immersed in a weighted generalized Robertson–Walker spacetime $I\times_\rho M_f^n$ is L_f-stable (see Theorem 3.3).

2. Preliminaries. In what follows, $\overline{M}^{n+1}$ denotes an $(n+1)$-dimensional Lorentzian manifold endowed with a timelike vector field V, $f:\overline{M}^{n+1}\to\mathbb{R}$ a smooth function and $\overline{M}_f^{n+1}$ the weighted Lorentzian manifold associated with $\overline{M}^{n+1}$ and f. If $\psi:\Sigma^n\to\overline{M}_f^{n+1}$ is a (connected) spacelike hypersurface in $\overline{M}_f^{n+1}$, then Σ^n is automatically orientable (see [O, p. 189]) and we can choose a globally defined unit timelike normal vector field N on Σ^n which is in the same time-orientation as V, that is, $\langle V,N\rangle<0$. Moreover, as already mentioned, we have an induced weighted Riemannian manifold $(\Sigma^n, \langle\ ,\ \rangle, e^{-f} d\sigma)$.

The weighted area functional of Σ^n is naturally defined by

$$\mathrm{vol}_f(\Sigma) = \int_\Sigma e^{-f}\, d\sigma.$$

The first and second variation formulas for the weighted area functional are well known in the case of hypersurfaces immersed in Riemannian spaces (see, for instanse, [CMZ]). In the present context, we have not found their proof in the current literature. So, for the sake of completeness, we will deduce them here.

For this, let $F : \Sigma^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ be a *variation* of Σ^n with compact support and fixed boundary, that is, $F(\cdot, t) : \Sigma^n \to \overline{M}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$, is a spacelike immersion and

(i) $F = \mathrm{Id}$ outside a compact subset of Σ^n;
(ii) $F(x, 0) = \psi(x)$ for all $x \in \Sigma^n$;
(iii) $F(x, t) = x$ for all $x \in \partial\Sigma$.

The vector field $F_t = \frac{\partial F}{\partial t}\big|_{t=0}$ restricted to Σ^n is called the *variational vector field* of the variation F. We note that

$$F_t = F_t^\top - \langle F_t, N\rangle N,$$

where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n. The variation F is called *normal* when $F_t^\top = 0$ on Σ^n.

LEMMA 2.1. *Let $\overline{M}_f^{n+1}$ be a weighted Lorentzian manifold and let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be a spacelike hypersurface. If $F : \Sigma^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ is a variation of Σ^n with compact support and fixed boundary, then*

$$\frac{d}{dt}\mathrm{vol}_f(F(\Sigma, t))\bigg|_{t=0} = n\int_\Sigma H_f \varphi e^{-f}\, d\sigma.$$

where $F_t = F_t^\top + \varphi N$ (thus $\varphi = -\langle F_t, N\rangle$). In particular, Σ^n is f-maximal if, and only if, $\frac{d}{dt}\mathrm{vol}_f(F(\Sigma, t))\big|_{t=0} = 0$ for every variation F with compact support and fixed boundary.

Proof. Let $(x_1, \dots, x_n)$ be a local coordinate system on Σ^n. Then $(F \circ x_1, \dots, F \circ x_n)$ is a local coordinate system on $\Sigma_t = F(\Sigma, t)$. Denote $F_{x_i}(t) = dF\big(\frac{\partial}{\partial x_i}\big)$ and consider

$$g_{ij}(t) = \langle F_{x_i}(t), F_{x_j}(t)\rangle \quad \text{and} \quad v(t) = \frac{\sqrt{\det(g_{ij}(t))}}{\sqrt{\det(g_{ij}(0))}}.$$

Note that $v(t)$ is well defined and independent of the choice of a local coordinate system on Σ^n. Furthermore, denoting by $d\sigma_t$ the standard volume element of Σ_t with respect to the metric induced from the ambient space, we have

$$\mathrm{vol}_f(F(\Sigma,t)) = \int_{\Sigma_t} e^{-f(p,t)}\,d\sigma_t = \int_{\Sigma} v(t)e^{-f(p,t)}\,d\sigma,$$

where $f(p,t)=f(F(p,t))$. Differentiating this, we obtain

$$\frac{d}{dt}\,\mathrm{vol}_f(F(\Sigma,t))\Big|_{t=0} = \int_{\Sigma} \frac{d}{dt}(v(t)e^{-f(x,t)})\Big|_{t=0}\,d\sigma. \tag{2.1}$$

Now, to evaluate $\frac{d}{dt}(v(t)e^{-f(p,t)})\big|_{t=0}$ at some point $p\in\Sigma^n$, we may choose a local coordinate system which is orthonormal at p. It is well known that $\frac{d}{dt}v(t)\big|_{t=0} = \mathrm{Div}\,F_t^T + nH\varphi$. Hence, we get

$$\begin{aligned}\frac{d}{dt}(v(t)e^{-f(x,t)})\Big|_{t=0} &\\ &= e^{-f(p,0)}\frac{d}{dt}v(t)\Big|_{t=0} - v(0)e^{-f(p,0)}\frac{d}{dt}f(F(p,t))\Big|_{t=0}\\ &= e^{-f}(\mathrm{Div}\,F_t^T + nH\varphi) - e^{-f}\langle\overline{\nabla} f, F_t\rangle\\ &= e^{-f}(\mathrm{Div}_f\,F_t^T + nH_f\varphi).\end{aligned} \tag{2.2}$$

Therefore, taking into account the weighted version of the divergence theorem (see [CR, Lemma 2.2]), we can use (2.1) and (2.2) to conclude the proof. ∎

Our aim is to study f-maximal spacelike hypersurfaces $\psi:\Sigma^n\to\overline{M}^{n+1}$ which maximize the weighted area functional for every normal variation of Σ^n with compact support and fixed boundary. Hence, in order to examine when an f-maximal spacelike hypersurface is actually a maximum of weighted area functional, one certainly needs to study the second variation $\frac{d^2}{dt^2}\,\mathrm{vol}_f(F(\Sigma,t))\big|_{t=0}$. In the next result, we compute the second variation formula of the weighted area functional.

LEMMA 2.2. *Let $\overline{M}_f^{n+1}$ be a weighted Lorentzian manifold and let $\psi:\Sigma^n\to\overline{M}_f^{n+1}$ be an f-maximal spacelike hypersurface. If $F:\Sigma^n\times(-\varepsilon,\varepsilon)\to\overline{M}^{n+1}$ is a normal variation of Σ^n with compact support and fixed boundary, then*

$$\frac{d^2}{dt^2}\,\mathrm{vol}_f(F(\Sigma,t))\Big|_{t=0} = \int_{\Sigma}\varphi\big(\Delta_f\varphi - (|A|^2+\overline{\mathrm{Ric}}_f(N,N))\varphi\big)e^{-f}\,d\sigma,$$

where $F_t=\varphi N$ (thus $\varphi=-\langle F_t,N\rangle$).

Proof. In what follows, we keep the notation established in the proof of Lemma 2.1. We have

$$\frac{d^2}{dt^2}\,\mathrm{vol}_f(F(\Sigma,t))\Big|_{t=0} = \int_{\Sigma}\frac{d^2}{dt^2}(v(t)e^{-f(x,t)})\Big|_{t=0}\,d\sigma. \tag{2.3}$$

It is well known that

$$\frac{d^2}{dt^2}v(t)\bigg|_{t=0} = \operatorname{Div}(\overline{\nabla}_{F_t}F_t)+n^2\varphi^2H^2-|\nabla\varphi|^2-(|A|^2+\overline{\operatorname{Ric}}(N,N))\varphi^2. \tag{2.4}$$

On the other hand, using the fact that Σ^n is f-maximal jointly with (2.4), with a straightforward computation we obtain

$$\begin{aligned}
&\frac{d^2}{dt^2}(v(t)e^{-f(x,t)})\bigg|_{t=0}\\
&\quad = e^{-f}\bigg(\frac{d^2}{dt^2}v(t)\bigg|_{t=0} - \langle\overline{\nabla}f, F_t\rangle\frac{d}{dt}v(t)\bigg|_{t=0} - v(0)\frac{d}{dt}\langle\overline{\nabla}f, F_t\rangle\bigg|_{t=0}\bigg)\\
&\qquad + \frac{d}{dt}e^{-f(x,t)}\bigg|_{t=0}\bigg(\frac{d}{dt}v(t)\bigg|_{t=0} - v(0)\langle\overline{\nabla}f, F_t\rangle\bigg)\\
&\quad = e^{-f}\big(\operatorname{Div}_f(\overline{\nabla}_{F_t}F_t) - |\nabla\varphi|^2 - (|A|^2 + \overline{\operatorname{Ric}_f}(N,N))\varphi^2\big).
\end{aligned}$$

Therefore, using once more the weighted divergence theorem and the fact that Σ^n is f-maximal we conclude our proof. ∎

3. Stability of f-maximal spacelike hypersurfaces. It follows from Lemma 2.2 that the second variation formula for the weighted area functional depends only on $\varphi \in C_0^\infty$. So, the following definition makes sense.

DEFINITION 3.1. Let $\overline{M}_f^{n+1}$ be a weighted Lorentzian manifold and let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be an f-maximal spacelike hypersurface. We say that Σ^n is *L_f-stable* if, for any compactly supported smooth function $\varphi \in C_0^\infty(\Sigma)$,

$$\frac{d^2}{dt^2}\operatorname{vol}_f(\Sigma)\bigg|_{t=0} = \int_\Sigma \varphi L_f\varphi e^{-f}\,d\sigma \le 0,$$

where the weighted Jacobi operator L_f is defined by (1.5).

To prove our main theorem, we will also need the following auxiliary result.

LEMMA 3.2. *Let $\overline{M}_f^{n+1}$ be a weighted Lorentzian manifold and let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be an f-maximal spacelike hypersurface. If there exists a positive smooth function $u \in C^\infty(\Sigma)$ such that $L_f u \le 0$, then Σ^n is L_f-stable.*

Proof. Assume that such a u exists and take $\varphi \in C_0^\infty(\Sigma)$. Then, we can choose $\eta \in C_0^\infty(\Sigma)$ satisfying $\varphi = \eta u$. Hence, from (1.5) we have

$$(3.1)\qquad \int_\Sigma \varphi L_f\varphi e^{-f}\,d\sigma = \int_\Sigma \eta u L_f(\eta u)e^{-f}\,d\sigma$$
$$= \int_\Sigma \big[\eta^2 u L_f u + \eta u^2 \Delta\eta + 2\eta u\langle\nabla u,\nabla\eta\rangle - \eta u^2\langle\nabla\eta,\nabla f\rangle\big]e^{-f}\,d\sigma$$
$$\le \int_\Sigma \big[\eta u^2 \Delta\eta + 2\eta u\langle\nabla u,\nabla\eta\rangle - \eta u^2\langle\nabla\eta,\nabla f\rangle\big]e^{-f}\,d\sigma$$
$$= \int_\Sigma \big[\eta u^2 \Delta\eta + \tfrac{1}{2}\langle\nabla u^2,\nabla\eta^2\rangle - \eta u^2\langle\nabla\eta,\nabla f\rangle\big]e^{-f}\,d\sigma.$$

On the other hand, it is not difficult to verify that

$$(3.2)\qquad \mathrm{Div}(u^2\nabla\eta^2) = \langle\nabla u^2,\nabla\eta^2\rangle + u^2\Delta\eta^2$$
$$= \langle\nabla u^2,\nabla\eta^2\rangle + 2\eta u^2\Delta\eta + 2u^2|\nabla\eta|^2.$$

Therefore, using once more the weighted version of the divergence theorem, from (3.1) and (3.2) we get

$$\int_\Sigma \varphi L_f\varphi e^{-f}\,d\sigma \le \int_\Sigma \big[\tfrac{1}{2}\,\mathrm{Div}(u^2\nabla\eta^2) - \eta u^2\langle\nabla\eta,\nabla f\rangle - u^2|\nabla\eta|^2\big]e^{-f}\,d\sigma$$
$$= \int_\Sigma \big[\tfrac{1}{2}\,\mathrm{Div}_f(u^2\nabla\eta^2) - u^2|\nabla\eta|^2\big]e^{-f}\,d\sigma$$
$$\le -\int_\Sigma u^2|\nabla\eta|^2 e^{-f}\,d\sigma \le 0,$$

and consequently Σ^n is L_f-stable. ■

Now, let M^n be a (connected) complete n-dimensional Riemannian manifold, $I\subset\mathbb{R}$ an open interval in $\mathbb{R}$ and $\rho: I\to\mathbb{R}$ a positive smooth function on I. We will denote by $\overline{M}^{n+1} = -I\times_\rho M^n$ the product manifold $I\times M^n$ endowed with the Lorentzian metric

$$(3.3)\qquad \langle\ ,\ \rangle = -\pi_I^*(dt^2) + (\rho\circ\pi_I)^2\pi_M^*(\langle\ ,\ \rangle_M),$$

where π_I and π_M denote the canonical projections from $I\times M^n$ onto each factor, $\langle\ ,\ \rangle_M$ is the Riemannian metric on the fiber M^n, and I is endowed with the metric $-dt^2$. The Lorentzian manifold $\overline{M}^{n+1}$ is called a *Lorentzian warped product* with base I, fiber M^n and warping function ρ.

When M^n has constant sectional curvature, the warped product $\overline{M}^{n+1} = -I\times_\rho M^n$ has been known in the literature as a Robertson–Walker (RW) spacetime, due to the fact that, for $n=3$, it is an exact solution of Einstein's field equations (see [O, Chapter 12]). After [ARS], the warped product $\overline{M}^{n+1} = -I\times_\rho M^n$ has usually been referred to as a *generalized Robertson–Walker* (GRW) *spacetime*, and we will stick to this terminology along this paper.

Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime and $f : \overline{M}^{n+1} \to \mathbb{R}$ a smooth function on $\overline{M}^{n+1}$. It follows from a splitting theorem due to Case (see [C, Theorem 1.2]) that if a weighted GRW spacetime $\overline{M}_f^{n+1}$ with bounded weight function f is such that $\overline{\mathrm{Ric}}_f(T,T) \geq 0$ for all timelike vector fields T, then f must be constant along I. Motivated by this result, here we will consider weighted GRW spacetimes $\overline{M}_f^{n+1}$ whose weight function f does not depend on $t \in I$, that is, $\langle \overline{\nabla} f, \partial_t \rangle = 0$; for simplicity, we will denote them by $\overline{M}_f^{n+1} = -I \times_\rho M_f^n$.

Recall that, since ∂_t is a globally defined timelike vector field on $\overline{M}_f^{n+1}$, there exists a unique unitary timelike normal vector field N globally defined on Σ^n which is in the same time-orientation as ∂_t, i.e. $\langle N, \partial_t \rangle \leq -1$. In this context, we will consider one particular smooth function, namely, the angle function $\Theta = \langle N, \partial_t \rangle$, where ∂_t stands for the unitary vector field which determines on $\overline{M}^{n+1} = -I \times_\rho M^n$ a codimension one foliation by totally umbilical slices $\{t\} \times M$.

Now, we will state and prove our main result concerning L_f-stability of spacelike hypersurfaces in a weighted GRW spacetime.

THEOREM 3.3. *Let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be an f-maximal spacelike hypersurface immersed into a weighted GRW spacetime $\overline{M}_f^{n+1} = -I \times_\rho M_f^n$.*

(a) *If $\rho'' \leq 0$ on Σ^n, then Σ^n is L_f-stable.*
(b) *If Σ^n is compact and $\rho'' \geq 0$ on Σ^n, then Σ^n is L_f-stable if and only if $\rho'' = 0$ on Σ^n.*
(c) *If Σ^n is compact and $\rho'' > 0$ on Σ^n, then Σ^n cannot be L_f-stable.*

Proof. First, we will prove (a). For this, let us consider on Σ^n the negative function $\tilde{\Theta} = \rho\Theta$. Since $\rho\partial_t$ is a conformal vector field in $\overline{M}_f^{n+1}$ with $\overline{\nabla}_X \rho\partial_t = \rho' X$ for every $X \in \mathfrak{X}(\Sigma)$, we can see that

$$\begin{aligned} X(\tilde{\Theta}) &= X\langle N, \rho\partial_t \rangle = \langle \overline{\nabla}_X N, \rho\partial_t \rangle + \langle N, \overline{\nabla}_X \rho\partial_t \rangle \\ &= -\langle AX, \rho\partial_t \rangle = -\langle X, A(\rho\partial_t^\top) \rangle. \end{aligned}$$

So, from the last equation we obtain $\nabla\tilde{\Theta} = -\rho A(\partial_t^\top)$, and from [CL, Proposition 2.1],

$$\Delta\tilde{\Theta} = n\rho\partial_t^\top(H) + n\rho' H - nN(p') + (|A|^2 + \overline{\mathrm{Ric}}(N,N))\tilde{\Theta}. \tag{3.4}$$

On the other hand, taking into account our restriction on the weight function, from (1.4) we get

$$\begin{aligned} n\partial_t^\top(H) &= \partial_t^\top \langle \overline{\nabla} f, N \rangle \\ &= \langle \overline{\nabla}_{\partial_t} \overline{\nabla} f, N \rangle + \Theta \langle \overline{\nabla}_N \overline{\nabla} f, N \rangle - \langle \overline{\nabla} f, A(\partial_t^\top) \rangle \\ &= \overline{\mathrm{Hess}}\, f(N, \partial_t) + \Theta\, \overline{\mathrm{Hess}}\, f(N,N) + \rho^{-1} \langle \nabla f, \nabla\tilde{\Theta} \rangle. \end{aligned} \tag{3.5}$$

Moreover, it is not difficult to verify that

$$\overline{\mathrm{Hess}}\, f(N,\partial_t) = -\rho^{-1}\rho'\langle\overline{\nabla} f, N\rangle = -n\rho^{-1}\rho' H. \tag{3.6}$$

It follows from (3.4)–(3.6) that

$$\Delta\tilde{\Theta} = \langle\nabla f, \nabla\tilde{\Theta}\rangle - nN(\rho') + (|A|^2 + \overline{\mathrm{Ric}}_f(N,N))\tilde{\Theta}.$$

Thus, from (1.3) we obtain

$$\begin{aligned}\Delta_f\tilde{\Theta} &= \Delta\tilde{\Theta} - \langle\nabla f, \nabla\tilde{\Theta}\rangle\\ &= -nN(\rho') + (|A|^2 + \overline{\mathrm{Ric}}_f(N,N))\tilde{\Theta}.\end{aligned}$$

Now, writing $N = N^* - \Theta\partial_t$, where N^* denotes the orthogonal projection of N onto the fiber M^n, we find that

$$N(\rho') = -\Theta\partial_t(\rho') = -\frac{\rho''}{\rho}\tilde{\Theta}.$$

Hence, we conclude that

$$L_f\tilde{\Theta} = n\frac{\rho''}{\rho}\tilde{\Theta}. \tag{3.7}$$

Therefore, we conclude that Σ^n is L_f-stable.

Now, let us consider (b). In this case, we have $C_0^\infty(\Sigma) = C^\infty(\Sigma)$. So, if Σ^n is L_f-stable, we obtain

$$0 \geq \int_\Sigma \tilde{\Theta}L_f\tilde{\Theta}e^{-f}\,d\sigma = \int_\Sigma n\frac{\rho''}{\rho}\tilde{\Theta}^2e^{-f}\,d\sigma \geq 0,$$

that is, $\rho'' = 0$ on Σ^n. The converse follows from item (a).

Finally, we prove (c). From the definition of L_f-stability we conclude that

$$\int_\Sigma \tilde{\Theta}L_f\tilde{\Theta}e^{-f}\,d\sigma = \int_\Sigma n\frac{\rho''}{\rho}\tilde{\Theta}^2e^{-f}\,d\sigma > 0.$$

Therefore, Σ^n cannot be L_f-stable. ∎

REMARK 3.4. Let $\overline{M}_f^{n+1} = -I\times_\rho M_f^n$ be a weighted GRW spacetime whose fiber M^n is compact. In particular, the slices in $\overline{M}_f^{n+1}$ are also compact. Assume in addition that $\rho'' \geq 0$ and that

$$\Omega = \{t \in I : \rho''(t) = 0\}$$

is a set of isolated points. Then, for every $t \in \Omega$ such that $\rho'(t) = 0$, the f-maximal slice $\Sigma_t = \{t\}\times M^n$ is L_f-stable.

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