Group-theoretical independence of
ℓ-adic Galois representations

by

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Introduction. Following Serre (cf. [19]), let us call an arbitrary family $(\rho_\ell : G \to G_\ell)_{\ell \in \mathbb{L}}$ of continuous homomorphisms of profinite groups *independent* if the induced homomorphism $\rho : G \to \prod_{\ell \in \mathbb{L}} \rho_\ell(G)$ is surjective. The family $(\rho_\ell : G \to G_\ell)_{\ell \in \mathbb{L}}$ is said to be *group-theoretically independent* if for all prime numbers $\ell_1 \neq \ell_2$ the groups $\rho_{\ell_1}(\mathrm{Gal}(K))$ and $\rho_{\ell_2}(\mathrm{Gal}(K))$ do not have a common finite simple group as a quotient. It is known that group-theoretical independence implies independence (cf. [19, Lemme 2]). The following results have been established in the series of papers [19], [8], [3], [4].

Let K be a field of characteristic $p \geq 0$ and X/K a separated algebraic scheme. Let $\mathbb{L}' = \mathbb{L} \smallsetminus \{p\}$. Let $q \in \mathbb{N}$ and let ρ_ℓ be the representation of $\mathrm{Gal}(K)$ on $H^q(X_{\overline{K}}, \mathbb{Q}_\ell)$.

  (I1) *If K is a finitely generated field of characteristic zero, then there exists a finite extension E/K such that the restricted family $(\rho_\ell|\mathrm{Gal}(E))_{\ell \in \mathbb{L}}$ is independent.*
  (I2) *If K is a function field over an algebraically closed field, then there exists a finite separable extension E/K such that $(\rho_\ell|\mathrm{Gal}(E))_{\ell \in \mathbb{L}'}$ is group-theoretically independent.*

In the important special case where K is a number field, (I1) was proved by Serre [19]. When $\mathrm{trdeg}(K/\mathbb{Q}) > 0$, (I1) was proved by Gajda and the author [8], answering a question of Serre (cf. [19, Section 3.2], [20, 10.1?]) and Illusie (cf. [11, 5.5]). (I2) was proved by Böckle, Gajda and the author [3] and independently by Cadoret and Tamagawa [4].

One cannot replace the word "independent" by "group-theoretically independent" in (I1): If K is a finitely generated field of characteristic zero,

then $\dim_{\mathbb{Q}_\ell}(H^2(\mathbb{P}^1_{\overline{K}}, \mathbb{Q}_\ell)) = 1$ and the action of $\mathrm{Gal}(K)$ on $H^2(\mathbb{P}^1_{\overline{K}}, \mathbb{Q}_\ell)$ is given by the inverse of the cyclotomic character $\varepsilon_{K,\ell} : \mathrm{Gal}(K) \to \mathbb{Z}_\ell^\times$ over K, and it can easily be seen from the prime number theorem in arithmetic progressions that the family of cyclotomic characters $(\varepsilon_{K,\ell})_{\ell \in \mathbb{L}}$ is *not* group-theoretically independent.

Assume for the moment that K is a finitely generated field without any restriction on its characteristic $p \geq 0$. In the light of the above facts the following question comes up naturally.

Are the cyclotomic characters the only obstruction to group-theoretical independence, i.e. is there a finite separable extension E/K such that $(\rho_\ell|\mathrm{Gal}(E(\mu_\infty)))_{\ell \in \mathbb{L}'}$ is group-theoretically independent?

If $p > 0$, then the answer is "yes", because then the prime field $\mathbb{F}$ of K is finite and hence $\overline{\mathbb{F}}$ agrees with the field $\mathbb{F}(\mu_\infty)$ obtained from $\mathbb{F}$ by adjoining all roots of unity; one can thus apply (I2) to $X_{\overline{\mathbb{F}}K}/\overline{\mathbb{F}}K$ to conclude that there is a finite separable extension E/K such that $(\rho_\ell|\mathrm{Gal}(\overline{\mathbb{F}}E)_{\ell \in \mathbb{L}})$ is group-theoretically independent. Somewhat surprisingly, the answer to the above question is "no" in the case $p = 0$ (cf. Corollary A.2, Remark A.3), as we shall see in the appendix. One can construct counterexamples from certain CM abelian varieties. We do have the following affirmative result, however.

MAIN THEOREM (cf. Corollary 3.2). *Let K be a finitely generated field of characteristic 0, and X/K a smooth projective variety. Let $q \in \mathbb{N}$ and let ρ_ℓ be the representation of $\mathrm{Gal}(K)$ on $H^q(X_{\overline{K}}, \mathbb{Q}_\ell)$. Then there are finite Galois extensions $k/\mathbb{Q}$ and F/K such that the family $(\rho_\ell|\mathrm{Gal}(k_{\mathrm{ab}}F))_{\ell \in \mathbb{L}}$ is group-theoretically independent, where k_{ab} stands for the maximal abelian Galois extension of k. Furthermore $(\rho_\ell|\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F))_{\ell \in \mathbb{L}}$ is group-theoretically independent, where $k_{\mathrm{ab}}^\dagger$ is the compositum $\prod_{\ell \in \mathbb{L}} k_{ab}^{(\ell)}$ with $k_{ab}^{(\ell)}/k$ being the maximal abelian extension of k which is unramified outside ℓ and of order prime to ℓ.*

In certain special cases it can be shown that group-theoretical independence is achieved already over the smaller extension $F_{\mathrm{cyc}}^\dagger = \prod_{\ell \in \mathbb{L}} F(\mu_\ell)$ where $F(\mu_\ell)$ is the field obtained from F by adjoining an ℓth root of unity. For example, if in the situation of the main theorem $q = 1$ and X is an abelian variety with $\mathrm{End}_{\overline{K}}(X) = \mathbb{Z}$ and of dimension 2, 6 or odd, then, based on Serre's open image theorem [17, 2.2.8], one can easily see that in this case one can choose the finite Galois extension F/K in such a way that $(\rho_\ell|\mathrm{Gal}(F_{\mathrm{cyc}}^\dagger))_{\ell \in \mathbb{L}}$ is group-theoretically independent. On the other hand, if $q = 1$, K is a number field and X/K is an absolutely simple abelian variety with complex multiplication over K, then $(\rho_\ell|\mathrm{Gal}(F_{\mathrm{cyc}}^\dagger))_{\ell \in \mathbb{L}}$ is group-theoretically independent for no finite Galois extension F/K (cf. Corollary A.2, Remark A.3).

There are three main ingredients to the proof: (1) We make strong use of certain concepts from group theory (profinite and algebraic). In particular we crucially use information about the structure of subgroups of $\mathrm{GL}_n(\mathbb{F}_\ell)$ (cf. [14] and [3, Section 3]) and about point groups of reductive algebraic groups defined over finite fields (cf. Proposition 1.1). (2) We use information about the ramification of the representations under consideration provided by constructibility and semistability theorems in étale cohomology (cf. [11]). Furthermore we make use of Caruso's solution of Serre's tame inertia conjecture (cf. [5]) in order to control the ramification of ρ_ℓ "at primes above ℓ". (3) Finally, we invoke finiteness results for étale fundamental groups from geometric class field theory (cf. [12] and [8, Section 2]).

NOTATION. Throughout this manuscript, $\mathbb{L}$ denotes the set of all prime numbers. If K is a field, then we denote by $\overline{K}$ an algebraic closure of K and by $\mathrm{Gal}(K)$ its absolute Galois group. We denote by K_{ab} the maximal abelian extension of K and by K_{solv} the maximal prosolvable extension of K. Furthermore we denote by $K_{\mathrm{cyc}} = K(\mu_\infty)$ the extension obtained from K by adjoining all roots of unity and write $K_{\mathrm{cyc}}^\dagger = \prod_{\ell \in \mathbb{L}} K(\mu_\ell)$. If K is a number field, then for $\ell \in \mathbb{L}$ we denote by $K_{\mathrm{ab}}^{(\ell)}$ the maximal abelian extension of K which is unramified outside ℓ and has degree prime to ℓ. Furthermore we define $K_{\mathrm{ab}}^\dagger := \prod_{\ell \in \mathbb{L}} K_{\mathrm{ab}}^{(\ell)}$.

A K-*variety* is a separated algebraic K-scheme which is reduced and irreducible. For a profinite group G we denote by $S_\ell(G)$ the normal subgroup of G generated by its ℓ-Sylow subgroups. If ℓ is clear from the context, then we write G^+ instead of $S_\ell(G)$.

1. Preliminaries on group theory. This section is devoted to the concepts from group theory used in this paper. The following proposition about point groups of connected algebraic groups over finite fields is probably well-known to experts. It will be applied later in the situation of the Main Theorem to suitable reductive envelopes of the images of the semisimplified mod-ℓ representations under consideration.

PROPOSITION 1.1. *Let F be a finite field of characteristic ℓ and $\underline{G}$ a connected algebraic group over F. Then the group $\underline{G}(F)/\underline{G}(F)^+$ is an* abelian *group of order prime to ℓ.*

Proof. Let $\underline{U}$ be the unipotent radical of $\underline{G}$ and $\underline{H} = \underline{G}/\underline{U}$. Let $\underline{S} = [\underline{H}, \underline{H}]$ be the derived group of $\underline{H}$. The group $\underline{U}(F)$ is a normal subgroup of $\underline{G}(F)$ of ℓ-power order. Hence $\underline{U}(F) \subset \underline{G}(F)^+ \subset \underline{G}(F)$. Furthermore $\underline{H}(F) = \underline{G}(F)/\underline{U}(F)$ because $\underline{U}$ is connected. It follows that $H(F)^+ = \underline{G}(F)^+/\underline{U}(F)$ and $\underline{G}(F)/\underline{G}(F)^+ \cong \underline{H}(F)/\underline{H}(F)^+$. Thus we can assume that $\underline{G}$ is reductive (i.e. $\underline{U}$ is trivial and $\underline{G} = \underline{H}$). Then $\underline{T} := \underline{G}/\underline{S}$ is a torus and

$\underline{G}(F)/\underline{S}(F)$ embeds into $\underline{T}(F)$. Hence $\underline{G}(F)/\underline{S}(F)$ is abelian of order prime to ℓ. This implies that $\underline{G}(F)^+ = \underline{S}(F)^+$.

Let $\underline{Z}$ be the centre of $\underline{G}$. Let $\widetilde{\underline{S}} \to \underline{S}$ be the simply connected covering of the semisimple group $\underline{S}$ and let I be the image of $\widetilde{\underline{S}}(F) \to \underline{S}(F)$. Then $\underline{S}(F)^+ = I$ by a theorem of Steinberg (cf. [22, 12.4, 12.6]). It thus suffices to show that $[\underline{G}(F), \underline{G}(F)] \subset I$. To prove this, let $a_1, a_2 \in \underline{G}(F)$ and $s = [a_1, a_2]$. Then $s \in \underline{S}(F)$ and we denote by $\overline{a_i}$ the image of a_i in $\underline{G}/\underline{Z}(F)$. We have central isogenies

$$\widetilde{\underline{S}} \xrightarrow{f} \underline{S} \xrightarrow{g} \underline{G}/\underline{Z}.$$

Choose for $i \in \{1, 2\}$ an $\tilde{a}_i \in \widetilde{\underline{S}}(\overline{F})$ such that $g \circ f(\tilde{a}_i) = \overline{a_i}$ and define $\tilde{s} = [\tilde{a}_1, \tilde{a}_2]$. There exist $z_i \in \underline{Z}(\overline{F})$ such that $f(\tilde{a}_i) = z_i a_i$. Thus

$$f(\tilde{s}) = [f(\tilde{a}_1), f(\tilde{a}_2)] = [z_1 a_1, z_2 a_2] = [a_1, a_2] = s.$$

For all $\sigma \in \mathrm{Gal}(F)$ and $i \in \{1, 2\}$ there exists $k_i \in \ker(g \circ f)(\overline{F})$ such that $\tilde{a}_i^\sigma = k_i \tilde{a}_i$. Hence $\tilde{s}^\sigma = [k_1 \tilde{a}_1, k_2 \tilde{a}_2] = \tilde{s}$, because the k_i lie in the centre of $\widetilde{\underline{S}}(\overline{F})$. It follows that $\tilde{s} \in \widetilde{\underline{S}}(F)$ is F-rational. Thus $s \in I$ as desired. ∎

DEFINITION 1.2. For $d \in \mathbb{N}$ we denote by $\mathfrak{B}(d)$ the class of all finite groups of order $\leq d$, and by $\mathfrak{Jor}(d)$ the class of all finite groups G which have an abelian normal subgroup N such that $G/N \in \mathfrak{B}(d)$. For $\ell \in \mathbb{L}$ denote by $\mathfrak{Lie}_\ell$ the class of all finite simple groups of Lie type in characteristic ℓ. Let $\mathfrak{Lie}_\ell(d)$ be the class of all finite groups G which have an abelian normal subgroup N such that $|N|$ is coprime to ℓ, $|N| \leq d$ and G/N is isomorphic to a finite product of groups in $\mathfrak{Lie}_\ell$.

Note that the product is allowed to be empty and thus the trivial group lies in $\mathfrak{Lie}_\ell(d)$ (but not in $\mathfrak{Lie}_\ell$). The following theorem about finite subgroups of $\mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ is a corollary to a result of Larsen and Pink [14], established in [3, Section 3]. It slightly generalizes [19, Thms. 3′ and 4]. We shall later apply it in order to understand the group-theoretical properties of the images of certain mod-ℓ representations.

THEOREM 1.3 (cf. [3, Section 3]). *For every $n \in \mathbb{N}$ there exists a constant $J'(n) \geq 5$ with the following property: For every $\ell \in \mathbb{L}$ and every finite subgroup G of $\mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ the group G/G^+ lies in $\mathfrak{Jor}(J'(n))$. Moreover, if $\ell > J'(n)$ and P denotes the maximal normal ℓ-subgroup of G^+, then G^+/P lies in $\mathfrak{Lie}_\ell(2^{n-1})$.*

Let G be a profinite group and H a finite simple (not necessarily non-abelian) group. We call H a *Jordan–Hölder factor* of G if there exists a closed normal subgroup G_1 of G, an open normal subgroup G_2 of G_1 and a continuous isomorphism $G_1/G_2 \cong H$. We denote by $\mathrm{JH}(G)$ the class of all Jordan–Hölder factors of G. Let $\mathrm{FSQ}(G)$ be the class of all finite simple

(not necessarily non-abelian) quotients of G. Then $\mathrm{FSQ}(G) \subset \mathrm{JH}(G)$. The proof of the following elementary lemma is left to the reader.

LEMMA 1.4.

(a) *If* $1 \to G' \to G \xrightarrow{\pi} G'' \to 1$ *is an exact sequence of profinite groups, then* $\mathrm{JH}(G) = \mathrm{JH}(G') \cup \mathrm{JH}(G'')$ *and* $\mathrm{FSQ}(G) \subset \mathrm{FSQ}(G') \cup \mathrm{FSQ}(G'')$.
(b) *Let* G *be a profinite group. Let* $(N_i)_{i \in I}$ *be a family of closed normal subgroups of* G *and* $N = \bigcap_{i \in I} N_i$. *Then* $\mathrm{JH}(G/N) \subset \bigcup_{i \in I} \mathrm{JH}(G/N_i)$.

Theorem 1.3 will frequently enter our considerations through the following remark.

REMARK 1.5. Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$ where $\ell \geq J'(n)$. Let P be the maximal normal ℓ-subgroup of G^+.

(a) The group G/G^+ lies in $\mathfrak{Jor}(J'(n))$ (cf. Theorem 1.3) and thus

$$\mathrm{JH}(G/G^+) \subset \mathfrak{B}(J'(n)) \cup \{\mathbb{Z}/p : p \in \mathbb{L} \smallsetminus \{\ell\}\}$$

by Lemma 1.4.
(b) The group G^+/P lies in $\mathfrak{Lie}_\ell(2^{n-1})$ (cf. Theorem 1.3) and thus

$$\mathrm{JH}(G^+) \subset \mathfrak{Lie}_\ell \cup \{\mathbb{Z}/p : p \in \mathbb{L} \smallsetminus \{\ell\}, p \leq 2^{n-1}\} \cup \{\mathbb{Z}/\ell\}$$

by Lemma 1.4. Furthermore $\mathrm{FSQ}(G^+)$ cannot contain groups of order prime to ℓ as G^+ is generated by its ℓ-Sylow subgroups, and consequently $\mathrm{FSQ}(G^+) \subset \mathfrak{Lie}_\ell \cup \{\mathbb{Z}/\ell\}$.

For technical reasons the following lemma will be useful.

LEMMA 1.6. *Let* G *be a finite subgroup of* $\mathrm{GL}_n(\mathbb{F}_\ell)$. *Assume that* $\ell > J'(n)$. *Let* N *be a normal subgroup of* G^+. *If* $\mathrm{JH}(G/N) \cap \mathfrak{Lie}_\ell = \emptyset$, *then* $N = N^+$.

Proof. Let P be the maximal normal ℓ-subgroup of G and consider the exact sequence of groups

$$1 \to N/P \cap N \to G^+/P \to G^+/NP \to 1.$$

The group G^+/NP is a quotient of G^+/N, which is in turn isomorphic to a normal subgroup of G/N. Hence, by Lemma 1.4, $\mathrm{JH}(G^+/NP) \cap \mathfrak{Lie}_\ell = \emptyset$. On the other hand, G^+/NP is a quotient of G^+/P. As $\ell > J'(n)$, we have

$$\mathrm{FSQ}(G^+/NP) \subset \mathrm{FSQ}(G^+/P) \subset \mathfrak{Lie}_\ell$$

(cf. Remark 1.5). It follows that $\mathrm{FSQ}(G^+/NP) = \emptyset$, hence G^+/NP is the trivial group. By the exact sequence above, $N/P \cap N \cong G^+/P$ is a group generated by its ℓ-Sylow subgroups. As $P \cap N$ is an ℓ-group, it follows that N is generated by its ℓ-Sylow subgroups as well, so $N = N^+$. $\blacksquare$

2. The monodromy groups of mod-ℓ representations. Let K be a field of characteristic zero and X/K a smooth projective variety. Fix $q \in \mathbb{N}$. Let ρ_ℓ (resp. $\overline{\rho}_\ell$) be the representation of $\mathrm{Gal}(K)$ on $H^q(X_{\overline{K}}, \mathbb{Q}_\ell)$ (resp. on $H^q(X_{\overline{K}}, \mathbb{F}_\ell)$). The following lemma explains the relation between the ℓ-adic monodromy groups $\rho_\ell(\mathrm{Gal}(K))$ and the mod-ℓ monodromy groups $\overline{\rho}_\ell(\mathrm{Gal}(K))$. The rest of this section is then devoted to images of mod-ℓ representations.

LEMMA 2.1. *There is a constant D such that for every prime number $\ell \geq D$ there is an epimorphism $\pi_\ell : \rho_\ell(\mathrm{Gal}(K)) \to \overline{\rho}_\ell(\mathrm{Gal}(K))$ such that $\pi_\ell \circ \rho_\ell = \overline{\rho}_\ell$ and $P_\ell := \ker(\pi_\ell)$ is a pro-ℓ group.*

Proof. By a theorem of Gabber (cf. [7]) there exists a constant D such that $H^q(X_{\overline{K}}, \mathbb{Z}_\ell)$ and $H^{q+1}(X_{\overline{K}}, \mathbb{Z}_\ell)$ are torsion free $\mathbb{Z}_\ell$-modules for all prime numbers $\ell \geq D$. For every $\ell \in \mathbb{L}$ there is an exact sequence of $\mathrm{Gal}(K)$-modules

$$H^q(X_{\overline{K}}, \mathbb{Z}_\ell) \xrightarrow{\ell} H^q(X_{\overline{K}}, \mathbb{Z}_\ell) \to H^q(X_{\overline{K}}, \mathbb{F}_\ell)$$
$$\to H^{q+1}(X_{\overline{K}}, \mathbb{Z}_\ell) \xrightarrow{\ell} H^{q+1}(X_{\overline{K}}, \mathbb{Z}_\ell).$$

We set $V_\ell := H^q(X_{\overline{K}}, \mathbb{Q}_\ell)$, $T_\ell := H^q(X_{\overline{K}}, \mathbb{Z}_\ell)$ and $W_\ell := H^q(X_{\overline{K}}, \mathbb{F}_\ell)$. Then, for all $\ell \in \mathbb{L}$ with $\ell > D$ the natural map $T_\ell \otimes \mathbb{F}_\ell \to W_\ell$ is an isomorphism, because $H^{q+1}(X_{\overline{K}}, \mathbb{Z}_\ell)[\ell] = 0$. Furthermore $V_\ell = T_\ell \otimes \mathbb{Q}_\ell$ (by definition), and the natural map $T_\ell \to V_\ell$ must be injective because T_ℓ is torsion free. We denote by ρ'_ℓ the representation of $\mathrm{Gal}(K)$ on the finitely generated free $\mathbb{Z}_\ell$-module T_ℓ. The canonical maps

$$\mathrm{GL}_{W_\ell}(\mathbb{F}_\ell) \xleftarrow{F_\ell} \mathrm{GL}_{T_\ell}(\mathbb{Z}_\ell) \xrightarrow{G_\ell} \mathrm{GL}_{V_\ell}(\mathbb{Q}_\ell)$$

induce by restriction epimorphisms

$$\overline{\rho}_\ell(\mathrm{Gal}(K)) \xleftarrow{f_\ell} \rho'_\ell(\mathrm{Gal}(K)) \xrightarrow{g_\ell} \rho_\ell(\mathrm{Gal}(K))$$

such that $f_\ell \circ \rho'_\ell = \overline{\rho}_\ell$ and $g_\ell \circ \rho'_\ell = \rho_\ell$. Furthermore $\ker(f_\ell)$ is pro-ℓ because $\ker(F_\ell)$ is pro-ℓ, and g_ℓ is injective because G_ℓ is injective. It follows that $\pi_\ell := f_\ell \circ (g_\ell)^{-1}$ is an epimorphism $\rho_\ell(\mathrm{Gal}(K)) \to \overline{\rho}_\ell(\mathrm{Gal}(K))$ such that $\rho_\ell \circ \pi_\ell = \overline{\rho}_\ell$, and $P_\ell := \ker(\pi_\ell)$ is pro-ℓ. ∎

For a non-archimedean place v of a number field K we shall denote by $\mathrm{char}(v)$ its residue characteristic. Recall the definition of $K_{\mathrm{ab}}^\dagger$ from the Notation section.

LEMMA 2.2. *Let K be a number field, X/K a smooth projective geometrically irreducible variety and $q \in \mathbb{N}$. Let $\overline{\rho}_\ell$ be the representation of $\mathrm{Gal}(K)$ on $H^q(X_{\overline{K}}, \mathbb{F}_\ell)$. Then there exists a finite Galois extension E/K such that $\overline{\rho}_\ell(\mathrm{Gal}(E_{\mathrm{ab}}^\dagger)) \subset \overline{\rho}_\ell(\mathrm{Gal}(K))^+$ for every $\ell \in \mathbb{L}$.*

Proof. We denote by ρ_ℓ the representation of $\mathrm{Gal}(K)$ on $H^q(X_{\overline{K}}, \mathbb{Q}_\ell)$. By [11, Cor. 2.3] there exists a finite set S of places of K such that for every non-archimedian place v of K outside S and every prime number $\ell \neq \mathrm{char}(v)$ the representation ρ_ℓ is unramified at v. Furthermore there exists a finite Galois extension K'/K such that for every non-archimedian place v' of K' above S and every prime number $\ell \neq \mathrm{char}(v')$ the group $\rho_\ell(I_{v'})$ is a pro-ℓ group (cf. [6], [2, 6.3.2]). After replacing K' by a larger finite Galois extension of K we can assume in addition that $\overline{\rho}_\ell(\mathrm{Gal}(K')) = \{e\}$ for all $\ell \leq D$ where D is the constant from Lemma 2.1. It follows (via Lemma 2.1) that for every place v' of K' and every $\ell \neq \mathrm{char}(v')$ the group $\overline{\rho}_\ell(I_{v'})$ is an ℓ-group, which is trivial if v' does not lie over S.

Let W_ℓ^{ss} be the semisimplification of the $\mathrm{Gal}(K)$-module $H^q(X_{\overline{K}}, \mathbb{F}_\ell)$ and $\overline{\rho}_\ell^{\mathrm{ss}}$ be the representation of $\mathrm{Gal}(K)$ on W_ℓ^{ss}. By the above we see that for every place v' of K' and every prime number $\ell \neq \mathrm{char}(v')$ the group $\overline{\rho}_\ell^{\mathrm{ss}}(I_{v'})$ is an ℓ-group, which is trivial if v' does not lie over S. Let $\ell_0 = \max(\mathrm{char}(w) : w \in S)$. Let $\ell \in \mathbb{L}$ with $\ell > \ell_0$. If w' is a place of K' with $\mathrm{char}(w') = \ell$, then by Caruso [5, Thm. 1.2] the weight of the tame inertia group $I^t(w')$ acting via the contragredient of the semisimplification of the restricted representation $\overline{\rho}_\ell^{\mathrm{ss}}|I(w')$ is in the interval $[0, eq]$ where e is the ramification index of $K'_{w'}/\mathbb{Q}_\ell$. Furthermore $\dim(W_\ell^{\mathrm{ss}})$ does not depend on ℓ. Altogether we see that $\overline{\rho}_\ell^{\mathrm{ss}}|\mathrm{Gal}(K')$ satisfies the conditions (a)–(d) of [23, Section 3.3]. By [23, Thm. 4] (or by [10, Thm. 2.3.5]) there exists, after replacing ℓ_0 by a larger constant, a finite Galois extension L/K' and for every prime number $\ell \geq \ell_0$ a reductive algebraic subgroup $\underline{G}_\ell/\mathbb{F}_\ell$ of $\underline{GL}_{W_\ell^{\mathrm{ss}}}$ with the following properties:

(1) $\overline{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}(L)) \subset \underline{G}_\ell(\mathbb{F}_\ell)$ for every prime number $\ell \geq \ell_0$.
(2) $\overline{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}(L))^+ = \underline{G}_\ell(\mathbb{F}_\ell)^+$ for every prime number $\ell \geq \ell_0$.

We can replace L by its Galois closure and enlarge ℓ_0 accordingly in order to assume L/K is Galois. It follows from (1), (2) and Proposition 1.1 that $\overline{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}(L))/\overline{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}(L))^+$ is abelian for every prime number $\ell \geq \ell_0$. The kernel P_ℓ of the natural epimorphism $g : \overline{\rho}_\ell(\mathrm{Gal}(L)) \to \overline{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}(L))$ is an ℓ-group; hence it lies in $\overline{\rho}_\ell(\mathrm{Gal}(L))^+$. Thus g induces an isomorphism

$$\overline{\rho}_\ell(\mathrm{Gal}(L))/\overline{\rho}_\ell(\mathrm{Gal}(L))^+ \cong \overline{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}(L))/\overline{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}(L))^+.$$

It follows that $\overline{\rho}_\ell(\mathrm{Gal}(L))/\overline{\rho}_\ell(\mathrm{Gal}(L))^+$ is abelian for every $\ell \geq \ell_0$. In particular

$$\overline{\rho}_\ell(\mathrm{Gal}(E_{\mathrm{ab}})) \subset \overline{\rho}_\ell(\mathrm{Gal}(L))^+ \subset \overline{\rho}_\ell(\mathrm{Gal}(K))^+$$

for every finite extension E/L and every $\ell \geq \ell_0$. We now choose E to be a finite Galois extension of K containing $L \cdot \prod_{\ell \leq \ell_0} K(\overline{\rho}_\ell)$. Then $\overline{\rho}_\ell(\mathrm{Gal}(E_{\mathrm{ab}})) \subset \overline{\rho}_\ell(\mathrm{Gal}(K))^+$ for every $\ell \in \mathbb{L}$. Moreover, for every $\ell \in \mathbb{L}$, the group $\overline{\rho}_\ell(I_v)$ is an ℓ-group and hence is contained in $\overline{\rho}_\ell(\mathrm{Gal}(K))^+$ for every place v of E

with $\mathrm{char}(v) \neq \ell$. Thus $\overline{\rho}(\mathrm{Gal}(E_{ab}^{(\ell)})) \subset \overline{\rho}_\ell(\mathrm{Gal}(K))^+$ for all $\ell \in \mathbb{L}$, so the assertion follows with the above choice of E. ∎

We shall now generalize Lemma 2.2 to the situation where the ground field K is an arbitrary finitely generated extension of $\mathbb{Q}$, possibly of transcendence degree ≥ 1. For this we use a specialization argument along with a finiteness theorem for unramified Jordan extensions from [8]; the latter theorem in turn relies on a finiteness theorem in geometric class field theory of Katz and Lang (cf. [12]) and on some finiteness results for geometric fundamental groups from SGA.

PROPOSITION 2.3. *Let $K/\mathbb{Q}$ be a finitely generated extension of fields. Let X/K be a smooth projective variety. For every prime number ℓ let $\overline{\rho}_\ell$ be the representation of $\mathrm{Gal}(K)$ on $H^q(X_{\overline{K}}, \mathbb{F}_\ell)$. Then there exist finite Galois extensions E/K and $k/\mathbb{Q}$ such that $\overline{\rho}_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger E)) \subset \overline{\rho}_\ell(\mathrm{Gal}(K))^+$ for every $\ell \in \mathbb{L}$.*

Proof. There exists a finite Galois extension K'/K such that $X_{K'}$ splits up into a disjoint sum of geometrically connected (smooth projective) K'-varieties. Once the proposition is true for every connected component of $X_{K'}/K'$ it will follow for X/K. We may thus assume right from the outset that X/K is geometrically connected.
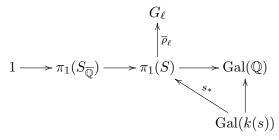
There exists a $\mathbb{Q}$-variety S with function field K. Moreover, by the usual spreading-out principles, there exists after replacing S by one of its dense open subschemes a smooth projective morphism $f : \mathscr{X} \to S$ with generic fibre X. The stalk of $f_*\mathscr{O}_{\mathscr{X}}$ at the generic point of S is zero because X/K is geometrically connected. Now, after replacing S by one of its non-empty open subschemes and shrinking $\mathscr{X}$ accordingly, we may assume that $f_*\mathscr{O}_{\mathscr{X}} = 0$, where $\mathscr{O}_{\mathscr{X}}$ stands for the structure sheaf of $\mathscr{X}$. Then f has geometrically connected fibres (cf. [9, 4.3.4]). Shrinking S and $\mathscr{X}$ once more one can assume that the étale sheaves $R^q f_* \mathbb{F}_\ell$ are lisse and compatible with any base change [11, Cor. 2.2]. In particular the representation $\overline{\rho}_\ell$ factors through [1] the étale fundamental group $\pi_1(S)$.

Let $G_\ell = \overline{\rho}_\ell(\pi_1(S))$ and let n be an upper bound for the dimensions of the $\mathbb{F}_\ell$-vector spaces $H^q(X_{\overline{K}}, \mathbb{F}_\ell)$ ($\ell \in \mathbb{L}$). The existence of such an upper bound is guaranteed by [11, Thm. 1.1]. Now $\overline{\rho}_\ell(\mathrm{Gal}(K))$ is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$ for all $\ell \in \mathbb{L}$. Thus there exists a constant $J'(n)$ such that $G_\ell/G_\ell^+ \in \mathfrak{Jor}(J'(n))$ for all $\ell \in \mathbb{L}$ (cf. Theorem 1.3). Hence, by [8, Prop. 2.2], there exists an open normal subgroup U of $\pi_1(S)$ such that $\overline{\rho}_\ell(U \cap \pi_1(S_{\overline{\mathbb{Q}}})) \subset G_\ell^+$ for all $\ell \in \mathbb{L}$. Let $s \in S$ be a closed point. Let S' be

---

[1] We take the étale fundamental group with respect to the geometric generic base point of S afforded by the choice of $\overline{K}$.

the finite étale Galois cover of S corresponding to U and pick a closed point $s' \in S'$ over s. Let E be the function field of S'.
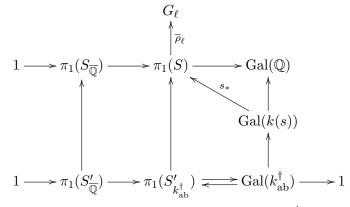
Consider the following commutative diagram of profinite groups:

$$
\begin{array}{ccccccc}
 & & & & G_\ell & & \\
 & & & & \uparrow{\scriptstyle\overline{\rho}_\ell} & & \\
1 & \longrightarrow & \pi_1(S_{\overline{\mathbb{Q}}}) & \longrightarrow & \pi_1(S) & \longrightarrow & \mathrm{Gal}(\mathbb{Q}) \\
 & & & & \swarrow{\scriptstyle s_*} & & \uparrow \\
 & & & & \mathrm{Gal}(k(s)) & &
\end{array}
$$

(The map s_* is well-defined only up to conjugation.) If $X_s = \mathscr{X} \times_S \mathrm{Spec}(k(s))$ is the special fibre of $\mathscr{X}$ over S, then, by the base change compatibility alluded to above, the representation $\overline{\rho}_\ell \circ s_*$ of $\mathrm{Gal}(k(s))$ on $H^q(X_{\overline{K}}, \mathbb{F}_\ell)$ is isomorphic to the representation of $\mathrm{Gal}(k(s))$ on $H^q(X_{s,\overline{k(s)}}, \mathbb{F}_\ell)$. Furthermore X_s is a smooth projective geometrically connected variety over the number field $k(s)$. By Lemma 2.2 there is a finite Galois extension $k/\mathbb{Q}$ containing $k(s)$ such that

$$
\overline{\rho}_\ell \circ s_*(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger)) \subset \overline{\rho}_\ell \circ s_*(\mathrm{Gal}(k))^+ \subset G_\ell^+.
$$

After replacing k by a finite extension we can also assume that $k \supset k(s')$. Now there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
 & & & & G_\ell & & & & \\
 & & & & \uparrow{\scriptstyle\overline{\rho}_\ell} & & & & \\
1 & \longrightarrow & \pi_1(S_{\overline{\mathbb{Q}}}) & \longrightarrow & \pi_1(S) & \longrightarrow & \mathrm{Gal}(\mathbb{Q}) & & \\
 & & \uparrow & & \uparrow & \swarrow{\scriptstyle s_*} & \uparrow & & \\
 & & & & & \mathrm{Gal}(k(s)) & & & \\
 & & & & & \uparrow & & & \\
1 & \longrightarrow & \pi_1(S'_{\overline{\mathbb{Q}}}) & \longrightarrow & \pi_1(S'_{k_{\mathrm{ab}}^\dagger}) & \rightleftarrows & \mathrm{Gal}(k_{\mathrm{ab}}^\dagger) & \longrightarrow & 1
\end{array}
$$

We already know that $\overline{\rho}_\ell(\pi_1(S'_{\overline{\mathbb{Q}}}))$ and $\overline{\rho}_\ell(s_*(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger)))$ are contained in G_ℓ^+. As $\pi_1(S'_{k_{\mathrm{ab}}^\dagger})$ is generated by $\pi_1(S'_{\overline{\mathbb{Q}}})$ and $s_*(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger))$ we conclude that $\overline{\rho}_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger E)) = \overline{\rho}_\ell(\pi_1(S'_{k_{\mathrm{ab}}^\dagger})) \subset G_\ell^+$ as desired. ∎

REMARK 2.4. In the situation of Proposition 2.3 it is easy to see (with the help of Remark 1.5 and Lemma 1.4) that

$$
\mathrm{JH}(\overline{\rho}_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger E))) \subset \mathfrak{Lie}_\ell \cup \{\mathbb{Z}/p : p \in \mathbb{L} \smallsetminus \{\ell\},\, p \le 2^{n-1}\} \cup \{\mathbb{Z}/\ell\}
$$

for all but finitely many primes $\ell \in \mathbb{L}$.

Note, however, that this does not rule out the possibility that for some small prime number p there exist infinitely many $\ell \in \mathbb{L}$ such that $\mathbb{Z}/p$ is a finite simple quotient of $\overline{\rho}_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger E))$. Hence Proposition 2.3 alone does not imply group-theoretical independence for the family $(\overline{\rho}_\ell|\mathrm{Gal}(k_{\mathrm{ab}}^\dagger E))_{\ell \in \mathbb{L}}$; we need additional arguments to establish the Main Theorem.

3. Independence results. In the following theorem we shall prove among other things that in the situation of Proposition 2.3 one can, after replacing E by a finite extension F which is Galois over K, achieve a very good control over the possible finite simple quotients of $\overline{\rho}_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F))$ and of $\rho_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F))$. In particular we shall see that for a suitable choice of F the groups $\overline{\rho}_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F))$ cannot have a finite simple quotient of order prime to ℓ any more. The argument is of a group-theoretical nature. The Main Theorem about group-theoretical independence along with some variants will then follow quite easily.

THEOREM 3.1. *Let $K/\mathbb{Q}$ be a finitely generated field extension. Let X/K be a smooth projective variety. Fix $q \in \mathbb{N}$. Let ρ_ℓ (resp. $\overline{\rho}_\ell$) be the representation of $\mathrm{Gal}(K)$ on $H^q(X_{\overline{K}}, \mathbb{Q}_\ell)$ (resp. on $H^q(X_{\overline{K}}, \mathbb{F}_\ell)$). Let $\ell_0 \in \mathbb{N}$. Then there are finite Galois extensions F/K and $k/\mathbb{Q}$ with the following properties:*

(a) *For every algebraic extension Ω/F and every $\ell \leq \ell_0$ in $\mathbb{L}$ the group $\overline{\rho}_\ell(\mathrm{Gal}(\Omega))$ is trivial and $\rho_\ell(\mathrm{Gal}(\Omega))$ is a pro-ℓ group.*
(b) *For every solvable Galois extension $\Omega/k_{\mathrm{ab}}^\dagger F$ and every $\ell \in \mathbb{L}$ we have $\overline{\rho}_\ell(\mathrm{Gal}(\Omega)) = \overline{\rho}_\ell(\mathrm{Gal}(\Omega))^+$, and there exists a closed normal pro-ℓ subgroup Q_ℓ of $\rho_\ell(\mathrm{Gal}(\Omega))$ such that $\rho_\ell(\mathrm{Gal}(\Omega))/Q_\ell \cong \overline{\rho}_\ell(\mathrm{Gal}(\Omega))$.*
(c) *For every solvable Galois extension $\Omega/k_{\mathrm{ab}}^\dagger F$ and every $\ell \in \mathbb{L}$ we have*

$$\mathrm{FSQ}(\overline{\rho}_\ell(\mathrm{Gal}(\Omega))) \subset \mathfrak{Lie}_\ell \cup \mathbb{Z}/\ell \quad \text{and} \quad \mathrm{FSQ}(\rho_\ell(\mathrm{Gal}(\Omega))) \subset \mathfrak{Lie}_\ell \cup \{\mathbb{Z}/\ell\}.$$

Proof. There exists $n \in \mathbb{N}$ such that

$$\dim_{\mathbb{Q}_\ell}(H^q(X_{\overline{K}}, \mathbb{Q}_\ell)) \leq \dim_{\mathbb{F}_\ell}(H^q(X_{\overline{K}}, \mathbb{F}_\ell)) \leq n$$

for all $\ell \in \mathbb{L}$ (cf. [11, Thm. 1.1]). We can assume right from the outset that $\ell_0 \geq \max(J'(n), D)$ where $J'(n)$ is the constant from Theorem 1.3 and D is the constant from Lemma 2.1. We put $G_\ell := \rho_\ell(\mathrm{Gal}(K))$ and $\overline{G}_\ell = \overline{\rho}_\ell(\mathrm{Gal}(K))$. For every $\ell \in \mathbb{L}$ the maximal normal pro-ℓ subgroup P_ℓ of G_ℓ is open.

By Proposition 2.3 there exists a finite Galois extension E/K with $\overline{\rho}_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger E)) \subset G_\ell^+$ for every $\ell \in \mathbb{L}$. Let $E' := E \cdot \prod_{\ell \leq \ell_0} \overline{K}^{\rho_\ell^{-1}(P_\ell)} \cdot \overline{K}^{\ker(\overline{\rho}_\ell)}$. Pick a prime number $\ell_1 > \max(\ell_0, [E' : K])$. Let $F := E' \cdot \prod_{\ell_0 < \ell \leq \ell_1} \overline{K}^{\ker(\overline{\rho}_\ell)}$. The E'/K and F/K are finite Galois extensions because P_ℓ is open and

normal in G_ℓ, and $\overline{G}_\ell$ is finite. Note that for every algebraic extension Ω/F the group $\rho_\ell(\mathrm{Gal}(\Omega))$ is pro-ℓ for every $\ell \le \ell_0$ in $\mathbb{L}$ and $\overline{\rho}_\ell(\mathrm{Gal}(\Omega))$ is trivial for every $\ell \le \ell_1$ in $\mathbb{L}$. In particular (a) holds true.

To prove (b) let Ω be a solvable Galois extension of $k_{\mathrm{ab}}^\dagger F$. We already know that (b) holds true for all $\ell \le \ell_0$. For $\ell > \ell_0$, by Lemma 2.1, the group $\rho_\ell(\mathrm{Gal}(\Omega))$ is an extension of $\overline{\rho}_\ell(\mathrm{Gal}(\Omega))$ by a pro-ℓ group. As we know that $\overline{\rho}_\ell(\mathrm{Gal}(\Omega))$ is trivial for all $\ell \le \ell_1$ it follows that $\rho_\ell(\mathrm{Gal}(\Omega))$ is pro-ℓ for all $\ell \le \ell_1$. Thus (b) holds true for every $\ell \le \ell_1$, and to establish (b) completely it suffices to prove

CLAIM. $\overline{\rho}_\ell(\mathrm{Gal}(\Omega)) = \overline{\rho}_\ell(\mathrm{Gal}(\Omega))^+$ *for all* $\ell > \ell_1$.

Let $\mathfrak{C}$ be the class of all prime cyclic groups. We shall now compute $\mathrm{JH}(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F/K))$ and then establish the Claim with the help of Lemma 1.6. By Lemma 1.4,

$$\mathrm{JH}(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F/K)) \subset \mathrm{JH}(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger K/K)) \cup \mathrm{JH}(\mathrm{Gal}(E'/K)) \cup \bigcup_{\ell_0 < \ell \le \ell_1} \mathrm{JH}(\overline{G}_\ell),$$

and $\mathrm{JH}(\mathrm{Gal}(E'/K)) \subset \mathfrak{B}([E'/K]) \subset \mathfrak{B}(\ell_1)$. Moreover $\mathrm{JH}(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger K/K)) \subset \mathfrak{C}$. If $\ell_0 < \ell \le \ell_1$, then $J'(n) < \ell$, and hence

$$\mathrm{JH}(\overline{G}_\ell) \subset \mathrm{JH}(\overline{G}_\ell^+) \cup \mathrm{JH}(\overline{G}_\ell/\overline{G}_\ell^+).$$

We have $\mathrm{JH}(\overline{G}_\ell^+) \subset \mathfrak{Lie}_\ell \cup \mathfrak{C}$ (cf. Remark 1.5), and

$$\mathrm{JH}(\overline{G}_\ell/\overline{G}_\ell^+) \subset \mathfrak{B}(J'(n)) \cup \mathfrak{C} \subset \mathfrak{B}(\ell_1) \cup \mathfrak{C}$$

(again cf. Remark 1.5). Altogether it follows that

$$\mathrm{JH}(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F/K)) \subset \mathfrak{C} \cup \mathfrak{B}(\ell_1) \cup \bigcup_{\ell_0 < \ell \le \ell_1} \mathfrak{Lie}_\ell.$$

For $\ell > \ell_1$ the groups in $\mathfrak{Lie}_\ell$ are non-commutative and generated by their ℓ-Sylow subgroups, and hence they are contained neither in $\mathfrak{C}$ nor in $\mathfrak{B}(\ell_1)$. Using a theorem of E. Artin about the orders of the finite simple groups of Lie type (cf. [19, Thm. 5], see also [1], [13]) we see that

$$\mathrm{JH}(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F/K)) \cap \mathfrak{Lie}_\ell = \emptyset \quad \text{for all } \ell > \ell_1.$$

We now prove the claim. Let $\ell > \ell_1$. Let $N_\ell = \overline{\rho}_\ell(\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F))$ and $M_\ell = \overline{\rho}_\ell(\mathrm{Gal}(\Omega))$. As $F \supset E$ by construction, we see that $N_\ell \subset G_\ell^+$. Moreover G_ℓ/N_ℓ is a quotient of $\mathrm{Gal}(k_{\mathrm{ab}}^\dagger F/K)$. Hence $\mathrm{JH}(G_\ell/N_\ell) \cap \mathfrak{Lie}_\ell = \emptyset$ for all $\ell > \ell_1$. Lemma 1.6 implies $N_\ell = N_\ell^+$. But then $M_\ell \subset N_\ell = N_\ell^+$, and moreover N_ℓ/M_ℓ is a quotient of $\mathrm{Gal}(\Omega/k_{\mathrm{ab}}^\dagger F)$. As $\Omega/k_{\mathrm{ab}}^\dagger F$ is solvable, we see that $\mathrm{JH}(N_\ell/M_\ell) \cap \mathfrak{Lie}_\ell = \emptyset$ for all $\ell > \ell_1$. Applying Lemma 1.6 once more, we find that $M_\ell = M_\ell^+$ for all $\ell > \ell_1$. This finishes the proof of the Claim and of part (b).

We now turn to part (c). If $\ell > \ell_1$, then (recall that $\ell_1 \geq \ell_0 \geq J'(n)$) $\mathrm{FSQ}(M_\ell) \subset \mathfrak{Lie}_\ell \cup \{\mathbb{Z}/\ell\}$ by Remark 1.5. If $\ell \leq \ell_1$, then we even have $\mathrm{FSQ}(M_\ell) = \emptyset$ by part (a). Thus $\mathrm{FSQ}(M_\ell) \subset \mathfrak{Lie}_\ell \cup \{\mathbb{Z}/\ell\}$ for all $\ell \in \mathbb{L}$. As $\rho_\ell(\mathrm{Gal}(\Omega))$ is an extension of M_ℓ by a pro-ℓ group, the statement about ρ_ℓ in part (c) is also true. ∎

COROLLARY 3.2. *Let $K/\mathbb{Q}$ be a finitely generated field extension. Let X/K be a smooth projective variety. Fix $q \in \mathbb{N}$. Let ρ_ℓ (resp. $\overline{\rho}_\ell$) be the representation of $\mathrm{Gal}(K)$ on $H^q(X_{\overline{K}}, \mathbb{Q}_\ell)$ (resp. on $H^q(X_{\overline{K}}, \mathbb{F}_\ell)$). Then there are finite Galois extensions F/K and $k/\mathbb{Q}$ such that for every solvable Galois extension $\Omega/k_{\mathrm{ab}}^\dagger F$ the families $(\rho_\ell|\mathrm{Gal}(\Omega))_{\ell \in \mathbb{L}}$ and $(\overline{\rho}_\ell|\mathrm{Gal}(\Omega))_{\ell \in \mathbb{L}}$ are group-theoretically independent.*

Proof. Let $\ell_0 = 5$. Let F/K and $k/\mathbb{Q}$ be finite Galois extensions such that the assertions (a)–(c) from Theorem 3.1 hold true. From (c) we have

$$\mathrm{FSQ}(\overline{\rho}_\ell(\mathrm{Gal}(\Omega))) \subset \mathfrak{Lie}_\ell \cup \{\mathbb{Z}/\ell\} \quad \text{and} \quad \mathrm{FSQ}(\rho_\ell(\mathrm{Gal}(\Omega))) \subset \mathfrak{Lie}_\ell \cup \{\mathbb{Z}/\ell\}$$

for all $\ell \in \mathbb{L}$. If $\ell \in \{2, 3\}$, then even

$$\mathrm{FSQ}(\overline{\rho}_\ell(\mathrm{Gal}(\Omega))) = \emptyset \quad \text{and} \quad \mathrm{FSQ}(\rho_\ell(\mathrm{Gal}(\Omega))) \subset \{\mathbb{Z}/\ell\}$$

by (a). The class $\mathfrak{Lie}_\ell$ consists of non-commutative groups, and $\mathfrak{Lie}_{\ell_1} \cap \mathfrak{Lie}_{\ell_2} = \emptyset$ for all prime numbers $5 \leq \ell_1 < \ell_2$ by E. Artin's theorem (cf. [19, Thm. 5], see also [1], [13]). Hence $\mathrm{FSQ}(\overline{\rho}_{\ell_1}(\mathrm{Gal}(\Omega))) \cap \mathrm{FSQ}(\overline{\rho}_{\ell_2}(\mathrm{Gal}(\Omega))) = \emptyset$ for all prime numbers $\ell_1 \neq \ell_2$, and thus $(\overline{\rho}_\ell)_{\ell \in \mathbb{L}}$ is group-theoretically independent. Similarly the assertion about $(\rho_\ell)_{\ell \in \mathbb{L}}$ is derived. ∎

REMARK 3.3. If in the situation of Corollary 3.2,

$$\Omega \in \{k_{\mathrm{ab}}^\dagger F, k_{\mathrm{ab}} F, k_{\mathrm{solv}} F, (kF)_{\mathrm{ab}}, (kF)_{\mathrm{solv}}\},$$

then $\Omega/k_{\mathrm{ab}}^\dagger F$ is solvable and hence the families

$$(\rho_\ell|\mathrm{Gal}(\Omega))_{\ell \in \mathbb{L}} \quad \text{and} \quad (\overline{\rho}_\ell|\mathrm{Gal}(\Omega))_{\ell \in \mathbb{L}}$$

are group-theoretically independent.

Appendix: A counterexample. The aim of this Appendix is to prove that one cannot replace $k_{\mathrm{ab}}^\dagger F$ by $k_{\mathrm{cyc}}^\dagger F$ (or by $k_{\mathrm{cyc}} F$) in Corollary 3.2. In fact certain abelian varieties with complex multiplication over a number field K provide a counterexample. Throughout this section let K be a number field and A/K an absolutely simple abelian variety with complex multiplication over K. For every $m \in \mathbb{N}$ we denote by $A[m] = \{x \in A(\overline{K}) : mx = 0\}$ the group of m-torsion points of A and for $\ell \in \mathbb{L}$ by $T_\ell(A) = \varprojlim_j A[\ell^j]$ the ℓ-adic Tate module of A. Let η_ℓ (resp. $\overline{\eta}_\ell$) be the representation of $\mathrm{Gal}(K)$ on $T_\ell(A)$ (resp. on $A[\ell]$). It is known that $\eta_\ell(\mathrm{Gal}(K))$ is abelian for all $\ell \in \mathbb{L}$.

THEOREM A.1. *For every prime number q there exists a set $S \subset \mathbb{L}$ of positive Dirichlet density such that $\mathbb{Z}/q \in \mathrm{FSQ}(\overline{\eta}_\ell(\mathrm{Gal}(K(\mu_\infty))))$ for every $\ell \in S$.*

Proof. There exists a finite extension K'/K such that $A_{K'}$ has good reduction everywhere (cf. [21, Thm. 7]).

For every number field F we let $\mathcal{O}_F$ be its ring of integers and $\mathrm{Spl}(F)$ the set of prime numbers ℓ which split completely in F. We always denote by $\underline{T}_F = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ the torus over $\mathbb{Q}$ obtained as the Weil restriction of the multiplicative group over F. Furthermore, for every prime $\mathfrak{p}$ of F we denote by $F_\mathfrak{p}$ the corresponding local field and by $U_\mathfrak{p}(F)$ the group of units in the integer ring of $F_\mathfrak{p}$. For any torus $\underline{T}/\mathbb{Q}$, $\ell \in \mathbb{L}$ and $n \geq 0$ we define, following Ribet [16, p. 77],

$$\underline{T}(1 + \ell^n \mathbb{Z}_\ell) := \{x \in \underline{T}(\mathbb{Q}_\ell) : v_\ell(\chi(x) - 1) \geq n$$
$$\text{for all } \chi \in \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(\underline{T}_{\overline{\mathbb{Q}}_\ell}, \mathbb{G}_{m, \overline{\mathbb{Q}}_\ell})\}$$

where v_ℓ denotes the unique extension to $\overline{\mathbb{Q}}_\ell$ of the canonical discrete valuation of the complete field $\mathbb{Q}_\ell$. We set

$$\underline{T}(\mathbb{Z}_\ell) = \{x \in \underline{T}(\mathbb{Q}_\ell) : v_\ell(\chi(x) - 1) \geq 0 \text{ for all } \chi \in \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(\underline{T}_{\overline{\mathbb{Q}}_\ell}, \mathbb{G}_{m, \overline{\mathbb{Q}}_\ell})\}$$

and $\underline{T}(\mathbb{F}_\ell) = \underline{T}(\mathbb{Z}_\ell)/\underline{T}(1 + \ell \mathbb{Z}_\ell)$. Note that $\underline{T}_F(\mathbb{Q}_\ell) = \prod_{\mathfrak{p}|\ell} F_\mathfrak{p}^\times$ and $\underline{T}_F(\mathbb{Z}_\ell) = \prod_{\mathfrak{p}|\ell} U_\mathfrak{p}(F)$ (cf. [16, Example 2.1]). We denote by I_F the idele group of F, define

$$I_F^1 = \left\{ (x_\mathfrak{p}) \in \prod_\mathfrak{p} F_\mathfrak{p}^\times : x_\mathfrak{p} \in U_\mathfrak{p}(F) \text{ for all } \mathfrak{p} \nmid \infty \right\}$$

and view $\underline{T}_F(\mathbb{Z}_\ell)$ as a subgroup of I_F^1.

Let $I \subset \mathbb{L}$ be a finite set of prime numbers that contains the primes dividing $[\mathcal{O}_E : E \cap \mathrm{End}_{K'}(A)]$ and the primes that are ramified in $K'E$. Let $\ell \in \mathbb{L} \setminus I$. Then the $\mathcal{O}_E \otimes \mathbb{Z}_\ell$-module $T_\ell(A)$ is free of rank 1 (cf. [21, Thm. 5]), and hence

$$\eta_\ell(\mathrm{Gal}(K')) \subset \mathrm{Aut}_{\mathcal{O}_E \otimes \mathbb{Z}_\ell}(T_\ell(A)) = (\mathcal{O}_E \otimes \mathbb{Z}_\ell)^\times = \underline{T}_E(\mathbb{Z}_\ell).$$

Thus η_ℓ factors to a map $\mathrm{Gal}(K'_{\mathrm{ab}}/K') \to \underline{T}_E(\mathbb{Z}_\ell)$, which is again denoted by η_ℓ. We compose η_ℓ with the Artin symbol $(-, K'_{\mathrm{ab}}/K')$ in order to obtain a map

$$\hat{\eta}_\ell : I_{K'} \to \mathrm{Gal}(K'_{\mathrm{ab}}/K') \xrightarrow{\eta_\ell} \underline{T}_E(\mathbb{Z}_\ell).$$

The image U of $I_{K'}^1$ under the norm residue symbol is open in $\mathrm{Gal}(K'_{\mathrm{ab}}/K')$, and the fixed field of U is the Hilbert class field H of K'. If $\mathfrak{p}$ is a prime of K' and $\mathfrak{p} \nmid \ell$, then η_ℓ is unramified at $\mathfrak{p}$ by [21, Thm. 1] because A has good reduction everywhere. Hence $\hat{\eta}_\ell(U_\mathfrak{p}) = \{e\}$. Thus $\hat{\eta}_\ell$ induces a map $\hat{\eta}'_\ell : \underline{T}_{K'}(\mathbb{Z}_\ell) = \prod_{\mathfrak{p}|\ell} U_\mathfrak{p}(K') \to \underline{T}_E(\mathbb{Z}_\ell)$ and $\mathrm{im}(\hat{\eta}'_\ell) = \eta_\ell(\mathrm{Gal}(H))$. By a reformulation due to Serre and Tate (cf. [21, Thm. 11,

Cor. 2]) of a theorem of Shimura and Taniyama there is a homomorphism $\psi : \underline{T}_{K'} \to \underline{T}_E$ of tori over $\mathbb{Q}$ such that $\hat{\eta}'(x) = \psi_\ell(x^{-1})$ for all $x \in \underline{T}_{K'}(\mathbb{Z}_\ell)$, where $\psi_\ell : \underline{T}_{K'}(\mathbb{Z}_\ell) \to \underline{T}_E(\mathbb{Z}_\ell)$ is the homomorphism induced by ψ. Thus $\eta_\ell(\mathrm{Gal}(H)) = \mathrm{im}(\psi_\ell)$. Let $\underline{T}/\mathbb{Q}$ be the image of ψ_ℓ. Ribet proved that $\dim(T) \geq 2$ (cf. [16, p. 87]). (It is known that $\underline{T}$ agrees with the Mumford-Tate group of A, but we will not need this fact.) Furthermore $\overline{\eta}_\ell(\mathrm{Gal}(H)) = \mathrm{im}(\overline{\psi}_\ell) \subset \underline{T}(\mathbb{F}_\ell)$ if $\overline{\psi}_\ell : \underline{T}_{K'}(\mathbb{F}_\ell) \to \underline{T}_E(\mathbb{F}_\ell)$ is the homomorphism induced by ψ.

By another theorem of Ribet (cf. [16, Thm. 2.4]) there is a constant C such that $c_\ell := [\underline{T}(\mathbb{F}_\ell) : \overline{\eta}_\ell(\mathrm{Gal}(H))] \leq C$ for all $\ell \in \mathbb{L} \smallsetminus I$. Furthermore, by [19], there exists a finite extension H'/H such that $(H'(\mu_{\ell^\infty}, A[\ell]))_{\ell \in \mathbb{L}}$ is a linearly disjoint sequence of extensions of H'. Now let $a \in \mathbb{N}$ be an exponent such that $q^a > C[H' : H]$. Let $L/\mathbb{Q}$ be a Galois extension such that $\underline{T} \times \mathrm{Spec}(L)$ is a split torus and such that $K'E(\mu_{q^a}) \subset L$. Then $\mathrm{Spl}(L) \cap I = \emptyset$. Let $\ell \in \mathrm{Spl}(L)$. Then $\underline{T} \times \mathbb{Q}_\ell$ is a split torus over $\mathbb{Q}_\ell$ because L can be embedded into $\mathbb{Q}_\ell$. Hence, if we set $d = \dim(\underline{T})$, then $\underline{T}(\mathbb{F}_\ell) = (\mathbb{F}_\ell^\times)^d$. If $\ell \in \mathrm{Spl}(L)$ does not divide the polarization degree π of A, then $H'(\mu_\ell) \subset H'(A[\ell])$ and

$$[H'(\mu_\ell) : H'][H'(A[\ell]) : H'(\mu_\ell)]c_\ell = (\ell - 1)^d,$$

$[H'(\mu_\ell) : H']$ divides $\ell - 1$, $c_\ell \leq C[H' : H]$, $q^a > C[H' : H]$, $d \geq 2$ and $\ell = 1 \bmod q^a$ because ℓ splits completely in $\mathbb{Q}(\mu_{p^a})$. This forces $[H'(A[\ell]) : H'(\mu_\ell)]$ to be divisible by q. Furthermore $\mathrm{Gal}(H'(\mu_{\ell^\infty})/H'(\mu_\ell)) \cong \mathbb{Z}_\ell$. It follows that $[H'(A[\ell], \mu_{\ell^\infty}) : H'(\mu_{\ell^\infty})]$ is still divisible by q. Finally, by the linear disjointness of the sequence $(H'(\mu_{\ell^\infty}, A[\ell]))_{\ell \in \mathbb{L}}$, one sees that $\prod_{\ell' \neq \ell} H'(\mu_{\ell'^\infty})$ is linearly disjoint from $H'(\mu_{\ell^\infty}, A[\ell])$ over H' for every $\ell \in \mathbb{L}$. It follows that $|\overline{\eta}_\ell(\mathrm{Gal}(H'(\mu_\infty)))| = [H'(\mu_\infty, A[\ell]) : H'(\mu_\infty)]$ is divisible by q for all but finitely many $\ell \in \mathrm{Spl}(L)$. Furthermore $\overline{\eta}_\ell(\mathrm{Gal}(K(\mu_\infty)))$ is a subgroup of $\overline{\eta}_\ell(\mathrm{Gal}(H'(\mu_\infty)))$, and hence $|\overline{\eta}_\ell(\mathrm{Gal}(K(\mu_\infty)))|$ is divisible by q for all $\ell \in \mathrm{Spl}(L)$. As $\overline{\eta}_\ell(\mathrm{Gal}(K(\mu_\infty)))$ is a finite abelian group, it follows that $\mathbb{Z}/q \in \mathrm{FSQ}(\overline{\eta}_\ell(\mathrm{Gal}(K(\mu_\infty))))$ for all $\ell \in \mathrm{Spl}(L)$. Finally, by Chebotarev, the set $S := \mathrm{Spl}(L)$ of prime numbers has a positive Dirichlet density. ∎

COROLLARY A.2. *For every finite extension E/K neither the restricted family $(\overline{\eta}_\ell|\mathrm{Gal}(E_{\mathrm{cyc}}))_{\ell \in \mathbb{L}}$ nor the restricted family $(\overline{\eta}_\ell|\mathrm{Gal}(E_{\mathrm{cyc}}^\dagger))_{\ell \in \mathbb{L}}$ is group-theoretically independent.*

Proof. Let E/K be a finite extension. We apply Theorem A.1 to A_E/E in order to conclude that $\mathbb{Z}/2 \in \mathrm{FSQ}(\overline{\eta}_\ell(\mathrm{Gal}(E_{\mathrm{cyc}})))$ for infinitely many $\ell \in \mathbb{L}$. As $\overline{\eta}_\ell(\mathrm{Gal}(E_{\mathrm{cyc}}))$ is a subgroup of the finite abelian group $\overline{\eta}_\ell(\mathrm{Gal}(E_{\mathrm{cyc}}^\dagger))$, we conclude that also $\mathbb{Z}/2 \in \mathrm{FSQ}(\overline{\eta}_\ell(\mathrm{Gal}(E_{\mathrm{cyc}}^\dagger)))$ for infinitely many $\ell \in \mathbb{L}$. ∎

REMARK A.3. Corollary A.2 holds accordingly with $\overline{\eta}_\ell$ replaced by η_ℓ. In fact, for every algebraic extension K'/K we have $\mathrm{FSQ}(\overline{\eta}_\ell(\mathrm{Gal}(K'))) \subset \mathrm{FSQ}(\eta_\ell(\mathrm{Gal}(K')))$ for all $\ell \in \mathbb{L}$, because $A[\ell] = T_\ell(A) \otimes \mathbb{F}_\ell$ for all $\ell \in \mathbb{L}$ and thus $\overline{\eta}_\ell(\mathrm{Gal}(K'))$ is a quotient of $\eta_\ell(\mathrm{Gal}(K'))$ for all $\ell \in \mathbb{L}$.

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