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# The set of regular values (in the sense of Clarke) of a Lipschitz map. A sufficient condition for rectifiability of class $C^3$

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**Abstract.** Let $n, N$ be positive integers such that $n < N$. We prove a result about the rectifiability of class $C^3$ of the set of regular values (in the sense of Clarke) of a Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^N$.

**1. Introduction and statement of the main result.** In this paper we prove a result about the rectifiability of class $C^3$ of the set of regular values (in the sense of Clarke) of a Lipschitz map

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N \quad (n < N).$$

Before we state it, let us recall some basic definitions.

A Borel subset $S$ of $\mathbb{R}^N$ is said to be $(\mathcal{H}^n, n)$ *rectifiable of class* $C^3$ (or simply *rectifiable of class* $C^3$) if there exist countably many $n$-dimensional submanifolds $M_j$ of $\mathbb{R}^N$ of class $C^3$ such that

$$\mathcal{H}^n\Big(S \setminus \bigcup_j M_j\Big) = 0.$$

Analogously one can define $(\mathcal{H}^n, n)$ rectifiable sets of class $C^k$ for each positive integer $k$. In particular, for $k = 1$ this notion is equivalent to that of $n$-rectifiable set, e.g. by [S, Lemma 11.1].

For $\gamma \in I(n, N) = \{\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n : 1 \le \gamma_1 < \cdots < \gamma_n \le N\}$ and $s \in \mathbb{R}^n$, let $\partial\varphi^\gamma(s)$ denote the *Clarke subdifferential* of the map

$$\varphi^\gamma := (\varphi^{\gamma_1}, \ldots, \varphi^{\gamma_n}) : \mathbb{R}^n \to \mathbb{R}^n,$$

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namely

$$\partial\varphi^\gamma(s) := \mathrm{co}\Big\{\lim_{i\to\infty} D\varphi^\gamma(s_i)\,\Big|\, D\varphi^\gamma(s_i) \text{ exists, } s_i \to s\Big\}$$

(see [CLSW, p. 133]). The set $\partial\varphi^\gamma(s)$ is said to be *nonsingular* if every matrix in $\partial\varphi^\gamma(s)$ is of rank $n$. Observe that $D\varphi^\gamma(s) \in \partial\varphi^\gamma(s)$ whenever $\varphi^\gamma$ is differentiable at $s$. In particular, $D\varphi^\gamma(s)$ is nonsingular provided $\partial\varphi^\gamma(s)$ is nonsingular. Define

$$\mathcal{R} := \{s \in \mathbb{R}^n \mid \partial\varphi^\gamma(s) \text{ is nonsingular for some } \gamma\}.$$

We can now state our theorem.

THEOREM 1.1. *Consider a Lipschitz map*

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N \qquad (n < N).$$

*Moreover let*

$$c_{1,i}, c_{2,i} : \mathbb{R}^n \to \mathbb{R} \setminus \{0\} \qquad (i = 1, \dots, n)$$

*be a family of locally bounded functions, let*

$$G_{1,i}, G_{2,i} : \mathbb{R}^n \to \mathbb{R}^N \quad (i = 1, \dots, n), \qquad H_{ij} : \mathbb{R}^n \to \mathbb{R}^N \quad (i, j = 1, \dots, n)$$

*be a family of Lipschitz maps and denote by $A$ the set of points $t \in \mathbb{R}^n$ satisfying the following conditions:*

(i) *The map $\varphi$ and all the maps $G_{1,i}$ are differentiable at $t$.*

(ii) *The equality*

$$D_i\varphi(t) = c_{1,i}(t)G_{1,i}(t) = c_{2,i}(t)G_{2,i}(t) \tag{1.1}$$

*holds for all $i = 1, \dots, n$.*

(iii) *One has*

$$D_jG_{1,i}(t) = c_{2,j}(t)H_{ij}(t) \tag{1.2}$$

*for all $i, j = 1, \dots, n$.*

*Also assume that*

(iv) *For almost every $a \in A$ there exists a nontrivial ball $B$ centered at $a$ and such that $\mathcal{L}^n(B \setminus A) = 0$.*

*Then $\varphi(A \cap \mathcal{R})$ is $(\mathcal{H}^n, n)$ rectifiable of class $C^3$.*

REMARK 1.2. As an immediate corollary of Theorem 1.1, we get this result: Let

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N, \qquad G_{1,i}, G_{2,i}, H_{ij} : \mathbb{R}^n \to \mathbb{R}^N \quad (i, j = 1, \dots, n)$$

be a family of Lipschitz maps and let

$$c_{1,i}, c_{2,i} : \mathbb{R}^n \to \mathbb{R} \setminus \{0\} \quad (i = 1, \dots, n),$$

be a family of bounded functions such that (1.1) and (1.2) hold almost everywhere in $\mathbb{R}^n$. Then $\varphi(\mathcal{R})$ is $(\mathcal{H}^n, n)$ rectifiable of class $C^3$.

Remark 1.3. Suppose each component $\varphi_i$ of $\varphi : \mathbb{R}^n \to \mathbb{R}^N$ belongs to $C^3(\mathbb{R}^n)$ and has uniformly bounded gradient $\nabla\varphi_i$. Moreover suppose the differential $D\varphi$ has rank $n$ at each point of $\mathbb{R}^n$. Then the assumptions of Theorem 1.1 are trivially satisfied upon setting

$$c_{1,i} := 1, \quad c_{2,i} := 1, \quad G_{1,i} := D_i\varphi, \quad G_{2,i} := D_i\varphi \quad (i = 1, \ldots, n)$$

and

$$H_{ij} := D^2_{ij}\varphi \quad (i, j = 1, \ldots, n)$$

with $A = \mathbb{R}^n$.

Rectifiable sets of class $C^k$ have been been introduced in [AS] and provide a natural setting for the description of singularities of convex functions and convex surfaces [A, AO]. More generally, they can be used to study the singularities of surfaces with generalized curvatures [AO]. Rectifiability of class $C^2$ is strictly related to Legendrian rectifiable subsets of $\mathbb{R}^N \times \mathbf{S}^{N-1}$ [Fu1, Fu2, D2, D3]. The level sets of a $W^{k,p}_{\mathrm{loc}}$ mapping between manifolds are rectifiable sets of class $C^k$ [BHS]. Applications of rectifiable sets of class $C^H$ (with $H \geq 2$) to geometric variational problems can be found in [D4].

Finally, we would like to explain the reasons of our interest in conditions (1.1) and (1.2). In the particular case when $n = 1$, such conditions arose naturally in the context of one-dimensional generalized Gauss graphs (see [AST, D1] for the basic definitions and results) and of two-storey towers of one-dimensional generalized Gauss graphs (see [D4]). Then it was natural to explore the question of how those assumptions could be generalized in order to get results about higher order rectifiability, including the case when $n \geq 2$. A general theorem for curves was provided in [D3], while in [D5] we started studying the case of general dimension by proving a result about rectifiability of class $C^2$. Further results in this direction can be found in [AS] and [Fu1, Fu2]. Roughly speaking, the very basic idea and the proof strategy in the present paper are the same as in [D5], namely we use the celebrated Whitney extension theorem to show that the image of $\varphi$ is captured, up to $\mathcal{H}^n$ measure 0, by countably many highly regular images of $\mathbb{R}^n$. More precisely, our main objective is to get a set of third order Whitney estimates which allows us to perform (countably many) extensions of class $C^3$ necessary to show that the image of $\varphi$ is $C^3$-rectifiable. This result is an outcome of our efforts to prove a general theorem about rectifiability of class $C^H$ in any dimension, which is the subject of our ongoing investigations.

## 2. Reduction to graphs

Remark 2.1. Under the hypotheses of Theorem 1.1, let $A'$ denote the set of $a \in A$ such that there exists a non-trivial ball $B$ centered at $a$ satisfying

$\mathcal{L}^n(B \setminus A) = 0$. One has

$$\mathcal{L}^n(A \setminus A') = 0 \tag{2.1}$$

by assumption (iv). Hence, it will be enough to prove that $\varphi(A' \cap \mathcal{R})$ is $(\mathcal{H}^n, n)$ rectifiable of class $C^3$.

Remark 2.2. By the main theorem of [D5], we already know that $\varphi(A \cap \mathcal{R})$ (hence also $\varphi(A' \cap \mathcal{R})$) is $(\mathcal{H}^n, n)$ rectifiable of class $C^2$.

Remark 2.3. Let $E$ be any subset of $\mathcal{R}$ and define

$$E^\gamma := \{s \in E \mid \partial\varphi^\gamma(s) \text{ is nonsingular}\}, \qquad \gamma \in I(n, N).$$

Then obviously

$$\bigcup_{\gamma \in I(n,N)} E^\gamma = E.$$

Remark 2.4. If $s \in \mathcal{R}^\gamma$, then by the Lipschitz inverse function theorem (e.g. [CLSW, Theorem 3.12]), there exist a neighborhood $U$ of $s$ (in $\mathbb{R}^n$) and a neighborhood $V$ of $\varphi^\gamma(s)$ (in $\mathbb{R}^n$) such that

- $V = \varphi^\gamma(U)$ and $\varphi^\gamma|U : U \to V$ is invertible;
- $(\varphi^\gamma|U)^{-1}$ is Lipschitz.

Let $\overline{\gamma}$ denote the multi-index in $I(N-n, N)$ which complements $\gamma$ in $\{1, \ldots, N\}$ in the natural increasing order and set (for $x \in \mathbb{R}^N$)

$$x^\gamma := (x^{\gamma_1}, \ldots, x^{\gamma_n}),$$
$$x^{\overline{\gamma}} := (x^{\overline{\gamma}_1}, \ldots, x^{\overline{\gamma}_{N-n}}).$$

Then the map

$$f := \varphi^{\overline{\gamma}} \circ (\varphi^\gamma|U)^{-1} : V \to \mathbb{R}^{N-n}$$

is Lipschitz and its graph

$$\mathcal{G}_f^\gamma := \{x \in \mathbb{R}^N \mid x^\gamma \in V \text{ and } x^{\overline{\gamma}} = f(x^\gamma)\}$$

coincides with $\varphi(U)$.

By the previous remarks, it will be enough to prove

Theorem 2.5. *Under the assumptions of Theorem* 1.1, *let* $\gamma \in I(n, N)$ *and consider a map*

$$g : \mathbb{R}^n \to \mathbb{R}^{N-n}$$

*of class* $C^2$. *Then* $\varphi((A' \cap \mathcal{R})^\gamma) \cap \mathcal{G}_g^\gamma$ *is* $(\mathcal{H}^n, n)$ *rectifiable of class* $C^3$.

Remark 2.6. The remainder of the paper is devoted to proving Theorem 2.5. With no loss of generality, we can restrict to the case of $\gamma = \{1, \ldots, n\}$.

## 3. Preliminaries

**3.1. Further reduction of the claim.** From now on, for simplicity,

$$\mathcal{G}_g^{\{1,\dots,n\}}, \qquad (A' \cap \mathcal{R})^{\{1,\dots,n\}}, \qquad \varphi^{\{1,\dots,n\}}$$

will be denoted by $\mathcal{G}_g$, $F$ and $\lambda$, respectively.

Define

$$L := \varphi^{-1}(\mathcal{G}_g) \cap F.$$

Without loss of generality, we can assume that $\mathcal{L}^n(L) < \infty$. Then, by a well-known regularity property of $\mathcal{L}^n$, for any given $\varepsilon > 0$ there exists a closed subset $L_\varepsilon$ of $\mathbb{R}^n$ with

$$L_\varepsilon \subset L, \qquad \mathcal{L}^n(L \setminus L_\varepsilon) \leq \varepsilon \tag{3.1}$$

(see e.g. [M, Theorem 1.10]). Moreover, since $L_\varepsilon$ is closed, one has

$$L_\varepsilon^* \subset L_\varepsilon \tag{3.2}$$

where $L_\varepsilon^*$ is the set of density points of $L_\varepsilon$. Recall that

$$\mathcal{L}^n(L_\varepsilon \setminus L_\varepsilon^*) = 0 \tag{3.3}$$

by a well-known result of Lebesgue. In the special case that $L$ has measure zero, we define $L_\varepsilon := \emptyset$, hence $L_\varepsilon^* := \emptyset$.

Observe that

$$\mathcal{G}_g \cap \varphi(F) \setminus \varphi(L_\varepsilon^*) \subset \varphi\big(\varphi^{-1}(\mathcal{G}_g) \cap F \setminus L_\varepsilon^*\big) = \varphi(L \setminus L_\varepsilon^*),$$

hence

$$\begin{aligned}\mathcal{H}^n(\mathcal{G}_g \cap \varphi(F) \setminus \varphi(L_\varepsilon^*)) \leq \mathcal{H}^n(\varphi(L \setminus L_\varepsilon^*)) &\leq \int_{L \setminus L_\varepsilon^*} J_n\varphi \, d\mathcal{L}^n \\ &\leq (\operatorname{Lip}\varphi)^n \mathcal{L}(L \setminus L_\varepsilon^*) \leq \varepsilon \, (\operatorname{Lip}\varphi)^n\end{aligned}$$

by the area formula (see [F, §3.2], [S, §8]) and (3.1)–(3.3). It follows that

$$\mathcal{H}^n\Big(\mathcal{G}_g \cap \varphi(F) \setminus \bigcup_{j=1}^{\infty} \varphi(L_{1/j}^*)\Big) = 0.$$

Thus, to prove Theorem 2.5, it suffices to show that $\varphi(L_\varepsilon^*)$ is $(\mathcal{H}^n, n)$ rectifiable of class $C^3$ for all $\varepsilon > 0$.

**3.2. Further notation.** Let us consider the projection

$$\Pi : \mathbb{R}^N \to \mathbb{R}^{N-n}, \qquad (x_1, \dots, x_N) \mapsto (x_{n+1}, \dots, x_N).$$

Moreover, set

$$\mathcal{R}_s^{(0)}(\sigma) := g(\lambda(\sigma)) - g(\lambda(s)) - \sum_{i=1}^n D_i g(\lambda(s))[\varphi^i(\sigma) - \varphi^i(s)] - \frac{1}{2}\sum_{i,j=1}^n D_{ij}^2 g(\lambda(s))[\varphi^i(\sigma) - \varphi^i(s)][\varphi^j(\sigma) - \varphi^j(s)],$$

$$\mathcal{R}_{i;s}^{(1)}(\sigma) := D_i g(\lambda(\sigma)) - D_i g(\lambda(s)) - \sum_{j=1}^n D_{ij}^2 g(\lambda(s))[\varphi^j(\sigma) - \varphi^j(s)],$$

$$\mathcal{R}_{ij;s}^{(2)}(\sigma) := D_{ij}^2 g(\lambda(\sigma)) - D_{ij}^2 g(\lambda(s)).$$

For $h = 1, 2$, let $G_h$ denote the $n \times n$ matrix field given by

$$[G_h(t)]_i^j := G_{h,i}^j(t), \quad t \in \mathbb{R}^n \quad (i, j = 1, \ldots, n).$$

Also let $H$ be the $n^2 \times n$ matrix field defined by

$$[H(t)]_{ij}^k := H_{ij}^k(t), \quad t \in \mathbb{R}^n \quad (i, j, k = 1, \ldots, n)$$

where the couples $ij$ (indexing the rows) are ordered lexicographically.

Then consider the $(n + n^2) \times (n + n^2)$ matrix field

$$M := \begin{bmatrix} G_1 & 0 \\ H & G_1 \otimes G_2 \end{bmatrix}$$

where $\otimes$ denotes the Kronecker product of matrices [HJ, Sect. 4.2].

For $l = 1, \ldots, N - n$, let $D^2 g^l$ denote the $\mathbb{R}^{n^2}$-valued field such that

$$[D^2 g^l(t)]^{ij} := D_{ij}^2 g^l(t), \quad t \in \mathbb{R}^n \quad (i, j = 1, \ldots, n)$$

where the lexicographical order is assumed.

Finally, given a matrix $X$ and an index $k$, denote by

$$R_k(X), \quad C_k(X)$$

the $k$th row and $k$th column of $X$, respectively.

## 4. The derivatives of $g$ in terms of $\{G_1, G_2, H\}$

Proposition 4.1. *Let $l \in \{1, \ldots, N - n\}$ and $s \in L_\varepsilon^*$. Then*

$$M(s)\big(Dg^l(\lambda(s)), D^2 g^l(\lambda(s))\big)^T = (G_1^{n+l}(s), H^{n+l}(s))^T \tag{4.1}$$

*where $G_1^{n+l}$ and $H^{n+l}$ are the the vector fields defined as follows:*

$$G_1^{n+l} := (G_{1,1}^{n+l}, \ldots, G_{1,n}^{n+l}),$$
$$H^{n+l} := [H_{ij}^{n+l}]_{i,j=1}^n \quad (\textit{in lexicographical order}).$$

*Proof.* First of all, observe that

$$g(\lambda(t)) = \Pi\varphi(t)$$

for all $t \in \varphi^{-1}(\mathcal{G}_g)$. Since $L^*_\varepsilon \subset A$, the two members of this equality are both differentiable at $s$. Moreover, $s$ is a limit point of $L_\varepsilon \subset \varphi^{-1}(\mathcal{G}_g)$. Hence

$$\sum_{j=1}^n D_j g(\lambda(s)) D_i\varphi^j(s) = \Pi D_i\varphi(s) \quad (i = 1, \ldots, n),$$

so

$$\sum_{j=1}^n D_j g(\lambda(s)) c_{1,i}(s) G^j_{1,i}(s) = c_{1,i}(s)\Pi G_{1,i}(s) \quad (i = 1, \ldots, n)$$

by (1.1). Since $c_{1,i}(s) \neq 0$ $(i = 1, \ldots, n)$, we get

$$\sum_{j=1}^n D_j g^l(\lambda(s)) G^j_{1,i}(s) = G^{n+l}_{1,i}(s) \quad (i = 1, \ldots, n), \tag{4.2}$$

i.e.

$$G_1(s) Dg^l(\lambda(s)) = G_1^{n+l}(s). \tag{4.3}$$

By the same argument, we can differentiate (4.2) to obtain

$$\sum_{j,k=1}^n D^2_{jk} g^l(\lambda(s)) D_m\varphi^k(s)\, G^j_{1,i}(s) + \sum_{j=1}^n D_j g^l(\lambda(s)) D_m G^j_{1,i}(s) = D_m G^{n+l}_{1,i}(s)$$

for all $i, m = 1, \ldots, n$. By (1.2),

$$\sum_{j,k=1}^n D^2_{jk} g^l(\lambda(s)) c_{2,m}(s) G^k_{2,m}(s) G^j_{1,i}(s) + \sum_{j=1}^n D_j g^l(\lambda(s)) c_{2,m}(s) H^j_{im}(s) = c_{2,m}(s) H^{n+l}_{im}(s)$$

for all $i, m = 1, \ldots, n$, so that

$$[G_1(s) \otimes G_2(s)] D^2 g^l(\lambda(s)) + H(s) Dg^l(\lambda(s)) = H^{n+l}(s). \tag{4.4}$$

We conclude by observing that the system of equalities (4.3) and (4.4) is equivalent to (4.1). ∎

We now investigate the properties of the matrix field $t \mapsto M(t)^{-1}$.

Proposition 4.2. *Let $s \in A$ be such that $D\lambda(s)$ is nonsingular (e.g. $s \in F$). Then there exists a nontrivial ball $B$, centered at $s$, such that:*

- *For all $t \in B$, the matrices $G_1(t)$, $G_2(t)$ and $M(t)$ are invertible and*

$$M(t)^{-1} = \begin{bmatrix} G_1(t)^{-1} & 0 \\ -[G_1(t)^{-1} \otimes G_2(t)^{-1}] H(t) G_1(t)^{-1} & G_1(t)^{-1} \otimes G_2(t)^{-1} \end{bmatrix}. \tag{4.5}$$

- *The map $t \mapsto M(t)^{-1}$, $t \in B$, is Lipschitz.*

*Proof.* One has

$$D\lambda(s)=\Big[\prod_{i=1}^{n}c_{1,i}(s)\Big]G_1(s)^T=\Big[\prod_{i=1}^{n}c_{2,i}(s)\Big]G_2(s)^T$$

by (1.1), hence $G_1(s)$ and $G_2(s)$ are nonsingular. Moreover,

$$\det M=\det G_1\det(G_1\otimes G_2)=(\det G_1)^{n+1}(\det G_2)^n \tag{4.6}$$

by [HJ, Sect. 4.2, Problem 1]. Thus $\det M(s)\neq 0$. Since the function $t\mapsto \det M(t)$ is continuous, there exists a nontrivial ball $B$ centered at $s$ and such that

$$|\det M(t)|\geq \frac{|\det M(s)|}{2}>0$$

for all $t\in B$. As a consequence, $M(t)$ is invertible at every $t\in B$. Formula (4.5) follows at once by observing that, for $t\in B$, the matrix $M(t)^{-1}$ has to be of the form (recall (4.6))

$$\begin{bmatrix} G_1(t)^{-1} & 0\\ X(t) & [G_1(t)\otimes G_2(t)]^{-1}\end{bmatrix}$$

with $X(t)$ satisfying

$$H(t)G_1(t)^{-1}+[G_1(t)\otimes G_2(t)]X(t)=0,$$

and finally recalling that

$$[G_1(t)\otimes G_2(t)]^{-1}=G_1(t)^{-1}\otimes G_2(t)^{-1}$$

(see [HJ, Corollary 4.2.11]). This concludes the proof of the first claim. The second one follows by observing that the entries of $M$ are Lipschitz. ∎

## 5. Whitney-type estimates

Proposition 5.1. *Let $s\in L^*_\varepsilon$ and $t\in A\cap\varphi^{-1}(\mathcal{G}_g)$ be such that*

$$\mathcal{H}^1([s;t]\setminus A)=0 \tag{5.1}$$

*where $[s;t]$ denotes the segment joining $s$ and $t$. Then*

$$\|\mathcal{R}^{(0)}_s(t)\|\leq\Big(\sup_{[s;t]}\|c_1\|\Big)\Big(\sup_{[s;t]}\|c_2\|\Big)\Lambda_s\|t-s\|^3$$

*where*

$$c_1:=(c_{1,1},\ldots,c_{1,n}),\qquad c_2:=(c_{2,1},\ldots,c_{2,n})$$

*and $\Lambda_s$ is a constant not depending on $t$.*

*Proof.* First of all, observe that:

- Since $s,t\in\varphi^{-1}(\mathcal{G}_g)$, one has $g(\lambda(s))=\Pi\varphi(s)$ and $g(\lambda(t))=\Pi\varphi(t)$.
- Consider the following parametrization of $[s;t]$:

$$\sigma:[0,1]\to\mathbb{R}^n,\qquad \rho\mapsto s+\rho(t-s).$$

Then the function $\rho \mapsto \varphi(\sigma(\rho))$ is Lipschitz, hence differentiable almost everywhere in $[0,1]$. Moreover (5.1) implies that

$$(\varphi\circ\sigma)'(\rho)=\sum_{i_1=1}^{n}(t^{i_1}-s^{i_1})D_{i_1}\varphi(\sigma(\rho))\qquad\text{for a.e. }\rho\in[0,1].$$

Recalling also (1.1), we obtain

$$\begin{aligned}\mathcal{R}_s^{(0)}(t)&=\Pi\varphi(t)-\Pi\varphi(s)-\sum_{i=1}^{n}D_ig(\lambda(s))[\varphi^i(t)-\varphi^i(s)]\\&\quad-\frac{1}{2}\sum_{i,j=1}^{n}D_{ij}^2g(\lambda(s))[\varphi^i(t)-\varphi^i(s)][\varphi^j(t)-\varphi^j(s)]\\&=\sum_{h=1}^{n}(t^h-s^h)\int_0^1\Big\{\Pi D_h\varphi(\sigma(\rho))-\sum_{i=1}^{n}D_ig(\lambda(s))D_h\varphi^i(\sigma(\rho))\\&\quad-\sum_{i,j=1}^{n}D_{ij}^2g(\lambda(s))[\varphi^i(\sigma(\rho))-\varphi^i(s)]D_h\varphi^j(\sigma(\rho))\Big\}\,d\rho,\end{aligned}$$

that is,

$$\mathcal{R}_s^{(0)}(t)=\sum_{h=1}^{n}(t^h-s^h)\int_0^1 c_{1,h}(\sigma(\rho))\Phi_{s,h}(\sigma(\rho))\,d\rho \tag{5.2}$$

where $\Phi_{s,h}$ denotes the Lipschitz map defined as follows:

$$\Phi_{s,h}:=\Pi G_{1,h}-\sum_{i=1}^{n}D_ig(\lambda(s))G_{1,h}^i-\sum_{i,j=1}^{n}D_{ij}^2g(\lambda(s))[\varphi^i-\varphi^i(s)]G_{1,h}^j. \tag{5.3}$$

Now, since $\Phi_{s,h}\circ\sigma$ is Lipschitz, it is differentiable almost everywhere in $[0,1]$ and

$$(\Phi_{s,h}\circ\sigma)'=\sum_{k=1}^{n}(t^k-s^k)(D_k\Phi_{s,h})\circ\sigma.$$

Moreover $\Phi_{s,h}(s)=0$, by (4.2). By (5.3) and recalling (1.2), we get

$$\begin{aligned}\Phi_{s,h}(\sigma(\rho))&=\Phi_{s,h}(\sigma(\rho))-\Phi_{s,h}(s)=\int_0^\rho(\Phi_{s,h}\circ\sigma)'\\&=\sum_{k=1}^{n}(t^k-s^k)\int_0^\rho(D_k\Phi_{s,h})\circ\sigma\\&=\sum_{k=1}^{n}(t^k-s^k)\int_0^\rho(c_{2,k}\circ\sigma)(\Psi_{s,hk}\circ\sigma)\end{aligned} \tag{5.4}$$

where $\Psi_{s,hk}$ is the Lipschitz map defined by

$$\Psi_{s,hk} := \Pi H_{hk} - \sum_{i=1}^{n} D_i g(\lambda(s)) H_{hk}^i - \sum_{i,j=1}^{n} D_{ij}^2 g(\lambda(s)) \{G_{2,k}^i G_{1,h}^j + [\varphi^i - \varphi^i(s)] H_{hk}^j\}.$$

Observe that

$$\Psi_{s,hk}(s) = \Pi H_{hk}(s) - \sum_{i=1}^{n} D_i g(\lambda(s)) H_{hk}^i(s) - \sum_{i,j=1}^{n} D_{ij}^2 g(\lambda(s)) G_{2,k}^i(s) G_{1,h}^j(s) = 0$$

by (4.4). Hence (for all $r \in [0,1]$)

$$\begin{aligned} \|\Psi_{s,hk}(\sigma(r))\| &= \|\Psi_{s,hk}(\sigma(r)) - \Psi_{s,hk}(s)\| \le \|\sigma(r) - s\| \operatorname{Lip} \Psi_{s,hk} \\ &= r\|t-s\| \operatorname{Lip} \Psi_{s,hk} \\ &\le \|t-s\| \Lambda_s \end{aligned}$$

with

$$\Lambda_s := \max_{h,k=1,\ldots,n} \operatorname{Lip} \Psi_{s,hk}.$$

Recalling (5.4), we obtain

$$\|\Phi_{s,h}(\sigma(\rho))\| \le \Big(\sup_{[s;t]} \|c_2\|\Big) \Lambda_s \|t-s\|^2.$$

The conclusion follows at once from (5.2). ∎

Proposition 5.2. *Let $s \in L_\varepsilon^*$. Then there exists a nontrivial ball $B$, centered at $s$, such that*

$$\|\mathcal{R}_{i;s}^{(1)}(t)\| \le \Big(\sup_{[s;t]} \|c_2\|\Big) \Sigma_s \|t-s\|^2 \quad (i = 1, \ldots, n)$$

*for all $t \in L_\varepsilon^* \cap B$ such that* (5.1) *is satisfied, where $c_2$ is defined as in Proposition* 5.1, *while $\Sigma_s$ is a constant not depending on $t$ or $i$.*

*Proof.* Since $s \in L_\varepsilon^* \subset F$, there exists a ball $B$ as in Proposition 4.2. Consider $t \in L_\varepsilon^* \cap B$ such that (5.1) is satisfied. Then (for $l = 1, \ldots, N-n$)

$$\begin{aligned} [\mathcal{R}_{i;s}^{(1)}(t)]^l &= D_i g^l(\lambda(t)) - D_i g^l(\lambda(s)) - \sum_{j=1}^{n} D_{ij}^2 g^l(\lambda(s)) [\varphi^j(t) - \varphi^j(s)] \\ &= R_i(G_1(t)^{-1}) \bullet G_1^{n+l}(t) - R_i(G_1(s)^{-1}) \bullet G_1^{n+l}(s) \\ &\quad - \sum_{j=1}^{n} D_{ij}^2 g^l(\lambda(s)) [\varphi^j(t) - \varphi^j(s)] \end{aligned}$$

by Propositions 4.1 and 4.2. Moreover, if $\sigma$ is the parametrization of $[s;t]$ defined above, the function

$$\Pi : \rho \mapsto R_i(G_1(\sigma(\rho))^{-1}) \bullet G_1^{n+l}(\sigma(\rho)), \qquad \rho \in [0,1],$$

is Lipschitz, hence differentiable almost everywhere in $[0,1]$. Recalling (5.1) and denoting by $G_1^{-1}$ the map $r \mapsto G_1(r)^{-1}$ (by a convenient abuse of notation), we obtain

$$\Pi'(\rho) = \sum_{q=1}^{n} (t^q - s^q)\{R_i(D_q G_1^{-1}) \bullet G_1^{n+l} + R_i(G_1^{-1}) \bullet D_q G_1^{n+l}\}(\sigma(\rho))$$

for a.e. $\rho \in [0,1]$. By the well-known formula for the derivative of the inverse matrix field (see [HJ, (6.5.7)]),

$$\begin{aligned}
\Pi'(\rho) &= \sum_{q=1}^{n} (t^q - s^q)\big\{R_i(G_1^{-1}) \bullet D_q G_1^{n+l} \\
&\qquad\qquad - R_i[G_1^{-1}(D_q G_1)G_1^{-1}] \bullet G_1^{n+l}\big\}(\sigma(\rho)) \\
&= \sum_{m,q=1}^{n} (t^q - s^q)\big\{[G_1^{-1}]_i^m D_q G_{1,m}^{n+l} - [G_1^{-1}(D_q G_1)G_1^{-1}]_i^m G_{1,m}^{n+l}\big\}(\sigma(\rho)) \\
&= \sum_{m,q=1}^{n} (t^q - s^q)\Big\{[G_1^{-1}]_i^m D_q G_{1,m}^{n+l} \\
&\qquad\qquad - \sum_{h,k=1}^{n} [G_1^{-1}]_i^h (D_q G_{1,h}^k)[G_1^{-1}]_k^m G_{1,m}^{n+l}\Big\}(\sigma(\rho))
\end{aligned}$$

for a.e. $\rho \in [0,1]$. Recalling (1.2), we get

$$\begin{aligned}
\Pi'(\rho) &= \sum_{m,q=1}^{n} c_{2,q}(\sigma(\rho))(t^q - s^q)\Big\{[G_1^{-1}]_i^m H_{mq}^{n+l} \\
&\qquad\qquad - \sum_{h,k=1}^{n} [G_1^{-1}]_i^h H_{hq}^k [G_1^{-1}]_k^m G_{1,m}^{n+l}\Big\}(\sigma(\rho))
\end{aligned}$$

for a.e. $\rho \in [0,1]$. It follows that

$$[\mathcal{R}_{i;s}^{(1)}(t)]^l = \sum_{q=1}^{n} (t^q - s^q) \int_0^1 c_{2,q}(\sigma(\rho))\Theta_{q;s}^l(\sigma(\rho))\, d\rho \tag{5.5}$$

where $\Theta_{q;s}^l : B \to \mathbb{R}$ is defined as

$$\Theta_{q;s}^l := \sum_{m=1}^{n} \Big\{[G_1^{-1}]_i^m H_{mq}^{n+l} - \sum_{h,k=1}^{n} [G_1^{-1}]_i^h H_{hq}^k [G_1^{-1}]_k^m G_{1,m}^{n+l} - D_{im}^2 g^l(\lambda(s)) G_{2,q}^m\Big\}.$$

One has

$$\begin{aligned}
D^2_{im} g^l(\lambda(s)) &= \sum_{c,d=1}^{n} [G_1(s)^{-1} \otimes G_2(s)^{-1}]^{cd}_{im} H^{n+l}_{cd}(s) \\
&\quad - \sum_{b,c,d,e=1}^{n} [G_1(s)^{-1} \otimes G_2(s)^{-1}]^{cd}_{im} H^b_{cd}(s) [G_1(s)^{-1}]^e_b G^{n+l}_{1,e}(s) \\
&= \sum_{c,d=1}^{n} [G_1(s)^{-1}]^c_i [G_2(s)^{-1}]^d_m H^{n+l}_{cd}(s) \\
&\quad - \sum_{b,c,d,e=1}^{n} [G_1(s)^{-1}]^c_i [G_2(s)^{-1}]^d_m H^b_{cd}(s) [G_1(s)^{-1}]^e_b G^{n+l}_{1,e}(s)
\end{aligned}$$

by Propositions 4.1 and 4.2. Hence

$$\begin{aligned}
\sum_{m=1}^{n} D^2_{im} g^l(\lambda(s)) G^m_{2,q}(s) &= \sum_{c,d=1}^{n} [G_1(s)^{-1}]^c_i H^{n+l}_{cd}(s)\, \delta_{dq} \\
&\quad - \sum_{b,c,d,e=1}^{n} [G_1(s)^{-1}]^c_i H^b_{cd}(s) [G_1(s)^{-1}]^e_b G^{n+l}_{1,e}(s)\, \delta_{dq} \\
&= \sum_{c=1}^{n} [G_1(s)^{-1}]^c_i H^{n+l}_{cq}(s) \\
&\quad - \sum_{b,c,e=1}^{n} [G_1(s)^{-1}]^c_i H^b_{cq}(s) [G_1(s)^{-1}]^e_b G^{n+l}_{1,e}(s),
\end{aligned}$$

so that

$$\Theta^l_{q;s}(s) = 0.$$

Moreover $\Theta^l_{q;s}$ is Lipschitz, by Proposition 4.2. Then, if we define

$$\Sigma_s := (N-n) \max_{\substack{q=1,\dots,n \\ l=1,\dots,N-n}} \operatorname{Lip} \Theta^l_{q;s},$$

we get

$$|\Theta^l_{q;s}(\sigma(\rho))| = |\Theta^l_{q;s}(\sigma(\rho)) - \Theta^l_{q;s}(s)| \le \frac{\Sigma_s}{N-n} \rho \|t-s\| \le \frac{\Sigma_s}{N-n} \|t-s\|$$

for all $q = 1, \dots, n$, all $l = 1, \dots, N-n$ and all $\rho \in [0,1]$. From (5.5) it finally follows that

$$\|\mathcal{R}^{(1)}_{i;s}(t)\| \le \sum_{l=1}^{N-n} |[\mathcal{R}^{(1)}_{i;s}(t)]^l| \le \Big(\sup_{[s;t]} \|c_2\|\Big) \Sigma_s \|t-s\|^2. \ \blacksquare$$

The estimate of the second order remainder term is established in the following result, an immediate consequence of Proposition 4.2 and (4.1).

PROPOSITION 5.3. *Let $s \in L_\varepsilon^*$. Then there exists a nontrivial ball $B$, centered at $s$, such that*

$$\|\mathcal{R}_{ij;s}^{(2)}(t)\| = \|D_{ij}^2 g(\lambda(t)) - D_{ij}^2 g(\lambda(s))\| \le \Gamma_s \|t-s\| \qquad (i,j=1,\dots,n)$$

*for all $t \in L_\varepsilon^* \cap B$, where $\Gamma_s$ is a constant not depending on $t$ or $i,j$.*

**6. Proof of Theorem 2.5.** As pointed out in Section 3.1, our task amounts to proving that $\varphi(L_\varepsilon^*)$ is $(\mathcal{H}^n, n)$ rectifiable of class $C^3$ (for all $\varepsilon > 0$).

For each positive integer $h$, define $\Gamma_{\varepsilon,h}$ as the set of $s \in L_\varepsilon^*$ such that

$$\|\mathcal{R}_s^{(0)}(t)\| \le h\|\lambda(t) - \lambda(s)\|^3 \tag{6.1}$$

and

$$\|\mathcal{R}_{i;s}^{(1)}(t)\| \le h\|\lambda(t)-\lambda(s)\|^2, \qquad \|\mathcal{R}_{ij;s}^{(2)}(t)\| \le h\|\lambda(t)-\lambda(s)\| \tag{6.2}$$

for all $i,j = 1,\dots,n$ and all $t \in L_\varepsilon^*$ satisfying

$$\|t-s\| \le 1/h.$$

PROPOSITION 6.1. *One has*

$$\bigcup_h \Gamma_{\varepsilon,h} = L_\varepsilon^*.$$

*Proof.* The inclusion $\bigcup_h \Gamma_{\varepsilon,h} \subset L_\varepsilon^*$ is obvious. In order to prove the opposite inclusion, consider $s \in L_\varepsilon^*$ and let $U$ and $V$ be as in Remark 2.4. Observe that

$$\begin{aligned} \|t-s\| &= \|(\lambda|U)^{-1}(\lambda(t)) - (\lambda|U)^{-1}(\lambda(s))\| \\ &\le \operatorname{Lip}(\lambda|U)^{-1}\,\|\lambda(t)-\lambda(s)\| \end{aligned} \tag{6.3}$$

for all $t \in U$.

Since $s \in A'$, there exists a nontrivial ball $B$ centered at $s$ such that

$$B \subset U, \qquad \mathcal{L}^n(B \setminus A) = 0.$$

By shrinking, if need be, we may also assume that $B$ is as in the claims of Propositions 5.2 and 5.3.

We now recall the following fact, proved in [D5]: given a null-measure subset $Z$ of $\mathbb{R}^n$ and $s \in \mathbb{R}^n$, one has $\mathcal{H}^1(Z \cap [s;t]) = 0$ for a.e. $t \in \mathbb{R}^n$.

For $Z := B \setminus A$, we get

$$\mathcal{H}^1([s;t] \setminus A) = \mathcal{H}^1(Z \cap [s;t]) = 0$$

for a.e. $t \in B$. Then Proposition 5.1 yields

$$\|\mathcal{R}_s^{(0)}(t)\| \le C\|t-s\|^3$$

for a.e. $t \in B \cap \varphi^{-1}(\mathcal{G}_g)$, where $C$ does not depend on $t$. By continuity we get

$$\|\mathcal{R}_s^{(0)}(t)\| \le C\|t-s\|^3$$

for all $t \in B \cap \varphi^{-1}(\mathcal{G}_g)$. Recalling (6.3) we conclude that

$$\|\mathcal{R}_s^{(0)}(t)\| \le C_0\|\lambda(t) - \lambda(s)\|^3, \quad C_0 := C[\mathrm{Lip}(\lambda|U)^{-1}]^3,$$

for all $t \in B \cap \varphi^{-1}(\mathcal{G}_g)$. Analogously, we can use Propositions 5.2 and 5.3, and (6.3), to deduce the existence of $C_1$ and $C_2$ which do not depend on $t$ and are such that

$$\|\mathcal{R}_{i;s}^{(1)}(t)\| \le C_1\|\lambda(t) - \lambda(s)\|^2 \quad (i = 1, \ldots, n),$$
$$\|\mathcal{R}_{ij;s}^{(2)}(t)\| \le C_2\|\lambda(t) - \lambda(s)\| \quad (i, j = 1, \ldots, n)$$

for all $t \in L_\varepsilon^* \cap B$. Hence $s \in \Gamma_{\varepsilon,h}$ provided $h$ is large enough. ∎

From Proposition 6.1 it follows that

$$\varphi(L_\varepsilon^*) = \bigcup_h \varphi(\Gamma_{\varepsilon,h}),$$

hence it will be enough to verify that

(6.4) $\varphi(\Gamma_{\varepsilon,h})$ is $(\mathcal{H}^n, n)$ rectifiable of class $C^3$

for all $\varepsilon$ and $h$.

To prove this, we first consider a countable measurable covering $\{Q_l\}_{l=1}^\infty$ of $\Gamma_{\varepsilon,h}$ such that $\operatorname{diam} Q_l \le 1/h$ for all $l$, and define

$$F_l := \overline{\lambda(\Gamma_{\varepsilon,h} \cap Q_l)}.$$

If $\xi, \eta \in F_l$, then there exist sequences $\{s_k\}, \{t_k\} \subset \Gamma_{\varepsilon,h} \cap Q_l$ such that

$$\lim_k \lambda(s_k) = \xi, \quad \lim_k \lambda(t_k) = \eta.$$

By (6.1) and (6.2), for all $k$,

$$\|\mathcal{R}_{s_k}^{(0)}(t_k)\| \le h\|\lambda(t_k) - \lambda(s_k)\|^3$$

and

$$\|\mathcal{R}_{i,s_k}^{(1)}(t_k)\| \le h\|\lambda(t_k) - \lambda(s_k)\|^2, \quad \|\mathcal{R}_{ij,s_k}^{(2)}(t_k)\| \le h\|\lambda(t_k) - \lambda(s_k)\|$$

for all $i, j = 1, \ldots, n$. Letting $k \to \infty$, we obtain

$$\Big\|g(\eta) - g(\xi) - \sum_{i=1}^n D_i g(\xi)(\eta^i - \xi^i) - \frac{1}{2}\sum_{i,j=1}^n D_{ij}^2 g(\xi)(\eta^i - \xi^i)(\eta^j - \xi^j)\Big\| \le h\|\eta - \xi\|^3,$$

$$\Big\|D_i g(\eta) - D_i g(\xi) - \sum_{j=1}^n D_{ij}^2 g(\xi)(\eta^j - \xi^j)\Big\| \le h\|\eta - \xi\|^2 \quad (i = 1, \ldots, n)$$

and

$$\|D_{ij}^2 g(\eta) - D_{ij}^2 g(\xi)\| \le h\|\eta - \xi\| \quad (i, j = 1, \ldots, n)$$

for all $\xi, \eta \in F_l$. By the Whitney extension theorem [St, Ch. VI, §2.3], each $g|F_l$ can be extended to a map in $C^{2,1}(\mathbb{R}^n, \mathbb{R}^{N-n})$. Then the Lusin type result of [F, §3.1.15] implies that $\varphi(\Gamma_{\varepsilon,h} \cap Q_l)$ is $(\mathcal{H}^n, n)$ rectifiable of class $C^3$. Finally, (6.4) follows by observing that

$$\varphi(\Gamma_{\varepsilon,h}) = \bigcup_l \varphi(\Gamma_{\varepsilon,h} \cap Q_l).$$

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