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Existence and uniqueness of group structures on covering spaces over groups

by

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Abstract. Let $f : X \to Y$ be a covering map from a connected space X onto a topological group Y and let $x_0 \in X$ be a point such that $f(x_0)$ is the identity of Y. We examine if there exists a group operation on X which makes X a topological group with identity x_0 and f a homomorphism of groups. We prove that the answer is positive in two cases: if f is an overlay map over a locally compact group Y, and if Y is locally compactly connected. In this way we generalize previous results for overlay maps over compact groups and covering maps over locally path-connected groups. Furthermore, we prove that in both cases the group structure on X is unique.

1. Introduction and main results. A covering map $f : X \to Y$ between topological groups, which is a homomorphism of groups as well, is called a *covering homomorphism*. In the theory of covering maps over topological groups the following question naturally arises:

> Let $f : X \to Y$ be a covering map from a connected space X onto a connected topological group Y and let $x_0 \in X$ be a point such that $f(x_0)$ is the identity of Y. Is it possible to define a group operation on X which makes X a topological group with identity x_0 and f a covering homomorphism?

The answer is positive (and known for a long time) if X is path-connected and Y is path-connected and locally path-connected [12, Theorem 79]. In that case the operation on X is defined by lifting of paths. In particular, any covering map $f : X \to Y$ from a connected space X onto a Lie group Y is a covering homomorphism. Recently, several authors have considered

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the question in the case of covering maps over arbitrary compact connected groups. Firstly, it was shown that the answer is positive if f is finite-sheeted ([6, Section 1 and 2] and [7, Theorem 1]). Previously, the positive answer was obtained for finite-sheeted covering maps over compact solenoidal groups, but using different methods [8, Theorem]. In [6] and [7] the investigation was motivated by applications to the theory of algebraic equations with continuous coefficients [6, Sections 3 and 4] and by finding algebraic criteria for triviality of covering maps over groups ([6, Section 4] and [7, Section 4]). We refer the reader to those references for more on this topic.

The finite-sheeted case for arbitrary compact connected groups was also proved independently in [2, Lemma 2.9]. In [2] and [7], f is presented as the inverse limit of a pull-back expansion of finite-sheeted covering maps $f_\lambda : X_\lambda \to Y_\lambda$ over compact connected Lie groups Y_λ, and the group operation on X is induced by those on X_λ.

Trying to solve the infinite-sheeted case, it was noticed that any covering homomorphism between topological groups is a special covering map, namely an overlay map ([3, Theorem 2.2] and [1, Corollary 3.8]). Then it was proved that for any overlay map $f : X \to Y$ from a connected space X onto a compact connected group Y there is a group structure on X with identity x_0 making f a covering homomorphism ([3, Theorem 2.4, Corollary 2.5] and [1, Corollary 7.3]). In [3] the proof was based on the fact that a covering map $f : X \to Y$ from a connected space X to a compact group Y is an overlay map if and only if f is the inverse limit of a pull-back expansion of covering maps $f_\lambda : X_\lambda \to Y_\lambda$ onto compact connected ANRs Y_λ [3, Theorem 2.4].

We recall that the notion of an overlay map was introduced by R. H. Fox in 1972 in an attempt to extend the classical classification theorem of covering space theory to arbitrary connected metric spaces ([4], [5]). Every overlay map is a covering map, and the converse holds in some cases: if Y is a connected locally connected paracompact space [9, Lemma 4] or if Y is a connected paracompact space and the number of sheets is finite [10, Theorem 1] (see also [5, Theorem 3] for the metric case). In the case of surjective maps between metrizable spaces, overlay maps are characterized as local isometries [1, Example 4.12 and Theorem 5.4]. Finally, the question was answered in the negative by constructing connected spaces X and infinite-sheeted covering maps $f : X \to \Sigma$ over solenoids Σ, i.e. compact connected 1-dimensional abelian groups, which do not admit a topological group structure on X making f a covering homomorphism [3, §3]. Note that the constructed maps $f : X \to \Sigma$ are covering maps which are not overlay maps.

The aim of this paper is to investigate the existence and uniqueness of a group structure on the total space X of a covering map $f : X \to Y$ over a group Y which makes f a covering homomorphism generalizing the results obtained for overlay maps over compact groups and covering maps over locally path-connected groups. To do that we introduce a new notion of "f-compactly connected" and its local version, and then consider two particular cases: overlay maps from connected, f-compactly openly connected spaces, and covering maps from connected, locally f-compactly connected spaces. We prove that in both cases there exists a group operation on X making X a topological group with identity x_0 and f a homomorphism (precise definitions of the relevant notions will be stated later):

THEOREM 1.1. *Let $f : X \to Y$ be an overlay map from a connected space X onto a topological group Y and let $x_0 \in X$ be such that $f(x_0)$ is the identity of Y. If X is f-compactly openly connected, then there exists an operation $\cdot$ on X such that $(X, \cdot)$ is a topological group with identity x_0 and f is a homomorphism of groups. In addition, if $x_0^\# \in X$ is another point such that $f(x_0^\#)$ is the identity of Y, and $\cdot^\#$ is the corresponding operation with identity $x_0^\#$, then the topological groups $(X, \cdot^\#)$ and $(X, \cdot)$ are isomorphic.*

THEOREM 1.2. *Let $f : X \to Y$ be a covering map from a connected space X onto a topological group Y and let $x_0 \in X$ be such that $f(x_0)$ is the identity of Y. If X is locally f-compactly connected, then there exists an operation $\cdot$ on X such that $(X, \cdot)$ is a topological group with identity x_0 and f is a homomorphism of groups. In addition, if $x_0^\# \in X$ is another point such that $f(x_0^\#)$ is the identity of Y, and $\cdot^\#$ is the corresponding operation with identity $x_0^\#$, then the topological groups $(X, \cdot^\#)$ and $(X, \cdot)$ are isomorphic.*

In addition, if Y is an abelian group, then so is X (Corollary 3.11). The group operation on X is given explicitly by lifting chains of open sets, following, in a way, the definition given by lifting paths for covering maps over connected locally path-connected groups. Since a connected total space X of a covering map f over a locally compact group is f-compactly openly connected, Theorem 1.1 implies that an overlay map from a connected space onto a locally compact group is a covering homomorphism (Corollary 3.13). In this way we generalize a previous result for overlay maps over compact groups. On the other hand, since path-connectedness implies compact connectedness, and local path-connectedness implies local compact connectedness, Theorem 1.2 generalizes the case of covering maps from path-connected spaces onto path-connected, locally path-connected groups. Hence we answer the starting question in the positive for overlay maps over locally compact groups and covering maps over locally compactly connected groups. In addition, we prove that in these cases the group operation is unique:

THEOREM 1.3. *Let $f : X \to Y$ be a covering map from a connected space X to a topological group Y and let $x_0 \in X$ be such that $f(x_0)$ is the identity of Y. If one of the following conditions holds:*

(1) *$f : X \to Y$ is an overlay map and Y is locally compact,*
(2) *X is locally f-compactly connected,*

then there exists a unique group structure on X making X a topological group with identity x_0 and f a homomorphism.

All topological groups in this paper are Hausdorff.

2. Basic notions. Let $f : X \to Y$ be a continuous map and let $B \subseteq Y$ be a non-empty open set in Y. We say that B is *evenly covered by f* if there exists a family $\mathcal{A}_B = (A_B^\sigma, \sigma \in S_B)$ of open sets in X, indexed by a set S_B, provided the following three conditions are fulfilled:

(C1) $f^{-1}(B) = \bigcup_{\sigma\in S_B} A_B^\sigma$,
(C2) $A_B^\sigma \cap A_B^\tau = \emptyset$ for $\sigma \neq \tau$,
(C3) $f|_{A_B^\sigma} : A_B^\sigma \to B$ is a homeomorphism.

In that case we also say that $\mathcal{A}_B = (A_B^\sigma, \sigma \in S_B)$ *evenly covers B* (*with respect to f*) and each A_B^σ, $\sigma \in S_B$, is called a *sheet* (or a *slice*) *over B*.

Let $\mathcal{B}$ be an open covering of Y and let $(\mathcal{A}_B, B \in \mathcal{B})$ be a collection of families $\mathcal{A}_B = (A_B^\sigma, \sigma \in S_B)$ of open sets in X. If $\mathcal{A}_B$ evenly covers B for every $B \in \mathcal{B}$, then $\mathcal{A} = (A_B^\sigma,\ B \in \mathcal{B},\ \sigma \in S_B)$ is an open covering of X and we say that $(\mathcal{A}, \mathcal{B})$ is a *covering pair* for $f : X \to Y$. Recall that a map $f : X \to Y$ is a *covering map* provided it admits a covering pair $(\mathcal{A}, \mathcal{B})$. Note that a covering map is an open surjection and a local homeomorphism.

LEMMA 2.1. *Let $(\mathcal{A}, \mathcal{B})$ be a covering pair for a covering map $f : X \to Y$. If $B, B' \in \mathcal{B}$ intersect, then* $\operatorname{card} S_B = \operatorname{card} S_{B'} = \operatorname{card} f^{-1}(\{y\})$ *for all $y \in B \cup B'$. If Y is connected, then all the fibers $f^{-1}(\{y\})$, $y \in Y$, and all the indexed sets S_B, $B \in \mathcal{B}$, have the same cardinality.*

Proof. Let $B, B' \in \mathcal{B}$ with $B \cap B' \neq \emptyset$. If $y \in B$, then the fiber $f^{-1}(\{y\})$ intersects each sheet A_B^σ, $\sigma \in S_B$, in a unique point. Consequently, $\operatorname{card} f^{-1}(\{y\}) = \operatorname{card} S_B$ is constant on B. Analogously, if $y \in B'$, then $\operatorname{card} f^{-1}(\{y\}) = \operatorname{card} S_{B'}$ is constant on B'. Since $B \cap B' \neq \emptyset$, it follows that $\operatorname{card} S_B = \operatorname{card} S_{B'} = \operatorname{card} f^{-1}(\{y\})$ for all $y \in B \cup B'$. Assume that Y is connected and let $y, y' \in Y$ be distinct. There exists a finite chain $B_1, \ldots, B_n \in \mathcal{B}$ such that $y \in B_1$, $y' \in B_n$ and $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$. Then $\operatorname{card} f^{-1}(\{y\}) = \operatorname{card} S_{B_1} = \operatorname{card} S_{B_2} = \cdots = \operatorname{card} S_{B_{n-1}} = \operatorname{card} S_{B_n} = \operatorname{card} f^{-1}(\{y'\})$. ∎

According to the previous lemma, if $(\mathcal{A}, \mathcal{B})$ is a covering pair for $f : X \to Y$ and Y is connected, then all $B \in \mathcal{B}$ have the same number of sheets and one can use the same index set S, with $\operatorname{card} S = s$, for all sets S_B, $B \in \mathcal{B}$. In this case $\mathcal{A} = (A_B^\sigma,\ B \in \mathcal{B},\ \sigma \in S)$ and we refer to S as the index set of the pair $(\mathcal{A}, \mathcal{B})$ and f is said to be an *s-sheeted* covering map.

Furthermore, depending on s being finite or infinite, we say that f is *finite-sheeted* or *infinite-sheeted*. In particular, if $f : X \to Y$ is a covering map from a connected space X, then f is s-sheeted for some cardinal s.

A covering pair $(\mathcal{A}, \mathcal{B})$ for a covering map $f : X \to Y$ is said to be an *overlay pair* for f provided the following additional condition is fulfilled:

(C4) If $B, B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$, then every $\sigma \in S_B$ admits a unique $\sigma' \in S_{B'}$ such that $A_B^\sigma \cap A_{B'}^{\sigma'} \neq \emptyset$.

A covering map $f : X \to Y$ is said to be an *overlay map* provided it admits an overlay pair $(\mathcal{A}, \mathcal{B})$. The notion of an overlay map was introduced by R. H. Fox in 1972 ([4], [5]). However, he considered only overlay maps over connected metric spaces. Later on the notion was extended to arbitrary connected spaces Y ([9]), and recently a different view on overlay maps over arbitrary spaces Y was given via star refinements of open covers [1]. We remark that in [9], besides (C4), it was required that the covering $\mathcal{B}$ be normal. Recall that an open covering is called *normal* (or *numerable*) if it admits a subordinated partition of unity. Since in a paracompact space each open covering is normal, the definitions given by Fox and those in [9] and [1] coincide if Y is paracompact and connected.

If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ are covering [overlay] pairs for $f : X \to Y$, we say that $(\mathcal{A}', \mathcal{B}')$ *refines* $(\mathcal{A}, \mathcal{B})$ and write $(\mathcal{A}', \mathcal{B}') \le (\mathcal{A}, \mathcal{B})$ if for every $B' \in \mathcal{B}'$ there exists $B \in \mathcal{B}$ such that $B' \subseteq B$, and for every $\sigma' \in S_{B'} = S_B$ there exists $\sigma \in S_B$ such that $A'^{\sigma'}_{B'} \subseteq A^\sigma_B$. If $(\mathcal{A}, \mathcal{B})$ is a covering [overlay] pair for f and an open covering $\mathcal{B}'$ refines $\mathcal{B}$, then naturally, by setting $A'^{\sigma}_{B'} = f^{-1}(B') \cap A^\sigma_B$ for $B' \subseteq B$, we get a covering [overlay] pair $(\mathcal{A}', \mathcal{B}')$ for f which refines $(\mathcal{A}, \mathcal{B})$.

It is known that a covering map $f : X \to Y$ over a connected paracompact space Y is an overlay map if f is finite-sheeted [10, Theorem 1] or if Y is locally connected [9, Lemma 4]. In both cases it is proved that for arbitrary covering pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ for f there exists an overlay pair $(\mathcal{A}'', \mathcal{B}'')$ which refines both $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$. However, the assumption that Y is connected does not play any essential role in those proofs, and it is easy to see that the following holds:

Lemma 2.2. *Let $f : X \to Y$ be a covering map over a paracompact space Y. If all the fibers of f are finite or if Y is locally connected, then f is an overlay map.*

Lemma 2.3 ([3, Theorem 2.2]). *Let $f : X \to Y$ be a covering map between topological groups. If f is a homomorphism of groups, then f is an overlay map.*

For a covering map $f : X \to Y$, a pair $(\mathcal{A}, \mathcal{B})$ consisting of a family $\mathcal{A}$ of open sets in X and of a family $\mathcal{B}$ of open sets in Y is said to be a *partial* *overlay pair* for f if $f|\bigcup\mathcal{A} : \bigcup\mathcal{A} \to \bigcup\mathcal{B}$ is an overlay map with overlay pair $(\mathcal{A},\mathcal{B})$. In that case we also say that $f : X \to Y$ is a *partial overlay map*.

LEMMA 2.4. *Let $f : X \to Y$ be a covering map and let $\mathcal{B}$ a family of open connected subsets of Y such that $B \cup B'$ is evenly covered by f for any $B, B' \in \mathcal{B}$ with $B \cap B' \neq \emptyset$. Then there exists a family $\mathcal{A}$ of open connected subsets of X such that $(\mathcal{A},\mathcal{B})$ is a partial overlay pair for f, and the family $\mathcal{A}$ is uniquely determined by $\mathcal{B}$ and f. If $\mathcal{B}$ is a covering of Y, then $(\mathcal{A},\mathcal{B})$ is an overlay pair for f.*

Proof. By assumption, each $B \in \mathcal{B}$ is evenly covered by f; let $\mathcal{A}$ be the family of all sheets over all $B \in \mathcal{B}$. Each $B \in \mathcal{B}$ is connected, so $\mathcal{A}$ consists of open connected subsets of X.

We claim that $f|\bigcup\mathcal{A} : \bigcup\mathcal{A} \to \bigcup\mathcal{B}$ is an overlay map with overlay pair $(\mathcal{A},\mathcal{B})$. Obviously $(\mathcal{A},\mathcal{B})$ is a covering pair for $f|\bigcup\mathcal{A}$. Let $B, B' \in \mathcal{B}$ with $B \cap B' \neq \emptyset$ and let $A \in \mathcal{A}$ be a sheet over B. Then there is a connected component A' of $f^{-1}(B')$ such that $A' \cap A \neq \emptyset$. Since f evenly covers $B \cup B'$, A' is unique. Hence $(\mathcal{A},\mathcal{B})$ is a partial overlay pair. Uniqueness of $\mathcal{A}$ follows from connectedness of members B of $\mathcal{B}$. If $\mathcal{B}$ is a covering of X, then $\mathcal{A}$ is a covering of X, since each $B \in \mathcal{B}$ is evenly covered by f. Hence, $(\mathcal{A},\mathcal{B})$ is an overlay pair for f. ∎

Note that the claim of Lemma 2.2 for the locally connected case easily follows from the previous lemma. Namely, in a paracompact locally connected space Y each open covering of Y has a star refinement consisting of open connected sets. Consequently, for a covering map $f : X \to Y$ we can find a covering $\mathcal{B}$ of Y consisting of open connected sets such that $B \cup B'$ is evenly covered by f for any $B, B' \in \mathcal{B}$ with $B \cap B' \neq \emptyset$, and the lemma applies.

A neighborhood V of the identity of a topological group is called *symmetric* if $V = V^{-1}$.

LEMMA 2.5. *Let $f : X \to Y$ be a covering map over a locally connected topological group Y, K a subset of Y, and U a neighborhood of the identity e of Y such that f evenly covers each Uy for $y \in K$. Then there exists a symmetric connected open neighborhood $V \subseteq U$ of e such that the family $\mathcal{B} = (Vy, y \in K)$ admits a partial overlay pair $(\mathcal{A},\mathcal{B})$ for f and $\mathcal{A}$ is uniquely determined by $\mathcal{B}$ and f.*

Proof. Choose a connected symmetric open neighborhood V of e such that $V^3 \subseteq U$ and set $\mathcal{B} = (Vy, y \in K)$. Suppose that $Vy \cap Vy' \neq \emptyset$ for $y, y' \in K$. Then there are $v_0, v_1 \in V$ such that $v_0 y = v_1 y'$. Hence $y' \in VVy$ and $Vy' \subseteq VVVy \subseteq Uy$. Now we get $Vy \cup Vy' \subseteq Uy$ and $Vy \cup Vy'$ is evenly covered by f. By Lemma 2.4 we get the desired partial overlay pair $(\mathcal{A},\mathcal{B})$ for f. ∎

For a covering map $f : X \to Y$, we say that a space X is *f-compactly connected* [*f-compactly openly connected*] if for any points $a, b \in X$ there exists a [an open] connected subset C of X such that $a, b \in C$ and the closure $\overline{f(C)}$ is compact. If $f : X \to Y$ is a covering map from a connected space X onto a compact space Y, then obviously X is f-compactly openly connected.

A space X is said to be *compactly connected* if X is id_X-compactly connected. Note that each compact connected space and each path-connected space is compactly connected.

The next lemma is a direct consequence of the Iwasawa structure theorem [11, §4.13].

Lemma 2.6. *Let X be a connected locally compact group. Then there exist a compact connected subgroup K and a subspace S containing the identity such that $X = SK$ and S is homeomorphic to the euclidean space $\mathbb{R}^n$ for some non-negative integer n, where $\mathbb{R}^0 = \{0\}$. In particular, X is homeomorphic to the direct product $K \times \mathbb{R}^n$.*

Proof. If X is compact, then $S = \{e\}$, e is the identity of X, and $K = X$ satisfy all the requirements for $n = 0$. Assume that X is not compact. According to the Iwasawa structure theorem [11, §4.13], X contains a compact connected subgroup K and subgroups $H_1, \ldots, H_n$ all isomorphic to the additive group $\mathbb{R}$ such that each $x \in X$ can be uniquely and continuously decomposed in the form $x = h_1 \cdots h_n k$, $h_i \in H_i$, $k \in K$. Denote by S the subspace $H_1 \cdots H_n$. Then S contains the identity e, $X = SK$ and S is homeomorphic to $\mathbb{R}^n$, $n \geq 1$. Since SK is homeomorphic to $K \times \mathbb{R}^n$, $n \geq 0$, we get the second claim of the lemma. ∎

Since the direct product of two compactly connected spaces is compactly connected, the preceding lemma implies:

Lemma 2.7. *A connected locally compact group is compactly connected.*

It is easy to see that if X is compactly connected, then it is f-compactly connected for any covering map $f : X \to Y$. We remark that connected covering spaces over solenoids constructed in [3] are not compactly connected, so there exist covering maps $f : X \to Y$ such that X is f-compactly connected but not compactly connected.

For a covering map $f : X \to Y$, the space X is said to be *locally f-compactly connected* if for any $a \in X$ and any neighborhood U of a there exists a neighborhood V of a with the property that for any $b \in V$ there exists a connected subset C of U such that $a, b \in C$ and $\overline{f(C)}$ is compact.

A space X is said to be *locally compactly connected* if X is locally id_X-compactly connected. Obviously, each locally path-connected space is locally compactly connected. If X is locally compactly connected, then X is locally connected. The converse holds if X is locally compact. In particular, the following holds:

LEMMA 2.8. *Let X be a path-connected locally compact group. Then the following statements are equivalent:*

(i) *X is locally path-connected.*
(ii) *X is locally compactly connected.*
(iii) *X is locally connected.*

Proof. (i)⇒(ii) and (ii)⇒(iii) are obvious, while (iii)⇒(i) follows from [13, Theorem 3]. ∎

We remark that there exist non-locally compact, connected groups that are locally compactly connected, but not locally path-connected. An example is the product $G_0 \times G_1$ of an infinite-dimensional normed space G_0 and a compact, connected, locally connected, non-locally path-connected group G_1. For G_1 one can take the Pontryagin dual of the infinite direct product of copies of the integer group.

LEMMA 2.9. *If a space X is connected and locally compactly connected, then X is compactly connected.*

Proof. Choose $a_0 \in X$ and let X_0 be the subset of X consisting of those $a \in X$ such that there exists a compact connected set C with $a_0, a \in C$. First note that X_0 is not empty, since $a_0 \in X_0$.

We claim that X_0 is clopen in X. Let $a \in X_0$ and let U be a neighborhood of a. Take a neighborhood $V \subseteq U$ of a obtained from the local compact connectedness of X. Then for each $b \in V$ there exists a connected compact set C with $a_0, b \in C$, which proves that V is contained in X_0. Hence, X_0 is open. Assume that $a \in \overline{X_0}$. Choose a neighborhood U of a and let V be a related neighborhood of a obtained from the local compact connectedness of X. Since $V \cap X_0 \neq \emptyset$, it follows that $a \in X_0$. Hence X_0 is closed. By assumption, X is connected, so $X_0 = X$ and we have the conclusion. ∎

If a space X is locally compactly connected, then it is locally f-compactly connected for any covering map $f : X \to Y$. The converse is considered in the next lemma.

LEMMA 2.10. *If X is locally f-compactly connected for some covering map $f : X \to Y$ over a regular space Y, then X is locally compactly connected.*

Proof. Pick $a \in X$ and a neighborhood U of a. Since f is a covering map over a regular space Y, X is regular as well. Hence, there is a neighborhood U_0 of a such that $\overline{U_0} \subseteq U$ and $f|\overline{U_0} : \overline{U_0} \to f(\overline{U_0})$ is a homeomorphism. By assumption, there is neighborhood V of a such that for any $b \in V$ there exists a connected subset C of U_0 with $a, b \in C$ and with $\overline{f(C)}$ compact.

Since $\overline{C} \subseteq \overline{U_0}$, the set $\overline{C}$ is homeomorphic to $\overline{f(C)}$ and is compact. Hence, V has the required property that for any $b \in V$ there is a connected set $C \subseteq U$ such that $a, b \in C$ and $\overline{C}$ is compact. ∎

3. Proofs of Theorems 1.1 and 1.2. Let $f : X \to Y$ be a covering map from a connected space X onto a topological group Y. Our aim is to prove that there is a group operation on X making X a topological group and f a group homomorphism in two cases: (i) f is an overlay map and X is f-compactly openly connected, (ii) X is locally f-compactly connected. To do that, we will follow, in a way, the idea given in [12, Theorem 79] for connected, locally path-connected spaces, but instead of lifting paths we will lift chains of open sets and use related finite sequences. Those sequences will play the role of paths and homotopies in covering space theory.

DEFINITION 3.1. Let $\mathcal{A}$ be a family of open sets in a space X. A sequence $a_0, \dots, a_n$ of points in X is called an $\mathcal{A}$-*sequence* if there exists a sequence $A_0, \dots, A_{n-1}$ of elements of $\mathcal{A}$ such that $a_i, a_{i+1} \in A_i$ for all $0 \le i \le n-1$.

The sequence $a_0, \dots, a_n$ obtained in the next lemma is called the $\mathcal{A}$-*lift* of the $\mathcal{B}$-sequence $y_0, \dots, y_n$ starting from a_0.

LEMMA 3.2. *Let $(\mathcal{A}, \mathcal{B})$ be a partial overlay pair for a covering map $f : X \to Y$ and let $y_0, \dots, y_n$ be a $\mathcal{B}$-sequence in Y. If $a_0 \in \bigcup \mathcal{A}$ with $f(a_0) = y_0$, then there exists a unique $\mathcal{A}$-sequence $a_0, \dots, a_n$ in X such that $f(a_i) = y_i$ for $0 \le i \le n$.*

Proof. Since $y_0, \dots, y_n$ is a $\mathcal{B}$-sequence, there exists a sequence $B_0, \dots, B_n$ of elements of $\mathcal{B}$ such that $y_i, y_{i+1} \in B_i$ for $0 \le i \le n-1$. Let $A_0 \in \mathcal{A}$ be unique such that $x_0 \in A_0$ and $f|A_0 : A_0 \to B_0$ is a homeomorphism. By the overlay property of $(\mathcal{A}, \mathcal{B})$ there are unique $A_1, \dots, A_{n-1} \in \mathcal{A}$ such that $A_i \cap A_{i+1} \neq \emptyset$ for $0 \le i \le n-2$, and $f|A_i : A_i \to B_i$, $1 \le i \le n-1$, is a homeomorphism. Since each $f|A_i$ is a homeomorphism, we have a unique $a_i \in A_i$ such that $f(a_i) = y_i$. Since $y_i, y_{i+1} \in B_i$ and $A_i \cap A_{i+1} \neq \emptyset$, it follows that $a_{i+1} \in A_i$ for $0 \le i \le n-1$. Hence, $a_0, \dots, a_n$ is an $\mathcal{A}$-sequence with the required property.

It remains to prove the uniqueness of $a_0, \dots, a_n$. Assume that $a'_0, \dots, a'_n$ is another $\mathcal{A}$-sequence such that $a'_0 = a_0$ and $f(a'_i) = y_i$ for $0 \le i \le n$. Then there exist $A'_0, \dots, A'_{n-1} \in \mathcal{A}$ such that $a'_i, a'_{i+1} \in A'_i$ for $0 \le i \le n-1$. For each $i = 0, \dots, n-1$ set $B'_i = f(A'_i) \in \mathcal{B}$ and note that $f|A'_i : A'_i \to B'_i$ is a homeomorphism and $y_i, y_{i+1} \in B'_i$. By induction on i we will prove $x_i = x'_i$ for $i = 0, \dots, n$. We have $a_0 = a'_0$ by assumption. Assume that $a_i = a'_i$. Then $A_i \cap A'_i \neq \emptyset$, and since $(\mathcal{A}, \mathcal{B})$ is an overlay pair, it follows that $f|A_i \cap A'_i : A_i \cap A'_i \to B_i \cap B'_i$ is a homeomorphism. On the other hand, $y_{i+1} \in B_i \cap B'_i$ and we conclude that $a_{i+1} = a'_{i+1}$, which proves the claim.

Note that in this way we also prove that the $\mathcal{A}$-sequence $a_0, \ldots, a_n$ does not depend on the choice of the elements B_i of $\mathcal{B}$ such that $y_i, y_{i+1} \in B_i$ for $0 \le i \le n-1$. ∎

Lemma 3.3. *Let V be a symmetric neighborhood of the identity of a topological group Y. Then $z \in yV \cap Vy$ if and only if $y \in zV \cap Vz$.*

Proof. Suppose that $z \in yV \cap Vy$. Then $y^{-1}z, zy^{-1} \in V$, and consequently $z^{-1}y, yz^{-1} \in V^{-1} = V$. Hence, $y \in zV \cap Vz$. The statement is self-symmetric and the converse also holds. ∎

Lemma 3.4. *Let K be a compact subset of a topological group Y and let U be a neighborhood of the identity e of Y. Then there exists an open symmetric neighborhood V of e such that $VyVy^{-1} \subseteq U$ for every $y \in K$.*

Proof. Assume the contrary, i.e. for each open symmetric neighborhood V of e there exists $y \in K$ such that $VyVy^{-1} \not\subseteq U$. Let $\mathcal{V}_e$ be the set of all open symmetric neighborhoods V of e. For $V \in \mathcal{V}_e$, consider the non-empty set $S_V = \{y \in K : VyVy^{-1} \not\subseteq U\}$. Note that $S_W \subseteq S_V$ for $W \subseteq V$. Since K is compact, the intersection $\bigcap\{\overline{S_V} : V \in \mathcal{V}_e\}$ is non-empty; choose y_0 in that intersection. Then there exists $V_0 \in \mathcal{V}_e$ such that $V_0^2 y_0 V_0^2 y_0^{-1} \subseteq U$. Since $y_0 \in \overline{S_{V_0}}$, there is $y_1 \in y_0V_0 \cap V_0y_0 \cap S_{V_0}$. Then $V_0y_1V_0y_1^{-1} \subseteq V_0V_0y_0V_0V_0^{-1}y_0^{-1} = V_0^2y_0V_0^2y_0^{-1} \subseteq U$, which contradicts $y_1 \in S_{V_0}$. ∎

From the previous lemma it follows that for any compact subset K of a topological group Y and any neighborhood U of the identity e of Y there exists an open symmetric neighborhood V of e such that $yVy^{-1} \subseteq U$ for every $y \in K$.

Let $(\mathcal{A}, \mathcal{B})$ be a partial overlay pair for a covering map $f : X \to Y$ from an f-compactly connected space X onto a group Y with identity e. Let $x_0 \in X$ with $f(x_0) = e$, and let C be a connected subset of $\bigcup \mathcal{A}$ such that $x_0 \in C$ and $\overline{f(C)}$ is compact.

To define $\mathcal{A}$-sequences which will play the role of paths and homotopies as mentioned above, we need neighborhoods V of e having the following property:

(∗) V is a symmetric open neighborhood of the identity of Y, and every VyV, for $y \in f(C)$, is contained in some $B \in \mathcal{B}$.

Suppose that (∗) holds for V. For $x \in C$, denote by W_x an open neighborhood of x such that $f|W_x$ is a homeomorphism onto $f(x)V \cap Vf(x)$ and $W_x \subseteq A$ for some $A \in \mathcal{A}$. We say that a sequence $x_0, \ldots, x_n$ of points in C is an *$\mathcal{A}$-sequence with respect to V* if $x_{i+1} \in W_{x_i}$ for $0 \le i \le n-1$. Note that $f(x_{i+1}) \in f(x_i)V \cap Vf(x_i)$ for $0 \le i \le n-1$.

A sequence $u_0, \dots, u_m$ in X is called an *expansion* of a sequence $x_0, \dots, x_n$ if there exist $0 = m_0 < m_1 < \dots < m_n = m$ such that $u_{m_i} = x_i$ for $i = 0, \dots, n$.

LEMMA 3.5. *Let $(\mathcal{A}, \mathcal{B})$ be a partial overlay pair for f and let $C \subseteq \bigcup \mathcal{A}$ be a connected subset of X such that $a_0 \in C$ and $\overline{f(C)}$ is compact. If V satisfies $(*)$, then, for every $a \in C$, there exists an $\mathcal{A}$-sequence $a_0, \dots, a_n$ in C with respect to V such that $a_n = a$.*

Proof. Let O be the set of all $a \in C$ such that there exists an $\mathcal{A}$-sequence from a_0 to a with respect to V satisfying the statement. Then O contains a_0 and is open in C.

We claim that O is closed in C. Let $u \in C$ and suppose that $W_u \cap O \neq \emptyset$. Choose $v \in W_u \cap O$. Then $f(v) \in f(u)V \cap Vf(u)$ and Lemma 3.3 implies $f(u) \in f(v)V \cap Vf(v)$. Since $v \in O$ and $f|W_v$ is a homeomorphism, and since $(\mathcal{A}, \mathcal{B})$ is a partial overlay pair, we have $u \in W_v \cap O$. So, O is clopen in C, and since C is connected, it follows that $O = C$. ∎

Next we define a group operation $\cdot$ on a covering space X under one of the conditions of Theorems 1.1 and 1.2. Since many parts are common for both theorems, for brevity we refer to Theorem 1.1 as *the first case* and to Theorem 1.2 as *the second* case.

Definition of a group operation on X. Take an overlay pair $(\mathcal{A}, \mathcal{B})$ for f in the first case and a covering pair $(\mathcal{C}, \mathcal{D})$ for f in the second. First note that a topological group is a regular space, which implies that X is f-compactly connected in the second case according to Lemmas 2.10 and 2.9. So, X is f-compactly connected in both cases. Hence, for $a, b \in X$ there is a connected subset C of X such that $x_0, a, b \in C$ and $\overline{f(C)}$ is compact. Since $\overline{f(C)f(C)}$ is compact, there exists a neighborhood U of e such that every Uy with $y \in \overline{f(C)f(C)}$ is contained in some $B \in \mathcal{B}$ in the first case, and in some $B \in \mathcal{D}$ in the second case. Then, in the second case, the family $(Uy, y \in \overline{f(C)f(C)})$ is evenly covered by f, and by Lemma 2.5 there exists an open symmetric connected neighborhood $U_0 \subseteq U$ of e such that $\mathcal{B} = (U_0 y, y \in \overline{f(C)f(C)})$ admits a partial overlay pair $(\mathcal{A}, \mathcal{B})$ for f, and $\mathcal{A}$ is uniquely determined by $\mathcal{B}$ and f. Since $f(C) \subseteq \overline{f(C)f(C)}$, the set C is contained in $\bigcup \mathcal{A}$. Rename this U_0 as U.

By Lemma 3.4, there exists an open symmetric neighborhood V of e such that $VyVy^{-1} \subseteq U$ for every $y \in \overline{f(C)f(C)}$. Since obviously $yV \cap Vy \subseteq VyV$ for $y \in \overline{f(C)f(C)}$, the neighborhood V satisfies $(*)$. Then there is an $\mathcal{A}$-sequence $b_0, \dots, b_n$ in C with respect to V such that $b_0 = x_0$ and $b_n = b$ by Lemma 3.5.

Since $f(a)f(b_{i+1}) \in f(a)f(b_i)V$ for $0 \le i \le n-1$, it follows that $f(a)f(b_0), \dots, f(a)f(b_n)$ is a $\mathcal{B}$-sequence. By Lemma 3.2, there is a unique

$\mathcal{A}$-sequence $u_0, \ldots, u_n$ such that $u_0 = a$ and $f(u_i) = f(a)f(b_i)$ for $0 \le i \le n-1$. We define $a \cdot b = u_n$.

Note that an overlay pair $(\mathcal{A}, \mathcal{B})$ is fixed and given in advance in the first case, while in the second case the partial overlay pair $(\mathcal{A}, \mathcal{B})$ depends on the set C. Further, $f(a \cdot b) = f(u_n) = f(a)f(b_n) = f(a)f(b)$.

We need to show that the operation $\cdot$ is well-defined.

LEMMA 3.6. *The product $a \cdot b$ does not depend on the choice of C or V whenever they fulfill the following requirement:*

(**) *C is a connected subset of $\bigcup \mathcal{A}$ such that $x_0, a, b \in C$, $\overline{f(C)}$ is compact, and V is a symmetric open neighborhood of $e \in Y$ such that for each $y \in f(C)f(C)$ there exists $B \in \mathcal{B}$ with $VyV \subseteq B$.*

In addition, in the second case, $a \cdot b$ does not depend on the choice of the partial overlay pair $(\mathcal{A}, \mathcal{B})$.

Proof. First we show that $a \cdot b$ does not depend on the choice of the $\mathcal{A}$-sequence $b_0, \ldots, b_n$ in C with respect to V such that $b_0 = x_0$ and $b_n = b$.

CLAIM 1. *Let $b'_0, \ldots, b'_k$ be an $\mathcal{A}$-sequence in C with respect to V such that $b'_0 = x_0$ and $b'_k = b$. If $u'_0, \ldots, u'_k$ is the $\mathcal{A}$-lift of $f(a)f(b'_0), \ldots, f(a)f(b'_k)$ starting from $u'_0 = a$, then $u'_k = u_n$.*

Choose in C an $\mathcal{A}$-sequence $a_0, \ldots, a_m$ with respect to V such that $a_0 = x_0$ and $a_m = a$. Since $f(a_i)f(b_0), \ldots, f(a_i)f(b_m)$ and $f(a_i)f(b'_0), \ldots, f(a_i)f(b'_k)$ are $\mathcal{B}$-sequences, each has a unique $\mathcal{A}$-lift starting from a_i. Denote by $a_i \cdot b$ and $a_i \cdot' b$ the points in X obtained from the sequences $b_0, \ldots, b_n$ and $b'_0, \ldots, b'_k$ respectively.

We will show $a_i \cdot b = a_i \cdot' b$ for every $i = 0, \ldots, m$, by induction on i. For $i = 0$, we have $a_0 = x_0$ and $a_0 \cdot b = b_n = b = b'_k = a_0 \cdot' b$. Assume that $a_i \cdot b = a_i \cdot' b$. To prove $a_{i+1} \cdot b = a_{i+1} \cdot' b$, it is sufficient to show that $a_{i+1} \cdot b$ and $a_{i+1} \cdot' b$ belong to the same sheet over $Vf(a_i)f(b)V \subseteq Uf(a_i)f(b_j)$ determined by $(\mathcal{A}, \mathcal{B})$. To do that, let $v_i = f(a_{i+1})f(a_i)^{-1}$ and $w_j = f(b_j)^{-1}f(b_{j+1})$ for $j = 0, \ldots, n-1$. Since $a_0, \ldots, a_m$ is an $\mathcal{A}$-sequence with respect to V, it follows that $f(a_{i+1}) \in f(a_i)V \cap Vf(a_i)$, and consequently $v_i = f(a_{i+1})f(a_i)^{-1} \in V$. Analogously, $w_j \in V$ for each $j = 0, \ldots, n-1$. We have a unique $\mathcal{A}$-lift $c_{i,0}, \ldots, c_{i,n}$ of the $\mathcal{B}$-sequence $f(a_i)f(b_0), \ldots, f(a_i)f(b_n)$ starting from $a_i = c_{i,0}$, and a unique $\mathcal{A}$-lift $c_{i+1,0}, \ldots, c_{i+1,n}$ of the $\mathcal{B}$-sequence $f(a_{i+1})f(b_0), \ldots, f(a_{i+1})f(b_n)$ starting from $a_{i+1} = c_{i+1,0}$.

Observe that the points $f(a_{i+1})f(b_j) = v_i f(a_i)f(b_j)$,

$$f(a_{i+1})f(b_{j+1}) = v_i f(a_i)f(b_{j+1}) = v_i f(a_i)f(b_j)w_j,$$

$f(a_i)f(b_j)$ and $f(a_i)f(b_{j+1}) = f(a_i)f(b_j)w_j$ all belong to $Vf(a_i)f(b_j)V$. Then by induction on j we see that $c_{i,j}$, $c_{i,j+1}$, $c_{i+1,j}$ and $c_{i+1,j+1}$ belong to the same sheet over $Vf(a_i)f(b_j)V$ determined by $(\mathcal{A}, \mathcal{B})$. Consequently,

$a_i \cdot b = c_{i,n}$ and $a_{i+1} \cdot b = c_{i+1,n}$ belong to the same sheet over $Vf(a_i)f(b_{n-1})V$ determined by $(\mathcal{A}, \mathcal{B})$. Analogously, $a_i \cdot' b$ and $a_{i+1} \cdot' b$ belong to the same sheet over $Vf(a_i)f(b_{n-1})V$ determined by $(\mathcal{A}, \mathcal{B})$. Since $f(a_{i+1} \cdot b) = f(a_{i+1})f(b) = f(a_{i+1} \cdot' b)$ and $a_i \cdot b = a_i \cdot' b$ by assumption, we get $a_{i+1} \cdot b = a_{i+1} \cdot' b$ and the inductive step is done. Hence, $a_n \cdot b = a_n \cdot' b$, which implies $u_n = u'_k$, which proves the claim.

Claim 2. *Suppose that C' and V' fulfill $(**)$ and let $b'_0, \ldots, b'_m$ be an $\mathcal{A}$-sequence in C' with respect to V' starting at $b'_0 = x_0$ and ending at $b'_m = b$. If $u'_0, \ldots, u'_m$ is the $\mathcal{A}$-lift of $f(a)f(b'_0), \ldots, f(a)f(b'_m)$ with $u'_0 = a$, then $u'_m = u_n$.*

Set $C'' = C \cup C'$. Then C'' is connected and $\overline{f(C'')} = \overline{f(C) \cup f(C')} = \overline{f(C)} \cup \overline{f(C')}$ is compact. Hence there is an open symmetric neighborhood V'' of e such that C'' and V'' fulfill $(**)$. Without loss of generality we may assume that $V'' \subseteq V \cap V'$. Since C is connected, we may apply Lemma 3.5 to the points b_i, b_{i+1} of C for each $i = 0, \ldots, n-1$, and get an $\mathcal{A}$-sequence in C with respect to V'' which expands $b_0, \ldots, b_n$. The expanded sequence is an $\mathcal{A}$-sequence in C with respect to V which starts at x_0 and ends at b. By Claim 1 the end point of the $\mathcal{A}$-lift starting from a of the $\mathcal{B}$-sequence related to the expanded sequence is equal to u_n. Analogously, we expand $b'_0, \ldots, b'_m$ to an $\mathcal{A}$-sequence in C' with respect to V''. Then the expanded sequence is also an $\mathcal{A}$-sequence with respect to V' which starts at $x_0 = b'_0$ and ends at $b'_m = b$. By Claim 1 the end point of the $\mathcal{A}$-lift starting from a of the related $\mathcal{B}$-sequence equals u'_m. Finally, the two expanded sequences are $\mathcal{A}$-sequences in $C \cup C'$ with respect to V'' from x_0 to b. Hence, applying Claim 1 once more, we get $u_n = u'_m$.

Note that the above arguments hold even in the second case for a fixed partial overlay pair $(\mathcal{A}, \mathcal{B})$. Now, we show that in the second case $a \cdot b$ does not depend on the choice of $(\mathcal{A}, \mathcal{B})$.

Claim 3. *In the second case, suppose C' and V' fulfill $(**)$ for a partial overlay pair $(\mathcal{A}', \mathcal{B}')$, and let $b'_0, \ldots, b'_m$ be an $\mathcal{A}'$-sequence in C' with respect to V' starting at $b'_0 = x_0$ and ending at $b'_m = b$. If $u'_0, \ldots, u'_m$ is the $\mathcal{A}'$-lift of $f(a)f(b'_0), \ldots, f(a)f(b'_m)$ with $u'_0 = a$, then $u'_m = u_n$.*

Set $C'' = C \cup C'$. Then C'' is connected and $\overline{f(C'')}$ is compact. Then there is a partial overlay pair $(\mathcal{A}_0, \mathcal{B}_0)$ and an open connected symmetric neighborhood V'' of e such that C'' and V'' fulfill $(**)$ for $(\mathcal{A}_0, \mathcal{B}_0)$. Without loss of generality we may assume that $V'' \subseteq V \cap V'$. Then C and V'' fulfill $(**)$ for $(\mathcal{A}, \mathcal{B})$ and C' and V'' fulfill $(**)$ for $(\mathcal{A}', \mathcal{B}')$.

Now consider the family $\mathcal{B}'' = (V''yV'', y \in \overline{f(C'')f(C'')})$. Each $V''yV''$ is connected and contained in some $B_0 \in \mathcal{B}_0$ by $(**)$. Denote by $(\mathcal{A}'', \mathcal{B}'')$ a partial overlay pair induced by $(\mathcal{A}_0, \mathcal{B}_0)$. Then C'' and V'' fulfill $(**)$

for $(\mathcal{A}'', \mathcal{B}'')$. Note that each member of $\mathcal{A}''$ is connected. Furthermore, if $y \in \overline{f(C)f(C)}$, then $V''yV'' \subseteq VyV$ is contained in some $B \in \mathcal{B}$ and each sheet over $V''yV''$ is contained in only one sheet over B. Analogously, if $y \in \overline{f(C')f(C')}$, then $V''yV'' \subseteq V'yV'$ is contained in some $B' \in \mathcal{B}'$ and each sheet over $V''yV''$ is contained in only one sheet over B'. Hence, each $\mathcal{A}''$-sequence in C with respect to V'' is an $\mathcal{A}$-sequence in C with respect to V'', and each $\mathcal{A}''$-sequence in C' with respect to V'' is an $\mathcal{A}'$-sequence in C' with respect to V''. Since C is connected, we may apply Lemma 3.5 to the points b_i, b_{i+1} of C, for each $i = 0, \ldots, n-1$, and get an $\mathcal{A}''$-sequence in C with respect to V'' starting at $b_0 = x_0$ and ending at $b_n = b$ which expands $b_0, \ldots, b_n$. Then the end point of the unique $\mathcal{A}''$-lift starting from a of the $\mathcal{B}''$-sequence related to the expanded sequence is equal to u_n. Namely, the expanded sequence is an $\mathcal{A}$-sequence in C with respect to V'', the $\mathcal{B}''$-sequence related to the expanded sequence is a $\mathcal{B}$-sequence, the $\mathcal{A}''$-lift obtained is also an $\mathcal{A}$-lift starting from a, and by Claim 2 we get the conclusion. Analogously, we expand $b'_0, \ldots, b'_m$ to an $\mathcal{A}''$-sequence in C' with respect to V''. Then the expanded sequence is also an $\mathcal{A}'$-sequence with respect to V'' which starts at $x_0 = b'_0$ and ends at $b'_m = b$. Similarly we conclude that the end point of the unique $\mathcal{A}''$-lift starting from a of the $\mathcal{B}''$-sequence related to the expanded sequence is equal to u'_m. Then the end point of the $\mathcal{A}'$-lift of the related $\mathcal{B}'$-sequence equals u'_m. Since the two expanded sequences are $\mathcal{A}''$-sequences in C'' with respect to V'' from x_0 to b, by Claim 1 we get $u_n = u'_m$. ∎

LEMMA 3.7. *The operation $\cdot$ is continuous.*

Proof. Pick $a, b \in X$ and let W be a neighborhood of $a \cdot b$.

In the first case, since X is f-compactly openly connected, the connected set C containing a, b, x_0 can be taken to be open. Pick V satisfying $(**)$. Then there is a symmetric open neighborhood $V_0 \subseteq V$ such that the sheet over $V_0 f(a)f(b)V_0$ containing $a \cdot b$ is contained in W. By the continuity of the group operation on Y there is a symmetric neighborhood $V_1 \subseteq V_0$ such that $V_1 f(a) V_1 V_1 f(b) V_1 \subseteq V_0 f(a) f(b) V_0$. Since C is open, there is a symmetric neighborhood $V^* \subseteq V_1$ such that the sheet W_a over $f(a)V^* \cap V^* f(a)$ containing a is contained in C, and the sheet W_b over $f(b)V^* \cap V^* f(b)$ containing b is contained in C. Obviously, V^* satisfies $(**)$.

We claim that $W_a \cdot W_b \subseteq W$. Let $a' \in W_a$ and $b' \in W_b$. Then $a', b' \in C$ and there is an $\mathcal{A}$-sequence $b_0, \ldots, b_n$ in C with respect to V^* such that $b_0 = x_0$ and $b_n = b$. Then $a \cdot b$ is the end point of the $\mathcal{A}$-lift of $f(a)f(b_0), \ldots, f(a)f(b_n)$ starting from a. Since b' and b belong to the same sheet over $f(b)V^* \cap V^* f(b)$, it follows that $b_0, \ldots, b_n, b'$ is also an $\mathcal{A}$-sequence in C with respect to V^*, and $a' \cdot b'$ is defined as the end point of the $\mathcal{A}$-lift of the $f(a')f(b_0), \ldots, f(a')f(b_n), f(a')f(b')$ starting from a'. Since a'

and a belong to the same sheet over $f(a)V^* \cap V^*f(a)$ and $f(a')f(b') \in V^*f(a)V^*V^*f(b)V^* \subseteq V_0f(a)f(b)V_0$, we conclude that $a' \cdot b'$ belongs to the sheet over $V_0f(a)f(b)V_0$ which contains $a \cdot b$. This implies $a' \cdot b' \in W$ as desired.

Next we deal with the second case. The initial setting is the same, but C may not be open and so we will need larger connected sets depending on points in neighborhoods of a and b. We follow the previous case until taking V_1. Then choose a symmetric neighborhood $V_2 \subseteq V_1$ so that $V_2^2 \subseteq V_1$. Let U_a and U_b be the sheets over $f(a)V_2 \cap V_2f(a)$ and $f(b)V_2 \cap V_2f(b)$ containing a and b respectively. According to Lemma 2.10, X is locally compactly connected. Then, applying the local compact connectedness to the open sets U_a and U_b, we find a symmetric neighborhood V^* with $V^* \subseteq V_2$ such that the sheets W_a over $f(a)V^* \cap V^*f(a)$ and W_b over $f(b)V^* \cap V^*f(b)$ containing a and b have the property that for each $a' \in W_a \subseteq U_a$ [$b' \in W_b \subseteq U_b$] there is a compact connected set $C_0 \subseteq U_a$ [$C_1 \subseteq U_b$] containing a and a' [b and b'].

We claim that $W_a \cdot W_b \subseteq W$. Let $a' \in W_a$ and $b' \in W_b$. Choose compact connected sets $C_0 \subseteq U_a$ and $C_1 \subseteq U_b$ such that $a, a' \in C_0$ and $b, b' \in C_1$. Since $\overline{f(C \cup C_0 \cup C_1)} = \overline{f(C)} \cup \overline{f(C_0)} \cup \overline{f(C_1)}$, $\overline{f(C \cup C_0 \cup C_1)}$ is compact. Next we show that, for each $y \in f(C \cup C_0 \cup C_1)f(C \cup C_0 \cup C_1)$, V^*yV^* is contained in some $B \in \mathcal{B}$. The case $y \in f(C)f(C)$ holds by assumption. We check only the cases $y \in f(C)f(C_0)$ and $y \in f(C_0)f(C_1)$, since all other cases are proved in a similar way. Let $y \in f(C)f(C_0)$ and choose $x \in C$ such that $y \in f(x)f(C_0)$. Since $f(C_0) \subseteq f(a)V_2$, we get $V^*yV^* \subseteq V^*f(x)f(a)V_2V^* \subseteq V_1f(x)f(a)V_1$ and the last term is contained in some $B \in \mathcal{B}$. Now consider the case $y \in f(C_0)f(C_1)$. Since $f(C_0)f(C_1) \subseteq V_2f(a)f(b)V_2$, it follows that $V^*yV^* \subseteq V^*V_2f(a)f(b)V_2V^* \subseteq V_1f(a)f(b)V_1$ and again the last term is contained in some $B \in \mathcal{B}$. We have shown that $C \cup C_0 \cup C_1$ and V^* satisfy $(**)$ for x_0, a, b, a', b'. Now using these data instead of C and V, arguing as in the first case, we conclude that $a' \cdot b' \in W$. ■

Lemma 3.8. *The operation $\cdot$ is associative.*

Proof. Let $a, b, c \in X$. We claim that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. First we choose a connected set C such that $x_0, a, b, c, a \cdot b \in C$ and $\overline{f(C)}$ is compact. By Lemma 3.7, the set $C_b = \{b \cdot x : x \in C\}$ is connected. By the definition of $\cdot$, it follows that $f(b \cdot x) = f(b)f(x)$. Hence, $f(C_b) = f(b)f(C)$, which implies that $\overline{f(C_b)}$ is compact. Then we have a neighborhood V of the identity of Y such that for each $y \in \overline{f(C \cup C_b)f(C \cup C_b)}$ there exists $B \in \mathcal{B}$ with $VyV \subseteq B$. Let V_1 be a symmetric open neighborhood of e such that $yV_1y^{-1} \subseteq V_1yV_1y^{-1} \subseteq V$ for every $y \in \overline{f(C)}$. Since $V_1 \subseteq V$, the sets C and V_1 fulfill $(**)$.

According to Lemma 3.6 the element $(a \cdot b) \cdot c$ is defined as follows. We choose an $\mathcal{A}$-sequence $b_0, \ldots, b_n$ in C with respect to V starting at $x_0 = b_0$

and ending at $b = b_n$, and an $\mathcal{A}$-sequence $c_0, \dots, c_k$ in C with respect to V_1 starting at $x_0 = c_0$ and ending at $c = c_k$ with the required properties. Then $a \cdot b$ is the end point of the unique $\mathcal{A}$-lift starting from a of the $\mathcal{B}$-sequence $f(a)f(b_0), \dots, f(a)f(b_n)$, and $(a \cdot b) \cdot c$ is the end point of the unique $\mathcal{A}$-lift starting from $a \cdot b$ of the the $\mathcal{B}$-sequence $f(a \cdot b)f(c_0), \dots, f(a \cdot b)f(c_k)$. Hence, $(a \cdot b) \cdot c$ is the end point of the $\mathcal{A}$-lift starting from a of the $\mathcal{B}$-sequence

$$(1) \qquad f(a)f(b_0), \dots, f(a)f(b_n) = f(a \cdot b) = f(a \cdot b)f(c_0), \dots, f(a \cdot b)f(c_k).$$

Now consider $a \cdot (b \cdot c)$. First note that $b \cdot c$ is the end point of the unique $\mathcal{A}$-lift starting from b of the $\mathcal{B}$-sequence $f(b)f(c_0), \dots, f(b)f(c_k)$. We claim that $b \cdot c_0, \dots, b \cdot c_k$ is an $\mathcal{A}$-sequence in C_b with respect to V. It is sufficient to show that $f(b)f(c_{i+1}) \in f(b)f(c_i)V \cap Vf(b)f(c_i)$ for $0 \le i \le k-1$. Since $c_0, \dots, c_k$ is an $\mathcal{A}$-sequence with respect to V_1, we have $f(c_{i+1}) \in f(c_i)V_1 \cap V_1 f(c_i)$ for each i. Hence, $(f(b)f(c_i))^{-1}(f(b)f(c_{i+1})) = f(c_i)^{-1}f(c_{i+1}) \in V_1 \subseteq V$ and

$$(f(b)f(c_{i+1}))(f(b)f(c_i))^{-1} = f(b)f(c_{i+1})f(c_i)^{-1}f(b)^{-1} \in f(b)V_1 f(b)^{-1} \subseteq V,$$

which proves $f(b)f(c_{i+1}) \in f(b)f(c_i)V \cap Vf(b)f(c_i)$.

We conclude that

$$x_0 = b_0, \dots, b_n = b \cdot c_0, b \cdot c_1, \dots, b \cdot c_k = b \cdot c$$

is an $\mathcal{A}$-sequence in $C \cup C_b$ with respect to V, and by the definition, $a \cdot (b \cdot c)$ is the end point of the unique $\mathcal{A}$-lift starting from a of the $\mathcal{B}$-sequence

$$(2) \qquad f(a)f(b_0), \dots, f(a)f(b_n) = f(a)f(b \cdot c_0), \dots, f(a)f(b \cdot c_k).$$

Note that the $\mathcal{B}$-sequences (1) and (2) coincide. Now both $a \cdot (b \cdot c)$ and $(a \cdot b) \cdot c$ are defined as the end point of the unique $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $f(a)f(b_0), \dots, f(a)f(b_n) = f(a)f(b)f(c_0), \dots, f(a)f(b)f(c_k)$ starting from a. Hence $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. ∎

LEMMA 3.9. *The element x_0 is a left identity for the operation $\cdot$ and each point $a \in X$ has a left inverse a^*.*

Proof. Let $a \in X$. We will prove that $x_0 \cdot a = a$, and that there exists $a^* \in X$ such that $a^* \cdot a = x_0$. Choose a connected set C containing x_0, a such that $\overline{f(C)}$ is compact. Then $\overline{(f(C))^{-1}} = (\overline{f(C)})^{-1}$ is compact and there is an open symmetric neighborhood V of e such that, for each $y \in \overline{f(C)(f(C))^{-1}f(C)} = \overline{f(C)}(\overline{f(C)})^{-1}\overline{f(C)}$, VyV is contained in some $B \in \mathcal{B}$. Then there exists an $\mathcal{A}$-sequence $a_0, \dots, a_m$ in C with respect to V starting at $a_0 = x_0$ and ending at $a_m = a$. Since $a_0, \dots, a_m$ is the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $f(x_0)f(a_0), \dots, f(x_0)f(a_m)$, it follows that $x_0 \cdot a = a_m = a$. This proves that x_0 is a left identity for $\cdot$.

Next we prove that a left inverse $a^* \in X$ of a exists. We will define points $u_{i,j}$ in X, for $0 \le i, j \le m$, such that $f(u_{ij}) = f(a_i)f(a)^{-1}f(a_j)$, $u_{m,j} = a_j$, $u_{i,m} = a_i$ and the points $u_{i,j}, u_{i,j+1}, u_{i+1,j}$ lie in some member $A \in \mathcal{A}$. Set

$z_{i,j} = f(a_i)f(a)^{-1}f(a_j)$, $0 \le i, j \le m$. Since $f(a_{i+1}) \in f(a_i)V \cap Vf(a_i)$, $0 \le i < m$, we get

$$z_{i,j+1} = f(a_i)f(a)^{-1}f(a_{j+1}) \in f(a_i)f(a)^{-1}f(a_j)V = z_{ij}V,$$
$$z_{i+1,j} = f(a_{i+1})f(a)^{-1}f(a_j) \in Vf(a_i)f(a)^{-1}f(a_j) = Vz_{ij},$$
$$z_{i+1,j+1} = f(a_{i+1})f(a)^{-1}f(a_{j+1}) \in Vf(a_i)f(a)^{-1}f(a_j)V = Vz_{ij}V.$$

By assumption, $Vz_{i,j}V$ is contained in some $B \in \mathcal{B}$, and we see that the points $z_{i,j}, z_{i,j+1}, z_{i+1,j}, z_{i+1,j+1}$ belong to some B. We will obtain the points u_{ij} by decreasing induction on i and j. First we set $u_{m,j} = a_j$ for each j. Since $a_0, \dots, a_m$ is an $\mathcal{A}$-sequence, $u_{m,j}, u_{m,j+1}$ belong to some member of $\mathcal{A}$. Assume that we have defined points $u_{k,l}$ with the required properties if $k > i$ or if $k = i$ and $l > j$. We define $u_{i,j}$ in the following way. If $j = m$, we set $u_{i,m} = a_i$ and note that $u_{i,m}$ and $u_{i+1,m} = a_{i+1}$ belong to some member of $\mathcal{A}$. If $j < m$, then, by the induction assumption, $u_{i,j+1}, u_{i,j+2}, u_{i+1,j+1}$ belong to some member $A' \in \mathcal{A}$, and $u_{i+1,j}, u_{i+2,j}, u_{i+1,j+1}$ belong to some member $A'' \in \mathcal{A}$. Since $z_{i,j+1}, z_{i+1,j}, z_{i+1,j+1} \in B$ and $u_{i+1,j+1} \in A' \cap A''$, it follows that $u_{i,j+1}, u_{i+1,j}, u_{i+1,j+1}$ belong to the same sheet A over B. Now, there exists a unique $u_{i,j} \in A$ such that $f(u_{i,j}) = z_{i,j}$ and the inductive step is done. The properties of the points $u_{i,j}$ imply that, for each $0 \le i \le m$, the sequence $u_{i,0}, \dots, u_{i,m}$ is the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $z_{i,0}, \dots, z_{i,m}$ starting at $u_{i,0}$, and for each $0 \le j \le m$, the sequence $u_{0,j}, \dots, u_{m,j}$ is the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $z_{0,j}, \dots, z_{m,j}$ starting at $u_{0,j}$. Finally, we set $a^* = u_{0,0}$. We claim that $a^* \cdot a = x_0$.

Choose a connected set C' which contains x_0, a, a^* such that $\overline{f(C')}$ is compact. Without loss of generality we may assume $C' \supseteq C$. Let V' be an open symmetric neighborhood of e such that, for each $y \in \overline{f(C')(f(C'))^{-1}f(C')}$, $V'yV'$ is contained in some $B \in \mathcal{B}$. Choose an $\mathcal{A}$-sequence $a'_0, \dots, a'_n$ in C' with respect to V' starting at $a'_0 = x_0$ and ending at $a'_n = a$. By the same procedure as above, we define points $v_{i,j}$ in X, for $0 \le i \le m$ and $0 \le j \le n$, such that $f(v_{ij}) = w_{i,j} = f(a_i)f(a)^{-1}f(a'_j)$, $v_{m,j} = a'_j$, $v_{i,n} = a_i$, and the points $v_{i,j}, v_{i,j+1}, v_{i+1,j}$ lie in some member $A \in \mathcal{A}$. As above, for each $0 \le i \le m$, the sequence $v_{i,0}, \dots, v_{i,n}$ is the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $w_{i,0}, \dots, w_{i,n}$ starting from $v_{i,0}$, and for each $0 \le j \le n$, the sequence $v_{0,j}, \dots, v_{m,j}$ is the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $w_{0,j}, \dots, w_{m,j}$ starting from $v_{0,j}$. In particular, $v_{0,0}, \dots, v_{0,n} = a_0 = x_0$ is the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $f(a)^{-1}f(a'_0), \dots, f(a)^{-1}f(a'_n)$ starting from $v_{0,0}$. Note that $f(u_{i,0}) = z_{i,0} = f(a_i)f(a)^{-1} = w_{i,0} = f(v_{i,0})$.

By decreasing induction on i, we will show that $u_{i,0} = v_{i,0}$ for $0 \le i \le m$. First note that $u_{m,0} = a_0 = x_0 = a'_0 = v_{m,0}$. Assume that $u_{i+1,0} = v_{i+1,0}$ and consider the points $u_{i,0}$ and $v_{i,0}$. Since $u_{i,0}$ and $u_{i+1,0}$ belong to some $A' \in \mathcal{A}$, $v_{i,0}$ and $v_{i+1,0}$ belong to some $A'' \in \mathcal{A}$ and $f(u_{i,0}), f(u_{i+1,0})$ belong to some $B \in \mathcal{B}$, we conclude that $u_{i,0}$ and $v_{i,0}$ belong to the same sheet $A \in \mathcal{A}$ over B. Hence, $u_{i,0} = v_{i,0}$. In particular, $u_{0,0} = v_{0,0}$.

It remains to prove that $v_{0,0} \cdot a = x_0$. By the definition of $\cdot$, $v_{0,0} \cdot a$ is the end point of the unique $\mathcal{A}$-lift starting from $v_{0,0}$ of the $\mathcal{B}$-sequence $f(a)^{-1}f(a'_0), \ldots, f(a)^{-1}f(a'_n)$. As we already remarked, $v_{0,0}, \ldots, v_{0,n} = x_0$ is the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $f(a)^{-1}f(a'_0), \ldots, f(a)^{-1}f(a'_n)$ starting from $v_{0,0}$, which implies $v_{0,0} \cdot a = x_0$. This proves that $a^* = u_{0,0} = v_{0,0}$ is a left inverse of a. ■

Proof of Theorem 1.1. Now $(X, \cdot)$ is a group by Lemmas 3.6, 3.8 and 3.9. Namely, since the operation $\cdot$ is associative, Lemma 3.9 implies that x_0 is the identity for $\cdot$, and a left inverse a^* of $a \in X$ is the inverse element of a.

To see the continuity of $a \mapsto a^{-1}$, let C be an open connected set containing x_0 and a such that $\overline{f(C)}$ is compact. Then, for each $y \in \overline{f(C)^{-1}f(C)}$, there is $B \in \mathcal{B}$ such that $VyV \subseteq B$. For a given neighborhood W of a^{-1}, we pick a symmetric open neighborhood $V_0 \subseteq V$ such that the sheet over $V_0 f(a)^{-1} V_0$ containing a^{-1} is contained in W. Since C is open, we can choose a symmetric open neighborhood $V_1 \subseteq V_0$ such that the sheet over $V_1 f(a) V_1$ containing a is contained in C. Let W_a be the sheet over $f(a)V_1 \cap V_1 f(a) \subseteq V_1 f(a) V_1$ containing a.

We claim that $W_a^{-1} \subseteq W$. Let $b \in W_a$. Then $f(b) \in f(a)V_1 \cap V_1 f(a))$. Choose an $\mathcal{A}$-sequence $a_0, \ldots, a_m$ in C with respect to V_1 starting at $a_0 = x_0$ and ending at $a_m = a$. By the definition of $a^* = a^{-1}$, the inverse a^{-1} is obtained as the end point of the $\mathcal{A}$-lift $a^*_0, \ldots, a^*_m$ starting from $a^*_0 = x_0$ of the $\mathcal{B}$-sequence $f(a)^{-1}f(a_m), \ldots, f(a)^{-1}f(a_0)$. Hence $a^{-1} = a^*_m$. Note that $a_0, \ldots, a_m, b$ is also an $\mathcal{A}$-sequence in C with respect to V_1, and analogously b^{-1} is defined as the end point of the $\mathcal{A}$-lift $b^*_0, \ldots, b^*_{m+1}$ starting from x_0 of a $\mathcal{B}$-sequence $f(b)^{-1}f(b), f(b)^{-1}f(a_m), \ldots, f(a)^{-1}f(a_0)$ starting from x_0. Since $f(b)^{-1} \in f(a)^{-1}V_1 \cap V_1 f(a)^{-1}$, there is $v \in V_1$ such that $f(b)^{-1} = vf(a)^{-1}$. Similarly, there is $v_i \in V_1$ such that $f(a_{i+1}) = f(a_i)v_{i+1}$ for $0 \leq i \leq m-1$. Hence,

$$f(b^*_{i+1}) = f(b)^{-1}f(a_{m-i}) = vf(a)^{-1}f(a_{m-i}),$$
$$f(a^*_{i-1}) = f(a)^{-1}f(a_{m-i+1}) = f(a)^{-1}f(a_{m-i})v_{m-i+1},$$

which implies that $f(b^*_{i+1}), f(a^*_i), f(a^*_{i-1}) \in V_1 f(a)^{-1} f(a_{m-i}) V_1 = V_1 f(a^*_i) V_1$ for $1 \leq i \leq m$. Since $a^*_0 = x_0 = b^*_0$, and a^*_1 and b^*_1 belong to the same sheet over V_1, it follows that $a^*_0 = x_0$, a^*_1 and b^*_2 belong to the same sheet over $V_1 f(a^*_1) V_1$. Then, by induction we can show that a^*_{m-1}, $a^*_m = a^{-1}$ and $b^*_{m+1} = b^{-1}$ belong to the same sheet over $V_1 f(a^*_m) V_1 = V_1 f(a)^{-1} V_1 \subseteq V_0 f(a)^{-1} V_0$, and consequently $b^{-1} \in W$. This completes the proof that $(X, \cdot)$ is a topological group with identity x_0 and $f : X \to Y$ is a homomorphism.

Finally, we show that the topological groups $(X, \cdot)$ and $(X, \cdot^{\#})$ are isomorphic. We will define an isomorphism $\varphi : (X, \cdot) \to (X, \cdot^{\#})$ such that $f\varphi = f$.

Define $\varphi : X \to X$ and $\psi : X \to X$ by $\varphi(a) = x_0^{\#} \cdot a$ and $\psi(a) = x_0 \cdot^{\#} a$, respectively. Then, by the continuity of $\cdot$ and $\cdot^{\#}$, φ and ψ are continuous maps. First we show that $\psi \circ \varphi = \mathrm{id}_X$. Take $a \in X$ and let C be a connected set containing $a, x_0, x_0^{\#}$ such that $\overline{f(C)}$ is compact. The sets $C_{x_0} = \{x_0 \cdot^{\#} x : x \in C\}$ and $C_{x_0^{\#}} = \{x_0^{\#} \cdot x : x \in C\}$ are connected with $\overline{f(C_{x_0})}$ and $\overline{f(C_{x_0^{\#}})}$ compact. Therefore, $C \cup C_{x_0} \cup C_{x_0^{\#}}$ is connected, $\overline{f(C \cup C_{x_0} \cup C_{x_0^{\#}})}$ is compact and there is a related symmetric neighborhood V. Let $a_0, \ldots, a_n = a$ be an $\mathcal{A}$-sequence in C with respect to V starting at $a_0 = x_0$. Then $\varphi(a) = x_0^{\#} \cdot a$ is the end point of the $\mathcal{A}$-lift $a_0^{\#}, \ldots, a_n^{\#}$ starting at $a_0^{\#} = x_0^{\#}$ of the $\mathcal{B}$-sequence $f(a_0), \ldots, f(a_n)$. Hence, $\varphi(a) = a_n^{\#}$. Note that $a_0^{\#}, \ldots, a_n^{\#}$ is an $\mathcal{A}$-sequence in $C_{x_0^{\#}}$ starting at $x_0^{\#}$ with respect to V, which implies that $\psi(\varphi(a)) = \psi(a_n^{\#}) = x_0 \cdot^{\#} a_n^{\#} = a$, by the uniqueness of $\mathcal{A}$-lifts starting from x_0. This proves $\psi \circ \varphi = \mathrm{id}_X$. Similarly, we have $\varphi \circ \psi = \mathrm{id}_X$. Hence φ is bijective.

To see that $\varphi : (X, \cdot) \to (X, \cdot^{\#})$ is a homomorphism, we show that $x \cdot b = x \cdot^{\#} \varphi(b)$ for any $x, b \in X$. Let $x, b \in X$ and consider a connected set C which contains $x, b, x_0, x_0^{\#}$ and $\overline{f(C)}$ is compact. Set $C_{x_0^{\#}} = \{x_0^{\#} \cdot u : u \in C\}$; as above, $C \cup C_{x_0^{\#}}$ is connected, $\overline{f(C \cup C_{x_0^{\#}})}$ is compact, and there is a related symmetric neighborhood V. Let $b_0, \ldots, b_m = b$ be an $\mathcal{A}$-sequence in C with respect to V starting at $b_0 = x_0$. Then $\varphi(b)$ is the end point of the $\mathcal{A}$-sequence $b_0^{\#}, \ldots, b_m^{\#}$ in $C_{x_0^{\#}}$ with respect to V starting at $b_0^{\#} = x_0^{\#}$ such that $f(b_i^{\#}) = f(b_i)$ for $0 \le i \le m$. Note that $x \cdot b$ is the end point of the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $f(x)f(b_0), \ldots, f(x)f(b_m)$ starting from x. On the other hand, $x \cdot^{\#} \varphi(b)$ is the end point of the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $f(x)f(b_0^{\#}), \ldots, f(x)f(b_m^{\#})$ starting from x. By the uniqueness of $\mathcal{A}$-lifts starting from x, we get $x \cdot b = x \cdot^{\#} \varphi(b)$.

Now, we prove that φ is a homomorphism: for all a, b,

$$\varphi(a \cdot b) = x_0^{\#} \cdot (a \cdot b) = (x_0^{\#} \cdot a) \cdot b = \varphi(a) \cdot b = \varphi(a) \cdot^{\#} \varphi(b).$$

We remark that we needed C to be open only in proving continuity of taking the inverse. ∎

Proof of Theorem 1.2. To see the continuity of taking the inverse, take $a \in X$ and let C be a connected set containing x_0 and a such that $\overline{f(C)}$ is compact. Then, for each $y \in \overline{f(C)^{-1} f(C)}$, there is $B \in \mathcal{B}$ such that $VyV \subseteq B$. For a given neighborhood W of a^{-1}, we choose a symmetric open neighborhood V_0 of e such that $V_0^2 \subseteq V$ and the sheet over $V_0 f(a)^{-1} V_0$

containing a^{-1} is contained in W. Let U_a be the sheet over $f(a)V_0 \cap V_0 f(a)$ containing a. Applying the local f-compact connectedness to U_a, we find a symmetric neighborhood $V_1 \subseteq V_0$ and the sheet W_a over $f(a)V_1 \cup V_1 f(a)$ containing a with the property that, for any $b \in W_a$, there exists a connected subset C_0 of U_a such that $a, b \in C_0$ and $\overline{f(C_0)}$ is compact.

We claim that $W_a^{-1} \subseteq W$. Let $b \in W_a$. Then there is a connected subset C_0 of U_a such that $a, b \in C_0$ and $\overline{f(C_0)}$ is compact. Now, $\overline{f(C \cup C_0)}$ is compact. Since $(V_0 f(a) \cap f(a)V_0)^{-1} = f(a)^{-1}V_0 \cap V_0 f(a)^{-1}$ and $V_0^2 \subseteq V$, we see that, for each $y \in f(C \cup C_0)^{-1} f(C \cup C_0)$, $V_0 y V_0$ is contained in some $B \in \mathcal{B}$. Let us consider only the case $y \in f(C_0)^{-1} f(C)$. Then there exists $x \in C$ such that $y \in f(C_0)^{-1} f(x)$. Since $f(C_0) \subseteq f(U_a) \subseteq f(a)V_0 \cap V_0 f(a)$, we get $f(C_0)^{-1} \subseteq V_0 f(a)^{-1}$, which implies that $V_0 y V_0 \subseteq V_0 V_0 f(a)^{-1} f(x) V_0 \subseteq V f(a)^{-1} f(x) V$ is contained in some $B \in \mathcal{B}$. The remaining part of the proof of continuity and also other parts of the proof are the same as in the preceding case. ■

Corollary 3.10. *Let Y be a connected compact group and X a connected space. If $f : X \to Y$ is an overlay map and $x_0 \in X$ is such that $f(x_0)$ is the identity of Y, then there exists a group operation on X making X a topological group with identity x_0 and f a homomorphism.*

Proof. Since $Y = f(X)$ is compact, X is f-compactly openly connected, and the conclusion follows from Theorem 1.1. ■

Corollary 3.11. *Let X be a connected space, $f : X \to Y$ a covering map to an abelian topological group Y and let $x_0 \in X$ be such that $f(x_0)$ is the identity of Y. Suppose that one of the following conditions holds:*

(1) *$f : X \to Y$ is an overlay map and X is f-compactly openly connected.*
(2) *X is locally f-compactly connected.*

Then there exists a group operation on X making X an abelian topological group with identity x_0 and f a homomorphism.

Proof. Let $\cdot$ be the group operation defined in Theorems 1.1 and 1.2. Let $a, b \in X$. We will show $a \cdot b = b \cdot a$, following the notation of Claim 1 in the proof of Lemma 3.6.

Let $v_i = f(a_{i+1}) f(a_i)^{-1}$ and $w_j = f(b_j)^{-1} f(b_{j+1})$ be from Claim 1. Then $v_i, w_j \in V$ and we see that the points $f(b_j)f(a_{i+1}) = f(a_{i+1})f(b_j) = v_i f(a_i) f(b_j)$, $f(b_{j+1}) f(a_{i+1}) = f(a_{i+1}) f(b_{j+1}) = v_i f(a_i) f(b_j) w_j$, $f(a_i)f(b_j) = f(b_j)f(a_i)$ and $f(b_{j+1})f(a_i) = f(a_i)f(b_{j+1}) = f(a_i)f(b_j)w_j$ all belong to $V f(a_i) f(b_j) V$. Then, much as in Claim 1, we can show that $a_i \cdot b_j = b_j \cdot a_i$ by induction, and consequently $a \cdot b = a_m \cdot b_n = b_n \cdot a_m = b \cdot a$. ■

Next we consider overlay maps over locally compact connected groups. First, we prove the following lemma.

LEMMA 3.12. *Let X be a connected space and $f : X \to Y$ a covering map to a locally compact group Y. Then X is f-compactly openly connected.*

Proof. According to Lemma 2.6 there exists a compact connected subgroup $K \subseteq Y$ such that Y is homeomorphic to the direct product of K and the euclidean space $\mathbb{R}^n$ for some $n \geq 0$. So, we identify Y with $K \times \mathbb{R}^n$, and $(y, \mathbf{0})$ with y for $y \in K$.

We claim that $f^{-1}(K)$ is connected. Choose $x_0 \in X$ such that $f(x_0)$ is the identity of Y and let X_0 be the connected component of $f^{-1}(K)$ containing x_0. We enumerate the connected components of $f^{-1}(K)$ as X_i. To simplify the reasoning, for each $\boldsymbol{r} \in \mathbb{R}^n$ we define a path $p_{\boldsymbol{r}} : [0,1] \to \mathbb{R}^n$ by $p_{\boldsymbol{r}}(t) = t\boldsymbol{r}$.

Then, for each point $(y, \boldsymbol{r}) \in K \times \mathbb{R}^n$, the path $q_{y,\boldsymbol{r}} : [0,1] \to Y$ given by $q_{y,\boldsymbol{r}}(t) = (y, p_{\boldsymbol{r}}(t))$ connects $y = (y, \mathbf{0}) \in K$ to $(y, \boldsymbol{r})$. For $y \in K$, $\boldsymbol{r} \in \mathbb{R}^n$ and $x \in f^{-1}(\{y\})$, let $\overline{q}_{y,\boldsymbol{r},x} : [0,1] \to X$ be the unique lift of the path $q_{y,\boldsymbol{r}}$ such that $\overline{q}_{y,\boldsymbol{r},x}(0) = x \in X_i$ and $f(\overline{q}_{y,\boldsymbol{r},x}(0)) = (y, \mathbf{0})$. Let Z_i be the set of all $z \in X$ defined by $z = \overline{q}_{y,\boldsymbol{r},x}(1)$ for some $x \in X_i \cap f^{-1}(\{y\})$. Then it is easy to see that $X = \bigcup_i Z_i$ and each Z_i is open. Moreover, tracing q reversely and considering the uniqueness of lift of a path from a given point, we see that each $x \in X$ belongs to a unique Z_i. Hence, the sets Z_i are open and disjoint. Now, the connectedness of X implies that $X = Z_0$ and $f^{-1}(K) = X_0$ is connected.

For $a \in X$, let $f(a) = (y, \boldsymbol{r})$. We choose a bounded open set $U \subseteq \mathbb{R}^n$ containing $\boldsymbol{r}$. Let $C_a = X_0 \cup \bigcup\{\mathrm{Im}(\overline{q}_{y,\boldsymbol{s},x}) : y \in K,\ \boldsymbol{s} \in U,\ f(x) = y\}$. Then C_a is open, connected and contains a. It is easy to see that $\overline{f(C_a)}$ is compact. Hence, if $a, b \in X$, then $C = C_a \cup C_b$ is an open connected set such that $a, b \in C$ and $\overline{f(C)}$ is compact. This proves that X is f-compactly openly connected. ∎

COROLLARY 3.13. *Let $f : X \to Y$ be an overlay map from a connected space X to a locally compact group Y and let $x_0 \in X$ be such that $f(x_0)$ is the identity of Y. Then there exists a group operation on X making X a topological group with identity x_0 and f a homomorphism. In addition, if Y is abelian, then so is X.*

Proof. According to Lemma 3.12, we can apply Theorem 1.1 and we get the first statement. The second one follows from Corollary 3.11. ∎

REMARK 3.14. Theorem 1.2 implies that any covering map $f : X \to Y$ from a connected space X onto a locally compactly connected group Y is an overlay map. Namely, in that case X is locally f-compactly connected and by Theorem 1.2, X admits a group structure making X a topological group and f a homomorphism. Then, f is an overlay map by Lemma 2.3.

4. Proof of Theorem 1.3. First we need some lemmas.

LEMMA 4.1. *Let $f : X \to Y$ be an overlay map from a connected space X onto a compact group Y. Denote by $\cdot$ and $\cdot'$ the group operations on X from Theorem* 1.1 *defined by overlay pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ for f, respectively, with identity x_0. If $(\mathcal{A}, \mathcal{B})$ refines $(\mathcal{A}', \mathcal{B}')$, then the operations $\cdot$ and $\cdot'$ coincide.*

Proof. Let $a, b \in X$. We claim $a \cdot b = a \cdot' b$. Let V and V' be related sets for $\cdot$ and $\cdot'$. Pick an open symmetric neighborhood V_0 of $e \in Y$ such that $V_0 \subseteq V \cap V'$. Choose an $\mathcal{A}$-sequence $b_0, \dots, b_n$ in X with respect to V_0 from $b_0 = x_0$ to $b_n = b$ and let $u_0, \dots, u_n$ be the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $f(a)f(b_0), \dots, f(a)f(b_n)$ from $u_0 = a$. Then $a \cdot b = u_n$. Since $(\mathcal{A}, \mathcal{B})$ refines $(\mathcal{A}', \mathcal{B}')$, $u_0, \dots, u_n$ is also the $\mathcal{A}'$-lift of the $\mathcal{B}'$-sequence $f(a)f(b_0), \dots, f(a)f(b_n)$ from $u_0 = a$. Hence, $a \cdot' b = u_n$, which shows that $a \cdot b = a \cdot' b$. ∎

LEMMA 4.2. *Let $f : X \to Y$ be a covering homomorphism from a connected group X to a compact group Y and let e_X be the identity for X. Then there exists an overlay pair $(\mathcal{A}, \mathcal{B})$ for f such that the group operation $\cdot$ on X from Theorem* 1.1 *defined by $(\mathcal{A}, \mathcal{B})$ and $x_0 = e_X$ coincides with the operation given on X.*

Proof. Denote the group operation on X by $\circ$. Since f is a covering homomorphism, there exist open symmetric neighborhoods W and U of x_0 and e respectively such that $f|W$ is a homeomorphism onto U and the open coverings $\mathcal{B} = \{yU : y \in Y\}$ and $\mathcal{A} = \{x \circ W : x \in X\}$ form an overlay pair $(\mathcal{A}, \mathcal{B})$ for f. Let $\cdot$ be the group operation on X defined by $(\mathcal{A}, \mathcal{B})$ and $x_0 = e_X$.

We claim that $a \circ b = a \cdot b$ for each $a, b \in X$. Let $V \subseteq U$ be a related open neighborhood of e for $\cdot$ and choose an $\mathcal{A}$-sequence $b_0, \dots, b_n$ in X with respect to V such that $b_0 = x_0$ and $b_n = b$. Let $u_0, \dots, u_n$ be the $\mathcal{A}$-lift of the $\mathcal{B}$-sequence $f(a)f(b_0), \dots, f(a)f(b_n)$ starting from $u_0 = a$. Then $a \cdot b = u_n$. Since $b_0, \dots, b_n$ is an $\mathcal{A}$-sequence, for each $0 \le i < n$ there exists $x_i \in X$ such that $b_i, b_{i+1} \in x_i \circ W$. Then $a \circ b_i, a \circ b_{i+1} \in (a \circ x_i) \circ W$, which shows that $a \circ b_0 = a, a \circ b_1, \dots, a \circ b_n = a \circ b$ is an $\mathcal{A}$-sequence and also the $\mathcal{A}$-lift of $f(a)f(b_0), \dots, f(a)f(b_n)$ starting from a. Consequently, $a \circ b = u_n = a \cdot b$. ∎

Now, we can prove the uniqueness of the group structure on X for a covering map $f : X \to Y$ from a connected compact space X onto a group Y.

LEMMA 4.3. *Let $f : X \to Y$ be a covering map from a connected compact space X to a group Y and let $x_0 \in X$ be such that $f(x_0)$ is the identity of Y. Then there exists a unique group operation on X making X a topological group with identity x_0 and f a homomorphism.*

Proof. Since X is compact, f is finite-sheeted. Then f is an overlay map and any two overlay pairs for f admit an overlay pair which is their common refinement [10, Theorem 1]. According to Corollary 3.10 there exists a group operation on X making X a topological group with identity x_0 and f a homomorphism. It remains to prove that the operation is unique.

Assume that $\circ$ and $*$ are two group operations on X with the required properties. According to Lemma 4.2, there exist overlay pairs $(\mathcal{A}', \mathcal{B}')$ and $(\mathcal{A}'', \mathcal{B}'')$ for f such that the group operation $\cdot'$ on X from Theorem 1.1 defined by $(\mathcal{A}', \mathcal{B}')$ and $x_0 = e_X$ coincides with $\circ$, and the group operation $\cdot''$ on X from Theorem 1.1 defined by $(\mathcal{A}'', \mathcal{B}'')$ and $x_0 = e_X$ coincides with $*$. Let $(\mathcal{A}, \mathcal{B})$ be an overlay pair for f which refines both $(\mathcal{A}', \mathcal{B}')$, and $(\mathcal{A}'', \mathcal{B}'')$, and let $\cdot$ be the group operation on X from Theorem 1.1 defined by $(\mathcal{A}, \mathcal{B})$ and x_0. According to Lemma 4.1, $\cdot$ coincides with $\cdot'$, and $\cdot$ coincides with $\cdot''$. Thus the group operations $\cdot'$ and $\cdot''$ coincide, and consequently $\circ$ and $*$ coincide. ∎

LEMMA 4.4. *Let $f : X \to Y$ be a covering homomorphism. If C is a compact subgroup of X, then the restriction $f|C : C \to f(C)$ is also a covering homomorphism.*

Proof. It is sufficient to find an open neighborhood V of e in $f(C)$ such that $f|C$ evenly covers V. Let U be a neighborhood of the identity of Y such that f evenly covers U. Let U_x be a neighborhood of $x \in \mathrm{Ker}(f)$ such that $f|U_x$ is a homeomorphism onto U. Then $f^{-1}(U) = \bigsqcup_{x\in \mathrm{Ker}(f)}(U_x)$. Since C is compact, the set $\{x \in \mathrm{Ker}(f) : U_x \cap C \neq \emptyset\}$ is finite. By shrinking U we may assume that this set equals $\mathrm{Ker}(f) \cap C$. Let $V = f(C) \cap U$ and $W_x = U_x \cap C$. Then, for each $x \in \mathrm{Ker}(f) \cap C$, $f|W_x$ is a homeomorphism onto V, the sets W_x are pairwise disjoint and $(f|C)^{-1}(V) = \bigsqcup_{x\in \mathrm{Ker}(f)\cap C}(W_x)$. Hence, $f|C$ evenly covers V. ∎

The converse of the previous lemma holds if C is compact and connected.

LEMMA 4.5. *Let $f : X \to Y$ be a covering homomorphism and let C be a connected compact subset of X such that C contains the identity of X and $f(C)$ is a subgroup of Y. Assume that $f|C : C \to f(C)$ is a covering map. Then C is a subgroup of X.*

Proof. Denote the group operation on X by $\circ$. Then there exist open symmetric neighborhoods W and U of the identities e_X and e_Y respectively such that $f|W$ is a homeomorphism onto U and the open coverings $\mathcal{B} = \{yU : y \in Y\}$ and $\mathcal{A} = \{x \circ W : x \in X\}$ form an overlay pair $(\mathcal{A}, \mathcal{B})$ for f. Since C is compact, $f|C$ is finite-sheeted and there exist open symmetric neighborhoods U' and $W'$of e_X in $f(C)$ and e_Y in C, respectively, such that $U' \subseteq U \cap f(C)$, $W' \subseteq W \cap C$ and $\mathcal{B}' = \{yU' : y \in f(C)\}$ and $\mathcal{A}' = \{x \circ W' : x \in C\}$ form an overlay pair $(\mathcal{A}', \mathcal{B}')$ for $f|C$. Let $\cdot$ be the group operation on C from Theorem 1.1 defined by $(\mathcal{A}', \mathcal{B}')$ and e_X.

To prove that $(C, \circ)$ is a group, it is sufficient to prove $a \cdot b = a \circ b$ for any $a, b \in C$. Let $V \subseteq U'$ be an open symmetric neighborhood of e_Y in $f(C)$ related to $\cdot$. Take $a, b \in C$ and let $b_0, \ldots, b_n$ be an $\mathcal{A}'$-sequence (in C) with respect to V with $b_0 = e_X$ and $b_n = b$. First note that $b_{i+1} \in b_i \circ W'$ for $0 \leq i < n-1$. Namely, $f(b_{i+1}) \in f(b_i)V \cap Vf(b_i) \subseteq f(b_i)U'$ and there exists $x_i \in C$ such that $b_i, b_{i+1} \in x_i \circ W'$. Since $b_i \circ W'$ is the only sheet over $f(b_i)U'$ which contains b_i, and $(\mathcal{A}', \mathcal{B}')$ is an overlay pair for $f|C$, it follows that $b_{i+1} \in b_i \circ W'$. Now, by induction on i it is easy to see that $a \circ b_i \in C$ for $0 \leq i \leq n$, since $a \circ b_{i+1} \in a \circ b_i \circ W'$. We conclude that $a = a \circ b_0, \ldots, a \circ b_n = a \circ b$ is an $\mathcal{A}'$-sequence. By the definition of $\cdot$, $a \cdot b$ is the end point of the unique $\mathcal{A}'$-lift starting from a of the $\mathcal{B}'$-sequence $f(a)f(b_0), \ldots, f(a)f(b_n)$. Consequently, $a \cdot b = a \circ b$. ∎

Since in each topological group the group operation and taking inverse are continuous maps, we get the following lemma.

Lemma 4.6. *Let P be a path-connected component of the identity of a topological group X. Then P is a subgroup of X.*

Lemma 4.7. *Let $f : X \to Y$ be a covering map onto a topological group Y and let P be the path-connected component of $x_0 \in X$. Suppose that there are group operations $\circ$ and $\cdot$ on X both making X a topological group with identity x_0 and f a homomorphism. Then $u \circ v = u \cdot v$ and $v \circ u = v \cdot u$ for all $u \in P$ and $v \in X$.*

Proof. Let $u \in P$ and $v \in X$. Let $p : [0,1] \to P$ be a path such that $p(0) = x_0$ and $p(1) = u$. Define $q : [0,1] \to Y$ by $q(t) = f(p(t))f(v)$. Then q is a path in Y from $q(0) = f(v)$ to $q(1) = f(u)f(v)$. Let $\overline{q} : [0,1] \to X$ be the unique lift of q with $\overline{q}(0) = v$. Let $p' : [0,1] \to X$ and $p'' : [0,1] \to X$ be defined by $p'(t) = p(t) \circ v$ and $p''(t) = p(t) \cdot v$. Since $f(p'(t)) = f(p''(t)) = f(p(t))f(v) = q(t)$ and $p'(0) = p''(0) = v = \overline{q}(0)$, by the uniqueness of lift, both p' and p'' are equal to $\overline{q}$. Hence $q(1) = p'(1) = p''(1)$, and so $u \circ v = u \cdot v$. By similar arguments we get $v \circ u = v \cdot u$. ∎

Lemma 4.8. *Let $f : X \to Y$ be a covering map onto a topological group Y and P be the path-connected component of $x_0 \in X$. If there exists a group structure on X making X a topological group with identity x_0 and f a homomorphism, then the group structure on P is unique.*

Proof. By Lemma 4.6, P is a subgroup, and the uniqueness follows from Lemma 4.7. ∎

Lemma 4.9. *Let $f : X \to Y$ be a covering map onto a topological group Y and P be the path-connected component of $x_0 \in X$. Suppose that group operations $\circ$ and $\cdot$ on X with identity x_0 making f a covering homomorphism coincide on a subgroup K of X and $X = P \circ K$. Then $\circ$ and $\cdot$ coincide.*

Proof. Let $u, v \in X$. By assumption, there are $x_1, x_2 \in P$ and $z_1, z_2 \in K$ such that $u = x_1 \circ z_1$ and $v = x_2 \circ z_2$. By Lemma 4.7 we have $u = x_1 \cdot z_1$ and $v = x_2 \cdot z_2$. Then there are $x \in P$ and $z \in K$ such that $z_1 \circ x_2 = x \circ z$. Again by Lemma 4.7 we get $z_1 \cdot x_2 = x \cdot z$. Now

$$\begin{aligned} u \circ v &= (x_1 \circ z_1) \circ (x_2 \circ z_2) = x_1 \circ (z_1 \circ x_2) \circ z_2 \\ &= x_1 \circ (x \circ z) \circ z_2 = (x_1 \circ x) \circ (z \circ z_2) = (x_1 \cdot x) \circ (z \cdot z_2), \end{aligned}$$

because $\circ$ and $\cdot$ coincide on P by Lemma 4.8 and on K by the assumption. Since $x_1 \cdot x \in P$, by Lemma 4.7 we get $(x_1 \cdot x) \circ (z \cdot z_2) = (x_1 \cdot x) \cdot (z \cdot z_2)$, which implies

$$\begin{aligned} u \circ v &= (x_1 \cdot x) \cdot (z \cdot z_2) = x_1 \cdot (x \cdot z) \cdot z_2 \\ &= x_1 \cdot (z_1 \cdot x_2) \cdot z_2 = (x_1 \cdot z_1) \cdot (x_2 \cdot z_2) = u \cdot v. \ \blacksquare \end{aligned}$$

LEMMA 4.10. *Let $f : X \to Y$ be a covering map from a connected, locally compactly connected space X onto a group Y and let $x_0 \in X$ be such $f(x_0)$ is the identity of Y. Then for any $a, b \in X$ there exist open sets $U_0, \dots, U_n$ and compact connected sets $C_0, \dots, C_n$ in X such that $x_0 \in C_0$, $b \in C_n$, and, for each i, $C_i \subseteq U_i$, $C_i \cap C_{i+1} \neq \emptyset$, the restriction $f|U_i$ is a homeomorphism and f evenly covers $f(U_i)$ and $f(a)f(U_i)$.*

Proof. Let $a \in X$. Let S be the set of all $b \in X$ such that there exist open sets $U_0, \dots, U_n$ and compact connected sets $C_0, \dots, C_n$ with the required properties. Then S is non-empty since $x_0 \in S$, and it is easy to see that S is open by the local compact connectedness of X.

To see that S is closed, let $u \in \overline{S}$. Take an open neighborhood U of u such that f evenly covers $f(U)$ and $f(a)f(U)$ and $f|U$ is a homeomorphism. By the local compact connectedness, there is a neighborhood V of u such that for each $x \in V$ there exists a connected compact subset of U which contains x and u. Since $u \in \overline{S}$, there is $x \in V \cap S$. Then there is a connected compact subset C of U which contains x and u. Since $x \in S$, there are open sets $U_0, \dots, U_n$ and compact connected sets $C_0, \dots, C_n$ with the required properties. Setting $U_{n+1} = U$ and $C_{n+1} = C$, we see that $u \in S$. Hence, S is clopen in X, and by the connectedness of X we get $S = X$. This proves that the conclusion holds for every $b \in X$. $\blacksquare$

Proof of Theorem 1.3. Since the existence of the required group operation on X has been proved in Corollary 3.13 and Theorem 1.2, it remains to show the uniqueness in both cases.

(1) Let $\circ$ and $\cdot$ be group operations on X both making X a topological group with identity x_0 and f a homomorphism. Denote by P the path-connected component of x_0. Since $(X, \circ)$ is a locally compact group, we can again apply Lemma 2.6 to $(X, \circ)$, i.e. $X = S \circ K$, where K is a connected compact subgroup and S is a subspace of Y containing x_0 which is homeo-

morphic to some $\mathbb{R}^n$. Then $S \subseteq P$ and $X = P \circ K$. Since K is a compact subgroup, $f|K : K \to f(K)$ is a covering map by Lemma 4.4. Applying Lemma 4.5 to $(X, \cdot)$ and $f|K : K \to f(K)$, we find that K is a subgroup of $(X, \cdot)$. Hence, $f|K$ is a covering homomorphism for both operations on K.

Now, by Lemma 4.3, the operations $\circ$ and $\cdot$ coincide on K. Finally, Lemma 4.9 implies that $\circ$ coincides with $\cdot$.

(2) Suppose that $\circ$ and $\cdot$ are group operations on X both making X a topological group with identity x_0 and f a homomorphism. Let $a, b \in X$.

We claim that $a \circ b = a \cdot b$. By Lemma 4.10 there exist open sets $U_0, \ldots, U_n$ and compact connected sets $C_0, \ldots, C_n$ in X such that $x_0 \in C_0$, $b \in C_n$, for each i we have $C_i \subseteq U_i$ and $C_i \cap C_{i+1} \neq \emptyset$, $f|U_i$ is a homeomorphism and f evenly covers $f(U_i)$ and $f(a)f(U_i)$. Set $b_0 = x_0$, $b_{n+1} = b$ and for each $0 \le i \le n-1$ choose $b_{i+1} \in C_i \cap C_{i+1}$. We show $a \circ b = a \circ b_{n+1} = a \cdot b_{n+1} = a \cdot b$ by induction on i. Obviously, $a \circ b_0 = a = a \cdot b_0$. Suppose that $a \circ b_i = a \cdot b_i$. Since $a \circ C_i$ and $a \cdot C_i$ are connected, each of the sets $a \circ C_i$ and $a \cdot C_i$ is contained in a unique sheet over $f(a)f(U_i)$. Now, the assumption $a \circ b_i = a \circ b_i$ implies that $a \circ b_{i+1}$ and $a \cdot b_{i+1}$ belong to the same sheet over $f(a)f(U_i)$. Since $f(a \circ b_{i+1}) = f(a \cdot b_{i+1})$, we conclude that $a \circ b_{i+1} = a \cdot b_{i+1}$, which proves the claim. ∎

Remark 4.11. Since path-connectedness implies compact connectedness, and local path-connectedness implies local compact connectedness, Theorem 1.2 implies a well-known result about inducing a group structure on a path-connected covering space over a path-connected locally path-connected group [12, Theorem 79].

Remark 4.12. 1. In Lemma 4.4, compactness is essential. We give an example to show this. Let $f : \mathbb{R}^2 \to \mathbb{T}^2$ be the universal covering map over the torus. Let S be a subgroup of $\mathbb{R}^2$ consisting of all elements on a line through the origin with an irrational slope. Then $f|S$ is injective, but $f(S)$ is not locally connected and hence $f|S$ is not a covering map.

2. To show the uniqueness of the group structures on a total space X of an overlay map $f : X \to Y$ the connectedness of X is essential. This seems to be obvious, but still we give an example. Let $f : X_0 \times Y \to Y$ be the projection onto a topological group Y. For instance, let X_0 be a discrete space of cardinality p^2 for some prime p. Then we have two distinct group structures on X_0, and f is a homomorphism for both group structures.

Our investigation brings about two questions. The first one is a redefinition of the starting question.

Question 1. *Let $f : X \to Y$ be an overlay map from a connected space X to a topological group Y. Is there a group operation on X such that X is a topological group and f is a homomorphism?*

Question 2. *Does there exist an overlay map $f : X \to Y$ from a connected space X to a topological group Y which admits two different group operations on X such that X is a topological group and f is a homomorphism?*

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