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Polynomial estimates on real and complex $L_p(\mu)$ spaces

by

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Abstract. In his commentary to Problem 73 of Mazur and Orlicz in the Scottish Book, L. A. Harris raised the following natural generalization: Let X be a Banach space, let $k_1, \dots, k_n$ be nonnegative integers whose sum is m and let $c(k_1, \dots, k_n; X)$ be the smallest number with the property that if L is any symmetric m-linear mapping of one real normed linear space into another, then $|L(x_1^{k_1} \dots x_n^{k_n})| \le c(k_1, \dots, k_n; X)\|\widehat{L}\|$, where $\widehat{L}$ is the m-homogeneous polynomial associated to L. In this paper, we give estimates in the case of a real $L_p(\mu)$ space using three different techniques and we get optimal results in some special cases.

1. Introduction and notation. If X is a Banach space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we let $\mathcal{L}^{\mathrm{s}}({}^m X, \mathbb{K})$ denote the Banach space of all *continuous symmetric m-linear forms* $L : X^m \to \mathbb{K}$ with the norm

$$\|L\| = \sup\{|L(x_1, \dots, x_m)| : \|x_1\| \le 1, \dots, \|x_m\| \le 1\}.$$

For simplicity, we write $\mathcal{L}^{\mathrm{s}}({}^m X)$ in place of $\mathcal{L}^{\mathrm{s}}({}^m X, \mathbb{K})$. A function $P : X \to \mathbb{K}$ is a *continuous m-homogeneous polynomial* if there is a continuous symmetric m-linear form $L : X^m \to \mathbb{K}$ for which $P(x) = L(x, \dots, x)$ for all $x \in X$. In this case it is convenient to write $P = \widehat{L}$. We let $\mathcal{P}({}^m X)$ denote the Banach space of all continuous m-homogeneous polynomials $P : X \to \mathbb{K}$ with the norm

$$\|P\| = \sup\{|P(x)| : \|x\| \le 1\}.$$

We write $L(x_1^{k_1} \dots x_n^{k_n})$ as shorthand for $L(x_1, \dots, x_1, \dots, x_n, \dots, x_n)$ where each x_i appears k_i times for $1 \le i \le n$, and $k_1 + \dots + k_n = m$.

It is known [2, Proposition 1.8] that if $L \in \mathcal{L}^{\mathrm{s}}({}^m X)$ and $\widehat{L}$ is the associated polynomial, then

$$\|L\| \le \frac{m^m}{m!}\|\widehat{L}\|$$

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and the constant $m^m/m!$ is best possible (see [2, Example 1.39]). This is the answer to Problem 73 of Mazur and Orlicz [13].

L. A. Harris in his commentary to Problem 73 raised the following natural generalization:

Let X be a Banach space, let $k_1, \ldots, k_n$ be nonnegative integers whose sum is m and let $c(k_1, \ldots, k_n; X)$ be the smallest number with the property that if L is any symmetric m-linear mapping of one real normed linear space into another, then

$$|L(x_1^{k_1} \ldots x_n^{k_n})| \leq c(k_1, \ldots, k_n; X)\|\widehat{L}\|.$$

Following the definition of the polarization constant introduced by S. Dineen [2], we define

$$c(X) = \limsup_{m\to\infty} c(k_1, \ldots, k_n; X)^{1/m},$$

which describes how the constant behaves asymptotically. This notation will be used only in the case where $x_1, \ldots, x_n$ are norm-one vectors with disjoint supports.

It is shown in [5, Theorem 1] that if only *complex* normed spaces are considered, then

$$c(k_1, \ldots, k_n; X) = \frac{k_1! \cdots k_n!}{k_1^{k_1} \cdots k_n^{k_n}} \frac{m^m}{m!}. \tag{1.1}$$

In the case of *real* normed linear spaces L. A. Harris has proved in [6, Corollary 7] (see also [10], [12]) that

$$|L(x_1^{k_1} \ldots x_n^{k_n})| \leq \sqrt{\frac{m^m}{k_1^{k_1} \cdots k_n^{k_n}}}\,\|\widehat{L}\| \tag{1.2}$$

for all nonnegative integers $k_1, \ldots, k_n$ with $k_1 + \cdots + k_n = m$.

It was also established by L. A. Harris (see [5, Theorem 6]) that in the case of *complex* L_p-space, $1 \leq p \leq \infty$, one has

$$\|L\| \leq \left(\frac{m^m}{m!}\right)^{|p-2|/p} \|\widehat{L}\|$$

when m is a power of 2. An improved estimate was given in [11, Theorem 2] by Y. Sarantopoulos, but it holds for a small range of p's. In the case of *real* or *complex* $L_p(\mu)$, for $1 \leq p \leq m'$, $1/m + 1/m' = 1$, he showed that

$$\|L\| \leq \frac{m^{m/p}}{m!}\|\widehat{L}\|. \tag{1.3}$$

L. A. Harris has also proved the following:

LEMMA 1.1 ([5, Theorem 1]). *Let $1 \le p \le \infty$ and $L \in \mathcal{L}^{s}({}^m\ell_p, \mathbb{C})$. If $x_1, \dots, x_n$ are norm-one vectors in ℓ_p with disjoint supports, then*

$$|L(x_1^{k_1} \dots x_n^{k_n})| \le \frac{k_1! \cdots k_n! m^{m/p}}{k_1^{k_1/p} \cdots k_n^{k_n/p} m!} \|\widehat{L}\|$$

for all nonnegative integers $k_1, \dots, k_n$ with $k_1 + \dots + k_n = m$.

Notice that (1.1) follows from the above estimate for $p = 1$, since every Banach space is isometric to a quotient of ℓ_1. The following example shows that the constant in Lemma 1.1 is best possible.

EXAMPLE 1.2. For any real or complex ℓ_p space and $x = (x_i) \in \ell_p$, let $\widehat{L} \in \mathcal{P}({}^m\ell_p)$ with $\widehat{L}(x) = x_1 \cdots x_m$. Take e_j, $j = 1, \dots, m$, to be the jth coordinate vector of ℓ_p and define

$$\begin{aligned} y_1 &= k_1^{-1/p}(e_1 + \dots + e_{k_1}), \\ y_i &= k_i^{-1/p}(e_{k_1+\dots+k_{i-1}+1} + \dots + e_{k_1+\dots+k_i}), \quad i = 2, \dots, n. \end{aligned}$$

Notice that $y_1, \dots, y_n$ are unit vectors in ℓ_p with disjoint supports. Moreover, we can easily check (see [11, Example 1]) that

$$|L(y_1^{k_1} \dots y_n^{k_n})| = \frac{k_1! \cdots k_n! \, m^{m/p}}{k_1^{k_1/p} \cdots k_n^{k_n/p} m!} \|\widehat{L}\|.$$

In the next two sections we provide some estimates of the constant $c(k_1, \dots, k_n; L_p(\mu))$, first for *complex* $L_p(\mu)$-spaces and then for *real* ones. In the *complex* case, we choose (x_i) to be a seminormalized unconditional basic sequence, which gives the case of x_i's, $i = 1, \dots, n$, having disjoint supports as a particular case. In general, the real case is more difficult. Consequently, we shall tackle this problem by using three different techniques, where $x_1, \dots, x_n$ are norm-one vectors with disjoint supports. The first technique is standard, using a well known polarization formula, while the second depends on a generalization of Clarkson's inequality (see [14]). The third one uses Hoeffding's inequality, which was particularly useful in [9] in order to get a lower bound on the radius of analyticity of a power series. The values for the constant $c(k_1, \dots, k_n; L_p(\mu))$ obtained by the third technique are the worst with few exceptions, but asymptotically this technique gives better results.

Each technique will also be used to obtain corresponding estimates for a seminormalized unconditional basic sequence (x_i) with appropriate additional constants. In the last section we thoroughly explain why and when each technique is useful. It seems reasonable to approach the problem using the type and cotype of the space, but the estimates for these constants are far from optimal.

2. The complex case. Recall that the nth *Rademacher function* r_n is defined on $[0,1]$ by $r_n(t) = \operatorname{sign}\sin 2^n\pi t$. Furthermore, for every natural number $n \geq 2$, the *generalized Rademacher functions* (s_j) are defined inductively as follows (see [1]): Let $a_1, \ldots, a_n$ be the complex nth roots of unity. For $j = 1, \ldots, n$ let $I_j = ((j-1)/n, j/n)$ and let I_{j_1,j_2} denote the j_2th open subinterval of length $1/n^2$ of I_{j_1} $(j_1, j_2 = 1, \ldots, n)$. Proceeding like this, it is clear how to define the interval $I_{j_1,\ldots,j_k}$ for any k. Now $s_1 : [0,1] \to \mathbb{C}$ is defined by setting $s_1(t) = a_j$ for $t \in I_j$, where $1 \leq j \leq n$. There is no harm in setting $s_k(t) = 1$ for all endpoints t. We shall need the following *polarization formula* (see [1]):

$$L(x_1, \ldots, x_m) = \frac{1}{m!}\int_0^1 s_1^{m-1}(t)\cdots s_m^{m-1}(t)\widehat{L}\Big[\sum_{i=1}^m s_i(t)x_i\Big]\,dt,$$

which can be generalized using the multinomial theorem and [1, Lemma 1] to get

Lemma 2.1. *Let the scalar field be $\mathbb{C}$, X a vector space and $L \in \mathcal{L}^{\mathrm{s}}({}^mX)$. If $x_1, \ldots, x_n \in X$, then*

$$L(x_1^{k_1}\ldots x_n^{k_n}) = \frac{k_1!\cdots k_n!}{m!}\int_0^1 s_1^{m-k_1}(t)\cdots s_n^{m-k_n}(t)\widehat{L}\Big(\sum_{i=1}^n s_i(t)x_i\Big)\,dt$$

for all nonnegative integers $k_1, \ldots, k_n$ with $k_1 + \cdots + k_n = m$.

Khinchin's inequality states that for every $0 < p < \infty$, there are $0 < A_p \leq B_p < \infty$ such that for all $n \in \mathbb{N}$ and all scalars $a_1, \ldots, a_n$,

$$A_p\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2} \leq \Big\|\sum_{i=1}^n a_i r_i\Big\|_{L_p} \leq B_p\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2},$$

where B_p depends not only on p but also on n, and from [8, Theorem & Corollary 3] is given by

$$B_p(n) = \begin{cases} \dfrac{(\int_0^1 |\sum_{i=1}^n r_i|^p\,dt)^{1/p}}{n^{1/2}} & \text{if } p \geq 3, \\[2ex] \dfrac{(\int_0^1 |\sum_{i=1}^n r_i|^p\,dt)^{1/p}}{n^{1/p}} & \text{if } 2 \leq p < 3. \end{cases}$$

Notice that this estimate for B_p also holds for the generalized Rademacher functions.

Definition 2.2. Let $0 < K < \infty$. A sequence (x_i) in a Banach space X is a *K-unconditional basic sequence* if for all n, all scalars $a_1, \ldots, a_n$ and all choices of $\epsilon_i = \pm 1$,

$$\Big\|\sum_{i=1}^n \epsilon_i a_i x_i\Big\| \leq K\Big\|\sum_{i=1}^n a_i x_i\Big\|.$$

DEFINITION 2.3. A sequence (x_i) in a Banach space X is called *seminormalized* if $\inf_i \|x_i\| > 0$ and $\sup_i \|x_i\| < \infty$.

If $\phi \in L_p(\mu)$, $2 < p < \infty$ and $\phi > 0$ almost everywhere, we set $d\mu_\phi = \phi^p d\mu$ and define $U_\phi f = f/\phi$ for $f \in L_p(\mu)$. By the Radon–Nikodým theorem, U_ϕ is an isometry of $L_p(\mu)$ onto $L_p(\mu_\phi)$. From [3, Proposition 3.4] we have

PROPOSITION 2.4. *Let (f_i) be a seminormalized K-unconditional basic sequence in $L_p(\mu)$, $2 < p < \infty$. If for every $\phi > 0$ with $\int_0^1 \phi(t)^p\,dt = 1$,*

$$C_\phi = \Big(\sum_{i=1}^{\infty} \|U_\phi f_i\|_{L_2(\mu_\phi)}^{2p(p-2)}\Big)^{(p-2)/2p} < \infty,$$

then (f_i) is equivalent to the usual basis of ℓ_p and for all n and all $a_1, \ldots, a_n$ we have

$$\Big\|\sum_{i=1}^{n} a_i f_i\Big\| \le K B_p C \Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p}, \tag{2.1}$$

where B_p is the Khinchin constant and $C = \sup C_\phi < \infty$.

PROPOSITION 2.5. *Let $2 < p < \infty$ and let $L \in \mathcal{L}^{\mathrm{s}}({}^m L_p(\mu), \mathbb{C})$. Suppose (x_i) is a seminormalized K-unconditional basic sequence in $L_p(\mu)$ and for every $\phi > 0$ with $\int_0^1 \phi(t)^p\,dt = 1$,*

$$C_\phi = \Big(\sum_{i=1}^{\infty} \|U_\phi x_i\|_{L_2(\mu_\phi)}^{2p(p-2)}\Big)^{(p-2)/2p} < \infty.$$

Then (x_i) is equivalent to the usual basis of ℓ_p and for all nonnegative integers $k_1, \ldots, k_n$ with $k_1 + \cdots + k_n = m$ we have

$$|L(x_1^{k_1} \ldots x_n^{k_n})| \le (K B_p C)^m \frac{k_1! \cdots k_n! m^{m/p}}{k_1^{k_1/p} \cdots k_n^{k_n/p} m!} \|\widehat{L}\|,$$

where B_p is the Khinchin constant and $C = \sup C_\phi < \infty$.

Proof. Using Lemma 2.1 and Proposition 2.4, we have

$$\begin{aligned}
&\Big|\frac{k_1^{k_1/p} \cdots k_n^{k_n/p}}{m^{m/p}} L(x_1^{k_1} \ldots x_n^{k_n})\Big| \\
&\quad = \Big|L\Big(\Big(\frac{k_1}{m}\Big)^{k_1/p} x_1^{k_1} \ldots \Big(\frac{k_n}{m}\Big)^{k_n/p} x_n^{k_n}\Big)\Big| \\
&\quad = \Big|\frac{k_1! \cdots k_n!}{m!} \int_0^1 s_1^{m-k_1}(t) \cdots s_n^{m-k_n}(t) \widehat{L}\Big(\sum_{i=1}^{n} \Big(\frac{k_i}{m}\Big)^{1/p} s_i(t) x_i\Big)\,dt\Big|
\end{aligned}$$

$$\le \frac{k_1!\cdots k_n!}{m!}\|\widehat{L}\|\int_0^1\Big\|\sum_{i=1}^n\Big(\frac{k_i}{m}\Big)^{1/p}s_i(t)x_i\Big\|_p^m\,dt$$

$$\le (KB_pC)^m\frac{k_1!\cdots k_n!}{m!}\|\widehat{L}\|\int_0^1\Big(\sum_{i=1}^n\frac{k_i}{m}\Big)^{m/p}\,dt$$

$$= (KB_pC)^m\frac{k_1!\cdots k_n!}{m!}\|\widehat{L}\|. \ \blacksquare$$

Remark 2.6. If in Proposition 2.5 we choose $x_1,\dots,x_n$ to have disjoint supports, then we get Harris' result of Lemma 1.1.

3. The real case

3.1. Using weights. If $r_i(t)=\operatorname{sign}\sin 2^i\pi t$ is the ith Rademacher function on [0,1], we shall need the following well known *polarization formula* ([2, Corollary 1.6], see also [11, Lemma 2]):

$$L(x_1,\dots,x_m)=\frac{1}{m!}\int_0^1 r_1(t)\cdots r_m(t)\widehat{L}\Big[\sum_{i=1}^m r_i(t)x_i\Big]\,dt,$$

which is generalized by the next result.

Lemma 3.1. *If X is a vector space over $\mathbb{K}$, $L\in\mathcal{L}^{\mathrm{s}}(^mX)$ and $x_1,\dots,x_n\in X$, then*

$$L(x_1^{k_1}\dots x_n^{k_n}) = \frac{1}{m!}\int_0^1 r_1(t)\cdots r_m(t)\widehat{L}\Big(\sum_{i=1}^{k_1}r_i(t)x_1+\cdots+\sum_{i=m-k_n+1}^{m}r_i(t)x_n\Big)\,dt$$

for all nonnegative integers $k_1,\dots,k_n$ with $k_1+\cdots+k_n=m$.

Theorem 3.2. *Let $1\le p\le\infty$ and let $L\in\mathcal{L}^{\mathrm{s}}(^mL_p(\mu),\mathbb{R})$. If $x_1,\dots,x_n$ are norm-one vectors in $L_p(\mu)$ with disjoint supports, then*

$$|L(x_1^{k_1}\dots x_n^{k_n})|\le\frac{k_1^{k_1}\cdots k_n^{k_n}}{m!}n^{m/p}\|\widehat{L}\| \tag{3.1}$$

for all nonnegative integers $k_1,\dots,k_n$ with $k_1+\cdots+k_n=m$.

Proof. From Lemma 3.1 we have

$$|L(x_1^{k_1}\dots x_n^{k_n})| \le \frac{k_1^{k_1}\cdots k_n^{k_n}}{m!}\|\widehat{L}\|\int_0^1\Big\|\sum_{i=1}^{k_1}r_i(t)\frac{x_1}{k_1}+\cdots+\sum_{i=m-k_n+1}^{m}r_i(t)\frac{x_n}{k_n}\Big\|_p^m\,dt.$$

But

$$\int_0^1 \Big\| \sum_{i=1}^{k_1} r_i(t)\frac{x_1}{k_1} + \cdots + \sum_{i=m-k_n+1}^{m} r_i(t)\frac{x_n}{k_n}\Big\|_p^m \, dt$$
$$\le \int_0^1 \bigg[\bigg(\frac{1}{k_1}\Big|\sum_{i=1}^{k_1} r_i(t)\Big|\bigg)^p + \cdots + \bigg(\frac{1}{k_n}\Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|\bigg)^p\bigg]^{m/p} dt \le n^{m/p},$$

and (3.1) follows. ∎

REMARK 3.3. For $k_1 = \cdots = k_n = 1$, the upper bound of $c(k_1,\ldots,k_n;\ell_p)$ from Lemma 1.1 and the lower bound from Theorem 3.2 give the same estimate $m^{m/p}/m!$.

Since by Stirling's formula $m! \sim \sqrt{2\pi}\, m^{m+1/2}e^{-m}$, the constant of Theorem 3.2 gives asymptotically $c(L_p(\mu)) = n^{1/p}e$.

Using (2.1) in the proof of Theorem 3.2, we get the following estimate for a seminormalized unconditional basic sequence (x_i) with appropriate additional constants.

PROPOSITION 3.4. *Let $2 < p < \infty$ and let $L \in \mathcal{L}^{\mathrm{s}}({}^m L_p(\mu),\mathbb{R})$. Suppose (x_i) is a seminormalized K-unconditional basic sequence in $L_p(\mu)$ and for every $\phi > 0$ with $\int_0^1 \phi(t)^p\,dt = 1$,*

$$C_\phi = \Big(\sum_{i=1}^{\infty} \|U_\phi x_i\|_{L_2(\mu_\phi)}^{2p(p-2)}\Big)^{(p-2)/2p} < \infty.$$

Then (x_i) is equivalent to the usual basis of ℓ_p and for all nonnegative integers $k_1,\ldots,k_n$ with $k_1+\cdots+k_n = m$ we have

$$|L(x_1^{k_1}\ldots x_n^{k_n})| \le (KB_pC)^m \frac{k_1^{k_1}\cdots k_n^{k_n}}{m!} n^{m/p}\|\widehat{L}\|,$$

where B_p is the Khinchin constant and $C = \sup C_\phi < \infty$.

Now, we compare some values of $c(k_1,\ldots,k_n;L_p(\mu))$ obtained from Theorem 3.2 with the corresponding ones already known from (1.2).

EXAMPLES 3.5. (a) For $m = 8$, $c(1,\ldots,1,2;L_8(\mu)) \simeq 0.0007$, while (1.2) gives 2048.

(b) $c(1,1,2;L_8(\mu)) \simeq 0.29$, while (1.2) gives 8.

(c) $c(1,1,6;L_4(\mu)) \simeq 10.4$, while (1.2) gives 19.

(d) For $m = 8$, $c(1,\ldots,1,2;L_2(\mu)) \simeq 0.24$, while (1.2) gives 2048.

3.2. Using Clarkson's inequalities. If $a_1, \ldots, a_n \in \mathbb{C}$, from [14, Theorem 5] we have Clarkson type inequalities

$$\Big(\int_0^1 \Big|\sum_{i=1}^n r_i(t)a_i\Big|^\lambda dt\Big)^{1/\lambda} \le \Big(\sum_{i=1}^n |a_i|^\lambda\Big)^{1/\lambda} \quad \text{if } 0 < \lambda \le 2,$$

$$\Big(\int_0^1 \Big|\sum_{i=1}^n r_i(t)a_i\Big|^\lambda dt\Big)^{1/\lambda} \le \Big(\sum_{i=1}^n |a_i|^{\lambda'}\Big)^{1/\lambda'} \quad \text{if } 2 \le \lambda < \infty,$$

where $\lambda' = \lambda/(\lambda-1)$ is the conjugate exponent of λ.

THEOREM 3.6. *Let $1 \le p \le \infty$ and let $L \in \mathcal{L}^{\mathrm{s}}({}^m L_p(\mu), \mathbb{R})$. If $x_1, \ldots, x_n$ are norm-one vectors in $L_p(\mu)$ with disjoint supports, then*

$$|L(x_1^{k_1} \ldots x_n^{k_n})| \le c(k_1, \ldots, k_n; L_p(\mu))\|\widehat{L}\|$$

for all nonnegative integers $k_1, \ldots, k_n$ with $k_1 + \cdots + k_n = m$, where

$$c(k_1, \ldots, k_n; L_p(\mu)) = \begin{cases} \dfrac{(k_1^{p-1} + \cdots + k_n^{p-1})^{m/p}}{m!} & \text{if } p \ge m, \\[2ex] \dfrac{n^{(m-p)/p}(k_1^{m-1} + \cdots + k_n^{m-1})}{m!} & \text{if } p \le m. \end{cases}$$

Proof. From Lemma 3.1 we have

$$\begin{aligned} |L(x_1^{k_1} \ldots x_n^{k_n})| &\le \frac{\|\widehat{L}\|}{m!} \int_0^1 \Big\|\sum_{i=1}^{k_1} r_i(t)x_1 + \cdots + \sum_{i=m-k_n+1}^{m} r_i(t)x_n\Big\|_p^m dt \\ &\le \frac{\|\widehat{L}\|}{m!} \int_0^1 \Big(\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^p + \cdots + \Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^p\Big)^{m/p} dt. \end{aligned}$$

Let $p \ge m$. Then $m/p \le 1$ and by Hölder's inequality

$$\begin{aligned} |L(x_1^{k_1} \ldots x_n^{k_n})| &\le \frac{\|\widehat{L}\|}{m!}\Big(\int_0^1 \Big|\sum_{i=1}^{k_1} r_i(t)\Big|^p dt + \cdots + \int_0^1 \Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^p dt\Big)^{m/p} \\ &\le \frac{(k_1^{p/p'} + \cdots + k_n^{p/p'})^{m/p}}{m!}\|\widehat{L}\| \quad (\text{since } p \ge 2) \\ &= \frac{(k_1^{p-1} + \cdots + k_n^{p-1})^{m/p}}{m!}\|\widehat{L}\|. \end{aligned}$$

Let $p \le m$. Then $m/p \ge 1$ and by Hölder's inequality

$$\begin{aligned} &|L(x_1^{k_1} \ldots x_n^{k_n})| \\ &\quad \le \frac{n^{m/p}\|\widehat{L}\|}{m!} \int_0^1 \Big(\frac{1}{n}\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^p + \cdots + \frac{1}{n}\Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^p\Big)^{m/p} dt \end{aligned}$$

$$\leq \frac{n^{m/p}\|\widehat{L}\|}{m!}\int_0^1 \frac{1}{n}\Big(\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^m + \cdots + \Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^m\Big)\,dt$$

$$= \frac{n^{(m-p)/p}\|\widehat{L}\|}{m!}\Big(\int_0^1\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^m\,dt + \cdots + \int_0^1\Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^m\,dt\Big)$$

$$\leq \frac{n^{(m-p)/p}(k_1^{m/m'} + \cdots + k_n^{m/m'})}{m!}\|\widehat{L}\| \quad (\text{since } m \geq 2)$$

$$= \frac{n^{(m-p)/p}(k_1^{m-1} + \cdots + k_n^{m-1})}{m!}\|\widehat{L}\|. \ ∎$$

REMARK 3.7. For $k_1 = \cdots = k_n = 1$, the upper bound of $c(k_1,\ldots,k_n;\ell_p)$ from Lemma 1.1 and the lower bound from Theorem 3.6, in the case $p \geq m$, give the same estimate $m^{m/p}/m!$.

Notice that the constant of Theorem 3.6 gives asymptotically $c(L_\infty(\mu)) = e$ and $c(L_p(\mu)) = n^{1/p}e$ for $1 \leq p < \infty$.

Using (2.1) in the proof of Theorem 3.6, we get the following estimate for a seminormalized unconditional basic sequence (x_i) with appropriate additional constants.

PROPOSITION 3.8. *Let $2 < p < \infty$ and let $L \in \mathcal{L}^{\mathrm{s}}({}^m L_p(\mu),\mathbb{R})$. Suppose (x_i) is a seminormalized K-unconditional basic sequence in $L_p(\mu)$ and for every $\phi > 0$ with $\int_0^1 \phi(t)^p\,dt = 1$,*

$$C_\phi = \Big(\sum_{i=1}^{\infty} \|U_\phi x_i\|_{L_2(\mu_\phi)}^{2p(p-2)}\Big)^{(p-2)/2p} < \infty.$$

Then (x_i) is equivalent to the usual basis of ℓ_p and for all nonnegative integers $k_1,\ldots,k_n$ with $k_1 + \cdots + k_n = m$ we have

$$|L(x_1^{k_1}\ldots x_n^{k_n})| \leq c(k_1,\ldots,k_n;L_p(\mu))\|\widehat{L}\|.$$

Here

$$c(k_1,\ldots,k_n;L_p(\mu)) = \begin{cases} (KB_pC)^m \dfrac{(k_1^{p-1} + \cdots + k_n^{p-1})^{m/p}}{m!} & \text{if } p \geq m,\\[2ex] (KB_pC)^m \dfrac{n^{(m-p)/p}(k_1^{m-1} + \cdots + k_n^{m-1})}{m!} & \text{if } p \leq m,\end{cases}$$

where B_p is the Khinchin constant and $C = \sup C_\phi < \infty$.

Now, we compare some values of $c(k_1,\ldots,k_n;L_p(\mu))$ obtained from Theorem 3.6 with the corresponding ones already known from (1.2).

EXAMPLES 3.9. (a) $c(4,4;L_8(\mu)) \simeq 0.81$, while (1.2) gives 16.

(b) $c(1,3;L_4(\mu)) = 1.1\bar{6}$, while (1.2) gives 3.08.

(c) $c(1,1,2;L_4(\mu)) = 0.41\bar{6}$, while (1.2) gives 8.

(d) $c(4,4;L_2(\mu)) \simeq 6.5$, while (1.2) gives 16.

(e) $c(1,3;L_2(\mu)) = 2.\bar{3}$, while (1.2) gives 3.08.
(f) $c(1,1,2;L_2(\mu)) = 1.25$, while (1.2) gives 8.

3.3. Using Hoeffding's inequality. Let $(\mathbb{R},\mathcal{M},\lambda)$ be the Lebesgue measure space and let $E \in \mathcal{M}$. If f is a measurable function on E, we define its distribution function $\lambda_f : (0,\infty) \to [0,\infty]$ by

$$\lambda_f(a) = \lambda(\{x \in E : |f(x)| > a\}),$$

where λ denotes Lebesgue measure. From [4], we have:

PROPOSITION 3.10. *If $\lambda_f(a) < \infty$ for every $a > 0$ and ϕ is a nonnegative Borel function on $(0,\infty)$, then*

$$\int_E \phi \circ |f|\, d\lambda = -\int_0^\infty \phi(a)\, d\lambda_f(a).$$

That is, the integrals of functions of $|f|$ on E can be reduced to Lebesgue–Stieltjes integrals.

The case we are interested in is $\phi(a) = a^p$, which gives

$$\int_E |f|^p\, d\lambda = -\int_0^\infty a^p\, d\lambda_f(a).$$

Integrating the right hand side by parts, we obtain

$$\int_E |f|^p\, d\lambda = p\int_0^\infty a^{p-1}\lambda_f(a)\, da. \tag{3.2}$$

The validity of this calculation becomes clear if we note that $a^p\lambda_f(a) \to 0$ as $a \to 0$ and $a \to \infty$ (since λ_f is strictly decreasing). In the following proposition the function f will be of the form $f(t) = r_1(t) + \cdots + r_k(t)$, $k \in \mathbb{N}$. Therefore, we need to find an upper bound for

$$\lambda_f(x) := \lambda_k(x) = P(|r_1(t) + \cdots + r_k(t)| \ge x).$$

This will be accomplished by using Hoeffding's inequality (see [7, Theorem 2]): If $X_1,\ldots,X_k$ are independent random variables with $a_i \le X_i \le b_i$ for every $i = 1,\ldots,k$ and with mean μ, then for all $x > 0$,

$$P(\overline{X_k} - \mu \ge x) \le e^{-2k^2x^2/\sum_{i=1}^k (b_i-a_i)^2}.$$

Since in our case $a_i = -1$, $b_i = 1$ and $\mu = 0$ for every i, it follows that

$$\begin{aligned} P(r_1(t) + \cdots + r_k(t) \ge x) &= P\left(\frac{r_1(t) + \cdots + r_k(t)}{k} \ge \frac{x}{k}\right) \\ &\le e^{-2k^2(x/k)^2/4k} = e^{-x^2/2k}. \end{aligned}$$

Hence,

$$\lambda_k(x) = P(|r_1(t) + \cdots + r_k(t)| \ge x) \le 2e^{-x^2/2k}. \tag{3.3}$$

Recall that the double factorial of a positive integer n is defined by

$$n!! = \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1 & \text{if } n \text{ is odd,}\\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2 & \text{if } n \text{ is even.}\end{cases}$$

LEMMA 3.11. *If $n\in\mathbb{N}$, then*

$$\Gamma\Big(\frac{n}{2}\Big) = \begin{cases} \dfrac{2}{n}\,2^{-n/2}n!! & \text{if } n \text{ is even,}\\ \dfrac{\sqrt{2\pi}}{n}\,2^{-n/2}n!! & \text{if } n \text{ is odd.}\end{cases}$$

Proof. If n is even, then

$$n!! = 2^{n/2}\Gamma\Big(\frac{n}{2}+1\Big) \;\Leftrightarrow\; n!! = 2^{n/2}\,\frac{n}{2}\Gamma\Big(\frac{n}{2}\Big) \;\Leftrightarrow\; \Gamma\Big(\frac{n}{2}\Big) = \frac{2}{n}2^{-n/2}n!!.$$

If n is odd, then

$$\begin{aligned} n!! = \pi^{-1/2}2^{n/2+1/2}\Gamma\Big(\frac{n}{2}+1\Big) &\;\Leftrightarrow\; n!! = \sqrt{\frac{2}{\pi}}\,2^{n/2}\,\frac{n}{2}\Gamma\Big(\frac{n}{2}\Big)\\ &\;\Leftrightarrow\; \Gamma\Big(\frac{n}{2}\Big) = \frac{\sqrt{2\pi}}{n}\,2^{-n/2}n!!. \;\blacksquare\end{aligned}$$

If $p>0$, $[p]$ denotes the integer part of p.

THEOREM 3.12. *Let $1\le p\le\infty$ and let $L\in\mathcal{L}^{\mathrm{s}}({}^mL_p(\mu),\mathbb{R})$. If $x_1,\dots,x_n$ are norm-one vectors in $L_p(\mu)$ with disjoint supports, then*

$$|L(x_1^{k_1}\dots x_n^{k_n})| \le c(k_1,\dots,k_n;L_p(\mu))\|\widehat{L}\|$$

for all nonnegative integers $k_1,\dots,k_n$ with $k_1+\dots+k_n=m$, where

$$c(k_1,\dots,k_n;L_p(\mu)) = \begin{cases} \dfrac{(\sqrt{2\pi}\,[p]!!\sum_{i=1}^n k_i^{[p]/2})^{m/[p]}}{m!} & \textit{if } p\ge m,\\ \dfrac{n^{(m-p)/p}\sqrt{2\pi}m!!\sum_{i=1}^n k_i^{m/2}}{m!} & \textit{if } p\le m.\end{cases} \tag{3.4}$$

Proof. From Lemma 3.1 we have

$$\begin{aligned} |L(x_1^{k_1}\dots x_n^{k_n})| &\le \frac{\|\widehat{L}\|}{m!}\int_0^1 \Big\|\sum_{i=1}^{k_1} r_i(t)x_1+\dots+\sum_{i=m-k_n+1}^{m} r_i(t)x_n\Big\|_p^m\,dt\\ &\le \frac{\|\widehat{L}\|}{m!}\int_0^1 \Big(\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^p+\dots+\Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^p\Big)^{m/p}\,dt.\end{aligned}$$

Let $p\ge m$. Since $p\ge[p]\ge m$ and $m/[p]\le 1$, using Hölder's inequality we have

$$|L(x_1^{k_1}\dots x_n^{k_n})| \le \frac{\|\widehat{L}\|}{m!}\int_0^1\Big(\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^{[p]}+\dots+\Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^{[p]}\Big)^{m/[p]}\,dt$$
$$\le \frac{\|\widehat{L}\|}{m!}\Big(\int_0^1\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^{[p]}\,dt+\dots+\int_0^1\Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^{[p]}dt\Big)^{m/[p]}.$$

Therefore, from (3.2) and (3.3) we get

$$\begin{aligned}
&|L(x_1^{k_1}\dots x_n^{k_n})|\\
&\quad\le \frac{\|\widehat{L}\|}{m!}\Big([p]\int_0^\infty x^{[p]-1}\lambda_{k_1}(x)\,dx+\dots+[p]\int_0^\infty x^{[p]-1}\lambda_{k_n}(x)\,dx\Big)^{m/[p]}\\
&\quad\le \frac{\|\widehat{L}\|}{m!}\Big([p]\int_0^\infty x^{[p]-1}2e^{-x^2/2k_1}\,dx+\dots+[p]\int_0^\infty x^{[p]-1}2e^{-x^2/2k_n}\,dx\Big)^{m/[p]}\\
&\quad= \frac{\|\widehat{L}\|}{m!}\bigg([p](2k_1)^{[p]/2}\Gamma\Big(\frac{[p]}{2}\Big)+\dots+[p](2k_n)^{[p]/2}\Gamma\Big(\frac{[p]}{2}\Big)\bigg)^{m/[p]}\\
&\quad= \frac{\big([p]2^{[p]/2}\Gamma([p]/2)\sum_{i=1}^n k_i^{[p]/2}\big)^{m/[p]}}{m!}\|\widehat{L}\|.
\end{aligned}$$

Now, an application of Lemma 3.11 yields

$$|L(x_1^{k_1}\dots x_n^{k_n})| \le \begin{cases} \dfrac{(2[p]!!\sum_{i=1}^n k_i^{[p]/2})^{m/[p]}}{m!}\|\widehat{L}\| & \text{if } [p] \text{ is even},\\[2ex] \dfrac{(\sqrt{2\pi}\,[p]!!\sum_{i=1}^n k_i^{[p]/2})^{m/[p]}}{m!}\|\widehat{L}\| & \text{if } [p] \text{ is odd}.\end{cases}\tag{3.5}$$

Let $p \le m$. Using Hölder's inequality and (3.2), (3.3), we get

$$\begin{aligned}
&|L(x_1^{k_1}\dots x_n^{k_n})|\\
&\quad\le \frac{n^{m/p}\|\widehat{L}\|}{m!}\int_0^1\Big(\frac{1}{n}\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^{p}+\dots+\frac{1}{n}\Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^{p}\Big)^{m/p}\,dt\\
&\quad\le \frac{n^{m/p}\|\widehat{L}\|}{m!}\int_0^1\frac{1}{n}\Big(\Big|\sum_{i=1}^{k_1} r_i(t)\Big|^{m}+\dots+\Big|\sum_{i=m-k_n+1}^{m} r_i(t)\Big|^{m}\Big)\,dt\\
&\quad\le \frac{n^{(m-p)/p}\|\widehat{L}\|}{m!}\Big(m\int_0^\infty x^{m-1}\lambda_{k_1}(x)\,dx+\dots+m\int_0^\infty x^{m-1}\lambda_{k_n}(x)\,dx\Big)\\
&\quad\le \frac{n^{(m-p)/p}\|\widehat{L}\|}{m!}\Big(m\int_0^\infty x^{m-1}2e^{-x^2/2k_1}\,dx+\dots+m\int_0^\infty x^{m-1}2e^{-x^2/2k_n}\,dx\Big)
\end{aligned}$$

$$= \frac{n^{(m-p)/p}\|\widehat{L}\|}{m!}\left(m(2k_1)^{m/2}\Gamma\left(\frac{m}{2}\right) + \cdots + m(2k_n)^{m/2}\Gamma\left(\frac{m}{2}\right)\right)$$
$$= \frac{n^{(m-p)/p} m 2^{m/2}\Gamma(m/2)\sum_{i=1}^n k_i^{m/2}}{m!}\|\widehat{L}\|.$$

An application of Lemma 3.11 yields

$$(3.6)\quad |L(x_1^{k_1}\dots x_n^{k_n})| \le \begin{cases} \dfrac{n^{(m-p)/p}\, 2m!!\sum_{i=1}^n k_i^{m/2}}{m!}\|\widehat{L}\| & \text{if } m \text{ is even,}\\[2ex] \dfrac{n^{(m-p)/p}\sqrt{2\pi}\, m!!\sum_{i=1}^n k_i^{m/2}}{m!}\|\widehat{L}\| & \text{if } m \text{ is odd.}\end{cases}$$

Finally, the estimate (3.4) follows from (3.5) and (3.6). ∎

Observe that in the *real* case the constant of Theorem 3.12 leads to $c(L_\infty(\mu)) = \sqrt{e}$, which is the best known asymptotic estimate and independent of n. Additionally, $c(L_p(\mu)) = n^{1/p}\sqrt{e}$ for $1 \le p < \infty$. By Lemma 1.1, the best asymptotic estimate for *complex* $L_\infty(\mu)$ spaces is 1.

Using (2.1) in the proof of Theorem 3.12, we get the following estimate for a seminormalized unconditional basic sequence (x_i) with appropriate additional constants.

Proposition 3.13. *Let $2 < p < \infty$ and let $L \in \mathcal{L}^{\mathrm{s}}({}^m L_p(\mu), \mathbb{R})$. Suppose (x_i) is a seminormalized K-unconditional basic sequence in $L_p(\mu)$ and for every $\phi > 0$ with $\int_0^1 \phi(t)^p\,dt = 1$,*

$$C_\phi = \Big(\sum_{i=1}^\infty \|U_\phi x_i\|_{L_2(\mu_\phi)}^{2p(p-2)}\Big)^{(p-2)/2p} < \infty.$$

Then (x_i) is equivalent to the usual basis of ℓ_p and for all nonnegative integers $k_1,\dots,k_n$ with $k_1+\dots+k_n = m$ we have

$$|L(x_1^{k_1}\dots x_n^{k_n})| \le c(k_1,\dots,k_n; L_p(\mu))\|\widehat{L}\|,$$

where

$$c(k_1,\dots,k_n;L_p(\mu)) = \begin{cases} (KB_pC)^m \dfrac{(\sqrt{2\pi}\,[p]!!\sum_{i=1}^n k_i^{[p]/2})^{m/[p]}}{m!} & \text{if } p \ge m,\\[2ex] (KB_pC)^m \dfrac{n^{(m-p)/p}\sqrt{2\pi}\, m!!\sum_{i=1}^n k_i^{m/2}}{m!} & \text{if } p \le m.\end{cases}$$

Once again B_p is the Khinchin constant and $C = \sup C_\phi < \infty$.

Now, we compare some values of $c(k_1,\dots,k_n;L_p(\mu))$ obtained from Theorem 3.12 with the corresponding ones already known from (1.2).

Examples 3.14. (a) $c(2,58,140;L_{180}(\mu)) \simeq 17.42\times 10^{27}$, while (1.2) gives 27.14×10^{27}.

(b) $c(2,58,140;L_{200}(\mu)) \simeq 15.42\times 10^{27}$, while (1.2) gives 27.14×10^{27}.

4. Conclusion. In this section we shall explain when each of the above techniques is useful, but the "rules" we are about to give are not strict, since for any fixed values of m and p there is a large variation for the values of n, k_i, $i = 1, \dots, n$.

First of all, note that when $1 \leq p \leq m'$ and $1/m + 1/m' = 1$, inequality (1.3) gives better estimates than any of these techniques.

Case $p = m$. If n is very small and the k_i's are (approximately) equal, we get the best estimates from Theorem 3.6 (Example 3.9a).

If n is very small and the k_i's have a large variation, Theorem 3.6 is best suited for very small values of m (Example 3.9b), but inequality (1.2) remains the best for larger m's.

If n is relatively large (not necessarily close to m), the best estimates are obtained from Theorem 3.6 for very small m's (Example 3.9c) and from Theorem 3.2 for larger m's (Example 3.5a).

Case $p > m$. For fixed m, as p increases, only the estimates of Theorem 3.2 decrease, therefore for large p 's we get the best estimates from Theorem 3.2 (Example 3.5b). If p is not very large (i.e. close to m), the "rules" of the case $p = m$ hold.

Case $p < m$. If n is very small and the k_i's are (approximately) equal, we prefer Theorem 3.6 (Example 3.9d). For large m's and very small p's, we get the best estimates from inequality (1.2).

If n is very small and the k_i's have a large variation, the best estimates are derived from (1.2) in most cases and sometimes Theorems 3.2 (Example 3.5c) and 3.6 (Example 3.9e) will do the job.

If n is relatively large (not necessarily close to m), Theorem 3.2 is the one we need (Example 3.5d), with few exceptions for very small values of m where the results from Theorem 3.6 are better (Example 3.9f).

Finally, observe that Theorem 3.12 gives the best known asymptotic estimate, which is independent of n and is obtained in the case of $L_\infty(\mu)$. Additionally, it is useful in both cases $p \geq m$ and $p < m$ for very large values of m and p, when $n \geq 3$ (but not close to m) and the k_i 's are neither equal nor have a very large variation (Examples 3.14a, 3.14b).

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