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Fourier coefficients of theta functions at cusps other than infinity

by

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1. Introduction. In this paper, we use the adelic theory to study the Fourier coefficients of twisted theta functions at different cusps, and in particular prove a conjecture of Goldfeld and Gunnells on their absolute values.

1.1. Fourier coefficients of integral-weight automorphic forms. The theory of Fourier expansions of automorphic forms at various cusps goes back to [Ro66] and [Ma83]. In sharp contrast to the rich theory for Fourier coefficients at infinity, the known results for the Fourier coefficients at finite cusps are very limited. For example, as far as we know, the problem of extending the Deligne bound on the Fourier coefficients to cusps other than infinity, with effective constants, is still unsolved. Using Fricke involutions, Asai [As76] gives an explicit formula of these Fourier coefficients for newforms of $\Gamma_0(N)$ when N is squarefree. This result was later generalized by Kojima [Ko79] to the case of $N = 4q$ for some prime q, and Asai's reasoning may be applied to any cusp which is related to infinity by a Fricke involution. A general explicit formula in terms of the corresponding adelic Whittaker function was presented in [GH11, Vol. 1]. In the works of Goldfeld, Hundley and Lee [GHL15], [Hu14] this description was used to address the question of multiplicativity at finite cusps.

The Fourier coefficients at cusps other than infinity are also of interest from the representation-theoretic point of view. Recall from [AL70] that for the sequence of p-power Fourier coefficients at infinity of a newform, there are essentially three possibilities, corresponding to whether the level N of the newform satisfies $p^2 \mid N$, $p \parallel N$ or $p \nmid N$. As described in [Ca73], these correspond to different possibilities for the local component at p of the corre-

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sponding automorphic representation. If $p^2 \nmid N$, then the isomorphism class of this representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ is completely determined by the level, character, and if $p \nmid N$, the eigenvalue of the Hecke operator T_p. On the other hand, if $p^2 \mid N$, then a wide range of representations is possible. Some of these possibilities can be distinguished from one another using Fourier coefficients at infinity of the original newform and all of its twists, but distinct supercuspidal representations cannot be distinguished in this way (see [BH06, Proposition 27.2]). On the other hand, given the Fourier coefficients of a modular form at all the cusps, one can compute all values of the Whittaker function attached to the corresponding adelic automorphic form, hence each local Whittaker function [GH11, Vol. 1, Theorem 4.13.3]. As the representation generated by the local Whittaker function at p is a model for the local constituent of the automorphic representation, this is, in principle, enough information to determine the representation completely. It would be interesting to see whether this heuristic argument can be made into an effective algorithm.

A first few steps in using adelic methods to study Fourier coefficients at various cusps were taken in [GHL15], [Hu14], where the multiplicative relations satisfied by these coefficients were studied. A next natural problem is to advance from integral weight to half-integral weight automorphic forms, and this is one of the motivations for our work. In this paper, we focus on the simplest and most fundamental types of such automorphic forms: theta functions. Because of a connection between such functions and the Weil representation, we are able to obtain formulae which are much more explicit and amenable to computer implementation (for example) than the formulae obtained in the integer-weight case. In this regard, our results are much stronger than a simple generalization of the formula presented in [GH11, Vol. 1].

1.2. Theta functions and their Fourier coefficients. The theory of theta functions has a long history, going back to Jacobi, and has a broad range of applications throughout various branches of mathematics (see, for example, [Mu07]).

In number theory, theta functions may be used to study representation numbers of quadratic forms (see [Iw97, Chapter 11]), or to shed light on the Shimura correspondence between modular forms of integral weights and those of half-integral weights (see [Shm73] [Shn75], [KS93] and [Wa81]). Moreover, in 1976, Serre and Stark [SS77] answered a question of Shimura, by showing that theta functions actually span the space of all modular forms of weight $1/2$ for $\Gamma_0(N)$.

An important development in the theory of theta functions was their interpretation in terms of a representation of the metaplectic group. This point of view goes back to [We64]. It leads naturally to a vast generalization of the Shimura correspondence, known as the theta correspondence, as well as a local analogue in the representation theory of groups over local fields. (See the survey article [Pr98] and its references.) For GL(2), the theory was significantly explicated by [Ge76], using explicit results of [Ku69]. The results of Serre and Stark were explained from this point of view by Gelbart and Piatetski-Shapiro [GPS80].

In this paper, we study the Fourier coefficients of modular forms of weight 1/2. Let Γ be a discrete group; for simplicity, we may assume that $-1 \in \Gamma$. Let $f(z)$ be a modular form of weight 1/2 with respect to Γ. Then $\mathrm{SL}_2(\mathbb{R})$ acts on $f(z)$ via the *weight*-1/2 *slash operator* $\big|^{\mathrm{Hol}}_{1/2}$, as defined in Section 2.2. To define the Fourier coefficients of $f(z)$ at a cusp $\mathfrak{a}$, one must choose a suitable "scaling" matrix, namely a matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}(2,\mathbb{R})$ which maps ∞ to $\mathfrak{a}$ and conjugates the stabilizer of $\mathfrak{a}$ in Γ to the stabilizer of ∞ in $\mathrm{SL}(2,\mathbb{Z})$. It is known that the function $f\big|^{\mathrm{Hol}}_{1/2}\sigma_{\mathfrak{a}}$ has a Fourier expansion supported on some additive shift of the integers:

$$f\big|^{\mathrm{Hol}}_{1/2}\sigma_{\mathfrak{a}}(z) = \sum_{n=0}^{\infty} A_f(\sigma_{\mathfrak{a}}, n+\kappa)e^{2\pi i(n+\kappa)z}$$

for some constant $\kappa = \kappa_{f,\mathfrak{a}} \in \mathbb{Q}\cap[0,1)$, called the *cusp parameter* of $f(z)$ at $\mathfrak{a}$ and actually depending only on the multiplier system of $f(z)$. We refer to the above coefficients $A_f(\sigma_{\mathfrak{a}}, n+\kappa)$ as the *Fourier coefficients of f at* $\mathfrak{a}$, defined relative to $\sigma_{\mathfrak{a}}$. We note that the values of $A_f(\sigma, n+\kappa)$ depend on the choice of σ, as addressed by Lemma 2.10, but their absolute values $|A_f(\sigma, n+\kappa)|$, especially their nonvanishing property, are independent of this choice.

More specifically, we will investigate the Fourier coefficients of theta functions. Each theta function which we consider is a modular form of weight 1/2 for the group $\Gamma_0(N)$ for a suitable value of N. Their multiplier systems are described in [Iw97], and it is not difficult to check that the resulting cusp parameters are trivial at all cusps. Our investigation was prompted and guided by some numerical computations and conjectures of Dorian Goldfeld and Paul Gunnells.

Let $\chi_{12} = \left(\frac{12}{\cdot}\right)$ be the primitive Dirichlet character modulo 12. Then the *twisted modular theta function*

$$\theta_{\chi_{12}}(z) = \sum_{n=-\infty}^{\infty} \chi_{12}(n)e^{2\pi i n^2 z}$$

is a modular form of weight 1/2 and level 576. Goldfeld and Gunnells computed the Fourier coefficients of $\theta_{\chi_{12}}$, and discovered that (for suitable scaling matrices) the sequence of its Fourier coefficients at every finite cusp was simply a scalar multiple of the sequence of the Fourier coefficients at infinity.

For the cusps which can be transported to infinity by a Fricke involution, this is expected, in view of the results of [As76] (see also [Ko79]). For the other cusps, the result is more surprising.

Goldfeld and Gunnells also considered the Fourier coefficients of the theta functions of higher levels. Motivated by their numerical computations, they proposed the following conjecture.

CONJECTURE 1.1 (Goldfeld–Gunnells). *Let* χ_{12} *be the primitive Dirichlet character modulo* 12 *defined above, let* χ_5 *be the unique nontrivial even Dirichlet character modulo* 5*, and consider the modular form*

$$\theta_{\chi_5\chi_{12}}(z) = \sum_{n=-\infty}^{\infty} \chi_5(n)\chi_{12}(n)e^{2\pi i n^2 z}$$

of weight $1/2$ *and level* 14400*. Let* $\mathfrak{a} = u/w \in \mathbb{Q}$ *with* $u, w \in \mathbb{Z}$ *relatively prime to each other and* $w \mid 14400$*, and let* $\sigma_{\mathfrak{a}} \in \mathrm{SL}(2,\mathbb{R})$ *be any scaling matrix for* $\mathfrak{a}$*. Then:*

(1) $A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n) = 0$ *unless* n *is a perfect square relatively prime to* 12*;*

(2) *if* $5 \nmid w$ *or* $5^2 \mid w$*, then*

$$|A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)| = \begin{cases} 2 & \text{if } \gcd(60,n) = 1, \\ 0 & \text{if } \gcd(60,n) > 1; \end{cases}$$

(3) *if* $5 \parallel w$ *and* $\gcd(12,n) = 1$*, then* $|A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)|$ *depends only on the image of* n^2 *in* $\mathbb{Z}/5\mathbb{Z}$*; more precisely,*

$$|A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)| = \begin{cases} 2a & \text{if } 5 \nmid n, \\ 4b & \text{if } 5 \mid n \end{cases} \qquad (\text{if } u/w \equiv \pm 1/5 \pmod 5),$$

$$|A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)| = \begin{cases} 2b & \text{if } 5 \nmid n, \\ 4a & \text{if } 5 \mid n \end{cases} \qquad (\text{if } u/w \equiv \pm 2/5 \pmod 5),$$

where

$$a = \frac{2}{\sqrt{5}} \sin\frac{4\pi}{5} = \frac{\sqrt{(10-2\sqrt{5})/5}}{2} \approx 0.52573,$$

$$b = \frac{2}{\sqrt{5}} \sin\frac{2\pi}{5} = \frac{\sqrt{(10+2\sqrt{5})/5}}{2} \approx 0.85065.$$

REMARK 1.2. As we discussed before, the fact that $|A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)|$ is independent of the choice of $\sigma_{\mathfrak{a}}$ is an easy consequence of Lemma 2.10.

This conjecture was further extended by Gunnells. Let $p \geq 5$ be a prime and χ_p (mod p) a nontrivial even character. Then he proposed analogous conjectures about $|A_{\theta_{\chi_{12}\chi_p}}(\sigma_{\mathfrak{a}}, n^2)|$. In particular, in the delicate case that $p \parallel w$, he predicted that the full sequence $(|A_{\theta_{\chi_{12}\chi_p}}(\sigma_{\mathfrak{a}}, n^2)|)_{n=1}^{\infty}$ is determined by a subsequence of length $(p+1)/2$. As $\mathfrak{a}$ ranges over all the finite cusps, Gunnells predicts that only $p-1$ distinct sequences should appear, each consisting of zeros of explicit integer polynomials. These $p-1$ sequences come in two classes. For each class there is an element corresponding to zero, and then a cycle of $(p-1)/2$ other values. As n runs through the nonzero squares modulo p, the sequence $(|A_{\theta_{\chi_{12}\chi_p}}(\sigma_{\mathfrak{a}}, n^2)|)$ runs through this cycle. It may start at any point in the cycle, and this accounts for the total of $p-1$ possibilities.

The desire to understand and to prove the conjecture of Goldfeld and Gunnells forms another motivation for our present paper. In particular, we explain their observations as special cases of our main theorem.

1.3. Main results and discussion of the Goldfeld–Gunnells conjecture. In this paper, we consider the theta functions twisted by certain Dirichlet characters, and use the adelic method to study their Fourier coefficients at cusps other than infinity. This work is motivated both by the previous results about integral weight modular forms, and by the computations and conjectures of Goldfeld and Gunnells on theta functions. Our main result, in its crudest form, is as follows (a detailed description can be found in Section 7).

THEOREM 1.3. *Let $M \geq 1$ and let $\chi_M \pmod{M}$ be an even Dirichlet character. Assume that either M itself or $M/2$ is a squarefree integer. Let $\mathfrak{a}$ be a finite cusp, and let $\sigma_{\mathfrak{a}} \in \mathrm{SL}(2,\mathbb{R})$ be any scaling matrix for $\mathfrak{a}$. Then $\theta_{\chi_M}\big|^{\mathrm{Hol}}_{1/2}\sigma_{\mathfrak{a}}(z)$ is a linear combination of the twisted theta functions*

$$\{\theta_{\chi_d}(z) : d \,|\, M,\ \chi_d \text{ is a Dirichlet character modulo } d\}.$$

In particular, if $A_{\theta_{\chi_M}}(\sigma_{\mathfrak{a}}, n)$ be the nth Fourier coefficient of θ_χ at $\mathfrak{a}$ with respect to the scaling matrix $\sigma_{\mathfrak{a}}$, then $A_{\theta_{\chi_M}}(\sigma_{\mathfrak{a}}, n) = 0$ unless n is a perfect square.

In the rest of this section, we illustrate how our results can be used to explain the computations of Goldfeld and Gunnells and to prove their conjecture.

For the twisted theta function $\theta_{\chi_{12}}$, Goldfeld and Gunnells' observation, that (for suitable scaling matrices) the sequence of its Fourier coefficients at every finite cusp was simply a scalar multiple of the sequence of the Fourier coefficients at infinity, suggests an adelic explanation. More precisely, it indicates that the theta function $\theta_{\chi_{12}}$ may correspond to an element of the Weil representation which is fixed, up to scalars, by a group which acts transitively on the cusps. In Example 7.10 we prove this with an explicit description of this scalar factor.

For the theta function $\theta_{\chi_{12}\chi_p}$ of higher twists, in Example 7.6 we again produce a group which acts transitively on the cusps. It no longer fixes the one-dimensional space spanned by our element of the Weil representation. Instead, it fixes a finite-dimensional space containing it, and this permits us to obtain explicit results concerning the Fourier coefficients at all the cusps.

For illustration, we consider the special case that $p = 5$ and explicitly establish the relevant conclusions in Conjecture 1.1. As in Conjecture 1.1, χ_5 is the unique primitive Dirichlet character modulo 5, and

$$\theta_{\chi_5\chi_{12}}(z) = \sum_{n=-\infty}^{\infty} \chi_5(n)\chi_{12}(n)e^{2\pi i n^2 z}$$

is a modular form of weight 1/2 and level 14400. Since the weight-1/2 slash operator (see Section 2.2) does not define a right action of $\mathrm{SL}(2,\mathbb{R})$ but of its double cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, it turns out to be more convenient to work with not $\Gamma_0(14400)$ but

$$\Gamma^{(120)} = \begin{pmatrix} 1 & \\ & 120 \end{pmatrix} \mathrm{SL}(2,\mathbb{Z}) \begin{pmatrix} 1 & \\ & 1/120 \end{pmatrix}.$$

As a conjugate of $\mathrm{SL}(2,\mathbb{Z})$ this group obviously acts transitively on the cusps, and it is easily verified that it provides a scaling matrix for every cusp relative to $\Gamma_0(14400)$. Now, let V be the three-dimensional complex vector space spanned by $\theta_{\chi_5\chi_{12}}$, $\theta_{\chi_5^0\chi_{12}}$ and $\theta^{(5)}_{\chi_{12}}$, where χ_5^0 is the principal Dirichlet character modulo 5, and

$$\theta^{(5)}_{\chi_{12}}(z) = \theta_{\chi_{12}}(z) - \theta_{\chi_5^0\chi_{12}}(z) = \sum_{\substack{n=-\infty \\ 5|n}}^{\infty} \chi_{12}(n)e^{2\pi i n^2 z}.$$

Then Theorem 7.7 constructs a map $M : \Gamma^{(120)} \to \mathrm{GL}(3,\mathbb{C})$ such that

$$\left[\theta_{\chi_5\chi_{12}}\big|^{\mathrm{Hol}}_{1/2}\sigma \quad \theta_{\chi_5^0\chi_{12}}\big|^{\mathrm{Hol}}_{1/2}\sigma \quad \theta^{(5)}_{\chi_{12}}\big|^{\mathrm{Hol}}_{1/2}\sigma\right] = \left[\theta_{\chi_5\chi_{12}} \quad \theta_{\chi_5^0\chi_{12}} \quad \theta^{(5)}_{\chi_{12}}\right] \cdot M(\sigma^{-1}) \tag{1.4}$$

for all $\sigma \in \Gamma^{(120)}$. As in the case of the half-integral weight slash operator, we emphasize that M is *not* a homomorphism but satisfies the twisted multiplicativity

$$M(\sigma_1)M(\sigma_2) \sim M(\sigma_1\sigma_2),$$

where we use $\sim$ to denote an equality between the two sides up to multiplication by a complex number of absolute value 1. The behavior of M is perhaps better understood by working with a metaplectic covering group. This point of view is presented in the body of the paper.

The formula (1.4) enables us to recover the Fourier coefficients of the twisted theta function $A_{\theta_{\chi_5\chi_{12}}}(\sigma, m)$, provided one can compute $M(\sigma^{-1})$ explicitly. In order to obtain an explicit formula for M, it is helpful to work with one prime at a time. If p is a prime, let

$$K_p^{(120)} = \begin{pmatrix} 1 & \\ & 120 \end{pmatrix} \mathrm{SL}(2, \mathbb{Z}_p) \begin{pmatrix} 1 & \\ & 1/120 \end{pmatrix}.$$

Clearly, $\Gamma^{(120)} \le K_p^{(120)}$ for each p. In Section 4 we explicitly construct the local component $M_p : K_p^{(120)} \to \mathrm{GL}(3, \mathbb{C})$ at every prime p; in particular, these local maps satisfy the twisted multiplicativity

$$M_p(\sigma_1)M_p(\sigma_2) \sim M_p(\sigma_1\sigma_2).$$

As we specialize the computations therein to this particular situation, we can show that M_p is trivial unless $p = 2, 3, 5$, and that the images of M_2 and M_3 are both the scalar matrices in $\mathrm{GL}(3, \mathbb{C})$ of absolute value 1. Hence, as we combine this observation with the product decomposition of the map M in Section 5, we have $M \sim M_5 : K_5^{(120)} \to \mathrm{GL}(3, \mathbb{C})$.

We now describe briefly how these results may be used to verify the Goldfeld–Gunnells conjecture regarding $\theta_{\chi_5\chi_{12}}$. As a scaling matrix for the cusp $\mathfrak{a} = u/w$, where $u, w \in \mathbb{Z}$ with $\gcd(u, w) = 1$ and $w \mid 14400$, we choose

$$\sigma_{\mathfrak{a}} := \begin{pmatrix} 120u/t & r/120 \\ 120w/t & s \end{pmatrix}$$

where we write $t = \gcd(w, 120)$ and choose $r, s \in \mathbb{Z}$ so that $120us - rw = t$.

If $25 \mid w$, then in $K_5^{(120)}$ we have the decomposition

$$\sigma_{\mathfrak{a}}^{-1} = \begin{pmatrix} 1 & -rt/(14400u) \\ & 1 \end{pmatrix} \begin{pmatrix} -t/(120u) & \\ & -120u/t \end{pmatrix} \cdot \begin{pmatrix} & 1/5 \\ -5 & \end{pmatrix} \begin{pmatrix} 1 & w/(25u) \\ & 1 \end{pmatrix} \begin{pmatrix} & 1/5 \\ -5 & \end{pmatrix};$$

if $25 \nmid w$, then in $K_5^{(120)}$,

$$\sigma_{\mathfrak{a}}^{-1} = \begin{pmatrix} 1 & -st/(120w) \\ & 1 \end{pmatrix} \begin{pmatrix} & 1/5 \\ -5 & \end{pmatrix} \begin{pmatrix} 24w/t & \\ & t/24w \end{pmatrix} \begin{pmatrix} 1 & -u/w \\ & 1 \end{pmatrix}.$$

In either case, we may apply direct computations via the twisted multiplicativity and Example 4.24. For convenience, write

$$v_{\mathfrak{a}} = \begin{cases} -\frac{rt}{14400u} & \text{if } 25 \mid w, \\ -\frac{st}{120w} & \text{if } 25 \nmid w, \end{cases}$$

and choose $\alpha \in \mathbb{R}$ so that $e^{i\alpha} = e_5(v_{\mathfrak{a}})$, where $e_p : \mathbb{Q}_p \to \mathbb{C}^\times$ is the usual additive character as defined in (4.2).

In the simple case that either $5 \nmid w$ or $25 \mid w$, we have

$$M(\sigma_{\mathfrak{a}}^{-1}) \sim \begin{pmatrix} \cos\alpha & * & * \\ i\sin\alpha & * & * \\ 0 & * & * \end{pmatrix},$$

so (1.4) implies that

$$\begin{aligned} \theta_{\chi_5\chi_{12}}\big|_{1/2}^{\mathrm{Hol}}\sigma_{\mathfrak{a}} &\sim (\cos\alpha)\theta_{\chi_5\chi_{12}} + i(\sin\alpha)\theta_{\chi_5^0\chi_{12}}, \\ A_{\theta_{\chi_5\chi_{12}},\mathfrak{a}}(\sigma_{\mathfrak{a}}, n^2) &\sim 2\chi_{12}(n)\big(\chi_5(n)\cos\alpha + i\chi_5^0(n)\sin\alpha\big). \end{aligned}$$

Hence

$$\begin{aligned} |A_{\theta_{\chi_5\chi_{12}},\mathfrak{a}}(\sigma_{\mathfrak{a}}, n)| &= 2|\chi_5(n)\chi_{12}(n)|\sqrt{\cos^2\alpha + \sin^2\alpha} \\ &= \begin{cases} 2 & \text{if } \gcd(60, n) = 1, \\ 0 & \text{if } \gcd(60, n) > 1. \end{cases} \end{aligned}$$

The case $5 \parallel w$ is more complicated. Write $v'_{\mathfrak{a}} = -2\pi u/w$, and as above choose $\beta \in \mathbb{R}$ so that $e^{i\beta} = e_5(v'_{\mathfrak{a}})$. Then direct computation gives

$$M(\sigma_{\mathfrak{a}}^{-1}) \sim \begin{pmatrix} (\sin\alpha\sin\beta)/\sqrt{5} - \chi_5(24w/t)\cos\alpha\cos\beta & * & * \\ -i(\cos\alpha\sin\beta)/\sqrt{5} - i\chi_5(24w/t)\sin\alpha\cos\beta & * & * \\ 4i(\sin\beta)/\sqrt{5} & * & * \end{pmatrix},$$

so (1.4) implies that

$$\begin{aligned} \theta_{\chi_5\chi_{12}}\big|_{1/2}^{\mathrm{Hol}}\sigma_{\mathfrak{a}} \sim{}& \left(\frac{\sin\alpha\sin\beta}{\sqrt{5}} - \chi_5\left(\frac{24w}{t}\right)\cos\alpha\cos\beta\right)\theta_{\chi_5\chi_{12}} \\ & - i\left(\frac{\cos\alpha\sin\beta}{\sqrt{5}} + \chi_5\left(\frac{24w}{t}\right)\sin\alpha\cos\beta\right)\theta_{\chi_5^0\chi_{12}} \\ & + \frac{4i\sin\beta}{\sqrt{5}}\theta_{\chi_{12}}^{(5)}. \end{aligned}$$

For $5 \mid n$, we have

$$\begin{aligned} A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2) &\sim \frac{8i\sin\beta}{\sqrt{5}}\chi_{12}(n), \\ |A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)| &= \frac{8|\chi_{12}(n)|}{\sqrt{5}}|\sin\beta| = \frac{8|\chi_{12}(n)|}{\sqrt{5}}\left|\sin\left(\frac{2\pi u}{w}\right)\right|. \end{aligned}$$

For $5 \nmid n$, we have

$$\begin{aligned} A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2) \sim{}& 2\left(\frac{\sin\alpha\sin\beta}{\sqrt{5}} - \chi_5\left(\frac{24w}{t}\right)\cos\alpha\cos\beta\right)\chi_5(n)\chi_{12}(n) \\ & - 2i\left(\frac{\cos\alpha\sin\beta}{\sqrt{5}} + \chi_5\left(\frac{24w}{t}\right)\sin\alpha\cos\beta\right)\chi_{12}(n), \end{aligned}$$

$$\begin{aligned}|A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)| &= 2|\chi_{12}(n)|\biggl(\biggl(\frac{\sin\alpha\sin\beta}{\sqrt{5}} - \chi_5\Bigl(\frac{24w}{t}\Bigr)\cos\alpha\cos\beta\biggr)^2 \\ &\qquad + \biggl(\frac{\cos\alpha\sin\beta}{\sqrt{5}} + \chi_5\Bigl(\frac{24w}{t}\Bigr)\sin\alpha\cos\beta\biggr)^2\biggr)^{1/2} \\ &= 2|\chi_{12}(n)|\sqrt{\frac{\sin^2\beta}{5} + \cos^2\beta} \\ &= 2|\chi_{12}(n)|\sqrt{\frac{1}{5}\sin^2\frac{2\pi u}{w} + \cos^2\frac{2\pi u}{w}}.\end{aligned}$$

Hence, if $\gcd(n, 12) = 1$, then in the case $u/w \equiv \pm 1/5 \pmod 5$ we have

$$|A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)| = \begin{cases} 2\sqrt{\frac{1}{5}\sin^2\frac{2\pi}{5} + \cos^2\frac{2\pi}{5}} = \frac{4}{\sqrt{5}}\sin\frac{4\pi}{5} & \text{if } 5 \nmid n, \\ \frac{8}{\sqrt{5}}\sin\frac{2\pi}{5} & \text{if } 5 \mid n, \end{cases}$$

while in the case $u/w \equiv \pm 2/5 \pmod 5$ we have

$$|A_{\theta_{\chi_5\chi_{12}}}(\sigma_{\mathfrak{a}}, n^2)| = \begin{cases} 2\sqrt{\frac{1}{5}\sin^2\frac{4\pi}{5} + \cos^2\frac{4\pi}{5}} = \frac{4|\chi(n)|}{\sqrt{5}}\sin\frac{2\pi}{5} & \text{if } 5 \nmid n, \\ \frac{8}{\sqrt{5}}\sin\frac{4\pi}{5} & \text{if } 5 \mid n. \end{cases}$$

This verifies the Goldfeld–Gunnells Conjecture 1.1 for the Fourier coefficients of $\theta_{\chi_5\chi_{12}}$.

1.4. Organization of the paper. In Section 2 we review the classical theory of the theta functions which we shall study. In Sections 3 and 4 we develop the relevant theory of local metaplectic groups and Weil representations, including the explicit formulae which are crucial for our aims in this paper. In Section 5 we review the relevant notions regarding the global metaplectic group. In Section 6 we define adelic theta functions corresponding to the classical counterparts reviewed in Section 2. The main theorems are proved in Section 7.

It may be noted that in this paper we have restricted attention to theta series attached to even Dirichlet characters whose conductors are either squarefree or equal to four times an odd squarefree number. However, it seems that the method extends to other characters in a natural way. We hope to return to this in future work.

2. The classical theory

2.1. The scaling matrices. Let $\mathcal{H}$ denote the upper half-plane. We shall make use of the classical action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{H}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \quad \biggl(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),\ z \in \mathcal{H}\biggr).$$

Let $\mathfrak{a}$ be a cusp for $\Gamma_0(N)$. Write $\Gamma_{\mathfrak{a}}$ for the stabilizer of $\mathfrak{a}$ in $\Gamma_0(N)$. Choose $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ with $\gamma\infty = \mathfrak{a}$. Then

$$\gamma^{-1}\Gamma_{\mathfrak{a}}\gamma \subset \Gamma_\infty = \left\langle \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right\rangle.$$

Moreover, $\gamma^{-1}\Gamma_{\mathfrak{a}}\gamma$ contains $\left(\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right)$. It follows that

$$\gamma^{-1}\Gamma_{\mathfrak{a}}\gamma = \left\langle \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 1 & m_{\mathfrak{a}} \\ & 1 \end{pmatrix} \right\rangle$$

for a unique positive integer $m_{\mathfrak{a}}$, called the *width* of $\mathfrak{a}$ (relative to $\Gamma_0(N)$). Note that the matrix

$$g_{\mathfrak{a}} = \gamma \begin{pmatrix} 1 & m_{\mathfrak{a}} \\ & 1 \end{pmatrix} \gamma^{-1}$$

is independent of the choice of γ, as the elements $\left(\begin{smallmatrix} 1 & m_{\mathfrak{a}} \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -m_{\mathfrak{a}} \\ & 1 \end{smallmatrix}\right)$ are not conjugate in $\mathrm{SL}(2,\mathbb{R})$. Hence $\Gamma_{\mathfrak{a}} = \langle -I, g_{\mathfrak{a}} \rangle$.

Now choose $\sigma \in \mathrm{SL}(2,\mathbb{R})$ such that

$$\sigma \cdot \infty = \mathfrak{a}, \qquad \sigma \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \sigma^{-1} = g_{\mathfrak{a}}.$$

Following [Iw97], we refer to σ as a "scaling matrix" for $\mathfrak{a}$. Clearly, σ is unique up to an element of the centralizer of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{SL}(2,\mathbb{R})$, which is the subgroup

$$\left\{ \begin{pmatrix} \varepsilon & \varepsilon t \\ & \varepsilon \end{pmatrix} : \varepsilon \in \{\pm 1\},\ t \in \mathbb{R} \right\}.$$

If $\mathfrak{a} = \infty$, then $m_{\mathfrak{a}} = 1$, $g_{\mathfrak{a}} = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$, and one can take σ to be the identity matrix.

If $\mathfrak{a} = u/w \in \mathbb{Q}$, then

$$m_{\mathfrak{a}} = \frac{N}{\gcd(w^2, N)}, \qquad g_{\mathfrak{a}} = I_2 + \frac{N}{\gcd(w^2,N)} \cdot \begin{pmatrix} -uw & u^2 \\ -w^2 & uw \end{pmatrix}.$$

As for the scaling matrix, a common choice is

$$\sigma_{\mathfrak{a}}^0 = \begin{pmatrix} u\sqrt{N/\gcd(w^2,N)} & \\ w\sqrt{N/\gcd(w^2,N)} & 1/(u\sqrt{N/\gcd(w^2,N)}) \end{pmatrix}.$$

If $N = M^2$, this simplifies to

$$\sigma_{\mathfrak{a}}^0 = \begin{pmatrix} uM/\gcd(w,M) & \\ wM/\gcd(w,M) & \gcd(w,M)/(uM) \end{pmatrix}. \tag{2.1}$$

This choice, however, is not suitable for our purpose. Instead, we would like to have our scaling matrix in a particular conjugate of $\mathrm{SL}(2,\mathbb{Z})$.

For any integer M, let

$$\Gamma^{(M)} := \begin{pmatrix} 1 & \\ & M \end{pmatrix} \mathrm{SL}(2,\mathbb{Z}) \begin{pmatrix} 1 & \\ & 1/M \end{pmatrix}. \tag{2.2}$$

LEMMA 2.3. *Let $M \geq 1$, and let $\mathfrak{a} = u/w \in \mathbb{Q}$ be a cusp for $\Gamma_0(M^2)$. Then there exists a scaling matrix σ of $\mathfrak{a}$ that lies in $\Gamma^{(M)}$. Explicitly, choose $r', s' \in \mathbb{Z}$ with*

$$uMs' - r'w = \gcd(M, w),$$

and write $t = \gcd(M, w)$; then we may take the scaling matrix

$$\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & \\ & M \end{pmatrix} \begin{pmatrix} uM/t & r' \\ w/t & s' \end{pmatrix} \begin{pmatrix} 1 & \\ & 1/M \end{pmatrix} = \begin{pmatrix} uM/t & r'/M \\ Mw/t & s' \end{pmatrix} \in \Gamma^{(M)}. \tag{2.4}$$

Proof. This follows from direct computations. ∎

REMARK 2.5. To illustrate the relation between the usual choice of scaling matrices and our choice, we have

$$\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}^{0} \begin{pmatrix} 1 & r't/(M^2u) \\ & 1 \end{pmatrix}.$$

In particular, the condition $\sigma \in \Gamma^{(M)}$ determines r' uniquely modulo uM/t, so the quantity $r't/(M^2u)$ is uniquely determined modulo $1/M$.

LEMMA 2.6. *Under the notation in Lemma 2.3, we have the decompositions*

$$\sigma_{\mathfrak{a}}^{-1} = \begin{pmatrix} 1 & -s't/wM \\ & 1 \end{pmatrix} \begin{pmatrix} & t/wM \\ -wM/t & \end{pmatrix} \begin{pmatrix} 1 & -u/w \\ & 1 \end{pmatrix}, \tag{2.7}$$

$$\begin{aligned} \sigma_{\mathfrak{a}}^{-1} = {} & \begin{pmatrix} 1 & -r't/(M^2u) \\ & 1 \end{pmatrix} \begin{pmatrix} -t/(uM) & \\ & -uM/t \end{pmatrix} \\ & \cdot \begin{pmatrix} & 1/M \\ -M & \end{pmatrix} \begin{pmatrix} 1 & w/(uM^2) \\ & 1 \end{pmatrix} \begin{pmatrix} & 1/M \\ -M & \end{pmatrix}. \end{aligned} \tag{2.8}$$

Proof. This follows from direct computations. ∎

REMARK 2.9. Let $p \mid M$ and

$$K_p^{(M)} := \begin{pmatrix} 1 & \\ & M \end{pmatrix} \mathrm{SL}(2,\mathbb{Z}_p) \begin{pmatrix} 1 & \\ & 1/M \end{pmatrix}.$$

The merit of the above lemma is that it gives an explicit decomposition of σ^{-1} within the group $K_p^{(M)}$. More precisely, if $v_p(w) \leq v_p(M)$, then each matrix in (2.7) is an element of $K_p^{(M)}$, while if $v_p(w) = 2$ and $v_p(M) = 1$, each matrix in (2.8) is an element of $K_p^{(M)}$. Here and throughout, v_p denotes the p-adic valuation.

2.2. The slash operators. We define the map $j : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{H} \to \mathcal{H}$ by

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz + d \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),\ z \in \mathcal{H}\right).$$

It satisfies the cocycle condition

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 \cdot z) j(\gamma_2, z).$$

We also define $\mathfrak{j}(\gamma, z) = j(\gamma, z)/|j(\gamma, z)|$. Observe that

$$\mathfrak{j}\left(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, i\right) = \mathfrak{j}\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, i\right) = e^{i\theta}$$

for all $x, y, \theta \in \mathbb{R}$ with $y > 0$.

For $g \in \mathrm{SL}(2,\mathbb{R})$ and $f : \mathcal{H} \to \mathbb{C}$, define

$$\bigl(f\big|_{1/2}^{\mathrm{Hol}} g\bigr)(z) = j(g, z)^{-1/2} f(g \cdot z), \qquad \bigl(f\big|_{1/2}^{\mathrm{Maa}} g\bigr)(z) = \mathfrak{j}(g, z)^{-1/2} f(g \cdot z).$$

Here, the square roots are defined to be the principal value, having an argument in $(-\pi/2, \pi/2)$. If $\mu(f)$ is the function defined by $[\mu(f)](z) = f(z)\,\mathrm{Im}(z)^{1/4}$, then for every $g \in \mathrm{SL}(2,\mathbb{R})$ we have

$$\mu\bigl(f\big|_{1/2}^{\mathrm{Hol}} g\bigr) = \mu(f)\big|_{1/2}^{\mathrm{Maa}} g.$$

Observe that although j and $\mathfrak{j}$ are cocycles, their square roots are not, because of the discontinuity of the principal value.

2.3. Modular forms of weight $1/2$**.** Let f be a modular form of weight $1/2$ and multiplier ϑ for $\Gamma_0(N)$, as in [Iw97, §§2.6, 2.7]. Thus for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have

$$f\big|_{1/2}^{\mathrm{Hol}} \gamma = \vartheta(\gamma)\left(\frac{c}{d}\right)\varepsilon_d f,$$

where $\left(\frac{c}{d}\right)$ denotes the quadratic residue symbol as defined in [Shm73] and

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4). \end{cases}$$

At every cusp $\mathfrak{a}$, let $\kappa = \kappa_{f,\mathfrak{a}}$ be the unique (rational) solution in $[0, 1)$ to the equation $e^{2\pi i\kappa} = \vartheta(g_{\mathfrak{a}})$. Then the function $e^{-2\pi i\kappa_{f,\mathfrak{a}} z} f\big|_{1/2}^{\mathrm{Hol}} \sigma_{\mathfrak{a}}$ is periodic of period 1 (see [Iw97, p. 43]), and the Fourier coefficients of f at $\mathfrak{a}$ relative to σ are the coefficients $A_f(\sigma, n + \kappa_{f,\mathfrak{a}})$ in its Fourier expansion:

$$\bigl(f\big|_{1/2}^{\mathrm{Hol}} \sigma\bigr)(z) = \sum_{n=0}^{\infty} A_f(\sigma, n + \kappa_{f,\mathfrak{a}}) e^{2\pi i(n+\kappa_{f,\mathfrak{a}})z}.$$

LEMMA 2.10. *We have*

$$A_f\left(\sigma\begin{pmatrix}\varepsilon & \\ & \varepsilon\end{pmatrix}, n+\kappa_{f,\mathfrak{a}}\right) = A_f(\sigma, n+\kappa_{f,\mathfrak{a}})\vartheta\begin{pmatrix}\varepsilon & \\ & \varepsilon\end{pmatrix},$$

$$A_f\left(\sigma\begin{pmatrix}1 & t\\ & 1\end{pmatrix}, n+\kappa_{f,\mathfrak{a}}\right) = A_f(\sigma, n+\kappa_{f,\mathfrak{a}})e^{2\pi i(n+\kappa_{f,\mathfrak{a}})t}.$$

Proof. This is obvious; for example, the latter comes immediately from

$$\sum_{n=0}^{\infty} A_f\left(\sigma\begin{pmatrix}1 & t\\ & 1\end{pmatrix}, n+\kappa_{f,\mathfrak{a}}\right)e^{2\pi i(n+\kappa_{f,\mathfrak{a}})z} = \left(f\big|_{1/2}^{\mathrm{Hol}}\sigma\begin{pmatrix}1 & t\\ & 1\end{pmatrix}\right)(z)$$
$$= \big(f\big|_{1/2}^{\mathrm{Hol}}\sigma\big)(z+t) = \sum_{n=0}^{\infty} A_f(\sigma, n+\kappa_{f,\mathfrak{a}})e^{2\pi i(n+\kappa_{f,\mathfrak{a}})(z+t)}. \blacksquare$$

2.4. The classical theta functions. Let $\chi \pmod M$ be an even Dirichlet character. Then the *classical twisted theta function* (cf. [Iw97, §10.5])

$$\theta_\chi(z) = \sum_{n=-\infty}^{\infty} \chi(n)e^{2\pi i n^2 z} \qquad (z \in \mathcal{H}) \tag{2.11}$$

is a cusp form of weight $1/2$ and level $4M^2$, and we have

$$\theta_\chi(\gamma z) = \chi(d)\chi_c(d)\varepsilon_d^{-1}(cz+d)^{1/2}\theta_\chi(z) \qquad \left(\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_0(4M^2)\right).$$

In other words,

$$\big(\theta_\chi\big|_{1/2}^{\mathrm{Hol}}\gamma\big)(z) = \chi(d)\chi_c(d)\varepsilon_d^{-1}\theta_\chi(z) \qquad (\gamma \in \Gamma_0(4M^2)).$$

In this paper, we consider the special cases that either M or $M/2$ is a squarefree integer, and study the Fourier coefficients of θ_χ at different cusps. The main result is Theorem 7.7.

3. Local metaplectic groups

3.1. Local metaplectic groups. Let $\mathbb{Q}_v$ be one of the completions of $\mathbb{Q}$. Thus $\mathbb{Q}_v = \mathbb{Q}_\infty = \mathbb{R}$ or $\mathbb{Q}_v$ is the p-adic numbers $\mathbb{Q}_p$ for some prime p. The Hilbert symbol on $\mathbb{Q}_v$ will be denoted by $(\ ,\ )_v$. As in [Ge76], we define the cocycle

$$\beta_v : \mathrm{SL}(2,\mathbb{Q}_v) \times \mathrm{SL}(2,\mathbb{Q}_v) \to \{\pm 1\},$$
$$(g_1, g_2) \mapsto \big(x(g_1), x(g_2)\big)_v\big(-x(g_1)x(g_2), x(g_1g_2)\big)_v s(g_1)s(g_2)s(g_1g_2),$$

where

$$x\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0, \end{cases}$$

$$s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c,d)_v & \text{if } v < \infty,\ cd \neq 0 \text{ and } \operatorname{ord}(c) \text{ is odd}, \\ 1 & \text{otherwise}. \end{cases}$$

In particular, in the Borel subgroup it simplifies to

$$\beta_v\left(\begin{pmatrix} a_1 & b_1 \\ & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ & d_2 \end{pmatrix}\right) = (a_1, d_2)_v,$$

and over $\mathrm{SL}(2,\mathbb{R})$ it simplifies to

$$\beta_\infty(g_1, g_2) = \big(x(g_1), x(g_2)\big)_\infty \big(-x(g_1)x(g_2), x(g_1g_2)\big)_\infty. \tag{3.1}$$

We may then define a double cover of $\mathrm{SL}(2,\mathbb{Q}_v)$, denoted $\widetilde{\mathrm{SL}}(2,\mathbb{Q}_v)$ and consisting of the set $\mathrm{SL}(2,\mathbb{Q}_v) \times \{\pm 1\}$ equipped with the operation

$$(g_1, \zeta_1)(g_2, \zeta_2) := \big(g_1g_2, \beta_v(g_1, g_2)\zeta_1\zeta_2\big). \tag{3.2}$$

The function $\mathrm{pr} : \widetilde{\mathrm{SL}}(2,\mathbb{Q}_v) \to \mathrm{SL}(2,\mathbb{Q}_v)$ given by $\mathrm{pr}(g,\zeta) = g$ is a homomorphism.

3.2. Generators for $\mathrm{SL}(2,\mathbb{Z}_p)$ and its preimage

Lemma 3.3. *The group $K_p = \mathrm{SL}(2,\mathbb{Z}_p)$ is generated by*

$$\left\{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbb{Z}_p\right\}. \tag{3.4}$$

Proof. Write H_p for the subgroup of $\mathrm{SL}(2,\mathbb{Z}_p)$ generated by (3.4). Then H_p contains $\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)$ for each x. Since

$$\begin{pmatrix} 1 & x-1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{x-1}{x} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix} = \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$$

for all $x \in \mathbb{Z}_p^\times$, it follows that H_p contains all the diagonal elements, and hence all the upper- and lower-triangular elements, of K_p.

Now consider an arbitrary element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of K_p. Note that either c or d is a unit. If d is a unit, then b/d and bc/d are both in $\mathbb{Z}_p$, and we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ & 1 \end{pmatrix}\begin{pmatrix} 1/d & \\ c & d \end{pmatrix} \in H_p.$$

If c is a unit, then a/c and d/c are both in $\mathbb{Z}_p$, and we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix}\begin{pmatrix} & -1/c \\ c & \end{pmatrix}\begin{pmatrix} 1 & d/c \\ & 1 \end{pmatrix} \in H_p. \blacksquare$$

3.3. The real metaplectic group. In our study of the real metaplectic group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, the basic tools are the metaplectic analogue of the classical Iwasawa decomposition and the corresponding slash operator. Recall that

$$\mathrm{SL}(2,\mathbb{R}) = B^+(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R}),$$

where

$$B^+(\mathbb{R}) = \left\{ \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} : y \in (0,\infty),\, x \in \mathbb{R} \right\} \le \mathrm{SL}(2,\mathbb{R}).$$

Hence our discussion starts with the metaplectic preimage of $\mathrm{SO}_2(\mathbb{R})$.

For $\theta \in \mathbb{R}$ define

$$\kappa_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}) \le \mathrm{SL}(2,\mathbb{R}), \qquad \widetilde{\kappa}_\theta = (\kappa_{2\theta}, \zeta(\theta)),$$

where ζ is the unique function $\mathbb{R} \to \{\pm 1\}$ which is periodic modulo 2π and satisfies

$$\zeta(\theta) = \begin{cases} 1 & \text{if } -\pi/2 \le \theta < \pi/2, \\ -1 & \text{if } \pi/2 \le \theta < 3\pi/2. \end{cases}$$

LEMMA 3.5. *The function $\theta \mapsto \widetilde{\kappa}_\theta$ is a homomorphism.*

Proof. Using (3.1), one checks that

$$\beta_\infty(\kappa_{2\theta_1}, \kappa_{2\theta_2}) = -1 \;\Leftrightarrow\; \frac{\zeta(\theta_1+\theta_2)}{\zeta(\theta_1)\zeta(\theta_2)} = -1$$

on a case-by-case basis. ■

We denote by $\widetilde{K}$ the set of the images $\widetilde{\kappa}_\theta$.

REMARK 3.6. If we define $\sqrt{z}\,'$ for $z \in \mathbb{C}^\times$ so that $\mathrm{Arg}(\sqrt{z}\,') \in [-\pi/2, \pi/2)$ for all $z \in \mathbb{C}^\times$, then the function $S^1 \to S^1 \times \{\pm 1\}$ defined by

$$e^{i\theta} \mapsto \widetilde{\kappa}_\theta \mapsto (e^{2i\theta}, \zeta(\theta))$$

is the restriction of the map

$$z \mapsto \left(z^2, \frac{\sqrt{z^2}\,'}{z} \right).$$

Observe that $\sqrt{z}\,'$ is the principal value of the square root of z except when $z \in (-\infty, 0)$.

Next we discuss the metaplectic phenomena over $B^+(\mathbb{R})$.

LEMMA 3.7. *The cocycle β_∞ is trivial on $B^+(\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and on $\mathrm{SL}(2,\mathbb{R}) \times B^+(\mathbb{R})$.*

Proof. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \qquad g_0 = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \in B^+(\mathbb{R}).$$

Then

$$gg_0 = \begin{pmatrix} * & * \\ cy^{1/2} & (cx+d)y^{-1/2} \end{pmatrix}, \qquad g_0 g = \begin{pmatrix} * & * \\ cy^{-1/2} & dy^{-1/2} \end{pmatrix}.$$

In particular,

$$x(g_0) = y^{1/2} > 0, \qquad x(g_0 g) = x(g_0)x(g), \qquad \operatorname{sgn} x(gg_0) = \operatorname{sgn} x(g).$$

Hence by definition,

$$\beta_\infty(g, g_0) = \big(x(g), x(g_0)\big)_\infty \big(-x(g)x(g_0), x(gg_0)\big)_\infty = \big(-x(g), x(gg_0)\big)_\infty = 1,$$
$$\beta_\infty(g_0, g) = \big(x(g_0), x(g)\big)_\infty \big(-x(g_0)x(g), x(g_0 g)\big)_\infty = \big(-x(g), x(g)\big)_\infty = 1. \ \blacksquare$$

REMARK 3.8. By the lemma, we have the injective homomorphism

$$B^+(\mathbb{R}) \hookrightarrow \widetilde{\mathrm{SL}}(2, \mathbb{R}), \qquad b \mapsto (b, 1),$$

which splits the covering map. We henceforth identify $B^+(\mathbb{R})$ with its image in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

Now we are ready to introduce the Iwasawa decomposition over $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

LEMMA 3.9. *Each element $\widetilde{g}$ of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ has a unique expression as $\widetilde{g} = b\widetilde{\kappa}$ with $b \in B^+(\mathbb{R})$ and $\widetilde{\kappa} \in \widetilde{K}$.*

Proof. This follows immediately from Lemma 3.7 and the analogous statement for $\mathrm{SL}(2, \mathbb{R})$. $\blacksquare$

Based on the above lemma, we are able to define the functions $b : \widetilde{\mathrm{SL}}(2, \mathbb{R}) \to B^+(\mathbb{R})$ and $\theta : \mathrm{SL}(2, \mathbb{R}) \to \mathbb{R}/2\pi\mathbb{Z}$ by

$$\widetilde{g} = b(\widetilde{g})\widetilde{\kappa}_{\theta(\widetilde{g})} \qquad (\widetilde{g} \in \widetilde{\mathrm{SL}}(2, \mathbb{R})).$$

LEMMA 3.10. *For $z = x + iy \in \mathcal{H}$, define*

$$b_z = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \in B^+(\mathbb{R}).$$

Then

$$b(\widetilde{g}) = b_{\mathrm{pr}(\widetilde{g})\cdot i}.$$

Proof. Clearly $\widetilde{g}\cdot z := \mathrm{pr}(\widetilde{g})\cdot z$ is an action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ on $\mathcal{H}$. The stabilizer of i is the preimage of $\mathrm{SO}_2(\mathbb{R})$, which is $\widetilde{K}$. Hence $\widetilde{g} \cdot i = b(\widetilde{g}) \cdot i$. And for each $z \in \mathcal{H}$, the matrix b_z can be described as the unique element of $B^+(\mathbb{R})$ mapping i to z. $\blacksquare$

In what follows, we shall continue to use the action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ on $\mathcal{H}$ by $\widetilde{g} \cdot z := \mathrm{pr}(\widetilde{g}) \cdot z$.

COROLLARY 3.11. *For any $\widetilde{g} \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$ and $z \in \mathcal{H}$, there exists $\theta(\widetilde{g}, z) \in \mathbb{R}/2\pi\mathbb{Z}$ such that $\widetilde{g} \cdot b_z = b_{\widetilde{g}\cdot z}\widetilde{\kappa}_{\theta(\widetilde{g},z)}$. In particular, we have the metaplectic Iwasawa decomposition $\widetilde{g} = b_{\widetilde{g}\cdot i}\widetilde{\kappa}_{\theta(\widetilde{g})}$.*

Specifically, $\theta(\widetilde{g}, z) = \theta(\widetilde{g} b_z)$, where the latter is defined using the Iwasawa decomposition as above. It is immediate from the definitions that the function $\theta : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{R}/2\pi\mathbb{Z}$ is a cocycle, i.e.,

$$\theta(\widetilde{g}_1\widetilde{g}_2, z) = \theta(\widetilde{g}_1, \widetilde{g}_2 \cdot z) + \theta(\widetilde{g}_2, z).$$

Lastly, we discuss the slash operator of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ on the functions over $\mathcal{H}$. Define $\widetilde{\mathfrak{j}} : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathcal{H} \to S^1$ by $\widetilde{\mathfrak{j}}(\widetilde{g}, z) = e^{i\theta(\widetilde{g},z)}$. Clearly, $\widetilde{\mathfrak{j}}$ is a cocycle, since θ is.

Lemma 3.12. *The cocycle* $\widetilde{\mathfrak{j}}(\widetilde{g}, z)$ *is always a square root of* $\mathfrak{j}(\mathrm{pr}(\widetilde{g}), z)$*:*

$$\widetilde{\mathfrak{j}}(\widetilde{g}, z)^2 = \mathfrak{j}(\mathrm{pr}(\widetilde{g}), z).$$

Proof. Write $\widetilde{g} b_z = b_{\widetilde{g}\cdot z}\widetilde{\kappa}_{\theta(\widetilde{g},z)}$. Then $\mathrm{pr}(\widetilde{g}) b_z = b_{\widetilde{g}\cdot z}\kappa_{2\theta(\widetilde{g},z)}$. But

$$\mathfrak{j}(\mathrm{pr}(\widetilde{g}), z) = \frac{\mathfrak{j}(\mathrm{pr}(\widetilde{g}) b_z, i)}{\mathfrak{j}(b_z, i)}$$

and $\mathfrak{j}(b_z, i) = 1$, so we get

$$\mathfrak{j}(\mathrm{pr}(\widetilde{g}), z) = \mathfrak{j}(\mathrm{pr}(\widetilde{g}) b_z, i) = e^{i2\theta(\widetilde{g},z)} = \widetilde{\mathfrak{j}}(\widetilde{g}, z)^2. \ \blacksquare$$

Clearly $\widetilde{\mathfrak{j}}(\widetilde{g}, z)$ is the principal value of the square root of $\mathfrak{j}(\mathrm{pr}(\widetilde{g}), z)$ if and only if $\theta(\widetilde{g}, z) \in (-\pi/2, \pi/2)$.

For $f : \mathcal{H} \to \mathbb{C}$, we now define the slash operator

$$\big(f\big|^{\sim}\widetilde{g}\big)(z) = \widetilde{\mathfrak{j}}(\widetilde{g}, z)^{-1} f(g \cdot z) = \widetilde{\mathfrak{j}}(\widetilde{g}, z)^{-1} f(\mathrm{pr}(\widetilde{g}) \cdot z) \qquad (\widetilde{g} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})).$$

Lemma 3.13. *The slash operator* $|^{\sim}$ *gives a well-defined right action of* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *on the space of all functions* $\mathcal{H} \to \mathbb{C}$.

Proof. This follows immediately from the fact that $\widetilde{\mathfrak{j}}$ is a cocycle. $\blacksquare$

Lemma 3.14. *Let* $f : \mathcal{H} \to \mathbb{C}$ *and* $g \in \mathrm{SL}(2,\mathbb{R})$*. Then*

$$\big(f\big|^{\sim}(g, 1)\big) = \big(f\big|^{\mathrm{Maa}}_{1/2} g\big). \tag{3.15}$$

Proof. Clearly $\widetilde{\mathfrak{j}}((g, 1), z)$ is the principal value of the square root of $\mathfrak{j}(g, z)$ if and only if $\theta((g, 1), z) \in (-\pi/2, \pi/2)$, which in turn is equivalent to $\zeta(\theta((g, 1), z)) = 1$.

Now in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ we have

$$(g, 1)b_z = (g, 1)(b_z, 1) = (gb_z, 1),$$

so by definition $\zeta(\theta((g, 1), z)) = 1$. This confirms that $\widetilde{\mathfrak{j}}((g, 1), z)$ is the principal value of the square root of $\mathfrak{j}(g, z)$. Hence

$$\big(f\big|^{\sim}(g, 1)\big) = \widetilde{\mathfrak{j}}((g, 1), z) f(g \cdot z) = \mathfrak{j}(g, z)^{-1/2} f(g \cdot z) = \big(f\big|^{\mathrm{Maa}}_{1/2} g\big). \ \blacksquare$$

Example 3.16. Let $\chi \pmod M$ be an even Dirichlet character, and consider the classical twisted theta function θ_χ as defined in (2.11). Define

$$\theta_\chi^{\mathrm{Maa}}(x+iy) = y^{1/4}\theta_\chi(x+iy).$$

Then for $\gamma \in \Gamma_0(4M^2)$,

$$\theta_\chi^{\mathrm{Maa}}(\gamma z) = \chi(d)\chi_c(d)\varepsilon_d^{-1}\mathfrak{j}(\gamma, z)^{1/2}\theta_\chi^{\mathrm{Maa}}(z).$$

Hence

$$\big(\theta_\chi^{\mathrm{Maa}}|^{\sim}(\gamma,1)\big)(z) = \big(\theta_\chi^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\gamma\big)(z) = \chi(d)\chi_c(d)\varepsilon_d^{-1}\theta_\chi^{\mathrm{Maa}}(z).$$

4. Local Weil representations

4.1. Local Weil representation. The Bruhat–Schwartz space of $\mathbb{Q}_v$ will be denoted $\mathcal{S}(\mathbb{Q}_v)$. It is the Schwartz space when $v=\infty$, and the space of all locally constant compactly supported functions when v is a prime. Following [GPS80] we consider the family of representations r^{ψ_v} of $\widetilde{\mathrm{SL}}(2,\mathbb{Q}_v)$ on $\mathcal{S}(\mathbb{Q}_v)$, indexed by the nontrivial characters ψ_v of $\mathbb{Q}_v$, and defined by

$$\left[r^{\psi_v}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, 1\right).\varphi\right](x) = \gamma(\psi_v)\widehat{\varphi}(x),$$

$$\left[r^{\psi_v}\left(\begin{pmatrix} 1 & b \\ & 1\end{pmatrix}, 1\right).\varphi\right](x) = \psi_v(bx^2)\varphi(x),$$

$$\left[r^{\psi_v}\left(\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}, 1\right).\varphi\right](x) = |a|^{1/2}\frac{\gamma(\psi_v)}{\gamma(\psi_{v,a})}\varphi(ax),$$

$$[r^{\psi_v}(I_2,\zeta)\varphi] = \zeta\cdot\varphi,$$

where the Fourier transform is given by

$$\widehat{\varphi}(x) = \alpha(\psi_v)\int_{\mathbb{Q}_v}\varphi(y)\psi_v(2xy)\,dy,$$

dy is the standard Haar measure over $\mathbb{Q}_v$, $\alpha(\psi_v)$ is the normalization factor such that $\widehat{\widehat{\varphi}}(x) = \varphi(-x)$, $\psi_{v,a}(x) = \psi_v(ax)$, and (1)

$$\gamma(\psi_{v,a}) = \begin{cases} e^{\frac{a}{|a|}\frac{\pi i}{4}} & \text{if } v=\infty \text{ and } \psi_\infty(x) = e^{2\pi i x},\\ \lim\limits_{m\to-\infty}\alpha(\psi_{v,a})\displaystyle\int_{p^m\mathbb{Z}_p}\psi_v(ay^2)\,dy & \text{if } v \text{ is a prime.}\end{cases}$$

Remark 4.1. The constant $\gamma(\psi_{v,a})$ is an eighth root of unity. This is obvious when $v=\infty$, and a result of Weil otherwise (cf. [Ge76, p. 36]).

(1) Note that the formula for $\gamma(\psi_{v,a})$ in [Ge76, p. 36] contains a typo.

Now we would like to explicitly describe the local Weil representation with respect to the additive character

$$e_v(x) = \begin{cases} e^{2\pi i x} & \text{if } v = \infty, \\ e^{-2\pi i \{x\}_p} & \text{if } v = p, \end{cases} \tag{4.2}$$

where, for every prime p, we denote by

$$\{\cdot\}_p : \mathbb{Q}_p \to \mathbb{Q}, \qquad \sum_{n=-N}^{\infty} a_n p^n \mapsto \sum_{n=-N}^{-1} a_n p^n,$$

the "p-adic fractional part" of $\mathbb{Q}_p$.

Proposition 4.3. *Let $a \in \mathbb{Q}_v^\times$. Then*

$$\alpha(e_{v,a}) = |2a|_v^{1/2}.$$

Proof. If v is a finite place, say at p, then as the test function we take $\varphi(x) = \mathbb{1}_{\mathbb{Z}_p}(x)$. By definition we have

$$\widehat{\varphi}(x) = \alpha(e_{p,a}) \int_{\mathbb{Z}_p} e_p(2axy)\, dy = \alpha(\psi_{p,a}) \mathbb{1}_{(2a)^{-1}\mathbb{Z}_p}(x),$$

so

$$\widehat{\widehat{\varphi}}(x) = \alpha(e_{p,a})^2 \int_{(2a)^{-1}\mathbb{Z}_p} e_p(2axy)\, dy = \frac{\alpha(e_{p,a})^2}{|2a|_p} \mathbb{1}_{\mathbb{Z}_p}(x).$$

Hence by definition, $\alpha(e_{p,a}) = |2a|_p^{1/2}$.

Now consider the case $v = \infty$. As the test function we take $\varphi(x) = e^{-\pi x^2}$; then

$$\widehat{\varphi}(x) = \alpha(e_{\infty,a}) \int_{-\infty}^{\infty} e^{-\pi y^2 + 4\pi a i x y}\, dy = \alpha(e_{\infty,a}) e^{-4\pi a^2 x^2},$$

$$\widehat{\widehat{\varphi}}(x) = \alpha(e_{\infty,a})^2 \int_{-\infty}^{\infty} e^{-4\pi a^2 y^2 + 4a\pi i x y}\, dy = \frac{\alpha(e_{\infty,a})^2}{|2a|} e^{-\pi x^2}.$$

Hence again $\alpha(e_{\infty,a}) = |2a|^{1/2}$. ∎

Next we evaluate $\gamma(e_{p,a})$.

Proposition 4.4. *Let $a \in \mathbb{Z}_p \setminus \{0\}$, say with the decomposition $a = \alpha p^r$ for some $r \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}_p^\times$. Then*

$$\gamma(e_{p,a}) = \begin{cases} \dfrac{1+i}{\sqrt{2}} \varepsilon_{-\alpha}^{-1} \left(\dfrac{2}{-\alpha}\right)^r & \textit{if } p = 2, \\ \varepsilon_p \left(\dfrac{-\alpha}{p}\right) & \textit{if } p \geq 3 \textit{ and } 2 \nmid r, \\ 1 & \textit{if } p \geq 3 \textit{ and } 2 \mid r, \end{cases}$$

where the factor $\varepsilon_{-\alpha}$ and the Kronecker symbols involving α are evaluated with respect to some $\alpha^* \in \mathbb{Z}$ such that $|\alpha - \alpha^*|_p$ is sufficiently small.

Proof. For $m \gg 1$ we have

$$\begin{aligned}
\gamma(e_{p,a}) &= \alpha(e_{p,a}) \int_{p^{-m}\mathbb{Z}_p} e_p(ay^2)\,dy \\
&= |2a|_p^{1/2} \int_{p^{-m}\mathbb{Z}_p} e(-\{ay^2\}_p)\,dy = |2a|_p^{1/2} p^m \int_{\mathbb{Z}_p} e\bigg(-\bigg\{\frac{ay^2}{p^{2m}}\bigg\}_p\bigg)\,dy \\
&= |2|_p^{1/2} p^{m-r/2} \int_{\mathbb{Z}_p} e\bigg(-\bigg\{\frac{\alpha y^2}{p^{2m-r}}\bigg\}_p\bigg)\,dy \\
&= |2|_p^{1/2} p^{r/2-m} \sum_{y\in\mathbb{Z}_p/p^{2m-r}\mathbb{Z}_p} e\bigg(-\frac{\alpha y^2}{p^{2m-r}}\bigg) \\
&= |2|_p^{1/2} p^{r/2-m} \sum_{y\in\mathbb{Z}/p^{2m-r}\mathbb{Z}} e\bigg(-\frac{\alpha y^2}{p^{2m-r}}\bigg).
\end{aligned}$$

To evaluate the inner quadratic Gauss sum, we quote the following famous result of Gauss:

$$\sum_{n=0}^{c-1} e\bigg(\frac{an^2}{c}\bigg) = \begin{cases} \varepsilon_c\Big(\dfrac{a}{c}\Big)\sqrt{c} & \text{if } 2\nmid c, \\ \varepsilon_a^{-1}(1+i)\Big(\dfrac{c}{a}\Big)\sqrt{c} & \text{if } a \text{ is odd and } 4\mid c, \\ 0 & \text{if } c\equiv 2 \pmod 4). \end{cases}$$

We start with the case $p = 2$. Since we have assumed $m \gg 1$, the quadratic Gauss sum becomes

$$\begin{aligned}
\sum_{y\in\mathbb{Z}/2^{2m-r}\mathbb{Z}} e\bigg(-\frac{\alpha y^2}{2^{2m-r}}\bigg) &= \varepsilon_{-\alpha}^{-1}(1+i)\bigg(\frac{2^{2m-r}}{-\alpha}\bigg)2^{m-r/2} \\
&= \varepsilon_{-\alpha}^{-1}(1+i)\bigg(\frac{2}{-\alpha}\bigg)^r 2^{m-r/2}.
\end{aligned}$$

Hence

$$\gamma(e_{2,a}) = \frac{1+i}{\sqrt{2}}\varepsilon_{-\alpha}^{-1}\bigg(\frac{2}{-\alpha}\bigg)^r.$$

Now we assume that $p \neq 2$. Then the above result on quadratic Gauss sums shows that

$$\gamma(e_{p,a}) = \varepsilon_{p^{2m-r}}\bigg(\frac{-\alpha}{p^{2m-r}}\bigg) = \varepsilon_{p^{2m-r}}\bigg(\frac{-\alpha}{p}\bigg)^r.$$

Since $p^2 \equiv 1 \pmod 4$, we have

$$\varepsilon_{p^{2m-r}} = \varepsilon_{p^r} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \text{ or } 2 \mid r, \\ i & \text{if } p \equiv 3 \pmod 4 \text{ and } 2 \nmid r. \end{cases}$$

Hence

$$\gamma(e_{p,a}) = \begin{cases} \left(\dfrac{-\alpha}{p}\right) & \text{if } p \equiv 1 \pmod 4 \text{ and } 2 \nmid r, \\ i\left(\dfrac{-\alpha}{p}\right) & \text{if } p \equiv 3 \pmod 4 \text{ and } 2 \nmid r, \\ 1 & \text{if } 2 \mid r. \ \blacksquare \end{cases}$$

4.2. The real Weil representation. In this section, we consider the real Weil representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

LEMMA 4.5. *Let* $\phi_\infty^0(x) = e^{-2\pi x^2}$. *Then*

$$r^{e_\infty}(\widetilde{\kappa}_\theta)\phi_\infty^0 = e^{-i\theta}\phi_\infty^0 \qquad (\theta \in \mathbb{R}).$$

Proof. We recall that

$$\widetilde{\kappa}_\theta = (\kappa_{2\theta}, \zeta(\theta)) = \left(\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \zeta(\theta)\right).$$

It is straightforward to verify that the lemma is valid if θ is $n\pi/2$ for some $n \in \mathbb{Z}$, so we may assume henceforth that $\sin 2\theta \neq 0$.

By direct computations we have

$$\left(\begin{pmatrix} 1 & \frac{\cos 2\theta}{\sin 2\theta} \\ & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} \frac{-1}{\sin 2\theta} & \\ & -\sin 2\theta \end{pmatrix}, 1\right)\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, 1\right)\left(\begin{pmatrix} 1 & \frac{\cos 2\theta}{\sin 2\theta} \\ & 1 \end{pmatrix}, 1\right)$$
$$= \left(\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, -\varepsilon\right),$$

where for simplicity we write $\varepsilon = \operatorname{sgn} \sin 2\theta$. Let

$$\begin{aligned} f(x) &= r^{e_\infty}\left(\begin{pmatrix} 1 & \frac{\cos 2\theta}{\sin 2\theta} \\ & 1 \end{pmatrix}\right)\phi_\infty^0(x) \\ &= e_\infty\left(\frac{\cos 2\theta}{\sin 2\theta}x^2\right)\phi_\infty^0(x) = \exp\bigl(-2\pi x^2(1 - i\cot 2\theta)\bigr). \end{aligned}$$

Then

$$\begin{aligned}
r^{e_\infty}(\widetilde{\kappa}_\theta)\phi^0_\infty(x) &= -\varepsilon\zeta(\theta) r^{e_\infty}\left(\begin{pmatrix} 1 & \frac{\cos 2\theta}{\sin 2\theta} \\ & 1\end{pmatrix}\right) r^{e_\infty}\left(\begin{pmatrix} \frac{-1}{\sin 2\theta} & 0 \\ 0 & -\sin 2\theta\end{pmatrix}\right) \\
&\quad \cdot r^{e_\infty}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) r^{e_\infty}\left(\begin{pmatrix} 1 & \frac{\cos 2\theta}{\sin 2\theta} \\ & 1\end{pmatrix}\right)\phi^0_\infty(x) \\
&= -\varepsilon\zeta(\theta) r^{e_\infty}\left(\begin{pmatrix} 1 & \frac{\cos 2\theta}{\sin 2\theta} \\ & 1\end{pmatrix}\begin{pmatrix} \frac{-1}{\sin 2\theta} & 0 \\ 0 & -\sin 2\theta\end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) f(x) \\
&= -\frac{\varepsilon\zeta(\theta)}{|\sin 2\theta|^{1/2}} e_\infty(x^2\cot 2\theta)\frac{\gamma(e_\infty)^2}{\gamma(e_{\infty,-1/\sin 2\theta})}\widehat{f}\left(-\frac{x}{\sin 2\theta}\right) \\
&= -\frac{\varepsilon\zeta(\theta)}{|\sin 2\theta|^{1/2}} e^{(2+\varepsilon)\pi i/4 + 2\pi i x^2\cot 2\theta}\widehat{f}\left(-\frac{x}{\sin 2\theta}\right) \\
&= \frac{\varepsilon\zeta(\theta)}{|\sin 2\theta|^{1/2}} e^{-\varepsilon\pi i/4 + 2\pi i x^2\cot 2\theta}\widehat{f}\left(-\frac{x}{\sin 2\theta}\right).
\end{aligned}$$

Now recall that if $\varphi_z(x) = e^{-2\pi z x^2}$ for some $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, then

$$\widehat{\varphi}_z(x) = \frac{1}{\sqrt{z}} e^{-2\pi x^2/z}.$$

In our case, we have

$$z = 1 + i\cot 2\theta = \frac{\sin 2\theta - i\cos 2\theta}{\sin 2\theta} = \frac{1}{|\sin 2\theta|} e^{2i\theta - \varepsilon\pi i/2},$$

and, according to our convention,

$$\frac{1}{z} = (\sin 2\theta)^2(1 + i\cot 2\theta), \qquad \sqrt{z} = \frac{\zeta(\theta)}{\sqrt{|\sin 2\theta|}} e^{i\theta - \varepsilon\pi i/4},$$

so

$$\widehat{f}(x) = \zeta(\theta)|\sin 2\theta|^{1/2} e^{\varepsilon\pi i/4 - i\theta} e^{-2\pi x^2(\sin 2\theta)^2(1+i\cot 2\theta)}.$$

Hence

$$\begin{aligned}
r^{e_\infty}(\widetilde{\kappa}_\theta)\phi^0_\infty(x) &= \frac{\zeta(\theta)}{|\sin 2\theta|^{1/2}} e^{-\varepsilon\pi i/4 + 2\pi i x^2\cot 2\theta}\widehat{f}\left(-\frac{x}{\sin 2\theta}\right) \\
&= e^{-i\theta} e^{-2\pi x^2} = e^{-i\theta}\varphi^0_\infty(x). \quad \blacksquare
\end{aligned}$$

4.3. The nonarchimedean Weil representation. I. We denote the characteristic function of $\mathbb{Z}_p$ by ϕ°_p.

LEMMA 4.6. *If $p > 2$ and $a \in \mathbb{Z}_p^\times$, then $r^{e_{p,a}}(\mathrm{SL}(2,\mathbb{Z}_p))$ fixes ϕ°_p.*

Proof. By Lemma 3.3, it suffices to check the assertion on the elements of the set (3.4), which is straightforward. $\blacksquare$

Take a nontrivial character $\mu : \mathbb{Z}_p^\times \to \mathbb{C}^\times$, and define

$$\phi_p^\mu = \mu \cdot \mathbb{1}_{\mathbb{Z}_p^\times} : \mathbb{Q}_p^\times \to \mathbb{C}.$$

PROPOSITION 4.7. *Let $p > 2$, and let*

$$f = \min\{m \geq 1 : \mu(1 + p^m \mathbb{Z}_p) = 1\}.$$

Then

$$r^{e_p}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\phi_p^\mu(y) = \frac{\mu^{-1}(2)\tau(\mu)}{p^f}\phi_p^{\mu^{-1}}(yp^f),$$

where

$$\tau(\mu) = \sum_{\substack{i=1 \\ (p,i)=1}}^{p^f-1} \mu(i) e_p\left(\frac{i}{p^f}\right).$$

Proof. One may decompose ϕ_p^μ as a linear combination of characteristic functions:

$$\phi_p^\mu = \sum_{\substack{i=1 \\ (p,i)=1}}^{p^f-1} \mu(i) \mathbb{1}_{i+p^f\mathbb{Z}_p}.$$

Then

$$\begin{aligned} r^{e_p}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\phi_p^\mu(y) &= \frac{1}{p^f} \sum_{\substack{i=1 \\ (p,i)=1}}^{p^f-1} \mu(i) e_p(2iy) \mathbb{1}_{p^{-f}\mathbb{Z}_p}(y) \\ &= \frac{\mu^{-1}(2)}{p^f} \sum_{\substack{i=1 \\ (p,i)=1}}^{p^f-1} \mu(i) e_p(iy) \mathbb{1}_{p^{-f}\mathbb{Z}_p}(y), \end{aligned}$$

where we have applied our previous results that

$$\gamma(e_p) = 1, \qquad \alpha(e_p) = 1.$$

Obviously the right-hand side vanishes if $y \notin p^{-f}\mathbb{Z}_p$. Moreover, if $y \notin p^{-f}\mathbb{Z}_p^\times$, then $i \mapsto e_p(iy)$ is constant on $1 + p^{f-1}\mathbb{Z}$, which causes the sum against μ to vanish. Thus the support of $r^{e_p}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).\phi_p^\mu$ is precisely $p^{-f}\mathbb{Z}_p^\times$. Further, a change of variables in i shows that

$$r^{e_p}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\phi_p^\mu(yy') = \mu(y')^{-1} r^{e_p}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\phi_p^\mu(y).$$

It follows that the function

$$y \mapsto r^{e_p}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\phi_p^\mu\left(\frac{y}{p^f}\right)$$

is a scalar multiple of $\phi_p^{\mu^{-1}}$, and the scalar is easily seen to be the value at $\mu^{-1}(2)\tau(\mu)/p^f$. ∎

Corollary 4.8. *With notation as before,*

$$r^{e_p}\left(\begin{pmatrix} 0 & p^{-f} \\ -p^f & 0 \end{pmatrix}\right).\phi_p^{\mu} = \left(\frac{-1}{p}\right)^f \frac{\tau(\mu)\mu^{-1}(2)}{p^{f/2}\gamma(e_{p,p^{-f}})}\phi_p^{\mu^{-1}}.$$

Proof. By definition,

$$\beta_p\left(\begin{pmatrix} p^{-f} & 0 \\ 0 & p^f \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)_p = (p^f,-1)_p(p^f,-p^f)_p = (p^f,p^f)_p = \left(\frac{-1}{p}\right)^f,$$

so in $\widetilde{\mathrm{SL}}(2,\mathbb{Q}_p)$ we have

$$\left(\begin{pmatrix} p^{-f} & 0 \\ 0 & p^f \end{pmatrix},1\right)\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},1\right) = \left(\begin{pmatrix} 0 & p^{-f} \\ -p^f & 0 \end{pmatrix},\left(\frac{-1}{p}\right)^f\right).$$

Hence it follows immediately from Proposition 4.7 and the definition of r^{e_p} on diagonal elements that

$$\begin{aligned} r^{e_p}\left(\begin{pmatrix} 0 & p^{-f} \\ -p^f & 0 \end{pmatrix}\right).\phi_p^{\mu} &= \left(\frac{-1}{p}\right)^f r^{e_p}\left(\begin{pmatrix} p^{-f} & 0 \\ 0 & p^f \end{pmatrix}\right) r^{e_p}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\phi_p^{\mu} \\ &= \left(\frac{-1}{p}\right)^f \frac{\tau(\mu)\mu^{-1}(2)}{p^{f/2}\gamma(e_{p,p^{-f}})}\phi_p^{\mu^{-1}}. \end{aligned}$$ ∎

Note that μ factors through $(\mathbb{Z}_p/p^f\mathbb{Z}_p)^\times \cong (\mathbb{Z}/p^f\mathbb{Z})^\times$, and that $\tau(\mu)$ is the Gauss sum of the (primitive) Dirichlet character mod p^f which it induces. Therefore, the quotient $\frac{\mu^{-1}(2)\tau(\mu)}{\gamma(e_{p,p^{-f}})p^{f/2}}$ is a root of unity.

4.4. The nonarchimedean Weil representation. II. Let p be a prime, $M \geq 1$, and

$$K_p^{(M)} := \begin{pmatrix} 1 & \\ & M \end{pmatrix} \mathrm{SL}(2,\mathbb{Z}_p) \begin{pmatrix} 1 & \\ & M^{-1} \end{pmatrix}.$$

Note that $K_p^{(M)}$ depends only on the p-adic valuation of M. By Lemma 3.3, the group $K_p^{(M)}$ is generated by

$$\left\{\begin{pmatrix} & 1/M \\ -M & \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1 & x/M \\ & 1 \end{pmatrix} : x \in \mathbb{Z}_p\right\}.$$

In addition, let $\widetilde{K}_p^{(M)}$ denote the preimage of $K_p^{(M)}$ in $\widetilde{\mathrm{SL}}(2,\mathbb{Q}_p)$.

In this section, for each prime p and for suitable values of M, we study a finite-dimensional subspace of $\mathcal{S}(\mathbb{Q}_p)$ which is invariant under the action of $\widetilde{K}_p^{(M)}$. Specifically, when $p = 2$ we consider $\widetilde{K}_2^{(4)}$ and $\widetilde{K}_2^{(8)}$, and when

$p > 2$ we consider $\widetilde{K}_p^{(p)}$. Our discussions and conclusions change dramatically according to whether $p \geq 3$ or $p = 2$. Hence we will consider these two cases separately.

In order to work explicitly we introduce some notation from elementary linear algebra. If V is a complex vector space of finite dimension n, $B = (\beta_1, \ldots, \beta_n)$ is an ordered basis for V, and v is a vector in V, then we write $[v]_B$ for the coordinate vector of v relative to B. Thus

$$[v]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Leftrightarrow \sum_{i=1}^{n} x_i \beta_i = v.$$

By identifying B with the row vector $[\beta_1 \ \ldots \ \beta_n]$, we may write this succinctly as $B \cdot [v]_B = v$. Similarly, if $T : V \to V$ is a linear operator, then $[T]_B \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ is the matrix satisfying

$$[T]_B [v]_B = [Tv]_B \qquad (v \in V).$$

4.4.1. *An injection.* Take a prime p and a positive integer m. Using the canonical isomorphism $\mathbb{Z}_p / p^m \mathbb{Z}_p \cong \mathbb{Z}/p^m\mathbb{Z}$, we may regard every function ϕ_0 over $\mathbb{Z}/p^m\mathbb{Z}$ as a function over $\mathbb{Z}_p$ which is constant on $p^m\mathbb{Z}_p$-cosets. We may then extend ϕ_0 to a function over $\mathbb{Q}_p$ by setting it equal to zero on $\mathbb{Q}_p \smallsetminus \mathbb{Z}_p$. This defines an injection

$$\iota_{p^m} : \mathrm{Map}(\mathbb{Z}/p^m\mathbb{Z}, \mathbb{C}) \hookrightarrow \mathcal{S}(\mathbb{Q}_p) \tag{4.9}$$

with image

$$\{\phi \in \mathcal{S}(\mathbb{Q}_p) : \mathrm{supp}(\phi) \subset \mathbb{Z}_p,\ \phi(x + p^m\mathbb{Z}_p) = \phi(x)\}.$$

4.4.2. *The case $p \geq 3$.* In this section, we let $p \geq 3$ and study the action of the local Weil representations of $\widetilde{K}_p^{(p)}$ on the p-dimensional complex vector space

$$V_p = \{\phi \in \mathcal{S}(\mathbb{Q}_p) : \mathrm{supp}(\phi) \subset \mathbb{Z}_p,\ \phi(x + p\mathbb{Z}_p) = \phi(x)\}, \tag{4.10}$$

which is the image of the map ι_p defined as in (4.9). Obviously, we have a natural decomposition into the subspaces of odd and even functions

$$V_p = V_p^+ \oplus V_p^-,$$

where

$$V_p^+ = \{\phi \in V_p : \phi(-x) = \phi(x)\}, \qquad V_p^- = \{\phi \in V_p : \phi(-x) = -\phi(x)\}.$$

To construct bases for V_p^+ and V_p^-, we fix $g_p \in \mathbb{Z}$ whose image in $(\mathbb{Z}/p\mathbb{Z})^\times$ generates this cyclic group. For every $1 \leq j \leq p-1$ define the character

$$\psi_{p,j} : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times, \qquad g_p \mapsto e\left(\frac{j}{p-1}\right).$$

We then let $\phi_p^{\psi_{p,j}} = \iota_p(\psi_{p,j}) \in \mathcal{S}(\mathbb{Q}_p)$. In particular,

$$\phi_p^{\psi_{p,p-1}} = \mathbb{1}_{\mathbb{Z}_p^\times} = \mathbb{1}_{\mathbb{Z}_p} - \mathbb{1}_{p\mathbb{Z}_p}.$$

Then V_p^+ has the ordered basis

$$B_p^+ = (\phi_p^{\psi_{p,2}}, \phi_p^{\psi_{p,4}}, \dots, \phi_p^{\psi_{p,p-1}}, \mathbb{1}_{p\mathbb{Z}_p})$$

of cardinality $(p+1)/2$, and V_p^- has the ordered basis

$$B_p^- = (\phi_p^{\psi_{p,1}}, \phi_p^{\psi_{p,3}}, \dots, \phi_p^{\psi_{p,p-2}})$$

of cardinality $(p-1)/2$.

Theorem 4.11. *The action of the local Weil representation $r_p^{e_p}$ of the group $\widetilde{K}_p^{(p)}$ preserves the vector spaces V_p^+ and V_p^- respectively. More precisely, write*

$$C_p^+ = \begin{pmatrix} \psi_{p,2}(1) & \psi_{p,4}(1) & \cdots & \psi_{p,p-1}(1) & 0 \\ \psi_{p,2}(2) & \psi_{p,4}(2) & \cdots & \psi_{p,p-1}(2) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_{p,2}\left(\frac{p-1}{2}\right) & \psi_{p,4}\left(\frac{p-1}{2}\right) & \cdots & \psi_{p,p-1}\left(\frac{p-1}{2}\right) & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{4.12}$$

$$C_p^- = \begin{pmatrix} \psi_{p,1}(1) & \psi_{p,3}(1) & \cdots & \psi_{p,p-2}(1) \\ \psi_{p,1}(2) & \psi_{p,3}(2) & \cdots & \psi_{p,p-2}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{p,1}\left(\frac{p-1}{2}\right) & \psi_{p,3}\left(\frac{p-1}{2}\right) & \cdots & \psi_{p,p-2}\left(\frac{p-1}{2}\right) \end{pmatrix}, \tag{4.13}$$

and for every $\gamma \in K_p^{(p)}$ write

$$\varrho_p^+(\gamma) = [r_p^{e_p}(\gamma)|_{V_p^+}]_{B_p^+}, \quad \varrho_p^-(\gamma) = [r_p^{e_p}(\gamma)|_{V_p^-}]_{B_p^-}.$$

Let $a \in \mathbb{Z}_p^\times$ and $b \in \mathbb{Z}_p$. Then

$$\varrho_p^+\begin{pmatrix} 1 & b/p \\ & 1 \end{pmatrix} = (C_p^+)^{-1} \begin{pmatrix} e_p\left(\frac{b}{p}\right) & & & & \\ & e_p\left(\frac{4b}{p}\right) & & & \\ & & \ddots & & \\ & & & e_p\left(\left(\frac{p-1}{2}\right)^2 \frac{b}{p}\right) & \\ & & & & 1 \end{pmatrix} C_p^+, \tag{4.14}$$

$$\varrho_p^-\begin{pmatrix}1 & b/p\\ & 1\end{pmatrix} = (C_p^-)^{-1}\begin{pmatrix}e_p\big(\frac{b}{p}\big) & & & \\ & e_p\big(\frac{4b}{p}\big) & & \\ & & \ddots & \\ & & & e_p\Big(\big(\frac{p-1}{2}\big)^2\frac{b}{p}\Big)\end{pmatrix}C_p^-, \tag{4.15}$$

$$\varrho_p^+\begin{pmatrix}a & \\ & 1/a\end{pmatrix} = \begin{pmatrix}\psi_{p,2}(a) & & & & \\ & \psi_{p,4}(a) & & & \\ & & \ddots & & \\ & & & \psi_{p,p-1}(a) & \\ & & & & 1\end{pmatrix}, \tag{4.16}$$

$$\varrho_p^-\begin{pmatrix}a & \\ & 1/a\end{pmatrix} = \begin{pmatrix}\psi_{p,1}(a) & & & \\ & \psi_{p,3}(a) & & \\ & & \ddots & \\ & & & \psi_{p,p-2}(a)\end{pmatrix}, \tag{4.17}$$

$$\varrho_p^+\begin{pmatrix} & 1/p\\ -p & \end{pmatrix} = \frac{1}{\varepsilon_p\sqrt{p}}\begin{pmatrix} & & \tau(\psi_{p,p-3})\psi_{p,p-3}^{-1}(2) & & \\ & \iddots & & & \\ \tau(\psi_{p,2})\psi_{p,2}^{-1}(2) & & & & \\ & & & -1 & 1\\ & & & p-1 & 1\end{pmatrix}, \tag{4.18}$$

$$\varrho_p^-\begin{pmatrix} & 1/p\\ -p & \end{pmatrix} = \frac{1}{\varepsilon_p\sqrt{p}}\begin{pmatrix} & & \tau(\psi_{p,p-2})\psi_{p,p-2}^{-1}(2)\\ & \iddots & \\ \tau(\psi_{p,1})\psi_{p,1}^{-1}(2) & & \end{pmatrix}. \tag{4.19}$$

REMARK 4.20. Let

$$B_p = (\phi_p^{\psi_{p,1}}, \phi_p^{\psi_{p,3}}, \dots, \phi_p^{\psi_{p,p-2}}, \phi_p^{\psi_{p,2}}, \phi_p^{\psi_{p,4}}, \dots, \phi_p^{\psi_{p,p-1}}, \mathbb{1}_{p\mathbb{Z}_p}), \tag{4.21}$$

which is a natural ordering on the union $B_p^- \cup B_p^+$, and hence an ordered basis for V_p. Then for every $\gamma \in K_p^{(p)}$ we write

$$\varrho_p(\gamma) = \begin{pmatrix}\varrho_p^-(\gamma) & \\ & \varrho_p^+(\gamma)\end{pmatrix} = [r_p^{e_p}(\gamma)|_{V_p}]_{B_p}.$$

Proof of Theorem 4.11. To prove the identities (4.14) and (4.15), we may construct alternative bases for $V_p^{\pm}$. For every $1 \le i \le (p-1)/2$, let

$$\mathbb{1}^{\pm}_{\boxed{i}} : \mathbb{Q}_p \to \mathbb{C}, \qquad x \mapsto \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p \text{ and } x \equiv i \pmod{p}, \\ \pm 1 & \text{if } x \in \mathbb{Z}_p \text{ and } x \equiv -i \pmod{p}, \\ 0 & \text{otherwise.} \end{cases}$$

Also, we write

$$\mathbb{1}_{\boxed{0}} = \mathbb{1}_{p\mathbb{Z}_p} : \mathbb{Q}_p \to \mathbb{C}, \qquad x \mapsto \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p \text{ and } x \equiv 0 \pmod{p}, \\ 0 & \text{otherwise.} \end{cases}$$

Then V_p^+ and V_p^- have the bases

$$A_p^+ = \{\mathbb{1}^+_{\boxed{1}}, \mathbb{1}^+_{\boxed{2}}, \dots, \mathbb{1}^+_{\boxed{(p-1)/2}}, \mathbb{1}_{\boxed{0}}\} \quad \text{and} \quad A_p^- = \{\mathbb{1}^-_{\boxed{1}}, \mathbb{1}^-_{\boxed{2}}, \dots, \mathbb{1}^-_{\boxed{(p-1)/2}}\}$$

respectively, and C_p^+ and C_p^- are the corresponding change-of-basis matrices, that is,

$$C_p^+[v]_{B_p^+} = [v]_{A_p^+}, \qquad C_p^-[v]_{B_p^-} = [v]_{A_p^-}.$$

Now (4.14) and (4.15) follow immediately from the definitions of $r_p^{e_p}$ and $\mathbb{1}_{\boxed{0}}$, and $\mathbb{1}^{\pm}_{\boxed{i}}$.

The identities (4.16) and (4.17) follow immediately from the definitions of $r_p^{e_p}$ and the elements of B_p, along with the fact that for $a \in \mathbb{Z}_p^\times$ we have

$$|a|_p = \gamma(e_p) = \gamma(e_{p,a}) = 1.$$

To prove (4.18) and (4.19), note that $\psi_{p,j}^{-1} = \psi_{p,p-1-j}$ for $1 \le j < p-1$. Therefore, by Corollary 4.8 we have

$$r_p^{e_p}\begin{pmatrix} & 1/p \\ -p & \end{pmatrix}\phi_p^{\psi_{p,j}} = \left(\frac{-1}{p}\right)\frac{\tau(\psi_{p,j})\psi_{p,j}^{-1}(2)}{p^{1/2}\gamma(e_{p,1/p})}\phi_p^{\psi_{p,p-1-j}}.$$

Further, direct calculation shows that

$$\gamma(e_{p,1/p}) = \varepsilon_p\left(\frac{-1}{p}\right)$$

and that

$$r_p^{e_p}\begin{pmatrix} & 1/p \\ -p & \end{pmatrix}\phi_p^\circ = \left(\frac{-1}{p}\right)\frac{\sqrt{p}}{\gamma(e_{p,1/p})}\mathbb{1}_{\boxed{0}} = \frac{\sqrt{p}}{\varepsilon_p}\mathbb{1}_{\boxed{0}},$$

$$r_p^{e_p}\begin{pmatrix} & 1/p \\ -p & \end{pmatrix}\mathbb{1}_{\boxed{0}} = \left(\frac{-1}{p}\right)\frac{1}{\sqrt{p}\,\gamma(e_{p,1/p})}\phi_p^\circ = \frac{1}{\varepsilon_p\sqrt{p}}\phi_p^\circ.$$

Then (4.18) and (4.19) follow. The rest of the conclusion follows from these explicit results, since the elements studied generate $K_p^{(p)}$. ∎

EXAMPLE 4.22. Consider the case $p = 3$. Then $\psi_{3,1}$ is the unique nontrivial character modulo 3 and is odd, so ϱ_3^- is a 1×1 matrix and so can be identified as a scalar function with

$$\left\{\begin{aligned} &\varrho_3^-\begin{pmatrix}1 & b/3\\ & 1\end{pmatrix} = e_3\left(\frac{b}{3}\right) \quad (b\in\mathbb{Z}_3),\\ &\varrho_3^-\begin{pmatrix}a & \\ & 1/a\end{pmatrix} = \psi_{3,1}(a) \quad (a\in\mathbb{Z}_3^\times),\\ &\varrho_3^-\begin{pmatrix} & 1/3\\ -3 & \end{pmatrix} = 1.\end{aligned}\right. \tag{4.23}$$

Example 4.24. Let $p=5$ and

$$\cos_5(2\pi x) = \frac{1}{2}\big(e_5(x)+e_5(-x)\big), \qquad \sin_5(2\pi x) = \frac{1}{2i}\big(e_5(x)-e_5(-x)\big),$$

where $e_p : \mathbb{Q}_p \to \mathbb{C}^\times$ is the usual additive character as defined in (4.2). Take $g_5 = 2$ as a generator of $(\mathbb{Z}/5\mathbb{Z})^\times$. Then $\psi_{5,2}$ is the unique primitive Dirichlet character modulo 5. By Theorem 4.11, we have

$$\varrho_5^+\begin{pmatrix}1 & b/5\\ & 1\end{pmatrix} = \begin{pmatrix}\cos_5(2\pi b/5) & i\sin_5(2\pi b/5) & 0\\ i\sin_5(2\pi b/5) & \cos_5(2\pi b/5) & 0\\ 0 & 0 & 1\end{pmatrix} \qquad (b\in\mathbb{Z}_5),$$

$$\varrho_5^+\begin{pmatrix}a & \\ & 1/a\end{pmatrix} = \begin{pmatrix}\chi_5(a) & & \\ & 1 & \\ & & 1\end{pmatrix} \qquad (a\in\mathbb{Z}_5^\times),$$

$$\varrho_5^+\begin{pmatrix} & 1/5\\ -5 & \end{pmatrix} = \begin{pmatrix}-1 & & \\ & -1/\sqrt{5} & 1/\sqrt{5}\\ & 4/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}.$$

4.4.3. *The case $p=2$.* In this section, we study the action of the local Weil representation of $\mathrm{SL}_2(\mathbb{Q}_2)$ on three finite-dimensional subspaces of $\mathcal{S}(\mathbb{Q}_2)$. We show that each is fixed by a conjugate of the preimage of $\mathrm{SL}_2(\mathbb{Z}_p)$. First consider the two-dimensional space

$$V_2 = V_2^+ = \{\phi : \mathbb{Q}_2 \to \mathbb{C} : \mathrm{supp}(\phi)\subset\mathbb{Z}_2,\ \phi(x+2y)=\phi(x)\ \forall x,y\in\mathbb{Z}_2\}, \tag{4.25}$$

with ordered basis $B_2 = B_2^+ = (\mathbb{1}_{\mathbb{Z}_2^\times}, \mathbb{1}_{2\mathbb{Z}_2})$. For consistency, we also define V_2^- to be the set of odd elements of V_2, i.e., the zero subspace $\{0\}$.

Theorem 4.26. *The action of the local Weil representation $r_2^{e_2}$ of $K_2^{(4)}$ preserves the vector space V_2^+. More precisely, for every $\gamma\in K_2^{(4)}$ write*

$$\varrho_2^+(\gamma) = [r_2^{e_2}(\gamma)|_{V_2^+}]_{B_2^+}.$$

Then

$$\varrho_2^+\left(\begin{pmatrix}1 & b/4\\ & 1\end{pmatrix}\right)=\begin{pmatrix}e_2(b/4) & \\ & 1\end{pmatrix}\qquad (b\in\mathbb{Z}_2),$$

$$\varrho_2^+\left(\begin{pmatrix}a & \\ & 1/a\end{pmatrix}\right)=-i\varepsilon_{-a}I_2\qquad (a\in\mathbb{Z}_2^\times),$$

$$\varrho_2^+\left(\begin{pmatrix} & 1/4\\ -4 & \end{pmatrix}\right)=\frac{1-i}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}.$$

Proof. The proof is analogous to that of Theorem 4.11, so we omit the details. ■

Next, consider the four-dimensional complex vector space

$$V_4=\{\phi\in\mathcal{S}(\mathbb{Q}_2):\operatorname{supp}(\phi)\subset\mathbb{Z}_2,\ \phi(x+4\mathbb{Z}_2)=\phi(x)\}=V_4^+\oplus V_4^-, \tag{4.27}$$

where

$$V_4^+=\{\phi\in V_4:\phi(-x)=\phi(x)\},\qquad V_4^-=\{\phi\in V_4:\phi(-x)=-\phi(x)\}$$

with ordered bases

$$B_4^+=(\phi_2^{\psi_{2,2}},\mathbb{1}_{2+4\mathbb{Z}_2},\mathbb{1}_{4+4\mathbb{Z}_2}),\qquad B_4^-=(\phi_2^{\psi_{2,1}})$$

respectively, where

$$\psi_{2,j}:(\mathbb{Z}/4\mathbb{Z})^\times\to\{\pm1\},\quad 3\mapsto(-1)^j\qquad (j=1,2),$$

and $\phi_2^{\psi_{2,j}}=\iota_4(\psi_{2,j})$ is the image under the injection (4.9) for $p=m=2$.

Theorem 4.28. *The action of the local Weil representation $r_2^{e_2}$ of $K_2^{(8)}$ preserves the vector spaces V_4^+ and V_4^- respectively. More precisely, for every $\gamma\in K_2^{(8)}$ write*

$$\varrho_4^+(\gamma)=[r_2^{e_2}(\gamma)|_{V_4^+}]_{B_4^+},\qquad \varrho_4^-(\gamma)=[r_2^{e_2}(\gamma)|_{V_4^-}]_{B_4^-}.$$

Then

$$\varrho_4^+\left(\begin{pmatrix}1 & b/8\\ & 1\end{pmatrix}\right)=\begin{pmatrix}e_2(b/8) & & \\ & e_2(b/2) & \\ & & 1\end{pmatrix}\qquad (b\in\mathbb{Z}_2),$$

$$\varrho_4^+\left(\begin{pmatrix}a & \\ & 1/a\end{pmatrix}\right)=-i\varepsilon_{-a}I_3\qquad (a\in\mathbb{Z}_2^\times),$$

$$\varrho_4^+\left(\begin{pmatrix} & 1/8\\ -8 & \end{pmatrix}\right)=\frac{1-i}{2\sqrt{2}}\begin{pmatrix}0 & -1 & 1\\ -2 & 1 & 1\\ 2 & 1 & 1\end{pmatrix},$$

$$\tag{4.29} \begin{cases} \varrho_4^-\left(\begin{pmatrix} 1 & b/8 \\ & 1 \end{pmatrix}\right) = e_2(b/8) \quad (b \in \mathbb{Z}_2), \\ \varrho_4^-\left(\begin{pmatrix} a & \\ & 1/a \end{pmatrix}\right) = -i\varepsilon_{-a}\psi_{2,1}(a) \quad (a \in \mathbb{Z}_2^\times), \\ \varrho_4^-\left(\begin{pmatrix} & 1/8 \\ -8 & \end{pmatrix}\right) = -\frac{1+i}{\sqrt{2}}. \end{cases}$$

REMARK 4.30. Let

$$\tag{4.31} B_4 = (\phi^{\psi_{2,1}}, \phi^{\psi_{2,2}}, \mathbb{1}_{2+4\mathbb{Z}_2}, \mathbb{1}_{4\mathbb{Z}_2}),$$

a natural ordering of $B_4^- \cup B_4^+$ and hence an ordered basis for V_4. Then for every $\gamma \in \widetilde{K}_2^{(8)}$ we write

$$\varrho_4(\gamma) = \begin{pmatrix} \varrho_4^-(\gamma) & \\ & \varrho_4^+(\gamma) \end{pmatrix} = [r_2^{e_2}(\gamma)|_{V_4}]_{B_4}.$$

Proof of Theorm 4.28. The proof is analogous to that of Theorem 4.11, so we omit the details. ∎

5. Global metaplectic group and Weil representation

5.1. Global metaplectic group. If $g = \{g_v\}_v$, $h = \{h_v\}_v \in \mathrm{SL}(2,\mathbb{A})$, then $\beta_v(g_v, h_v) = 1$ for all but finitely many v (see [Ge76, Proposition 2.8]). Set

$$\beta(g,h) = \prod_v \beta_v(g_v, h_v).$$

Here v runs over the places on $\mathbb{Q}$. Then $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ is defined as $\mathrm{SL}(2,\mathbb{A}) \times \{\pm 1\}$ equipped with the product

$$(g_1, \zeta_1)(g_2, \zeta_2) := \big(g_1 g_2, \beta(g_1, g_2)\zeta_1\zeta_2\big),$$

where $g_1, g_2 \in \mathrm{SL}(2,\mathbb{A})$ and $\zeta_1, \zeta_2 \in \{\pm 1\}$. For each v, we have the embedding $i_v : \mathrm{SL}(2,\mathbb{Q}_v) \hookrightarrow \mathrm{SL}(2,\mathbb{A})$. The definition is that, for $g_v \in \mathrm{SL}(2,\mathbb{Q}_v)$ and w a place of $\mathbb{Q}$, the component $i_v(g_v)_w$ of $i_v(g_v)$ at w is g_v if $v = w$, and the identity matrix I_2 otherwise. Now, for all w, the cocycle β_w is trivial on $\{I_2\} \times \mathrm{SL}(2,\mathbb{Q}_v)$ and $\mathrm{SL}(2,\mathbb{Q}_v) \times \{I_2\}$, which implies that the restriction of the global cocycle β to the image of $\mathrm{SL}(2,\mathbb{Q}_v) \times \mathrm{SL}(2,\mathbb{Q}_v)$ in $\mathrm{SL}(2,\mathbb{A})$ is precisely the local cocycle β_v. It follows that i_v extends to an embedding $\widetilde{i}_v : \widetilde{\mathrm{SL}}(2,\mathbb{Q}_v) \hookrightarrow \widetilde{\mathrm{SL}}(2,\mathbb{A})$ defined by

$$\widetilde{i}_v(g,\zeta) = (i_v(g), \zeta) \quad (g \in \mathrm{SL}(2,\mathbb{Q}_v),\ \zeta \in \{\pm 1\}).$$

We shall also make use of the embedding

$$\widetilde{i}_{\mathrm{diag}} : \mathrm{SL}(2,\mathbb{Q}) \hookrightarrow \widetilde{\mathrm{SL}}(2,\mathbb{A}), \qquad \gamma \mapsto (i_{\mathrm{diag}}(\gamma), s_{\mathbb{A}}(\gamma))$$

as described in [Ge76, p. 23], where

$$s_{\mathbb{A}} = \prod_v s_v,$$

and $i_{\mathrm{diag}} : \mathrm{SL}(2,\mathbb{Q}) \hookrightarrow \mathrm{SL}(2,\mathbb{A})$ is the diagonal embedding. We also let

$$\widetilde{i}_{\mathrm{f}}(\gamma) = \widetilde{i}_\infty(\gamma^{-1},1)\widetilde{i}_{\mathrm{diag}}(\gamma) \in \widetilde{\mathrm{SL}}(2,\mathbb{A}) \quad (\gamma \in \mathrm{SL}(2,\mathbb{Q})).$$

Observe that $\widetilde{i}_{\mathrm{f}}$ is *not* a homomorphism. Rather, it satisfies

$$\widetilde{i}_{\mathrm{f}}(\gamma_1)\widetilde{i}_{\mathrm{f}}(\gamma_2) = \widetilde{i}_{\mathrm{f}}(\gamma_1\gamma_2)\cdot(I_2,\beta_\infty(\gamma_2^{-1},\gamma_1^{-1})) \quad (\gamma_1,\gamma_2 \in \mathrm{SL}(2,\mathbb{Q})). \tag{5.1}$$

Indeed, $\widetilde{i}_{\mathrm{f}}(\gamma_1)$ commutes with $\widetilde{i}_\infty(\gamma_2)$ since either one or the other of them has the identity matrix at each place. Hence

$$\begin{aligned}\widetilde{i}_{\mathrm{f}}(\gamma_1)\widetilde{i}_{\mathrm{f}}(\gamma_2) &= \widetilde{i}_{\mathrm{f}}(\gamma_1)\widetilde{i}_\infty(\gamma_2^{-1},1)\widetilde{i}_{\mathrm{diag}}(\gamma_2) = \widetilde{i}_\infty(\gamma_2^{-1},1)\widetilde{i}_{\mathrm{f}}(\gamma_1)\widetilde{i}_{\mathrm{diag}}(\gamma_2)\\ &= \widetilde{i}_\infty(\gamma_2^{-1},1)\widetilde{i}_\infty(\gamma_1^{-1},1)\widetilde{i}_{\mathrm{diag}}(\gamma_1)\widetilde{i}_{\mathrm{diag}}(\gamma_2)\\ &= \widetilde{i}_\infty(\gamma_2^{-1}\gamma_1^{-1},\beta_\infty(\gamma_2^{-1},\gamma_1^{-1}))\widetilde{i}_{\mathrm{diag}}(\gamma_1\gamma_2)\\ &= \widetilde{i}_\infty(\gamma_2^{-1}\gamma_1^{-1},1)\widetilde{i}_{\mathrm{diag}}(\gamma_1\gamma_2)\big(I_2,\beta_\infty(\gamma_2^{-1},\gamma_1^{-1})\big).\end{aligned}$$

Notice that

$$\widetilde{i}_{\mathrm{f}}(\gamma) = \big(i_{\mathrm{f}}(\gamma),\beta_\infty(\gamma^{-1},\gamma)s_{\mathbb{A}}(\gamma)\big) \quad (\gamma \in \mathrm{SL}(2,\mathbb{Q})),$$

where $i_{\mathrm{f}} : \mathrm{SL}(2,\mathbb{Q}) \to \mathrm{SL}(2,\mathbb{A})$ is the diagonal embedding at the finite places only. By [Ge76, Proposition 2.8], the restriction of $\widetilde{i}_p$ to $\mathrm{SL}(2,\mathbb{Z}_p)$ is a homomorphism for $p > 2$ (cf. [Ge76, p. 19]). It follows that the inclusion

$$i_S : \prod_{v\in S}\mathrm{SL}(2,\mathbb{Q}_v) \times \prod_{p\notin S}\mathrm{SL}(2,\mathbb{Z}_p) \hookrightarrow \mathrm{SL}(2,\mathbb{A})$$

extends to a homomorphism

$$\widetilde{i}_S : \prod_{v\in S}\widetilde{\mathrm{SL}}(2,\mathbb{Q}_v) \times \prod_{p\notin S}\mathrm{SL}(2,\mathbb{Z}_p) \hookrightarrow \widetilde{\mathrm{SL}}(2,\mathbb{A})$$

for any finite set S of places of $\mathbb{Q}$ which contains $\{2,\infty\}$. The kernel of this homomorphism is

$$\ker\widetilde{i}_S = \Big\{(I_2,\varepsilon_v)_{v\in S} \times (I_2)_{p\notin S} : \prod_{v\in S}\varepsilon_v = 1\Big\}.$$

Moreover,

$$\widetilde{\mathrm{SL}}(2,\mathbb{A}) = \bigcup_S \operatorname{im}\widetilde{i}_S,$$

with the union ranging over finite sets S of places of $\mathbb{Q}$ which contain $\{2,\infty\}$.

5.2. Global Weil representation. The adelic Bruhat–Schwartz space $\mathcal{S}(\mathbb{A})$ consists of all finite linear combinations of functions $\prod_v \varphi_v$, where φ_v is in $\mathcal{S}(\mathbb{Q}_v)$ for all v, and $\varphi_p = \phi_p^\circ$ is the characteristic function of $\mathbb{Z}_p$ for all but finitely many primes p.

For any finite set S of places of $\mathbb{Q}$, the injection

$$\bigotimes_{v\in S}\varphi_v \mapsto \bigotimes_{v\in S}\varphi_v \otimes \bigotimes_{p\notin S}\phi_p^\circ$$

sends $\bigotimes_{v\in S}\mathcal{S}(\mathbb{Q}_v)$ to a subspace of $\mathcal{S}(\mathbb{A})$. The action of $\widetilde{\mathrm{SL}}(2,\mathbb{Q}_v)$ on $\mathcal{S}(\mathbb{Q}_v)$ induces an action on $\mathcal{S}(\mathbb{A})$ for all v. Moreover, the action of $\{I_2\}\times\{\pm 1\}\subset \widetilde{\mathrm{SL}}(2,\mathbb{Q}_v)$ is the same (scalar multiplication) for all v. By Lemma 4.6, $\mathrm{SL}(2,\mathbb{Z}_p)$ fixes ϕ_p° for all but finitely many p. To be precise, if

$$\psi(\{x_v\}) = \prod_v e_v(ax_v)$$

for some $a\in\mathbb{Q}^\times$, then $\mathrm{SL}(2,\mathbb{Z}_p)$ fixes ϕ_p° for all $p>2$ such that $a\in\mathbb{Z}_p^\times$.

Take S a finite set of places containing ∞ and all primes p such that $\mathrm{SL}(2,\mathbb{Z}_p)$ does not fix ϕ_p°, and let $\widetilde{\mathrm{SL}}(2,\mathbb{A})_S$ denote the subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ consisting of elements (g,ζ) with $g=(g_v)\in\mathrm{SL}(2,\mathbb{A})$ and $\zeta\in\{\pm1\}$ such that $g_p\in\mathrm{SL}(2,\mathbb{Z}_p)$ for $p\notin S$. Notice that $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ is the union of the subgroups $\widetilde{\mathrm{SL}}(2,\mathbb{A})_S$ thus defined. Further, the formula

$$\begin{aligned}\big[r_S^\psi\big(\widetilde{i}_S((g_v,\zeta_v)_{v\in S}\times(k_p,1)_{p\notin S})\big)\big].\Big[\bigotimes_{v\in S}\varphi_v\otimes\bigotimes_{p\notin S}\phi_p^\circ\Big]&\\ =\bigotimes_v[r^{\psi_v}(g_v,\zeta_v)].\varphi_v\otimes\bigotimes_{p\notin S}\phi_p^\circ&\end{aligned}$$

gives a well-defined action of $\widetilde{\mathrm{SL}}(2,\mathbb{A})_S$ on $\mathcal{S}(\mathbb{A})$. Taken together, these formulae give a well-defined action r^ψ of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ on $\mathcal{S}(\mathbb{A})$. In particular, let $\gamma\in\mathrm{SL}(2,\mathbb{Q})$ and $\phi=\prod_v\phi_v\in\mathcal{S}(\mathbb{A})$. Choose a finite set S of places including ∞ such that $\phi_p=\phi_p^\circ$ and $\gamma\in\mathrm{SL}(2,\mathbb{Z}_p)$ for every $p\notin S$. Then we have the decomposition

$$r^e(\widetilde{i}_{\mathrm{f}}(\gamma)).\phi = s_{\mathbb{A}}(\gamma)\beta_\infty(\gamma^{-1},\gamma)\cdot\phi_\infty\cdot\Big(\prod_{p\in S} r^{e_p}(\gamma,1).\phi_p\Big)\cdot\Big(\prod_{p\notin S}\phi_p^\circ\Big). \tag{5.2}$$

6. The adelic theta functions. For any $\varphi\in\mathcal{S}(\mathbb{A})$ and any additive character $\psi:\mathbb{Q}\backslash\mathbb{A}\to\mathbb{C}$, define

$$\Theta_{\mathrm{ad}}^\psi(\varphi;\widetilde{g}) := \sum_{\xi\in\mathbb{Q}}[r^\psi(\widetilde{g}).\varphi](\xi).$$

It follows from [Ge76, Proposition 2.33] that

(6.1)
$$\Theta_{\mathrm{ad}}^\psi(\varphi;\widetilde{i}_{\mathrm{diag}}(\gamma)\widetilde{g}) = \Theta_{\mathrm{ad}}^\psi(\varphi;\widetilde{g})\qquad(\varphi\in\mathcal{S}(\mathbb{A}),\ \widetilde{g}\in\widetilde{\mathrm{SL}}(2,\mathbb{A}),\ \gamma\in\mathrm{SL}(2,\mathbb{Q})).$$

The function $\Theta_{\mathrm{ad}}^\psi$ is then an intertwining map from the representation r^ψ

to the representation of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ on automorphic forms by right translation, namely

$$\Theta^{\psi}_{\mathrm{ad}}(\varphi;\widetilde{g}_1\widetilde{g}_2)=\Theta^{\psi}_{\mathrm{ad}}(r^{\psi}(\widetilde{g}_2).\varphi;\widetilde{g}_1)\qquad(\widetilde{g}_1,\widetilde{g}_2\in\widetilde{\mathrm{SL}}(2,\mathbb{A}),\ \varphi\in\mathcal{S}(\mathbb{A})).\tag{6.2}$$

We now construct an adelic theta function corresponding to the classical theta functions θ_χ as defined in (2.11).

The first step is to define an element in $\mathcal{S}(\mathbb{A})$ corresponding to the Dirichlet character χ. It will be useful to carry out this construction not only for even Dirichlet characters, but for all even periodic arithmetic functions. First, let $f:\mathbb{Z}/M\mathbb{Z}\to\mathbb{C}$ be an even function which is *factorizable*, in the sense that one may write f as

$$f(m)=\prod_{p\mid M}f_p(m),\tag{6.3}$$

where f_p is a function over $\mathbb{Z}/p^{v_p(M)}\mathbb{Z}$ and $v_p(M)$ is the p-adic valuation of M, i.e., the integer such that $p^{v_p(M)}\,\|\,M$. Then we write $\phi_p^{f_p}=\iota_{p^{v_p(M)}}(f_p)$ as the function over $\mathbb{Q}_p$ induced by f_p.

REMARK 6.4. Recall that ϕ_p^{μ} was previously defined for μ a character of $\mathbb{Z}_p^{\times}$. Thus, if $\chi:(\mathbb{Z}/p\mathbb{Z})^{\times}\to\mathbb{C}^{\times}$ is a character, then ϕ_p^{χ} can be defined either by viewing χ as a character of $\mathbb{Z}_p^{\times}$ and using the earlier definition, or by extending χ by zero to a multiplicative function $\mathbb{Z}/p^m\mathbb{Z}\to\mathbb{C}$ and using the above definition. However, one readily checks that the two definitions are consistent with one another.

Recall that

$$\phi^{\circ}_{\infty}(u)=e^{-2\pi u^2},\qquad \phi^{\circ}_p=\mathbb{1}_{\mathbb{Z}_p}.$$

Now define

$$\phi^f_v=\begin{cases}\phi^{\circ}_v & \text{if } v=\infty \text{ or } v\nmid M,\\ \phi_p^{f_p} & \text{if } v=p\mid M,\end{cases}$$

$$\phi^f(\{x_v\}_v)=\prod_v\phi^f_v(x_v)\qquad(\{x_v\}_v\in\mathbb{A}).$$

This is an element of the adelic Bruhat–Schwartz space $\mathcal{S}(\mathbb{A})$. Notice that the individual factors f_p in the factorization (6.3) are not uniquely determined. (Each is determined only up to a nonzero scalar.) Nevertheless, ϕ^f is uniquely determined by f. Also, the mapping $f\mapsto\phi^f$ is linear in f_p for each p. Hence it extends linearly to the vector space of all even functions $\mathbb{Z}/M\mathbb{Z}\to\mathbb{C}$ (which is spanned by factorizable elements).

REMARK 6.5. This construction may be understood conceptually as follows: the set

$$\bigcup_{M\in\mathbb{Z},\,M>0}\mathrm{Map}(\mathbb{Z}/M\mathbb{Z},\mathbb{C})\tag{6.6}$$

is canonically identified with $C^\infty(\widehat{\mathbb{Z}},\mathbb{C})$, where $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ is the profinite completion of $\mathbb{Z}$. This, in turn, is canonically identified with the set

$$\Big\{\phi \in \mathcal{S}(\mathbb{A}_{\mathrm{f}}) : \mathrm{supp}(\phi) \subset \prod_p \mathbb{Z}_p\Big\},$$

where $\mathbb{A}_{\mathrm{f}}$ denotes the finite adèles. Constructing an injection $\mathcal{S}(\mathbb{A}_{\mathrm{f}}) \hookrightarrow \mathcal{S}(\mathbb{A})$ is as simple as deciding what to put at ∞, and here we have chosen ϕ_∞°. However, [Iw97, Section 10.5] suggests that an embedding using ϕ_∞° is only the correct choice for even Dirichlet characters. This is the motivation for restricting to the even elements of the set (6.6).

Next, for every even arithmetic function f we define its associated holomorphic theta function

$$\theta_f(z) = \sum_{n=-\infty}^{\infty} f(n) e^{2\pi i n^2 z} \qquad (z \in \mathcal{H}),$$

and its Maass form version $\theta_f^{\mathrm{Maa}}(z) = y^{1/4}\theta_f(z)$. Obviously θ_f has a linear dependence upon f. Notice that if f is a Dirichlet character, this recovers the previous definition, and that for general f, the function θ_f is a linear combination of theta functions attached to Dirichlet characters. Notice also that

$$A_{\theta_f}(I_2, n) = \begin{cases} f(0) & \text{if } n = 0, \\ 2f(m) & \text{if } n = m^2 \text{ for some } m \neq 0, \\ 0 & \text{if } n \text{ is not a square.} \end{cases}$$

Lemma 6.7. *Let $z \in \mathcal{H}$, and let $e = \prod_v e_v \in \mathrm{Hom}(\mathbb{Q} \setminus \mathbb{A}, \mathbb{C}^\times)$ with e_v as in* (4.2)*. Let M be a positive integer, and f an even arithmetic function with period M. Then*

$$\varTheta_{\mathrm{ad}}^e(\phi^f; \widetilde{i}_\infty(b_z, 1)) = \theta_f^{\mathrm{Maa}}(z).$$

Proof. It suffices to prove the identity in the special case when f is factorizable, since both sides are linear in f and factorizable functions span the space of even arithmetic function with period M. For $y > 0$, we write

$$\phi_y^f(x) = r^e\bigg(\widetilde{i}_\infty\begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\bigg)\phi^f(x).$$

Then by definition

$$\begin{aligned}\varTheta_{\mathrm{ad}}^e(\phi^f; \widetilde{i}_\infty(b_z, 1)) &= \sum_{\xi \in \mathbb{Q}} \bigg[r^e\bigg(\widetilde{i}_\infty\bigg(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\bigg), 1\bigg)\phi^f\bigg](\xi) \\ &= \sum_{\xi \in \mathbb{Q}} \bigg[r^e\bigg(\widetilde{i}_\infty\bigg(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}, 1\bigg)\phi_y^f\bigg)\bigg](\xi) = \sum_{\xi \in \mathbb{Q}} e_\infty(x\xi^2)\phi_y^f(\xi).\end{aligned}$$

Now

$$\begin{aligned}\phi_y^f(x) &= r^{e_\infty}\left(\begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\right)\phi_\infty^f(x_\infty)\cdot\prod_p \phi_p^f(x_p)\\ &= y^{1/4}\phi_\infty^f(y^{1/2}x_\infty)\cdot\prod_p\phi_p^f(x_p), = y^{1/4}e^{-2\pi y x_\infty^2}\prod_p\phi_p^f(x_p),\end{aligned}$$

so

$$\begin{aligned}\Theta_{\mathrm{ad}}^e(\phi^f;\widetilde{i}_\infty(b_z,1)) &= y^{1/4}\sum_{\xi\in\mathbb{Q}} e_\infty(x\xi^2)\phi_y^f(\xi)\\ &= y^{1/4}\sum_{\xi\in\mathbb{Q}} e^{2\pi i x\xi^2}e^{-2\pi y\xi^2}\prod_p\phi_p^f(\xi).\end{aligned}$$

For every $\xi\in\mathbb{Q}$, direct computations show that

$$\prod_p \phi_p^f(\xi) = \begin{cases} 0 & \text{if } \xi\notin\mathbb{Z},\\ f(\xi) & \text{if } \xi\in\mathbb{Z}.\end{cases}$$

Hence

$$\begin{aligned}\Theta_{\mathrm{ad}}^e(\phi^f;\widetilde{i}_\infty(b_z,1)) &= y^{1/4}\sum_{n\in\mathbb{Z}} f(n)e^{2\pi i x n^2-2\pi y n^2}\\ &= y^{1/4}\sum_{n\in\mathbb{Z}} f(n)e^{2\pi i n^2(x+iy)} = \theta_f^{\mathrm{Maa}}(x+iy). \blacksquare\end{aligned}$$

THEOREM 6.8. *We have*

$$\Theta_{\mathrm{ad}}^e(\phi^f;\widetilde{i}_\infty(\widetilde{g}_\infty)) = \big(\theta_f^{\mathrm{Maa}}|^{\sim}\widetilde{g}\big)(i)\qquad (\widetilde{g}_\infty\in\widetilde{\mathrm{SL}}(2,\mathbb{R})).$$

Proof. Let $\widetilde{g}_\infty\in\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Then in Corollary 3.11 we have shown the Iwasawa decomposition

$$\widetilde{g}_\infty = \begin{pmatrix} y^{1/2} & xy^{-1/2}\\ & y^{-1/2}\end{pmatrix}\widetilde{\kappa}_\theta = b_{x+iy}\widetilde{\kappa}_\theta$$

for some $x\in\mathbb{R}$, $y>0$ and $\theta\in\mathbb{R}$. In particular, $\widetilde{\mathrm{j}}(\widetilde{g}_\infty,i)=e^{i\theta}$. Hence

$$\begin{aligned}\Theta_{\mathrm{ad}}^e(\phi^f;\widetilde{i}_\infty(\widetilde{g}_\infty)) &= \sum_{\xi\in\mathbb{Q}}[r^e(\widetilde{i}_\infty(\widetilde{g}_\infty))\phi^f](\xi) = \sum_{\xi\in\mathbb{Q}}[r^e(\widetilde{i}_\infty(b_{x+iy}))r^{e_\infty}(\widetilde{\kappa}_\theta)\phi^f](\xi)\\ &= e^{-i\theta}\sum_{\xi\in\mathbb{Q}}[r^e(\widetilde{i}_\infty(b_{x+iy}))\phi^f](\xi) = e^{-i\theta}\Theta_{\mathrm{ad}}^e(\phi^f,\widetilde{i}_\infty(b_{x+iy}))\\ &= e^{-i\theta}\theta_f^{\mathrm{Maa}}(x+iy) = \widetilde{\mathrm{j}}(\widetilde{g}_\infty,i)^{-1}\theta_f^{\mathrm{Maa}}(\widetilde{g}_\infty\cdot i)\\ &= (\theta_f^{\mathrm{Maa}}|^{\sim}\widetilde{g}_\infty)(i). \blacksquare\end{aligned}$$

COROLLARY 6.9. *If* $\gamma\in\mathrm{SL}(2,\mathbb{Q})$ *then*

$$\big(\theta_f^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\gamma\big)(z) = s_{\mathbb{A}}(\gamma)\beta_\infty(\gamma,\gamma^{-1})\Theta_{\mathrm{ad}}^e\big(r^e(\widetilde{i}_{\mathrm{f}}(\gamma^{-1})).\phi^f;\widetilde{i}_\infty(b_z,1)\big).$$

Proof. By (3.15), we have

$$\big(\theta_f^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\gamma\big)(z) = \big(\theta_f^{\mathrm{Maa}}\big|^{\sim}(\gamma,1)\big)(z) = \Theta_{\mathrm{ad}}^{e}(\phi^f;\widetilde{i}_\infty(\gamma b_z,1)).$$

But Θ_{ad}^{e} is invariant on the left by $\widetilde{i}_{\mathrm{diag}}(\gamma^{-1})$, as shown in (6.1), so the right-hand side is equal to

$$\Theta_{\mathrm{ad}}^{e}\big(\phi^f;\widetilde{i}_{\mathrm{f}}(\gamma^{-1})\widetilde{i}_\infty(b_z,1)\big) = \Theta_{\mathrm{ad}}^{e}\big(\phi^f;\widetilde{i}_\infty(b_z,1)\widetilde{i}_{\mathrm{f}}(\gamma^{-1})\big).$$

Applying (6.2) completes the proof. ∎

7. Fourier coefficients of classical theta functions. In this section, let M be an even positive integer such that $M/2$ is squarefree, and choose

$$\epsilon : \{p : p \mid M\} \to \{\pm\} \quad \text{such that} \quad \prod_{p\mid M} \epsilon(p) = +. \tag{7.1}$$

(The motivation for the restriction of ϵ comes from [Iw97, Section 10.5].)

Pointwise multiplication gives an isomorphism between $\mathcal{S}(\mathbb{A})$ and the restricted tensor product $\bigotimes_v' \mathcal{S}(\mathbb{Q}_v)$, by which we identify the two spaces. Recall that we defined vector spaces V_p, V_p^+, V_p^- for all primes p in (4.10) and (4.25), and V_4, V_4^+, V_4^- in (4.27). Thus, we have defined $V_{p^{v_p(M)}}^{\epsilon(p)}$ for each prime p dividing M. Now we define

$$\mathcal{S}_M^{\epsilon} := \bigotimes_{p\mid M} V_{p^{v_p(M)}}^{\epsilon(p)} \otimes \bigotimes_{v\nmid M} \phi_v^{\circ} \subset \mathcal{S}(\mathbb{A}),$$

and define $\Gamma^{(2M)}$ as in Section 2.2. Then it follows from Theorems 4.11, 4.26, 4.28 and the decomposition (5.2) that

$$s_{\mathbb{A}}(\gamma^{-1})\beta_\infty(\gamma^{-1},\gamma)r^e(i_{\mathrm{f}}(\gamma^{-1}),1).\phi \in \mathcal{S}_M^{\epsilon} \quad (\gamma \in \Gamma^{(M)},\ \phi \in \mathcal{S}_M^{\epsilon}),$$

Moreover, if ϕ is a pure tensor, then so is $s_{\mathbb{A}}(\gamma^{-1})\beta_\infty(\gamma^{-1},\gamma)r^e(i_{\mathrm{f}}(\gamma^{-1}),1).\phi$.

Next, let $\mathcal{F}_M$ be the set of arithmetic functions with period M, which can be identified with $\mathrm{Map}(\mathbb{Z}/M\mathbb{Z},\mathbb{C})$. Then it follows from the Chinese Remainder Theorem that pointwise multiplication is an isomorphism between $\bigotimes_{p\mid M}\mathcal{F}_{p^{v_p(M)}}$ and $\mathcal{F}_M$. Let $\mathcal{F}_M^+$ and $\mathcal{F}_M^-$ denote the subspaces of $\mathcal{F}_M$ consisting of even and odd elements respectively, and let $\mathcal{F}_M^{\epsilon}$ denote the subspace spanned by $\{\prod_{p\mid M} f_p : f_p \in \mathcal{F}_{p^{v_p(M)}}^{\epsilon(p)}\}$. Then we have a linear isomorphism $\mathcal{F}_M^{\epsilon} \to \mathcal{S}_M^{\epsilon}$ such that each pure tensor $f = \prod_p f_p \in \mathcal{F}_M^{\epsilon}$ corresponds to

$$\phi^f := \bigotimes_{p\mid M} \iota_{p^{v_p(M)}}(f_p) \otimes \bigotimes_{v\nmid M} \phi_v^{\circ} \in \mathcal{S}_M^{\epsilon}.$$

It follows that there exists a map $\rho_M^{\epsilon} : \Gamma^{(2M)} \times \mathcal{F}_M^{\epsilon} \to \mathcal{F}_M^{\epsilon}$ such that

$$s_{\mathbb{A}}(\gamma^{-1})\beta_\infty(\gamma^{-1},\gamma)r^e(i_{\mathrm{f}}(\gamma^{-1}),1).\phi^f = \phi^{\rho_M^{\epsilon}(\gamma^{-1},f)} \quad (f \in \mathcal{F}_M^{\epsilon},\ \gamma \in \Gamma^{(2M)}).$$

By Corollary 6.9, this implies

$$\theta_f^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\gamma = \theta_{\rho_M^\epsilon(\gamma^{-1},f)} \qquad (f \in \mathcal{F}_M^\epsilon,\ \gamma \in \Gamma^{(2M)}).$$

Therefore

$$A_{\theta_f}(\gamma, n) = A_{\theta_{\rho_M^\epsilon(\gamma^{-1},f)}}(I_2, n) = \rho_M^\epsilon(\gamma^{-1}, f)(n).$$

Now, for $\phi = \bigotimes_v \phi_v \in \mathcal{S}(\mathbb{A})$, let $\phi_{\mathrm{f}} = \bigotimes_p \phi_p \in \mathcal{S}(\mathbb{A}_{\mathrm{f}})$ be its finite part. Then it follows immediately from the definitions that $f(n) = \phi_{\mathrm{f}}^f(i_{\mathrm{f}}(n))$. Thus

$$A_{\theta_f}(\gamma, n) = s_{\mathbb{A}}(\gamma^{-1})\beta_\infty(\gamma^{-1},\gamma)\prod_{p|M}[r_p^{e_p}(\gamma^{-1},1).\iota_{p^{v_p(M)}}(f_p)](n).$$

Next, recall that for each prime p and both signs ϵ, in Theorems 4.11 and 4.28 we fixed an ordered basis $B^\epsilon_{p^{v_p(M)}}$ for $V^\epsilon_{p^{v_p(M)}}$, and defined $\varrho^\epsilon_{p^{v_p(M)}}(\gamma) = \big[r_p^{e_p}(\gamma)\big|_{V^\epsilon_{p^{v_p(M)}}}\big]_{B^\epsilon_{p^{v_p(M)}}}$. Thus

$$[r_p^{e_p}(\gamma,1)\iota_{p^{v_p(M)}}(f_p)](n) = B^\epsilon_{p^{v_p(M)}}(n)\cdot \varrho^\epsilon_{p^{v_p(M)}}(\gamma^{-1})\cdot[\iota_{p^{v_p(M)}}(f_p)]_{B^\epsilon_{p^{v_p(M)}}}.$$

Here we think of $B^\epsilon_{p^{v_p(M)}}$ as a row vector of Schwartz functions, or as a row-vector-valued function, and for $h \in V^\epsilon_{p^{v_p(M)}}$ we denote by $[h]_{B^\epsilon_{p^{v_p(M)}}}$ the coordinate (column) vector of h with respect to the basis $B^\epsilon_{p^{v_p(M)}}$.

Finally, let $\mathrm{B}^\epsilon_{p^{v_p(M)}}$ denote the preimage of $B^\epsilon_{p^{v_p(M)}}$ under the linear isomorphism $\iota_{p^{v_p(M)}}$. Thus it is an ordered basis for $\mathcal{F}^\epsilon_{p^{v_p(M)}}$ and, in the notation of Section 4.4, it is given by

$$\mathrm{B}^\epsilon_{p^{v_p(M)}} = \begin{cases} [\psi_{2,2}\ \mathbb{1}_{2\mathbb{Z}}], & p=2,\ v_p(M)=1,\ \epsilon=+,\\ \emptyset, & p=2,\ v_p(M)=1,\ \epsilon=-,\\ [\psi_{2,2}\ \mathbb{1}_{2+4\mathbb{Z}}\ \mathbb{1}_{4\mathbb{Z}}], & p=2,\ v_p(M)=2,\ \epsilon=+,\\ [\psi_{2,1}], & p=2,\ v_p(M)=2,\ \epsilon=-,\\ [\psi_{p,2}\ \psi_{p,4}\ \ldots\ \psi_{p,p-1}\ \mathbb{1}_{p\mathbb{Z}}], & p\neq 2,\ \epsilon=+,\\ [\psi_{p,1}\ \psi_{p,3}\ \ldots\ \psi_{p,p-2}], & p\neq 2,\ \epsilon=-. \end{cases} \tag{7.2}$$

Combining all of this, we obtain

THEOREM 7.3. *Let $M \geq 1$ be an even positive integer such that $M/2$ is squarefree, and let ϵ be as given in* (7.1). *Take $f \in \mathcal{F}_M^\epsilon$ and $\gamma \in \Gamma^{(2M)}$. Then*

$$A_{\theta_f}(\gamma,n) = s_{\mathbb{A}}(\gamma^{-1})\beta_\infty(\gamma^{-1},\gamma)\prod_{p|M}\mathrm{B}^{\epsilon(p)}_{p^{v_p(M)}}(n)\cdot\varrho^{\epsilon(p)}_{p^{v_p(M)}}(\gamma^{-1})\cdot[f_p]_{\mathrm{B}^{\epsilon(p)}_{p^{v_p(M)}}}.$$

REMARK 7.4. In the key special case when f is a Dirichlet character χ (mod M), we write $\chi = \prod_p \chi_p$. For each p, χ_p is one of the characters $\psi_{p,j}$ for some $1 \leq j \leq p-1$, as defined in Section 4.4, $\epsilon(p) = \chi_p(-1)$, and $[\chi_p]_{\mathrm{B}^{\epsilon(p)}_{p^{v_p(M)}}}$ is the standard basis vector $e_{\lceil j/2\rceil}$.

REMARK 7.5. Suppose that M_1 is odd and squarefree, and ϵ_1 is a function $\{p\,|\,M_1\}\to\{\pm\}$ with $\prod_{p|M_1}\epsilon_1(p)=+$. Define $M=2M_1$ and $\epsilon:\{p\,|\,M\}\to\{\pm\}$ by $\epsilon(2)=+$ and by $\epsilon(p)=\epsilon_1(p)$ for $p\neq 2$. Then $\mathcal{F}^{\epsilon_1}_{M_1}$ is a subspace of $\mathcal{F}^{\epsilon}_{M}$. Combining this remark with the previous one we may apply the theorem to Dirichlet characters modulo M_1 as well. Note, however, that the space $\mathcal{F}^{\epsilon_1}_{M_1}$ is not preserved by $\Gamma^{(2M)}$ or $\Gamma^{(M)}$, owing to the fact that ϕ_2° is not fixed by $\widetilde{K}_2^{(2^n)}$ for any n. Hence our analysis of the theta function attached to a Dirichlet character modulo M_1 must necessarily involve theta functions attached to arithmetic functions which are only periodic modulo M. This is the reason for the restriction to even M.

EXAMPLE 7.6. Let $p\geq 5$ be a prime, $\chi_p \pmod p$ an even Dirichlet character, and $\chi=\chi_{12}\chi_p$. Then for $\sigma\in\Gamma^{(24p)}$ we have

$$\begin{aligned}A_{\theta_\chi}(\sigma,n^2)&=2\chi_{12}(n)s_2(\sigma^{-1})s_3(\sigma^{-1})s_p(\sigma^{-1})\\&\quad\cdot\varrho_2^-(\sigma^{-1})\varrho_3^-(\sigma^{-1})B_p^+(n)\varrho_p^+(\sigma^{-1})[\chi]_{B_p^+}.\end{aligned}$$

Alternatively, for every $n\geq 0$ choose $i(n)\geq 1$ such that $i(n)=(p+1)/2$ if $p\,|\,n$ and that otherwise $i(n)$ is the unique element of $\{1,\dots,(p-1)/2\}$ satisfying $i(n)^2\equiv n^2 \pmod p$. Let $e_{i(n)}$ be the p-dimensional column vector whose only nonzero entry is the $i(n)$th and equals to one, and let C_p^+ be as defined in (4.12). Then

$$\begin{aligned}A_{\theta_\chi}(\sigma,n^2)&=2\chi_{12}(n)s_2(\sigma^{-1})s_3(\sigma^{-1})s_p(\sigma^{-1})\\&\quad\cdot\varrho_2^-(\sigma^{-1})\varrho_3^-(\sigma^{-1})^t e_{i(n)}C_p^+\varrho_p^+(\sigma^{-1})[\psi]_{B_p^+}.\end{aligned}$$

This formula readily explains the observations made by Gunnells, as discussed following Conjecture 1.1.

7.1. Explicit action on vector-valued forms. In this section we give another formulation of our results. To do so, we briefly recall the Kronecker product of matrices and its connection with tensor product operators. If A and B are matrices of sizes $m\times n$ and $p\times q$, then the *Kronecker product* $A\otimes B$ is the $mp\times nq$ matrix C defined by

$$a_{ij}b_{kl}=c_{p(i-1)+k,q(j-1)+l},$$

where a_{ij}, denotes the i,j entry of the matrix A, and b_{kl},c_{rs} are defined likewise for B and C. If we think of matrices as representing operators with respect to ordered bases, then the Kronecker product corresponds to taking the tensor product of operators, combined with a choice of how to combine ordered bases of two spaces to form an ordered basis of the tensor product. Explicitly, if $\mathcal{B}=(v_1,\dots,v_n)$ and $\mathcal{C}=(w_1,\dots,w_m)$ are ordered bases for two spaces V and W, while T and L are linear endomorphisms of V and W respectively, then $[T]_{\mathcal{B}}\otimes[L]_{\mathcal{C}}$ is the matrix of $T\otimes W\in\mathrm{End}(V\otimes W)$ with respect to the ordered basis

$$(v_1 \otimes w_1, \dots, v_1 \otimes w_m, v_2 \otimes w_1, \dots, v_2 \otimes w_m, \dots, v_n \otimes w_1, \dots, v_n \otimes w_m).$$

Clearly, $A \otimes B$ is not, in general, equal to $B \otimes A$. One may think of $A \otimes B$ and $B \otimes A$ as two matrices obtained by writing the same operator in two different sets of coordinates, obtained from two distinct orderings of the same basis. In particular, they are conjugate by a permutation matrix. Clearly, this extends to products of arbitrary length, and it is a routine check that the Kronecker product is associative.

Now, fix an even positive integer M with $M/2$ squarefree. Recall that pointwise multiplication defines an isomorphism $\bigotimes_{p|M} \mathcal{F}_{p^{v_p(M)}} \to \mathcal{F}_M$. We use this isomorphism to identify the two spaces. We also identify $\mathcal{F}_M^\epsilon$ with $\bigotimes_{p|M} \mathcal{F}_p^{\epsilon(p)}$, for each $\epsilon : \{p \mid M\} \to \{\pm 1\}$. Recall $\mathrm{B}^\epsilon_{p^{v_p(M)}}$ from (7.2), and define $\mathrm{B}_{p^{v_p(M)}}$ to be the preimage of $B_{p^{v_p(M)}}$ under $\iota_{p^{v_p(M)}}$. Then we obtain bases

$$\mathrm{B}_M^\epsilon = \Big\{ \bigotimes_{p|M} \phi_p : \phi_p \in \mathrm{B}^{\epsilon(p)}_{p^{v_p(M)}},\ \forall p \mid M \Big\},$$

$$\mathrm{B}_M = \Big\{ \bigotimes_{p|M} \phi_p : \phi_p \in \mathrm{B}_{p^{v_p(M)}},\ \forall p \mid M \Big\}$$

for $\mathcal{F}_M^\epsilon$ and $\mathcal{F}_M$ respectively, and the convention employed in defining the Kronecker product (combined with the natural order on the primes $p \mid M$) determines orders on B_M and B_M^ϵ. Note that B_M contains all the Dirichlet characters modulo M (as well as some other elements). Number the elements $\mathrm{B}_M = \{\xi_{M,1}, \dots, \xi_{M,M}\}$. Also, we define

$$\overrightarrow{\theta_M} = (\theta_{\xi_{M,1}}, \dots, \theta_{\xi_{M,M}}).$$

Further, we write

$$\varrho_M = \bigotimes_{p|M} \varrho_{p^{v_p(M)}} \qquad \text{(Kronecker product)}.$$

THEOREM 7.7. *Let $\gamma \in \Gamma^{(2M)}$. Then*

$$\overrightarrow{\theta_M}^{\mathrm{Maa}} \Big|^{\mathrm{Maa}}_{1/2} \gamma = s_{\mathbb{A}}(\gamma^{-1}) \beta_\infty(\gamma, \gamma^{-1}) \overrightarrow{\theta_M} \cdot \varrho_M(\gamma^{-1}).$$

REMARK 7.8. We could also fix $\epsilon : \{p \mid M\} \to \{\pm\}$, attach a vector-valued theta function to B_M^ϵ, and prove an analogous result involving the Kronecker product $\varrho_M^\epsilon = \bigotimes_{p|M} \varrho^{\epsilon(p)}_{p^{v_p(M)}}$. The matrix ϱ_M is a block matrix with block ϱ_M^ϵ for each $\epsilon : \{p \mid M\} \to \{\pm 1\}$. (We permit the degenerate case of a 0×0 block when $v_2(M) = 1$ and $\epsilon(2) = -$.) The tuple B_M is obtained by concatenating the tuples B_M^ϵ, $\epsilon : \{p \mid M\} \to \{\pm 1\}$, in a certain order. This is then inherited by $\overrightarrow{\theta_M}$. We remark that a certain proportion of the functions

$\xi_{M,i}$ will lie in subspaces attached to the functions $\epsilon : \{p \mid M\} \to \{\pm\}$ with $\prod_{p|M} \epsilon = -$. For such i, we have $\theta_{\xi_{M,i}} = 0$.

Proof of Theorem 7.7. Each element ξ of B_M factors as $\prod_p \xi_p$ with $\xi_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ for each p and $\xi_p \equiv 1$ if $p \nmid M$. Hence ξ corresponds to $\phi^\xi = \phi_\infty^\circ \cdot \prod_p \phi_p^{\xi_p} \in \mathcal{S}_M$. The individual factors ξ_p are unique up to shifting nonzero scalars among them, and the function ϕ^ξ is uniquely determined by ξ. Further, $\phi_p^{\xi_p} = \mathbb{1}_{\mathbb{Z}_p}$ for all $p \nmid M$. By Corollary 6.9, for every $\xi \in \mathrm{B}_M$ we have

$$\theta_\xi^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\gamma(z) = s_{\mathbb{A}}(\gamma^{-1})\beta_\infty(\gamma,\gamma^{-1})\Theta_{\mathrm{ad}}^e\Big(\phi_\infty^\circ \prod_p (r_p^{e_p}(\gamma^{-1},1)\phi_p^{\xi_p}); \widetilde{i}_\infty(\gamma b_z)\Big).$$

If $p \nmid M$, then $\theta_{\xi,p} = \mathbb{1}_{\mathbb{Z}_p}$ and γ^{-1} acts on it trivially. Hence

$$\begin{aligned}\theta_\xi^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\gamma(z) &= s_{\mathbb{A}}(\gamma^{-1})\beta_\infty(\gamma,\gamma^{-1})\\ &\quad\cdot \Theta_{\mathrm{ad}}^e\Big(\Big(\prod_{v\nmid M}\phi_v^\circ\Big)\Big(\prod_{p|M}(r_p^{e_p}(\gamma^{-1},1)\phi_p^{\xi_p})\Big); \widetilde{i}_\infty(\gamma b_z)\Big).\end{aligned}$$

Now, $\varrho_p(\gamma^{-1})$ is, by definition, the matrix of $r_p^{e_p}(\gamma^{-1},1)$ acting on V_p with respect to the ordered basis B_p. It then follows from the definition of the Kronecker product that $\varrho_M(\gamma^{-1})$ is the matrix of $r^e(i_{\mathrm{f}}(\gamma^{-1}),1)$ acting on $\mathcal{S}_M$ with respect to the basis B_M, and the result follows from the linearity of Θ_{ad}^e. ∎

THEOREM 7.9. *Let χ (mod M) be an even Dirichlet character and $\sigma \in \Gamma^{(2M)}$. Then $A_{\theta_\chi}(\sigma, n) = 0$ unless $n \geq 0$ is a perfect square. Further, let the M-dimensional column vector $[\chi]_M$ be the coordinate of χ with respect to the basis B_M. Then*

$$A_{\theta_\chi}(\sigma, n^2) = \begin{cases} 2s_{\mathbb{A}}(\sigma^{-1})\beta_\infty(\sigma,\sigma^{-1})\,\mathrm{B}_M(n)\cdot\varrho_M(\sigma^{-1})\cdot[\chi]_M & \text{if } n \geq 1,\\ s_{\mathbb{A}}(\sigma^{-1})\beta_\infty(\sigma,\sigma^{-1})\,\mathrm{B}_M(0)\cdot\varrho_M(\sigma^{-1})\cdot[\chi]_M & \text{if } n = 0.\end{cases}$$

Proof. Since

$$\overrightarrow{\theta_M}^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\sigma = s_{\mathbb{A}}(\sigma^{-1})\beta_\infty(\sigma,\sigma^{-1})\overrightarrow{\theta_M}\cdot\varrho_M(\sigma^{-1}),$$

we have

$$\theta_\chi^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\sigma = \big(\overrightarrow{\theta_M}^{\mathrm{Maa}}\big|_{1/2}^{\mathrm{Maa}}\sigma\big)\cdot[\chi]_M = s_{\mathbb{A}}(\sigma^{-1})\beta_\infty(\sigma,\sigma^{-1})\overrightarrow{\theta_M}\cdot\varrho_M(\sigma^{-1})\cdot[\chi]_M.$$

As we compare the mth Fourier coefficients on both sides, we have

$$A_{\theta_\chi}(\sigma, m) = s_{\mathbb{A}}(\sigma^{-1})\beta_\infty(\sigma,\sigma^{-1})(A_{\theta_\xi}(I_2, m))_{\xi\in B_M}\cdot\varrho_M(\sigma^{-1})\cdot[\chi]_M.$$

Now by the definition of θ_ξ,

$$A_{\theta_\xi}(I_2, n) = \begin{cases} 0 & \text{if } m \text{ is not a perfect square},\\ 2\xi(n) & \text{if } m = n^2 \text{ for some } n \geq 1,\\ \xi(0) & \text{if } m = 0.\end{cases}$$

Hence $A_{\theta_\chi}(\sigma, m) = 0$ unless $m = n^2$ for some $n \geq 0$, and we have the required formula for $A_{\theta_\chi}(\sigma, n^2)$. ∎

EXAMPLE 7.10. Let $\chi_{12} = \left(\frac{12}{\cdot}\right)$ be the primitive Dirichlet character modulo 12. Define ϱ_4^- and ϱ_3^- as in (4.29) and (4.23) respectively. By Theorem 7.7, we have

$$\theta_{\chi_{12}}\big|_{1/2}^{\mathrm{Hol}}\sigma = s_{\mathbb{A}}(\gamma)\beta_\infty(\gamma^{-1},\gamma)\varrho_4^-(\sigma^{-1})\varrho_3^-(\sigma^{-1})\theta_{\chi_{12}} \qquad (\sigma \in \Gamma^{(24)}).$$

In particular, let $\mathfrak{a} = u/w$ be a cusp of $\Gamma_0(576)$. Then

$$A_{\theta_{\chi_{12}}}(\sigma_{\mathfrak{a}}, n) = \begin{cases} 2s_2(\sigma_{\mathfrak{a}}^{-1})s_3(\sigma_{\mathfrak{a}}^{-1})\varrho_4^-(\sigma_{\mathfrak{a}}^{-1})\varrho_3^-(\sigma_{\mathfrak{a}}^{-1}) & \text{if } n = m^2 \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{\theta_{\chi_{12}}}(\sigma_{\mathfrak{a}}^0, n) = e\left(-\frac{m^2 wr}{24u[24,w]}\right)A_{\theta_\chi}(\sigma_{\mathfrak{a}}, n),$$

where we choose $r, s \in \mathbb{Z}$ such that $24us - wr = \gcd(24, w)$, and the scaling matrices $\sigma_{\mathfrak{a}}^0$ and $\sigma_{\mathfrak{a}}$ are as given in (2.1) and (2.4) respectively.

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