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SOME ISOMORPHIC PROPERTIES IN $K(X,Y)$ AND IN PROJECTIVE TENSOR PRODUCTS

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Abstract. We study the (DPrcp) property and the Gelfand–Phillips properties in spaces of compact operators. Moreover we give some sufficient conditions implying that the projective tensor product of two Banach spaces is sequentially right (SR) or has the L-limited property. We introduce the dual (SR*) property and we give a characterization of it, also showing that it is intermediate between the (V*) and the (RDP*) properties. Finally, we study the Bourgain–Diestel property (BD) and the (RDP*) property in the space $K_{w^*\text{-}w}(X^*,Y)$.

1. Introduction and notation. In this note we study lifting results for isomorphic properties from some Banach spaces to spaces of compact operators and to projective tensor products.

In Section 2, after giving the suitable definitions, we provide a new condition implying that $K(X,Y)$ has the (DPrcp) property and relate it to known properties. We also study the (GP) property in $K(X,Y)$.

In Section 3 we investigate and relate two properties recently introduced in [16] and [25], namely the L-limited property and the sequential right (for short (SR)) property which is intermediate between Pełczyński's property (V) and the reciprocal Dunford–Pettis property. We also give a sufficient condition implying that the projective tensor product of two Banach spaces has the (SR) property. Moreover, we introduce and characterize the (SR*) property, dual in a sense to the (SR) property.

Finally, we study the Bourgain–Diestel property and the (RDP*) property in the space $K_{w^*\text{-}w}(X^*,Y)$ of w^*-w-continuous compact operators from X^* into Y.

Our notation is standard. Throughout, X, Y, Z, E, F denote Banach spaces, X^* the dual space and B_X the closed unit ball of X. The closed unit ball B_{X^*} is always equipped with the weak* topology. We use the symbol $L(X,Y)$ for the space of (bounded linear) operators from X into Y equipped

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with the operator norm, while $K(X,Y)$ denotes the subspace of compact operators.

We write $X \otimes_\pi Y$ for the completed projective tensor product of X and Y.

2. (DPrcp) and the Gelfand–Phillips property in $K(X,Y)$. In this section, we first give a sufficient condition implying that (DPrcp) lifts from the Banach spaces X^* and Y to the space $K(X,Y)$. Then we compare our result with Theorem 3.8 of [15]. We give a sufficient condition implying that a closed subspace of $K(X,Y)$ has the Gelfand–Phillips property. We recall the following definitions.

Definition 1. A bounded subset K in a Banach space X is called *limited* (respectively *Dunford–Pettis*) if

$$\lim_n \sup_{x\in K} |x_n^*(x)| = 0$$

for every weak* null (respectively weakly null) sequence (x_n^*) in X^*.

We say that $A \subset X^*$ is an *L-set* if every weakly null sequence (x_n) in X converges (to 0) uniformly on A.

It is easy to see that every relatively compact subset of X is limited (and the converse is true if X is separable [6, Ex. 14, p. 116]), every limited set is a Dunford–Pettis set and every Dunford–Pettis subset of a dual Banach space is an L-set, but the converses of these assertions are, in general, false. Dunford–Pettis sets are conditionally weakly compact (see [3]). It is also easy to see that a bounded operator $T \in L(X,Y)$ sends limited (resp. Dunford–Pettis) sets to limited (resp. Dunford–Pettis) sets. Moreover T^* sends L-sets in Y^* to L-sets in X^*.

Definition 2 ([12], [26]). A Banach space X has (DPrcp) if every Dunford–Pettis subset of X is relatively compact.

A Banach space X has the *Gelfand–Phillips* (for short (GP)) *property* if every limited subset is relatively compact.

It is easy to see that the (GP) and (DPrcp) properties are inherited by closed subspaces. Every Schur space (where weakly convergent sequences are norm convergent) has (DPrcp). It is also known (see [12]) that a dual Banach space has (DPrcp) if and only if it has the weak Radon–Nikodym property if and only if its predual does not contain a copy of l_1.

Banach spaces having the (GP) property include, among others, separable Banach spaces, separably complemented spaces, reflexive Banach spaces, spaces with weak* sequentially compact dual balls, $C(K)$ spaces, where K is both compact and sequentially compact, and spaces with (DPrcp).

In the following we shall use the well known Dunford–Pettis (DP) property (X has this property if every weakly compact operator $T : X \to Y$ is completely continuous, i.e. it sends weakly Cauchy sequences to norm convergent ones). Equivalently, X has the (DP) property if, for every weakly null sequence (x_n) in X and every weakly null sequence (x_n^*) in X^*, $\lim_n\langle x_n, x_n^*\rangle = 0$ [6, p. 113].

To obtain our first result we need the following well known

THEOREM 3 ([18]). *Let X be a Banach space without a copy of l_1. Let $M \subset K(X,Y)$ be such that*

(1) *for every $x \in X$, $M(x) = \{T(x) : T \in M\}$ is relatively compact in Y;*
(2) *M is weakly norm sequentially equicontinuous, that is,*

$$\lim_n \sup_{T\in M} \|T(x_n)\| = 0$$

for every weakly null sequence $(x_n) \subset X$.

Then M is relatively compact.

THEOREM 4. *Let X and Y be Banach spaces such that X^* and Y have* (DPrcp). *Assume that, for every $T \in L(X, Y^{**})$, for every weakly null sequence $(x_n) \subset X$, the sequence $(T(x_n))$ is an L-set. Then $K(X,Y)$ has* (DPrcp).

Proof. Let $M \subset K(X,Y)$ be a Dunford–Pettis set. Then, for every $x \in X$, $M(x)$ is a Dunford–Pettis set in Y. Since Y has (DPrcp), it is also a relatively compact set. So M satisfies condition (1) of Theorem 3. Moreover X does not contain a copy of l_1 since X^* has (DPrcp). By Theorem 3, M will be relatively compact if condition (2) is satisfied. Suppose it is not. Then there are a positive number ϵ, a weakly null sequence $(x_n) \subset X$ and a sequence $(T_n) \subset M$ such that

$$\|T_n(x_n)\| > \epsilon \quad \forall n \in \mathbb{N}.$$

For every $y^* \in Y^*$, the set $\{T_n^*(y^*) : n \in \mathbb{N}\}$ is also a Dunford–Pettis subset of X^*, hence it is relatively compact. Since (x_n) is weakly null, for every $y^* \in Y^*$, it follows that

$$\langle T_n(x_n), y^*\rangle = \langle x_n, T_n^*(y^*)\rangle \to 0.$$

So the sequence $(T_n(x_n))$ is weakly null.

Now we claim that $(T_n(x_n))$ is a Dunford–Pettis subset of Y. Let (y_n^*) be a weakly null sequence in Y^*. The sequence $(x_n \otimes y_n^*)$ is weakly convergent in $X \otimes_\pi Y^*$. Indeed, let $T \in (X \otimes_\pi Y^*)^* = L(X, Y^{**})$. By assumption $(T(x_n))$ is an L-set in Y^{**}. Since (y_n^*) is weakly null in Y^*,

$$\langle T, x_n \otimes y_n^*\rangle = \langle T(x_n), y_n^*\rangle \to 0.$$

Since $X \otimes_\pi Y^*$ embeds into $(K(X,Y))^*$, the sequence $(x_n \otimes y_n^*)$ is also weakly null in $(K(X,Y))^*$. Then, since (T_n) is a Dunford–Pettis set, we get

$$\lim_n \langle T_n(x_n), y_n^*\rangle = \lim_n \langle T_n, x_n \otimes_\pi y_n^*\rangle = 0.$$

So we have proved that $(T_n(x_n))$ is a Dunford–Pettis set, hence, again by the (DPrcp) property of Y, it must be relatively compact. Since it is a weakly null sequence, it is norm null. This contradiction proves the claim. ∎

Let us now present an example of Banach spaces X and Y satisfying the assumptions of Theorem 4 and such that $L(Y^*, X^*) \neq K(Y^*, X^*)$. Let E be the first Bourgain–Delbaen space [2]. It has the Schur property, hence (DPrcp). Since $E^* = M([0,1])$, on the one hand it contains a complemented copy of l_1, in particular $L(E^*, l_1) \neq K(E^*, l_1)$, on the other hand E^{**} is isomorphic to a $C(K)$ space, hence it has the Dunford–Pettis property. Let

$$X = l_p \times c_0, \qquad Y = E \times l_q, \qquad 1 < q < p < \infty.$$

Obviously X does not contain l_1, hence X^* has (DPrcp), and Y, being a product of (DPrcp) spaces, inherits this property. Since E^* is complemented in Y^* and l_1 identifies with a closed subspace of X^*, $L(E^*, l_1)$ can be identified with a closed subspace of $L(Y^*, X^*)$, hence $L(Y^*, X^*) \neq K(Y^*, X^*)$.

Let us verify the last assumption of Theorem 4. Let $T : X \to Y^{**} = E^{**} \times l_q$ be a continuous linear operator and let

$$p_1 : Y^{**} \to E^{**}, \qquad p_2 : Y^{**} \to l_q$$

be the natural projections. Define $T_1 : l_p \to E^{**}$ and $T_2 : l_p \to l_q$ by

$$\begin{aligned} T_1(a) &= p_1(T(a,0)) \quad \forall a \in l_p, \\ T_2(a) &= p_2(T(a,0)) \quad \forall a \in l_p. \end{aligned}$$

Obviously for every $x = (a,b) \in l_p \times c_0$ one has

$$\begin{aligned} T(a,b) &= T(a,0) + T(0,b) = (p_1(T(a,0)), p_2(T(a,0))) + T(0,b) \\ &= (T_1(a), T_2(a)) + T(0,b). \end{aligned}$$

Let $(x_n) = (a_n, b_n)$ be a weakly null sequence in X. Since c_0 and E^{**} have (DP), both $(T(0,b_n))$ and $(T_1(a_n))$ are Dunford–Pettis sets. Moreover, by Pitt's Theorem, T_2 is a compact operator, so $(T_2(a_n))$ is norm null, hence a Dunford–Pettis set. It follows that $(T(x_n)) = (T(a_n, b_n))$ is a Dunford–Pettis set, hence an L-set. ∎

If X^* has the Schur property, it has (DPrcp) and obviously satisfies the last assumption in Theorem 4. Hence Theorem 4 implies [12, Cor. 9]. It is strictly stronger, as shown by the above example. In passing, let us recall that X^* has the Schur property if and only if X has (DP) and does not contain a copy of l_1 [6, p. 212].

Proposition 5 below shows that Theorem 4 also implies [15, Theorem 3.8], where the last assumption in Theorem 4 is replaced by (c) of Proposition 5. The above example shows that Theorem 4 is strictly stronger than [15, Theorem 3.8].

Proposition 5. *Consider the following assertions:*

(a) *for every* $T \in L(X, Y^{**})$ *and for every weakly null sequence* $(x_n) \subset X$, $(T(x_n))$ *is an L-set;*
(b) $L(X, Y^{**}) = K(X, Y^{**})$;
(c) $L(Y^*, X^*) = K(Y^*, X^*)$.

Then (b)⇒(c)⇒(a). *If moreover* X *and* Y^* *do not contain copies of* l_1, *then* (a)⇒(b).

Proof. (a)⇒(b). Assume that X and Y^* do not contain copies of l_1. Let $T \in L(X, Y^{**})$. Let $(x_n) \subset X$ be a weakly null sequence. By hypothesis, $(T(x_n))$ is an L-set in Y^{**}. Since Y^* does not contain l_1, $(T(x_n))$ is relatively compact [8]. Since $(T(x_n))$ is weakly null, it is norm null. Since X does not contain l_1, T is compact by Odell's Theorem [22].

(b)⇒(c). Let $T \in L(Y^*, X^*)$. Then $S = T^*_{|X} : X \to Y^{**}$ is compact by assumption; so S^* is compact and $S^*_{|Y^*}$ coincides with T.

(c)⇒(a). Let $T \in L(X, Y^{**})$, let (x_n) be a bounded sequence in X, and let (y_m^*) be a weakly null sequence in Y^*. Then, identifying Y^* with a closed subspace of Y^{***}, we get

$$\sup_n |\langle T(x_n), y_m^*\rangle| = \sup_n |\langle x_n, T^*(y_m^*)\rangle| \le \sup_n \|x_n\|_X \|T^*(y_m^*)\|_{X^*}.$$

By (c) the sequence $(T^*(y_m^*))$ is norm null, which proves that $(T(x_n))$ is an L-set. Since every weakly convergent sequence is bounded, (a) is satisfied. ∎

For later use in Corollary 20, let us add the next proposition as a comment on the above assumption (c).

Proposition 6. *Let* X *and* Z *be Banach spaces such that* $L(Z, X^*) = K(Z, X^*)$. *Then* X *or* Z *does not contain* l_1.

Proof. Suppose that X contains a copy of l_1. Hence X^* does not have the Compact Range Property (for short (CRP)) (we refer the reader to [19] for the definition). Let T be an integral operator from Z into X^*. By assumption T is compact. By [4, Theorem 4.9, (a)⇔(f)], Z does not contain l_1. ∎

Using Theorem 3, a condition is given in [24] implying that a closed subset of $K(X, Y)$ has the (GP) property. Similarly, using Theorem 7 below we shall give a condition implying that a closed subspace of $K(X, Y)$ has (GP).

Theorem 7 ([13]). *Let* X *be a Banach space such that* X^* *has the Gelfand–Phillips property. Let* $M \subset K(X, Y)$ *be such that*

(1) *for every* $x \in X$, $M(x) = \{T(x) : T \in M\}$ *is relatively compact in* Y;
(2) *for every weak* null sequence* $(x_n^{**}) \subset X^{**}$, $\lim_n \sup_{T\in M} \|(T^{**}(x_n^{**}))\|_Y = 0$.

Then M *is relatively compact.* ∎

Definition 8. An operator $T : X \to Y$ is called *limited completely continuous* if it sends limited weakly null sequences to norm null sequences.

Theorem 9. *Let X^* be a dual space with the* (GP) *property and let E be a closed subspace of $K(X,Y)$. Assume that, for every $x^{**} \in X^{**}$, the evaluation map $\phi_{x^{**}} : K(X,Y) \to Y$ defined by $\phi_{x^{**}}(T) := T^{**}(x^{**})$ is limited completely continuous on E. Then E has the* (GP) *property.*

Proof. Let $M \subset E$ be a limited set. In order to prove that it is relatively compact we use Theorem 7. Condition (1) of that theorem follows from the last assumption. Indeed, as recalled after Definition 1, M is conditionally weakly compact, i.e. every sequence (T_n) in M has a weakly Cauchy subsequence (which we still denote by (T_n)).

Let $x \in X$ and assume that $(T_n(x))$ is not norm Cauchy. So there exist $\epsilon > 0$ and two subsequences (n_k), (m_k) increasing to ∞ such that $\|T_{n_k}(x) - T_{m_k}(x)\|_Y > \epsilon$. Since $(T_{n_k} - T_{m_k})$ is limited and weakly null, this contradicts the last assumption on ϕ_x.

Now, we verify condition (2) of Theorem 7. For every $T \in M$, the adjoint operator T^* is compact, i.e. $T^*(B_{Y^*})$ is relatively compact, so a limited set in X^*. Let $(x_n^{**}) \subset X^{**}$ be a weak* null sequence. Then

$$\|\phi_{x_n^{**}}(T)\| = \sup_{y^* \in B_{Y^*}} |\langle x_n^{**}, T^*(y)\rangle| \to 0.$$

So $(\phi_{x_n^{**}})$ is a pointwise norm null sequence of continuous linear operators. Using our assumptions, [24] implies that $(\phi_{x_n^{**}})$ converges uniformly on limited sets, i.e.

$$\lim_n \sup_{T \in M} \|\phi_{x_n^{**}}(T)\|_Y = 0,$$

which is condition (2) of Theorem 7. ∎

Remark 10. Assume that Y has (GP). It follows from the definitions that the last assumption of Theorem 9 is satisfied for $E = K(X,Y)$, for every Banach space X. On the other hand, if this last assumption is satisfied for $E = x_0^* \otimes Y$ for some $x_0^* \in X^*$ (X^* is not supposed to have (GP)) then E, hence Y, has (GP). Indeed, let $x_0 \in X$ be such that $\langle x_0, x_0^*\rangle \neq 0$. Then ϕ_{x_0} is an identification between $K(X',Y)$ and Y where $X' = \operatorname{span} x_0$, hence the assumption on ϕ_{x_0} means that limited weakly null sequences in Y are norm null; since every limited set in Y is weakly conditionally compact, this implies the claim.

It is proved in [7, Theorem 3.1] that the injective product $Z \otimes_\epsilon Y$ has (GP) if and only if Z and Y have (GP). Here we get

Corollary 11. *$K(X,Y)$ has* (GP) *if and only if X^* and Y have* (GP).

Proof. If X^* and Y have (GP), the claim follows from the first statement of Remark 10 and Theorem 9 applied to $E = K(X,Y)$. The converse is obvious since X^* and Y are isomorphic to closed subspaces of $K(X,Y)$. ∎

3. Lifting isomorphic properties to projective tensor products of Banach spaces. In this section we focus on some recently introduced isomorphic properties, namely the sequential right and L-limited properties. We study in particular the lifting of these properties from two Banach spaces to their projective tensor product.

The *right topology* on a Banach space X is the restriction of the Mackey topology $\tau(X^{**}, X^*)$ to X and it is also the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$-compact sets of X^* [21].

A sequence (x_n) in a Banach space X is right null if and only if it is a Dunford–Pettis set and it is weakly null [5]. Hence a bounded operator from X into Y sends right null sequences in X into right null sequences in Y.

DEFINITION 12 ([16]). A bounded subset K of X^* is a *right set* if for every right null sequence $(x_n) \subset X$ one has

$$\lim_n \sup_{x^* \in K} |x^*(x_n)| = 0.$$

If $T : X \to Y$ is a bounded operator, then T^* sends right sets in Y^* to right sets in X^*.

PROPOSITION 13. *The following assertions are equivalent:*

(a) *every bounded set in X^* is a right set;*
(b) *every right null sequence in X is norm null;*
(c) *X has* (DPrcp).

Proof. (a)⇒(b). This is obvious from Definition 12.

(b)⇒(c). Suppose that $A \subset X$ is a Dunford–Pettis set that is not relatively compact. Since a Dunford–Pettis set is conditionally weakly compact, there exist an $\epsilon > 0$, a weakly Cauchy sequence $(x_n) \subset A$ and two subsequences (x_{n_k}), (x_{m_k}), with (n_k) and (m_k) increasing to ∞, such that $\|x_{n_k} - x_{m_k}\| > \epsilon$ for every k. The sequence $(x_{n_k} - x_{m_k})$ is weakly null and it is a Dunford–Pettis sequence, hence it is right null [5] but not norm null, which ends the proof.

(c)⇒(a). Let K be a bounded set in X^* and let (x_n) be a right null sequence in X, hence a Dunford–Pettis set. Since X has (DPrcp), (x_n) is relatively compact, therefore norm null. It follows that

$$\sup_{x^* \in K} |x^*(x_n)| \le \sup_{x^* \in K} \|x^*\| \, \|x_n\| \to 0,$$

i.e. K is a right set. ∎

DEFINITION 14 ([21]). A bounded operator $T : X \to Y$ is *pseudo weakly compact* if it sends right null sequences in X to norm null sequences in Y.

A Banach space X is *sequentially right* (for short (SR)) if every pseudo weakly compact operator $T : X \to Y$ is weakly compact.

THEOREM 15 ([16]). *X is* (SR) *if and only if every right set in X^* is relatively weakly compact.* ■

COROLLARY 16. *A Banach space X is reflexive if and only if it has* (DPrcp) *and is* (SR).

Proof. Obviously a reflexive Banach space X is (SR). Moreover, since X^* does not contain l_1, $X^{**} = X$ has (DPrcp). The converse follows from Proposition 13 and Theorem 15. ■

In the following theorem we give a condition implying that the projective tensor product of two sequentially right spaces enjoys the same property.

THEOREM 17. *If X and Y are* (SR) *and $L(X, Y^*) = K(X, Y^*)$, then $X \otimes_\pi Y$ is* (SR).

Proof. Let $H \subset (X \otimes_\pi Y)^* = L(X, Y^*) = K(X, Y^*)$ be a right set. In order to prove our claim it is enough, by Theorem 15, to prove that H is relatively weakly compact. So let (T_n) be a sequence in H. We must find a subsequence (T_{n_k}) weakly converging to some $T \in K(X, Y^*)$. By Rainwater's Theorem and Ruess and Stegall's Theorem [23], it is enough to show that

$$\lim_k \langle T_{n_k}^{**}(x^{**}), y^{**} \rangle = \langle T^{**}(x^{**}), y^{**} \rangle$$

for all $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$.

(a) For $y^{**} \in Y^{**}$, $(T_n^*(y^{**})) = (\phi_{y^{**}}(T_n))$ where $\phi_{y^{**}}$ is the evaluation operator $K(X, Y^*) \to X^*$ defined by $\phi_{y^{**}}(T) = T^*(y^{**})$. Since (T_n) is a right set, so is $(\phi_{y^{**}}(T_n)) \subset X^*$. Since X is sequentially right, $(T_n^*(y^{**}))$ is relatively weakly compact in X^* by Theorem 15. Similarly, for every $x^{**} \in X^{**}$, $(T_n^{**}(x^{**}))$ is relatively compact in Y^* since Y is sequentially right.

(b) Let $F := \overline{\operatorname{span}\{T_n(x) : x \in X,\, n \in \mathbb{N}\}} \subset Y^*$. Observe that F is separable since the T_n's are compact, so that F^* is separable when equipped with its weak* topology. By definition the T_n's (respectively the T_n^{**}'s) send X (respectively X^{**}) into F; hence $T_n^*(y^{**})$, $y^{**} \in Y^{**}$, only depends on $\pi(y^{**})$ where π is the quotient map: $Y^{**} \to F^*$. Let D be a countable set in Y^{**} such that $\pi(D)$ is weak* dense in F^*. Since every sequence $(T_n^*(y^{**}))$, $y^{**} \in D$, is relatively weakly compact in X^*, by the diagonal procedure there is a subsequence (T_{n_k}) such that the following limit exists:

$$\lim_k \langle x^{**}, T_{n_k}^*(y^{**}) \rangle = \lim_k \langle T_{n_k}^{**}(x^{**}), y^{**} \rangle, \quad x^{**} \in X^{**},\ y^{**} \in D.$$

(c) By (a), for every $x^{**} \in X^{**}$, $(T_{n_k}^{**}(x^{**}))$ is relatively weakly compact in F. The existence of the last limit in (b) implies that the weak limit points of $(T_{n_k}^{**}(x^{**}))$ (which belong to F) coincide on $\pi(D)$, hence coincide in F. This proves the existence of $w_{x^{**}} \in Y^*$ such that

$$\langle w_{x^{**}}, y^{**} \rangle = \lim_k \langle T_{n_k}^{**}(x^{**}), y^{**} \rangle, \quad x^{**} \in X^{**},\ y^{**} \in Y^{**}.$$

In particular, for $y \in Y$, by the end of (b), $\langle w_{x^{**}}, y\rangle = \lim_k \langle T_{n_k}^{**}(x^{**}), y\rangle = \langle x^{**}, T_{n_k}^*(y)\rangle$. Since by (a), $(T_{n_k}^*(y))$ is relatively weakly compact in X^*, this means that $(T_{n_k}^*(y))$ weakly converges in X to some $z_y \in X^*$.

(d) Let $S : X^{**} \to Y^*$ be the bounded operator defined by

$$S(x^{**}) = w_{x^{**}}, \quad x^{**} \in X^{**}.$$

We claim that S is w^*-w^*-continuous. Indeed, let (x_α^{**}) be a net in X^{**} converging to 0 for the weak* topology of X^{**} and let $y \in Y$. By the last assertion of (c),

$$\lim_\alpha \langle S(x_\alpha^{**}), y\rangle = \lim_\alpha \langle w_{x_\alpha^{**}}, y\rangle = \lim_\alpha \langle x_\alpha^{**}, z_y\rangle = 0.$$

Let $T = S_{|X}$, hence $T : X \to Y^*$ is compact by assumption. We claim that $S = T^{**}$. Indeed, for $x^{**} \in X^{**}$ let (x_α) be a net in X converging to x^{**} for the weak* topology of X^{**}. Then, for every $y \in Y$, the w^*-w^*-continuity of S implies

$$\langle T^{**}(x^{**}), y\rangle = \lim_\alpha \langle T(x_\alpha), y\rangle = \lim_\alpha \langle S(x_\alpha), y\rangle = \langle S(x^{**}), y\rangle.$$

Hence, for all $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$,

$$\lim_k \langle T_{n_k}^{**}(x^{**}), y^{**}\rangle = \langle S(x^{**}), y^{**}\rangle = \langle T^{**}(x^{**}), y^{**}\rangle,$$

which ends the proof. ∎

We recall that a Banach space X has (RDP) (*Reciprocal Dunford–Pettis property*) if every completely continuous operator $T : X \to Y$ is weakly compact [21]. By Rosenthal's Theorem a space without a copy of l_1 has (RDP).

A Banach space X has *Pełczyński's property* (V) if every unconditionally converging operator $T : X \to Y$ is weakly compact [20]. See [9], [17], and [14] for more on these properties. The above definitions easily imply (see [16, Corollaries 3.3 and 3.20]) that

$$(V) \Rightarrow (SR) \Rightarrow (RDP).$$

Corollary 18. *Let* $1 < p \le \infty$. *Then* $l_p \otimes_\pi c_0$ *and* $l_p \otimes_\pi Y$, *where* Y *is the second Bourgain–Delbaen space, are sequentially right.*

Proof. The spaces c_0 and l_p, $1 < p \le \infty$, have Pełczyński's property (V) (see [20] for $p = \infty$), hence they are sequentially right by [16, Corollaries 3.3 and 3.20]. By [16, Example 3.34], Y is also sequentially right. Thus the first assumption of Theorem 17 is fulfilled. Since $Y^* = c_0^* = l_1$ has the Schur property, the second assumption follows if every $T : l_p \to l_1$ is weakly compact. This is obvious if $1 < p < \infty$; if $p = \infty$, this holds because every $T : l_\infty \to l_1$ is unconditionally converging since l_1 contains no copy of c_0 and l_∞ has property (V) [20]. ∎

REMARK 19. Since the second Bourgain–Delbaen space does not have Pełczyński's property (V), Corollary 18 cannot be deduced from [14, Theorem 2].

Part (a) of the corollary below generalizes Corollary 18. In view of Corollary 20 we recall [6, p. 212] that if a Banach space Y has (DP) and does not contain l_1, then Y^* is a Schur space.

COROLLARY 20.

(a) *If X and Y are sequentially right and one of X^* or Y^* is a Schur space, then $L(X, Y^*) = K(X, Y^*)$ and $X \otimes_\pi Y$ is sequentially right.*
(b) *If $X \otimes_\pi Y$ is sequentially right and $L(X, Y^*) = K(X, Y^*)$, then X and Y are sequentially right and at least one of them does not contain l_1.*

Proof. Observe that X and Y play symmetric roles.

(a) Assuming that Y^* has the Schur property, every weakly compact operator $T : X \to Y^*$ is compact. Hence by Theorem 17 we only have to verify that every $T : X \to Y^*$ is weakly compact. Since Y^* is a Schur space, it has (DPrcp), hence right null sequences in Y^* are norm null. Since T sends right null sequences to right null ones, hence to norm null ones, T is pseudo weakly compact. Since X is sequentially right, T is weakly compact.

(b) The first assumption implies that the closed subspaces X, Y are sequentially right. The second assumption and Proposition 6 imply that X or Y does not contain l_1. ■

We now consider analogues of Definitions 12, 14 and Theorems 15, 17 with Dunford–Pettis sequences replaced by limited ones.

DEFINITION 21 ([25]). A subset A of a dual space X^* is *L-limited* if every weakly null and limited sequence (x_n) in X converges (to 0) uniformly on A. A Banach space X has the *L-limited property* if every L-limited set in X^* is relatively weakly compact.

THEOREM 22. *If X and Y have the L-limited property and $L(X, Y^*) = K(X, Y^*)$ then $X \otimes_\pi Y$ has the same property.*

Proof. This follows by adaptating the proof of Theorem 17. Indeed, it is enough to replace (in the introduction and in (a)) "right set" by "L-limited set" and "X is sequentially right" by "X has the L-limited property". ■

Our next characterization of the L-limited property uses the following

THEOREM 23 ([25, Theorem 2.8]). *The following assertions are equivalent:*

(a) *X has the L-limited property;*
(b) *for every Banach space Y a bounded operator $T : X \to Y$ is limited completely continuous if and only if it is weakly compact.*

Proposition 24. *A Banach space X has the L-limited property if and only if it is sequentially right and has the Grothendieck property* (*i.e. weak* convergent sequences in X^* are weakly convergent*).

Proof. Suppose that X has the L-limited property. Let (x_n) be a limited and weakly null sequence in X. Then it is a Dunford–Pettis set by Definition 1, hence a right null sequence by [5]. Let $T : X \to Y$ be a pseudo weakly compact operator. So $(T(x_n))$ is norm null by definition. Then T is limited completely continuous by Definition 8. Hence, by Theorem 23, it is weakly compact. By Definition 14, X is sequentially right. Moreover, by [25, Theorem 2.10], X has the Grothendieck property.

Conversely, let (x_n) be a right null sequence. Then it is weakly null and a Dunford–Pettis set by [5]. Since X has the Grothendieck property, limited sets and right sets in X coincide. Thus, every limited completely continuous operator is pseudo weakly compact, and since X is sequentially right, it is a weakly compact operator by Definition 14; therefore X has the L-limited property, again by Theorem 23. ∎

We now introduce the (SR*) property, which plays the same role with respect to the sequential right property as Pełczyński's property (V*), respectively (RDP*) (both defined in Remark 28 below), plays with respect to (V), respectively (RDP). Definition 25 and Proposition 27 are analogues of Definitions 12, 14 and Theorem 15.

Definition 25. A bounded subset K of a Banach space X is a *right* set* if for every right null sequence (x_m^*) in X^*,

$$\lim_m \sup_{x\in K} |x_m^*(x)| = 0.$$

A Banach space X has the (SR*) *property* if every right* set in X is relatively weakly compact.

If X has the (SR) property, it is easy to see that X^* has the (SR*) property since a right* set in X^* is also a right set.

Remark 26. In the following example, X has (SR*) but X^* fails the (SR) property. Let $X = (\sum l_\infty^n)_1$. It has property (V*) [1, Theorem 9], hence it has the (SR*) property (see Remark 28 below). By [10], X contains a complemented copy of l_1, so X^* contains l_∞, hence is not weakly sequentially complete. Therefore X^* is not sequentially right by [16, Corollary 3.26].

Proposition 27. *The following properties are equivalent:*

(a) *X has the* (SR*) *property;*
(b) *for every Y, every pseudo weakly compact operator $T^* : X^* \to Y^*$ is weakly compact.*

Proof. (a)⇒(b). Let $T : Y \to X$ be such that $T^* : X^* \to Y^*$ is a pseudo weakly compact operator. We claim that T is weakly compact (hence T^* is too). Indeed, since X has (SR*) it is enough to verify that $T(B_Y)$ is a right* set. So let (x_m^*) be a right null sequence in X^*. Then, since T^* is pseudo weakly compact,

$$\lim_n \sup_{y \in B_Y} |\langle x_m^*, T(y)\rangle| = \lim_n \|T^*(x_m^*)\|_{Y^*} = 0.$$

(b)⇒(a). Let (x_n) be a bounded sequence in X. Define $S : l_1 \to X$ by

$$S(y) := \sum_{n=1}^{\infty} y_n x_n \quad \forall y = (y_n) \in l_1.$$

It is obviously a bounded operator and $S^* : X^* \to l_\infty$ is defined by $S^*(x^*) = (x^*(x_n))$.

Let now $K \subset X$ be a right* set and let $(x_n)_n$ be a sequence in K. We must show that $(x_n) = (S(e_n))$ is relatively weakly compact. This holds as soon as S^* is weakly compact, since then S is also weakly compact. By the assumption, it is enough to show that S^* is pseudo weakly compact, i.e., by definition, that S^* sends right null sequences (x_m^*) in X^* to norm null sequences in l_∞. So, let (x_m^*) be such a sequence. Since (x_n) is a right* set, Definition 25 implies

$$\lim_m \|S^*(x_m^*)\|_{l_\infty} = \lim_m \sup_n |x_m^*(x_n)| = 0,$$

and the proof is complete. ■

Remark 28. We recall that X has (V*) if and only if, for every $T : Y \to X$ such that T^* is unconditionally convergent, T^* is weakly compact.

A Banach space X has the (RDP*) property if every Dunford–Pettis subset is relatively weakly compact [11]. Moreover X has (RDP*) if and only if, for every $T : Y \to X$ such that T^* is completely continuous, T^* is weakly compact [11, Remark 1]. So the same arguments as for (V) ⇒ (SR) ⇒ (RDP) imply

$$(\mathrm{V}^*) \Rightarrow (\mathrm{SR}^*) \Rightarrow (\mathrm{RDP}^*).$$

We end by considering another well known similar property.

Definition 29 ([3]). A Banach space X has the *Bourgain–Diestel* (for short (BD)) *property* if every limited subset of X is relatively weakly compact.

In particular (RDP*) implies (BD).

Theorem 30. *If X and Y have the* (BD) (*respectively* (RDP*)) *property, and if $K_{w^*\text{-}w}(X^*, Y) = L_{w^*\text{-}w}(X^*, Y)$, then $K_{w^*\text{-}w}(X^*, Y)$ has the same property.*

Proof. The two proofs are similar so we only give the one involving the (RDP*) property. Let (h_n) be a Dunford–Pettis sequence in $K_{w^*\text{-}w}(X^*,Y)$; hence (h_n) is weakly conditionally compact and we may suppose that it is a weakly Cauchy sequence. So, for $x^* \in X^*$, the sequence $(h_n(x^*))$ is a Dunford–Pettis sequence in Y that is also weakly Cauchy. Since Y has the (RDP*) property, $(h_n(x^*))$ has a weak limit in Y. Define $h : X^* \to Y$ by

$$h(x^*) := w\text{-}\lim_n h_n(x^*).$$

Obviously, $h \in L(X^*,Y)$. As $K_{w^*\text{-}w}(X^*,Y)$ is identified with $K_{w^*\text{-}w}(Y^*,X)$ via the mapping $T \to T^*$, we may similarly define $k : Y^* \to X$ by

$$k(y^*) := w\text{-}\lim_n h_n^*(y^*).$$

It is easy to prove that $h = k^*$, hence h is w^*-w-continuous. Then $h \in L_{w^*}(X^*,Y)$, which is $K_{w^*\text{-}w}(X^*,Y)$ by assumption. Since $\langle h_n(x^*), y^*\rangle \to \langle h(x^*), y^*\rangle$ for all $x^* \in X^*$ and $y^* \in Y^*$, (h_n) is weakly converging to h by Rainwater's Theorem, which ends the proof. ■

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