STUDIA MATHEMATICA 235 (2) (2016)

Permutations of $\mathbb{Z}^d$ with restricted movement

by

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Abstract. We investigate dynamical properties of the set of permutations of $\mathbb{Z}^d$ with *restricted movement*, i.e., permutations π of $\mathbb{Z}^d$ such that $\pi(\mathbf{n}) - \mathbf{n}$ lies, for every $\mathbf{n} \in \mathbb{Z}^d$, in a prescribed finite set $\mathsf{A} \subset \mathbb{Z}^d$. For $d = 1$, such permutations occur, for example, in restricted orbit equivalence (cf., e.g., Boyle and Tomiyama (1998), Kammeyer and Rudolph (1997), or Rudolph (1985)), or in the calculation of determinants of certain bi-infinite multi-diagonal matrices. For $d \geq 2$ these sets of permutations provide natural classes of multidimensional shifts of finite type.

1. Introduction. Let $d \geq 1$, and let $S^\infty(\mathbb{Z}^d)$ be the group of all permutations of the integer lattice $\mathbb{Z}^d$. We fix a finite set $\mathsf{A} \subset \mathbb{Z}^d$ (always assumed to be nonempty) and write Π_{A} for the set of all permutations $\pi \colon \mathbf{n} \mapsto \pi(\mathbf{n})$ of $\mathbb{Z}^d$ which satisfy

$$\omega_{\mathbf{n}}^{(\pi)} := \pi(\mathbf{n}) - \mathbf{n} \in \mathsf{A} \quad \text{for every } \mathbf{n} \in \mathbb{Z}^d. \tag{1.1}$$

In view of (1.1) we may regard Π_{A} as a subset of the space $\prod_{\mathbf{n}\in\mathbb{Z}^d}(\mathbf{n} + \mathsf{A})$ which is obviously closed, and hence compact, in the product topology on $\prod_{\mathbf{n}\in\mathbb{Z}^d}(\mathbf{n} + \mathsf{A})$.

Every $\pi \in \Pi_{\mathsf{A}}$ is determined by the point $\omega^{(\pi)} = (\omega_{\mathbf{n}}^{(\pi)})_{\mathbf{n}\in\mathbb{Z}^d} \in \mathsf{A}^{\mathbb{Z}^d}$. Furthermore, the set $\Omega_{\mathsf{A}} = \{\omega^{(\pi)} : \pi \in \Pi_{\mathsf{A}}\}$ is a closed subset of the compact space $\mathsf{A}^{\mathbb{Z}^d}$, and the map $\pi \mapsto \omega^{(\pi)}$ from Π_{A} to Ω_{A} is a homeomorphism.

In view of the one-to-one correspondence between Π_{A} and Ω_{A} it will be convenient to write $\pi^{(\omega)} \in \Pi_{\mathsf{A}}$ the for the permutation corresponding to an element $\omega \in \Omega_{\mathsf{A}}$. Then $\omega = \omega^{(\pi^{(\omega)})}$ and $\pi^{(\omega^{(\pi)})} = \pi$.

PROPOSITION 1.1. *Let ς be the $\mathbb{Z}^d$-action on itself by translation, given by $\varsigma^{\mathbf{m}}(\mathbf{n}) = \mathbf{n} + \mathbf{m}$, and let $\sigma \colon \mathbf{m} \to \sigma^{\mathbf{m}}$ be the shift action $(\sigma^{\mathbf{m}}\omega)_{\mathbf{n}} = \omega_{\mathbf{n}+\mathbf{m}}$ of $\mathbb{Z}^d$ on $\mathsf{A}^{\mathbb{Z}^d}$.*

2010 *Mathematics Subject Classification*: 37A35, 37B10, 37B50.

Key words and phrases: infinite permutations of the integers, permutations with restricted movement, parity cocycle.
Received 8 January 2015; revised 3 August 2016.
Published online 14 October 2016.

DOI: 10.4064/sm8498-8-2016

(1) *For every* $\mathbf{m} \in \mathbb{Z}^d$, *the set* $\Pi_{\mathsf{A}} \subset S^\infty(\mathbb{Z}^d)$ *is invariant under the inner automorphism* $\pi \mapsto \mathrm{Ad}_{\varsigma^{\mathbf{m}}}(\pi) = \varsigma^{\mathbf{m}}\pi\varsigma^{-\mathbf{m}}$ *of* $S^\infty(\mathbb{Z}^d)$.
(2) *For every* $\omega \in \Omega_{\mathsf{A}}$ *and* $\mathbf{m} \in \mathbb{Z}^d$, $\mathrm{Ad}_{\varsigma^{\mathbf{m}}}(\pi) = \pi^{(\sigma^{-\mathbf{m}}\omega)}$. *Hence* $\Omega_{\mathsf{A}} \subset \mathsf{A}^{\mathbb{Z}^d}$ *is shift-invariant.*
(3) *The topological* $\mathbb{Z}^d$*-dynamical systems* $(\Pi_{\mathsf{A}}, \mathrm{Ad}_\varsigma)$ *and* $(\Omega_{\mathsf{A}}, \sigma)$ *are topologically conjugate.*
(4) *For every* $\mathbf{b} \in \mathbb{Z}^d$ *set* $\mathsf{A}+\mathbf{b} = \{\mathbf{a}+\mathbf{b} : \mathbf{a} \in \mathsf{A}\}$. *Then* $\Pi_{\mathsf{A}+\mathbf{b}} = \varsigma^{\mathbf{b}}\Pi_{\mathsf{A}}$ *and* $\Omega_{\mathsf{A}+\mathbf{b}} = \Omega_{\mathsf{A}}+(\dots,\mathbf{b},\mathbf{b},\mathbf{b},\dots)$. *Furthermore, the systems* $(\Omega_{\mathsf{A}}, \sigma)$ *and* $(\Omega_{\mathsf{A}+\mathbf{b}}, \sigma)$ (*and hence* $(\Pi_{\mathsf{A}}, \mathrm{Ad}_\varsigma)$ *and* $(\Pi_{\mathsf{A}+\mathbf{b}}, \mathrm{Ad}_\varsigma)$) *are topologically conjugate.*

Proof. For every $\pi \in S^\infty(\mathbb{Z}^d)$ and $\mathbf{m} \in \mathbb{Z}^d$, the permutation $\mathrm{Ad}_{\varsigma^{\mathbf{m}}}(\pi)$ satisfies $\mathrm{Ad}_{\varsigma^{\mathbf{m}}}(\pi)(\mathbf{n}) = \pi(\mathbf{n}-\mathbf{m})+\mathbf{m}$. Hence $\mathrm{Ad}_{\varsigma^{\mathbf{m}}}(\pi^{(\omega)})(\mathbf{n}) = \omega_{\mathbf{n}-\mathbf{m}} + \mathbf{n} - \mathbf{m} + \mathbf{m} = \omega_{\mathbf{n}-\mathbf{m}} + \mathbf{n} = (\sigma^{-\mathbf{m}}\omega)_{\mathbf{n}} + \mathbf{n} = \pi^{(\sigma^{-\mathbf{m}}\omega)}(\mathbf{n})$. Since $\pi(\mathbf{n}) - \mathbf{n} \in \mathsf{A}$ for every $\mathbf{n} \in \mathbb{Z}^d$ if and only if $\mathrm{Ad}_{\varsigma^{\mathbf{m}}}(\pi)(\mathbf{n}) - \mathbf{n} = \pi(\mathbf{n}-\mathbf{m}) - \mathbf{n} + \mathbf{m} \in \mathsf{A}$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, the set Π_{A} is Ad_ς-invariant. This proves (1) and (2), and (3)–(4) are obvious. ∎

In Sections 2–4 we restrict our attention to the case where $d = 1$ and A is a finite interval in $\mathbb{Z}$, which for simplicity we assume to be of the form $\mathsf{A}_K = \{0,\dots,K\}$ with $K \geq 1$ (cf. Proposition 1.1(4)). For notational economy we set

$$\Pi_K := \Pi_{\mathsf{A}_K} \quad \text{and} \quad \Omega_K := \Omega_{\mathsf{A}_K} \subset \mathsf{A}_K^{\mathbb{Z}}. \tag{1.2}$$

In Sections 2 and 3 we prove the following results:

- For every $\omega \in \Omega_K$ there exists an integer $a(\omega) \in \mathsf{A}_K$ such that
$$\Big|\sum_{n=m}^{m+N-1} \omega_n - Na(\omega)\Big| < K^2$$
for every $m \in \mathbb{Z}$ and $N \geq 1$ ((2.18) and Corollary 2.10). This integer can be viewed as the average 'shift' of $\mathbb{Z}$ effected by the permutation $\pi^{(\omega)} \in S^\infty(\mathbb{Z})$.
- The space Ω_K is a shift of finite type (abbreviated as *SFT*) which is not irreducible. Its irreducible components are given by
$$\Omega_{K,l} = \{\omega \in \Omega_K : a(\omega) = l\}, \qquad l \in \mathsf{A}_K$$
(Proposition 2.7).
- For $0 \leq l \leq K$, the *SFT*'s $\Omega_{K,l}$ and $\Omega_{K,K-l}$ are topologically conjugate (Proposition 2.12).
- $|\Omega_{K,0}| = |\Omega_{K,K}| = 1$, and for $l = 1,\dots,K-1$, the topological entropy $h(\Omega_{K,l})$ satisfies
$$(1 - l/K)\log(l+1) \leq h(\Omega_{K,l}) \leq \log(l+1)$$
(Theorem 3.3).

- For $0 \le l < K/2$, $h(\Omega_{K,l}) \le h(\Omega_{K,l+1})$ (Proposition 3.5).

In Section 4 we consider periodic points of the *SFT* Ω_K. If $\omega \in \Omega_K$ is periodic with period p, say, then the p-tuple

$$\pi^{(\omega)}_{(m,p)} := (\pi^{(\omega)}(m) \pmod p), \dots, \pi^{(\omega)}(m+p-1) \pmod p))$$

is, for every $m \in \mathbb{Z}$, a permutation of $(0, \dots, p-1)$. What is the parity (or sign) of this permutation? In Theorem 4.1 we prove that there exists a continuous cocycle $\mathsf{s}\colon \mathbb{Z} \times \Omega_K \to \{\pm 1\}$ which describes the parities of all these permutations. Together with the function $a\colon \Omega_K \to \mathsf{A}_K$ in (2.3), this parity cocycle s can be used to express the determinants of certain circulant-like matrices appearing in entropy calculations for algebraic actions of the discrete Heisenberg group (cf. [7, Section 8] and Example 4.4).

In Section 5 we consider permutations of $\mathbb{Z}^d$ with $d \ge 2$ and prove the following results:

- If A is a finite subset of $\mathbb{Z}^d$, $d \ge 2$, then Ω_{A} has positive entropy if and only if $|\mathsf{A}| \ge 3$ (Theorem 5.2).
- Let $d \ge 2$, and let $\mathsf{A} \subset \mathbb{Z}^d$ be a finite subset. Then Ω_{A} is topologically mixing if and only if $\mathsf{D} = \mathsf{A} - \mathsf{A}$ is not contained in a one-dimensional subspace of $\mathbb{R}^d$ (Theorem 5.6).

Finally, in Section 6, we return to one of the simplest $\mathbb{Z}^2$-*SFT*'s arising from permutations of $\mathbb{Z}^2$ with restricted movement, the space Ω_{A} with $\mathsf{A} = \{(0,0), (1,0), [0](0,1)\}$. This space had appeared already in Example 5.5, and we describe its dynamical properties (like entropy and the logarithmic growth-rate of the number of its periodic points) in greater detail.

2. Permutations of $\mathbb{Z}$ with bounded movement. We set $d = 1$. Fix $K \ge 1$, let $\mathsf{A}_K = \{0, \dots, K\}$, and write, as usual, σ instead of σ^1 for the shift $(\sigma\omega)_k = \omega_{k+1}$ on $\mathsf{A}^{\mathbb{Z}}$ (cf. Proposition 1.1). For $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \mathsf{A}_K^{\mathbb{Z}}$ and $m \in \mathbb{Z}$ we define

$$\tilde{\omega}_m = \omega_m + m. \tag{2.1}$$

In the notation of Section 1 we set $\Omega_K = \{\omega^{(\pi)} : \pi \in \Pi_K\} \subset \mathsf{A}_K^{\mathbb{Z}}$. Then Ω_K is the subshift of $\mathsf{A}_K^{\mathbb{Z}}$ defined by the following condition: for every $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Omega_K$, the map $\pi^{(\omega)}\colon n \mapsto \tilde{\omega}_n$, $n \in \mathbb{Z}$, is a permutation of $\mathbb{Z}$.

For the following lemma we recall a basic definition: if A is a finite alphabet and $L \ge 1$, then a subshift $\Omega \subset A^{\mathbb{Z}}$ is an L-*step SFT* if there exists a set $F \subset A^{L+1}$ of *forbidden words* such that Ω is the set of all sequences $\omega \in A^{\mathbb{Z}}$ not containing any of the words in F. The proof of the following lemma is immediate.

LEMMA 2.1. *For every $K \ge 1$ the subshift $\Omega_K \subset \mathsf{A}_K^{\mathbb{Z}}$ has the following properties:*

(1) *Let* $\omega \in \Omega_K$.

(a) *For any distinct* $m, n \in \mathbb{Z}$, $\tilde{\omega}_m \neq \tilde{\omega}_n$.

(b) *For every* $n \in \mathbb{Z}$, $n \in \{\tilde{\omega}_{n-K}, \dots, \tilde{\omega}_n\}$.

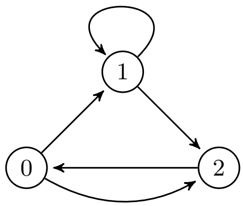
(2) *An element* $\omega \in \mathsf{A}_K^{\mathbb{Z}}$ *lies in* Ω_K *if and only if it satisfies the conditions* (a) *and* (b) *in* (1)*: for every* $n \in \mathbb{Z}$, *the set* $\{\tilde{\omega}_{n-K}, \dots, \tilde{\omega}_n\}$ *has* $K+1$ *distinct elements and contains* n. *In particular,* Ω_K *is a* K*-step SFT.*

(3) *If* $\omega \in \Omega_K$ *is periodic with period* p, *then*

$$\pi^{(\omega)}_{(m,p)} := \big(\tilde{\omega}_m \pmod p, \dots, \tilde{\omega}_{m+p-1} \pmod p\big) \tag{2.2}$$

is, for every $m \in \mathbb{Z}$, *a permutation of* $(0, \dots, p-1)$.

Example 2.2. If $K = 1$, $\Omega_1 = \{(\dots 0, 0, 0, \dots), (\dots, 1, 1, 1, \dots)\}$. For $K = 2$, Ω_2 is the union of the fixed points $(\dots 0, 0, 0, \dots)$, $(\dots, 2, 2, 2, \dots)$ and the mixing *SFT* determined by the directed graph



More generally, the following is true.

Lemma 2.3.

(1) *For every* $K \geq 1$, *the SFT* Ω_K *is* K*-step, but not* $(K-1)$*-step.*

(2) *For every* $K \geq 1$ *and* $\omega \in \Omega_K$ *set*

$$a(\omega) = |\{k > 0 : \tilde{\omega}_{-k} \geq 0\}| = |\{k = 1, \dots, K : \tilde{\omega}_{-k} \geq 0\}|. \tag{2.3}$$

Then

$$\Omega_{K,l} = \{\omega \in \Omega_K : a(\omega) = l\} \tag{2.4}$$

is, for every $l = 0, \dots, K$, *a closed, shift-invariant subset of* Ω_K *which is a* $(K-1)$*-step SFT.*

(3) *The map* $a \colon \Omega_K \to \mathsf{A}_K$ *defined by* (2.3) *is continuous.*

For the proofs of Lemma 2.3 and Theorem 4.1, Figure 1 will be useful, where we assume that $p > K \geq 2$.

Consider the following subsets of $\mathbb{Z}^2$ illustrated in Figure 1 (note that the first coordinate in this picture increases as one moves to the right, while the second increases as one moves down):

$$\begin{aligned}
\text{the square}\quad & Q=\{(k_1,k_2):0\le k_i<p\},\\
\text{the triangles}\quad & A=\{(k_1,k_2):0\le k_1<K,\ k_1-K\le k_2<0\},\\
& B=\{(k_1,k_2):0\le k_1<K,\ 0\le k_2\le k_1\},\\
& D=\{(k_1,k_2):p-K\le k_1<p,\ p-K\le k_2\le k_1\},\\
& A'=\{(k_1,k_2):p\le k_1<p+K,\ k_1-K\le k_2<p\},\\
& A^*=\{(k_1,k_2):0\le k_1<K,\ p-K+k_1\le k_2<p\},\\
\text{the trapezoid}\quad & C=\{(k_1,k_2):K\le k_1<p,\\
& \qquad k_1-K\le k_2\le \min(k_1,p-K-1)\}.
\end{aligned}$$

We fix $\omega\in\Omega_K$, set

$$S(\omega)=\{(\tilde\omega_k,k):k\in\mathbb{Z}\}\subset\mathbb{Z}^2, \tag{2.5}$$

and denote by $\tilde A=A\cap S(\omega)$, $\tilde B=B\cap S(\omega),\ldots,\tilde A'=A'\cap S(\omega)$, $\tilde C=C\cap S(\omega)$, and $\tilde Q=Q\cap S(\omega)=\tilde B\cup\tilde C\cup\tilde D$ the intersections of these sets with $S(\omega)$ (note that $A^*\cap S(\omega)=\emptyset$).

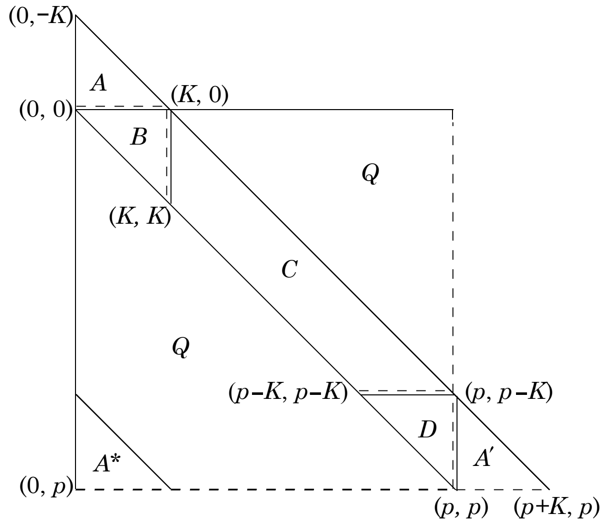


Fig. 1.

Proof of Lemma 2.3. We start by proving the first assertion in (2). Fix $\omega\in\Omega_K$, define $S(\omega)$ by (2.5), and consider the sets $\tilde Q,\tilde A,\ldots,\tilde A',\tilde C$ defined above. According to the definition of Ω_K,

$$\{(m,n):n\in\mathbb{Z}\}\cap S(\omega)=\{(m,m-k):0\le k\le K\}\cap S(\omega)=\{(m,\tilde\omega_m)\}$$

for every $m\in\mathbb{Z}$. Similarly, $|\{(m,n):m\in\mathbb{Z}\}\cap S(\omega)|=\{(n+k,n):0\le k\le K\}\cap S(\omega)|=1$ for every $n\in\mathbb{Z}$. It follows that $|\tilde A\cup\tilde B|=|\tilde D\cup\tilde A'|=K$

and $|\tilde{A} \cup \tilde{Q}| = |\tilde{Q} \cup \tilde{A}'| = p$. Since $|\tilde{A}| = a(\omega)$ (cf. (2.3)), we obtain $a(\omega) = |\tilde{A}| = |\tilde{A}'| = a(\sigma^p\omega)$.

As $p > K$ is arbitrary, $a(\omega) = a(\sigma\omega)$ for every $\omega \in \Omega_K$, which proves that the sets $\Omega_{K,l}$, $l = 0, \dots, K$, are shift-invariant.

In order to prove (1) we observe that any $\omega \in \Omega_K$ with $\omega_{-K} = \dots = \omega_{-1} = 0$ satisfies $a(\omega) = 0$ and thus lies in $\Omega_{K,0}$. A point $\omega' \in \Omega_K$ with $\omega_{-K+1} = \dots = \omega_{-1} = 0$ and $\omega_0 = 1$ satisfies $a(\sigma\omega') = 1$ and hence $\omega' \in \Omega_{K,1}$. If Ω_K were $(K-1)$-step, there would exist a point $\omega'' \in \Omega_K$ with $\omega''_{-K} = \dots = \omega''_{-1} = 0$ and $\omega''_0 = 1$, and hence with $0 = a(\omega'') \neq a(\sigma\omega'') = 1$. This would contradict the shift-invariance of $a(\cdot)$ shown above.

Having verified (1), we return to (2) by showing that each $\Omega_{K,l}$ is $(K-1)$-step (it is obviously K-step). If $l = 0$ or $l = K$, then $\Omega_{K,l}$ consists of a single fixed point and is therefore K-step. If $0 < l < K$ we observe that a point $\omega \in \Omega_K$ lies in $\Omega_{K,l}$ if and only if, for every $n \in \mathbb{Z}$, either

$$|\{k = 1, \dots, K-1 : \omega_{n-k} > k\}| = l \quad \text{and} \quad \omega_n = 0, \tag{2.6}$$

or

$$\begin{aligned} |\{k = 1, \dots, K-1 : \omega_{n-k} > k\}| = l-1, \quad \omega_n > 0, \quad \text{and} \\ \omega_n \neq \omega_{n-k} - k \quad \text{for } k = 1, \dots, K-1. \end{aligned} \tag{2.7}$$

This proves that $\Omega_{K,l}$ is $(K-1)$-step.

Finally, we note that the continuity of the map $a\colon \Omega_K \to \mathsf{A}$ claimed in (3) is an immediate consequence of (2.3). ∎

We will re-code $\Omega_{K,l}$ as a one-step *SFT* $X_{K,l}$ with alphabet

$$\mathsf{B}_{K,l} = \{\mathsf{a} \subset \mathsf{A}_{K-1} : |\mathsf{a}| = l\}, \tag{2.8}$$

the set of all l-element subsets of $\mathsf{A}_{K-1} = \{0, \dots, K-1\}$. To visualize $X_{K,l}$ we adapt a simile from [9, Section 2.1].

Definition 2.4 (The *SFT* $X_{K,l}$—a strange office). Consider an office with K desks, numbered from 0 to $K-1$, and arranged side by side. At each desk there is a clerk who can handle at most one file at any given time. At regular intervals each clerk checks his desk. If he finds a file there he passes it on to his neighbour on the left. If the clerk sitting at the leftmost desk (the desk 0) finds a file on his desk he carries it over to one of the empty desks, chosen at random (like everybody else, he may simultaneously receive a file from his neighbour on the right).

The *SFT* $X_{K,l}$ corresponds to the *modus operandi* of this office if there are a total of l files in circulation.

To make this description more formal we view $\mathsf{B}_{K,l}$ as the set of possible positions of l files on the K desks and define a $(\mathsf{B}_{K,l} \times \mathsf{B}_{K,l})$-matrix $\mathsf{M} = \mathsf{M}_{K,l}$ with entries in $\{0, 1\}$ by setting, for all $\mathsf{a}, \mathsf{b} \in \mathsf{B}_{K,l}$, $\mathsf{M}(\mathsf{a}, \mathsf{b}) = 1$ if and only if one of the following conditions is satisfied:

(M1) $0 \notin \mathsf{a}$ and $\mathsf{b} = \mathsf{a} - 1 := \{a - 1 : a \in \mathsf{a}\}$ (i.e., there is no file on desk 0, and each file is moved one step to the left),

(M2) $0 \in \mathsf{a}$ and $\mathsf{b} = (\mathsf{a}' - 1) \cup \{j\}$ for some $j \in \mathsf{A}_{K-1} \smallsetminus (\mathsf{a}' - 1)$, where $\mathsf{a}' = \mathsf{a} \smallsetminus \{0\}$ (i.e., there is a file on desk 0 which gets dropped on the floor; then all the other files are moved one position to the left, and the file on the floor is placed on an empty desk).

We shall prove that $\Omega_{K,l}$ is conjugate to the *SFT*

$$X_{K,l} = \{(\mathsf{a}_n)_{n\in\mathbb{Z}} \in \mathsf{B}_{K,l}^{\mathbb{Z}} : \mathsf{M}(\mathsf{a}_n, \mathsf{a}_{n+1}) = 1 \text{ for every } n \in \mathbb{Z}\} \tag{2.9}$$

defined by the transition matrix M.

Proposition 2.5. *Let $\phi_{K,l}\colon \Omega_{K,l} \to \mathsf{B}_{K,l}^{\mathbb{Z}}$ be the map given by*

$$\phi_{K,l}(\omega)_n = \pi_1(A \cap S(\sigma^n\omega)) \tag{2.10}$$

for every $\omega \in \Omega_{K,l}$ and $n \in \mathbb{Z}$, where $A \subset \mathbb{Z}^2$ is the triangle appearing in Figure 1, $S(\sigma^n\omega)$ is defined in (2.5), and $\pi_1\colon \mathbb{Z}^2 \to \mathbb{Z}$ is the first coordinate projection. This map has the following properties:

(1) *$\phi_{K,l}$ is shift-equivariant and injective.*
(2) *$\phi_{K,l}(\Omega_{K,l}) = X_{K,l}$.*

Proof. The shift-equivariance of $\phi_{K,l}$ is obvious from (2.10). The injectivity of $\phi_{K,l}$ follows from (2.6)–(2.7): for every $\omega \in \Omega_{K,l}$ and $n \in \mathbb{Z}$,

$$\begin{aligned} &\omega_n = 0 \quad \text{if } 0 \notin \phi_{K,l}(\omega)_n, \\ &\omega_n \in (\phi_{K,l}(\omega)_{n+1} + 1) \smallsetminus \phi_{K,l}(\omega)_n \quad \text{otherwise.} \end{aligned} \tag{2.11}$$

Since this holds for every $n \in \mathbb{Z}$, the sequence $\phi_{K,l}(\omega)$ determines ω, which proves (1).

(2) If a sequence $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \mathsf{A}_K^{\mathbb{Z}}$ satisfies the conditions (2.6)–(2.7) for every $n \in \mathbb{Z}$, then the sets $\phi_{K,l}(\omega)_n$, $n \in \mathbb{Z}$, satisfy (M1)–(M2). This shows that $\phi_{K,l}(\Omega_{K,l}) \subset X_{K,l}$. Conversely, if $(\mathsf{a}_n)_{n\in\mathbb{Z}} \in X_{K,l}$, and if we set

$$\omega_n = 0 \quad \text{if } 0 \notin \mathsf{a}_n, \quad \text{and} \quad \omega_n \in (\mathsf{a}_{n+1} + 1) \smallsetminus \mathsf{a}_n \quad \text{otherwise} \tag{2.12}$$

for every $n \in \mathbb{Z}$, we obtain an element $\omega = (\omega_n) \in \Omega_{K,l}$ with $\phi_{K,l}(\omega) = (\mathsf{a}_n)_{n\in\mathbb{Z}}$. This proves (2). ∎

Proposition 2.6. *Let M be the transition matrix of $X_{K,l}$ in* (M1)–(M2). *For every state $\mathsf{a} \in \mathsf{B}_{K,l}$ of $X_{K,l}$, the* follower set $\mathbf{f}(\mathsf{a}) = \{\mathsf{b} \in \mathsf{B}_{K,l} : \mathsf{M}(\mathsf{a}, \mathsf{b}) = 1\}$ *of a is given by*

$$\mathbf{f}(\mathsf{a}) = \{\mathsf{b} \in \mathsf{B}_{K,l} : \mathsf{b} \supset (\mathsf{a} - 1) \cap A_{K-1}\} \tag{2.13}$$

and satisfies

$$|\mathbf{f}(\mathsf{a})| = \begin{cases} K - l + 1 & \text{if } 0 \in \mathsf{a}, \\ 1 & \text{otherwise.} \end{cases} \tag{2.14}$$

The predecessor set $\mathbf{p}(\mathsf{a}) = \{\mathsf{b} \in \mathsf{B}_{K,l} : \mathsf{M}(\mathsf{b},\mathsf{a}) = 1\}$ *of* a *is given by*

$$\mathbf{p}(\mathsf{a}) = \{\mathsf{b} \in \mathsf{B}_{K,l} : \mathsf{b} \subset \{0\} \cup (\mathsf{a}+1)\}, \tag{2.15}$$

and satisfies

$$|\mathbf{p}(\mathsf{a})| = \begin{cases} 1 & \text{if } K-1 \in \mathsf{a}, \\ l+1 & \text{otherwise.} \end{cases} \tag{2.16}$$

Proof. Let $\mathsf{a} \in \mathsf{B}_{K,l}$. If $0 \notin \mathsf{a}$, then $\mathsf{a} - 1 = \{a-1 : a \in \mathsf{a}\}$ is the only follower of a. If $0 \in \mathsf{a}$ we let $\mathsf{a}' = \mathsf{a} \smallsetminus \{0\}$ and set $\mathsf{b} = (\mathsf{a}'-1) \cup \{j\}$ for some $j \in \mathsf{A}_K \smallsetminus (\mathsf{a}'-1)$. Clearly, there are $|\mathsf{A}_K \smallsetminus (\mathsf{a}'-1)| = K-l+1$ possibilities for doing this, and every b obtained in this manner is a follower of a.

To describe the predecessors of a we first assume that $K-1 \notin \mathsf{a}$ and set $\mathsf{a}'' = (\mathsf{a}+1) \cup \{0\}$. This set has $l+1$ elements, and every $\mathsf{b} \subset \mathsf{A}_K$ obtained by removing one of the elements of a'' is a predecessor of a. Since there are $l+1$ possibilities for doing this, $\mathbf{p}(\mathsf{a}) = l+1$. If, on the other hand, $K-1 \in \mathsf{a}$, then the set $\mathbf{p}(\mathsf{a})$ in (2.15) has the single element $\mathsf{b} = \{0\} \cup ((\mathsf{a} \smallsetminus \{K-1\})+1) \in \mathsf{B}_{K,l}$. ∎

PROPOSITION 2.7. *For every* $K \ge 2$ *and* $l \in \{1,\dots,K-1\}$, *the SFT* $\Omega_{K,l}$ *is irreducible and aperiodic.*

Proof. In view of the isomorphism $\phi_{K,l}\colon \Omega_{K,l} \to X_{K,l}$ it suffices to prove the analogous assertion for the *SFT* $X_{K,l}$ in (2.9). The state

$$\mathsf{e} = \{0,\dots,l-1\} \in \mathsf{B}_{K,l} \tag{2.17}$$

satisfies $|(\mathsf{e}+1) \smallsetminus \mathsf{e}| = 1$, so that $\mathsf{e} \in \mathbf{f}(\mathsf{e})$ (and hence $\mathsf{e} \in \mathbf{p}(\mathsf{e})$) by condition (M2). Furthermore there obviously exists a *path* (i.e., a finite sequence of allowed transitions) from every $\mathsf{b} \in \mathsf{B}_{K,l}$ to e. Conversely, if $\mathsf{b} = \{b_1,\dots,b_l\}$ with $b_1 < \dots < b_l$, there is a path $\mathsf{e} = \{0,\dots,l-1\} \to \{0,1,\dots,l-2, b_1+l-1\} \to \{0,\dots,l-3, b_1+l-2, b_2+l-2\} \to \dots \to \{0, b_1+1,\dots,b_{l-1}+1\} \to \mathsf{b}$ of length l. This shows that $X_{K,l}$ is irreducible and aperiodic (since it contains a fixed point). ∎

PROPOSITION 2.8. *For every* $K \ge 1$ *and* $l \in \{0,\dots,K\}$ *there exists a continuous map* $b_{K,l}\colon \Omega_{K,l} \to \mathbb{Z}$ *such that*

$$\omega_0 = l + b_{K,l}(\omega) - b_{K,l}(\sigma\omega) \tag{2.18}$$

for every $\omega \in \Omega_{K,l}$.

Proof. Fix K,l and $\omega \in \Omega_{K,l}$. Recall the triangle $A = \{(k_1,k_2) : 0 \le k_1 < K,\ k_1 - K \le k_2 < 0\} \subset \mathbb{Z}^2$ from Figure 1 and for every $m \in \mathbb{Z}$ set $A_m = A + (m,m)$. By (2.4), $a(\sigma^n\omega) = |A \cap S(\sigma^n\omega)| = |A_n \cap S(\omega)| = l$ for every $n \in \mathbb{Z}$. Hence $\sum_{m=1}^{N} |A_m \cap S(\omega)| = Nl$ for every $N \ge 1$.

For any $m,n \in \mathbb{Z}$, $(n,\tilde{\omega}_n) \in A_m$ if and only if $\omega_n > 0$ and $m = n+1,\dots,n+\omega_n$. In other words, $\sum_{m\in\mathbb{Z}} 1_{A_m}(n,\tilde{\omega}_n) = \sum_{m=1}^{K} 1_{A_{n+m}}(n,\tilde{\omega}_n) = \omega_n$, where $1_{A_m}\colon \mathbb{Z}^2 \to \mathbb{Z}$ is the indicator function of A_m.

Let $N > K$ (it actually suffices to choose $N > 0$, but with $N > K$ one can use Figure 1 to see what is going on here). Then

$$
\begin{aligned}
(2.19)\qquad Nl &= \sum_{m=1}^{N} |A_m \cap S(\omega)| = \sum_{m=1}^{N}\sum_{n\in\mathbb{Z}} 1_{A_m}(n,\tilde{\omega}_n) \\
&= \sum_{n\in\mathbb{Z}}\sum_{m=1}^{N} 1_{A_m}(n,\tilde{\omega}_n) \\
&= \sum_{n=0}^{N-1}\sum_{m=1}^{K} 1_{A_{n+m}}(n,\tilde{\omega}_n) + \sum_{n=-K+1}^{-1}\sum_{m=1}^{K-1} 1_{A_m}(n,\tilde{\omega}_n) \\
&\quad - \sum_{n=N-K+1}^{N-1}\sum_{m=1}^{K-1} 1_{A_{N+m}}(n,\tilde{\omega}_n) \\
&= \sum_{n=0}^{N-1}\omega_n + \sum_{n=-K+1}^{-1}\sum_{m=1}^{K-1} 1_{A_m}(n,\tilde{\omega}_n) \\
&\quad - \sum_{n=N-K+1}^{N-1}\sum_{m=1}^{K-1} 1_{A_{N+m}}(n,\tilde{\omega}_n).
\end{aligned}
$$

For every $\omega \in \Omega_{K,l}$ we set

$$
\begin{aligned}
(2.20)\qquad b_{K,l}(\omega) &= \sum_{m=1}^{\infty} |A \cap A_m \cap S(\omega)| = \sum_{m=1}^{K-1} |A \cap A_m \cap S(\omega)| \\
&= \sum_{n=-K+1}^{-1}\sum_{m=1}^{K-1} 1_{A_m}(n,\tilde{\omega}_n).
\end{aligned}
$$

Then

$$
\begin{aligned}
b_{K,l}(\sigma^N\omega) &= \sum_{m=1}^{K-1} |A \cap A_m \cap S(\sigma^N\omega)| = \sum_{m=1}^{K-1} |A_N \cap A_{N+m} \cap S(\omega)| \\
&= \sum_{n=N-K+1}^{N-1}\sum_{m=1}^{K-1} 1_{A_{N+m}}(n,\tilde{\omega}_n),
\end{aligned}
$$

and (2.19) shows that

$$
(2.21)\qquad \sum_{n=0}^{N-1}\omega_n = Nl + b_{K,l}(\omega) - b_{K,l}(\sigma^N\omega)
$$

for every $\omega \in \Omega_{K,l}$ and $N \geq 0$. By setting $N = 1$ in (2.21) we obtain (2.18). ∎

Corollary 2.9. *Let $K \geq 2$, and let $\omega \in \Omega_K$ be a periodic point with period p, say. Then $l := \frac{1}{p}\sum_{n=0}^{p-1}\omega_n \in \mathsf{A}_K$ and $\omega \in \Omega_{K,l}$.*

Corollary 2.10. *For every $K \geq 1$, $l \in \{0, \dots, K\}$, and $\omega \in \Omega_{K,l}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega_n = l. \tag{2.22}$$

Remarks 2.11. (1) Equation (2.18) shows that the cocycle $\boldsymbol{\omega}\colon \mathbb{Z} \times \Omega_{K,l} \to \mathbb{Z}$ defined by

$$\boldsymbol{\omega}(n,\omega) = \begin{cases} \sum_{k=0}^{n-1} \omega_k & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\omega(-n, \sigma^n \omega) & \text{if } n < 0 \end{cases}$$

is cohomologous to the homomorphism $n \mapsto ln$, with transfer function $b_{K,l}$ given by (2.20). In the context of bounded topological orbit equivalence, an analogous formula appears in [2, Lemma 2.6 and Theorem 2.3(2)].

(2) When combined with Proposition 1.1(4), Corollary 2.10 is equivalent to [8, Theorem 1]. In particular, the integer $l = a(\omega)$ in (2.3)–(2.4) determines, for every $\omega \in \Omega_K$, the position of the 'main diagonal' of the bi-infinite permutation matrix associated with ω (cf. [8, p. 526]).

We end this section with a few comparisons between the *SFT's* $\Omega_{K,l}$ (or $X_{K,l}$) for different values of K and l.

Proposition 2.12. *For every $K \geq 2$ and $l \in \{1, \dots, K-1\}$ we have:*

(1) $\Omega_{K,l} \subset \Omega_{K+1,l}$.
(2) $\Omega_{K,l} + 1 = \{(\omega_m + 1)_{m\in\mathbb{Z}} : \omega \in \Omega_{K,l}\} \subset \Omega_{K+1,l+1}$.
(3) *The SFT's $\Omega_{K,l}$ and $\Omega_{K,K-l}$ are topologically conjugate.*

Proof. The first two assertions are obvious. For the third, we define a shift-equivariant bijection $\Phi\colon \mathsf{A}_K^{\mathbb{Z}} \to \mathsf{A}_K^{\mathbb{Z}}$ by setting

$$\Phi(\omega)_m = K - \omega_{-m} \tag{2.23}$$

for every $\omega \in \Sigma_K$ and $m \in \mathbb{Z}$. It is clear that $\Phi(\Omega_K) = \Omega_K$. In order to check that $\Phi(\Omega_{K,l}) = \Omega_{K,K-l}$ we take another look at Figure 1: if $\omega \in \Omega_{K,l}$, then $|\tilde{A}| = |A \cap S(\omega)| = l$. Since $|(A \cap S(\omega)) \cup (B \cap S(\omega))| = K$, we obtain $|B \cap S(\omega)| = K - l$. Finally, we note that $|A \cap S(\omega)| = |B \cap S(\Phi(\omega))|$ and $|B \cap S(\omega)| = |A \cap S(\Phi(\omega))|$, so that $\Phi(\omega) \in \Omega_{K,K-l}$ if and only if $\omega \in \Omega_{K,l}$. ∎

Remark 2.13. The conjugacy between $\Omega_{K,l}$ and $\Omega_{K,K-l}$ can, of course, be also expressed in terms of the *SFT's* $X_{K,l}$ and $X_{K,K-l}$. For every $\mathsf{a} \in \mathsf{B}_{K,l}$ we set

$$\mathsf{a}^* = \{K - 1 - j : j \in (\mathsf{A}_{K-1} \smallsetminus \mathsf{a})\}. \tag{2.24}$$

If $\mathsf{a} \in \mathsf{B}_{K,l}$ then $\mathbf{p}(\mathsf{a}^*) = \{\mathsf{b}^* : \mathsf{b} \in \mathbf{f}(\mathsf{a})\} = \mathbf{f}(\mathsf{a})^*$ and $\mathbf{f}(\mathsf{a}^*) = \{\mathsf{b}^* : \mathsf{b} \in \mathbf{p}(\mathsf{a})\} = \mathbf{p}(\mathsf{a})^*$. The corresponding shift-equivariant isomorphism $\Psi_{K,l}$:

$X_{K,l} \to X_{K,K-l}$ is given by

$$\Psi_{K,l}(\mathsf{a})_m = \mathsf{a}^*_{-m} \tag{2.25}$$

for every $(\mathsf{a}_n) \in X_{K,l}$ and $m \in \mathbb{Z}$.

3. Entropy

LEMMA 3.1. *For any K, l with $0 < l < K$, the topological entropy of $\Omega_{K,l}$ satisfies $0 < h(\Omega_{K,l}) \leq \log(l+1)$.*

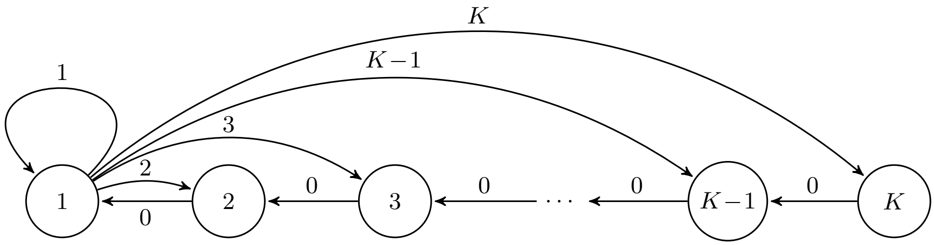
Proof. Since $\Omega_{K,l}$ and $X_{K,l}$ are topologically conjugate by Proposition 2.5, their entropies coincide. Clearly, $h(X_{K,l}) > 0$ since $X_{K,l}$ contains the 'diamond' consisting of the paths $\mathsf{e} \to \mathsf{e} \to \mathsf{e}$ and $\mathsf{e} \to \{0, 1, \dots, l-2, l\} \to \mathsf{e}$ (cf. (2.17)), and $h(X_{K,l}) \leq \log(l+1)$ since every state $\mathsf{a} \in \mathsf{B}_{K,l}$ has at most $l+1$ predecessors (cf. (2.16)). ∎

Proposition 2.12 shows that

$$h(\Omega_{K,l}) \leq h(\Omega_{K+1,l}) \quad \text{and} \quad h(\Omega_{K,l}) \leq h(\Omega_{K+1,l+1})$$

whenever $K > l > 0$. Our next aim is to investigate $\lim_{K\to\infty} h(\Omega_{K,l})$ for every $l \geq 1$.

EXAMPLE 3.2 (The *SFT* $\Omega_{K,1}$). For every $K \geq 1$, $h(\Omega_{K,1})$ is equal to $\log \beta_K$, where β_K is the largest root of the polynomial $f_K(x) = x^K - x^{K-1} - \dots - 1$. Hence $\lim_{K\to\infty} h(\Omega_{K,1}) = \log 2$. Indeed, the *SFT* $X_{K,1}$ (which is conjugate to $\Omega_{K,1}$) has the form



and is described by the $(K \times K)$-transition matrix

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

with characteristic polynomial f_K. The largest root β_K of f_K satisfies $1 = \beta_K^{-1} + \dots + \beta_K^{-K}$. Now, both $\beta_K \to 2$ and $h(\Omega_{K,1}) = h(X_{K,1}) = \log \beta_K \to \log 2$ as $K \to \infty$.

The following theorem yields an analogous result for arbitrary l.

THEOREM 3.3. *For any* K, l *with* $0 < l < K$, $(1 - l/K)\log(l+1) \leq h(\Omega_{K,l}) \leq \log(l+1)$. *In particular,* $\lim_{K\to\infty} h(\Omega_{K,l}) = \log(l+1)$ *for every* $l \geq 1$.

Proof. Since $\Omega_{K,l}$ is topologically conjugate to $X_{K,l}$ by Proposition 2.5, it will suffice to prove the corresponding assertion for the *SFT* $X_{K,l}$. From Proposition 2.7 we know that $h(X_{K,l}) \leq \log(l+1)$.

Fix K, l and consider the state $\mathsf{e} = \{0, \ldots, l-1\}$ in (2.17). We are interested in the number of paths of length K in $X_{K,l}$ which begin and end in e. For this it will be convenient to work from right to left, starting from e: by (2.16), e has $l+1$ predecessors $\mathsf{a}_{-1}^{(i)}$, $i = 1, \ldots, l+1$, say, each of which has a maximal element $\leq l$. If $l < K-1$ we can repeat this argument and obtain, for each $\mathsf{a}_{-1}^{(i)}$, $l+1$ predecessors. This second generation of predecessors has maximal elements which are all $\leq l+1$. After repeating this $K-l$ times we have a total of $(l+1)^{K-l}$ distinct allowed paths of length $K-l+1$ in $X_{K,l}$, all of which have e as their final state. If v is such a path with initial state b, say, we can extend it to the left by choosing l successive predecessors of b until we arrive at e (as explained in the proof of Proposition 2.7).

This construction results in $(l+1)^{K-l}$ distinct allowed paths of length $K+1$ in $X_{K,l}$, all of which begin and end in e. Since we can concatenate these paths arbitrarily (overlapping in the symbol e), we have proved that $h(X_{K,l}) \geq \frac{1}{K}\log((l+1)^{K-l}) = \frac{K-l}{K}\log(l+1)$, as claimed. The last assertion is a trivial consequence of this. ∎

REMARK 3.4. According to Proposition 2.12(3), $h(\Omega_{K,l}) = h(\Omega_{K,K-l})$ for every $K \geq 1$ and $l = 0, \ldots, K$ (where $h(\Omega_{K,l}) = 0$ if $l = 0$ or $l = K$). This allows us to symmetrize the first inequality in Theorem 3.3 and to conclude that

$$\frac{1}{K}\max((K-l)\log(l+1), l\log(K-l+1)) \leq h(\Omega_{K,l}) \leq \log(l+1)$$

for every $K \geq 1$ and $l = 0, \ldots, K$.

For reasons of symmetry one would also expect that $h(\Omega_{K,l})$ is maximal if l lies in the middle of the range $\{0, \ldots, K\}$, i.e., if $|K/2 - l| \leq 1/2$. Our next proposition shows that this is indeed the case.

PROPOSITION 3.5. *For all* K, l *with* $K \geq 2$ *and* $1 \leq l \leq K/2$,

$$h(\Omega_{K,l-1}) \leq h(\Omega_{K,l}). \tag{3.1}$$

For the proof of Proposition 3.5 we need additional notation. For every finite set $\mathsf{u} \subset \mathbb{Z}$ and every $j \in \mathbb{Z}$ we set $\mathsf{u} + j = \{i + j : i \in \mathsf{u}\}$.

Fix K, l with $K \geq 2$ and $0 \leq l \leq K$. For any pair u, v of disjoint (and possibly empty) subsets of $\mathbb{Z}$ we set

$$\mathsf{B}_{K,l}^{[\mathsf{u},\mathsf{v}]} = \begin{cases} \{\mathsf{a} \in \mathsf{B}_{K,l} : \mathsf{u} \subset \mathsf{a} \text{ and } \mathsf{a} \cap \mathsf{v} = \emptyset\} & \text{if } (\mathsf{u} \cup \mathsf{v}) \subset \mathsf{A}_{K-1}, \\ \emptyset & \text{otherwise,} \end{cases} \tag{3.2}$$

and

$$X_{K,l}^{[\mathsf{u},\mathsf{v}]} = X_{K,l} \cap (\mathsf{B}_{K,l}^{[\mathsf{u},\mathsf{v}]})^{\mathbb{Z}}. \tag{3.3}$$

Both $\mathsf{B}_{K,l}^{[\mathsf{u},\mathsf{v}]}$ and $X_{K,l}^{[\mathsf{u},\mathsf{v}]}$ are empty whenever $|\mathsf{u}| > l$, and $\mathsf{B}_{K,l}^{[\emptyset,\emptyset]} = \mathsf{B}_{K,l}$ and $X_{K,l}^{[\emptyset,\emptyset]} = X_{K,l}$. If $|\mathsf{u}| = 1$ with $\mathsf{u} = \{j\}$ for some $j \in \mathsf{A}_{K-1}$, we write $\mathsf{B}_{K,l}^{[j,\mathsf{v}]}$ and $X_{K,l}^{[j,\mathsf{v}]}$ instead of $\mathsf{B}_{K,l}^{[\{j\},\mathsf{v}]}$ and $X_{K,l}^{[\{j\},\mathsf{v}]}$. The case where $|\mathsf{v}| = 1$ will be treated similarly.

We set

$$\begin{aligned} \Omega_{K,l}^{(0)} &= \{\omega = (\omega_n) \in \Omega_{K,l} : \omega_n \neq 0 \text{ for every } n \in \mathbb{Z}\}, \\ \Omega_{K,l}^{(K)} &= \{\omega = (\omega_n) \in \Omega_{K,l} : \omega_n \neq K \text{ for every } n \in \mathbb{Z}\}. \end{aligned} \tag{3.4}$$

If $\phi = \phi_{K,l} \colon \Omega_{K,l} \to X_{K,l}$ is the shift-equivariant isomorphism defined in (2.10), then

$$\phi(\Omega_{K,l}^{(0)}) = X_{K,l}^{[0,\emptyset]} \quad \text{and} \quad \phi(\Omega_{K,l}^{(K)}) = X_{K,l}^{[\emptyset,K-1]}.$$

The reason for our interest in these subshifts is that

$$\Omega_{K,l}^{(0)} \simeq \Omega_{K-1,l-1}, \qquad \Omega_{K,l}^{(K)} \simeq \Omega_{K-1,l},$$

and hence

$$X_{K,l}^{[0,\emptyset]} \simeq X_{K-1,l-1} \quad \text{and} \quad X_{K,l}^{[\emptyset,K-1]} \simeq X_{K-1,l}.$$

This shows that (3.1) is equivalent to

$$h(X_{K+1,l}^{[0,\emptyset]}) \leq h(X_{K+1,l}^{[\emptyset,K]}) \tag{3.5}$$

for every $K \geq 1$ and $l \leq (K+1)/2$.

We set $Y = X_{K+1,l}^{[0,\emptyset]}$, $Z = X_{K+1,l}^{[\emptyset,K]}$, and write Y_N and Z_N for the sets of all paths of length N in Y and Z, respectively. Since $h(Y) = \lim_{N\to\infty} \frac{1}{N} \log |Y_N|$ and $h(Z) = \lim_{N\to\infty} \frac{1}{N} \log |Z_N|$, we have to investigate the growth rates of the cardinalities $|Y_N|$ and $|Z_N|$ as $N \to \infty$.

We start with Y_N and write, for all disjoint finite sets $\mathsf{u}, \mathsf{v} \subset \mathbb{Z}$, $Y_N^{[\mathsf{u},\mathsf{v}]} \subset Y_N$ for the set of all paths of length N ending in an element of $\mathsf{B}_{K+1,l}^{[\mathsf{u},\mathsf{v}]}$ (note that this set will be empty if $\mathsf{u} \cup \mathsf{v} \not\subset \{1, \dots, K\}$ or $|\mathsf{u}| \geq l$).

LEMMA 3.6. *For every* $N \geq 1$ *and every* $\mathsf{u} \subset \{1, \dots, K\}$,

$$|Y_{N+1}^{[\mathsf{u},\emptyset]}| = (K - l - |\mathsf{u}| + 1)|Y_N^{[\{1\}\cup(\mathsf{u}+1),\emptyset]}| + \sum_{j\in\{1\}\cup(\mathsf{u}+1)} |Y_N^{[(\{1\}\cup(\mathsf{u}+1))\smallsetminus\{j\},\emptyset]}|. \tag{3.6}$$

Proof. If $|\mathsf{u}| \geq l$, then $\mathsf{B}_{K+1,l}^{[\{0\}\cup\mathsf{u},\emptyset]} = \emptyset$ and hence $Y_N^{[\mathsf{u},\emptyset]} = \emptyset$ for every $N \geq 0$. If $|\mathsf{u}| < l$, then every $y \in Y_{N+1}^{[\mathsf{u},\emptyset]}$ has the form $y = (\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_{N-1}, \mathsf{a}_N)$ with $\mathsf{a}_i \in \mathsf{B}_{K,l}^{[0,\emptyset]}$ for $i = 0, \dots, N$, $\mathsf{a}_N \supset \mathsf{u}$, and $\mathsf{a}_{N-1} \in \mathbf{p}(\mathsf{a}_N)$. Since both a_{N-1} and a_N contain 0, equation (2.15) implies that one of the following conditions is satisfied:

(i) $\{1\} \cup (\mathsf{u} + 1) \subset \mathsf{a}_{N-1}$, in which case $|\mathbf{f}(\mathsf{a}_{N-1})| = K - l + 2$ and every successor of a_{N-1} (including a_N, of course) is of the form

$$((\mathsf{a}_{N-1} \smallsetminus \{0\}) - 1) \cup \{j\} \text{ for some } j \in \{0, \dots, K\} \smallsetminus (\mathsf{a}_{N-1} - 1),$$

(ii) $\{1\} \cup (\mathsf{u} + 1) \not\subset \mathsf{a}_{N-1}$, but $\mathsf{a}_{N-1} \supset (\{1\} \cup (\mathsf{u} + 1)) \smallsetminus \{j\}$ for some $j \in \{1\} \cup (\mathsf{u} + 1)$. In this case $|\mathbf{f}(\mathsf{a}_{N-1})| = 1$ and $\mathbf{f}(\mathsf{a}_{N-1}) = \{\mathsf{a}_N\}$.

If $K \notin \mathsf{u}$, we obtain

$$\begin{aligned}|Y_{N+1}^{[\mathsf{u},\emptyset]}| &= (K - l + 2)|Y_N^{[\{1\}\cup(\mathsf{u}+1),\emptyset]}| + \sum_{j\in\{1\}\cup(\mathsf{u}+1)} |Y_N^{[(\{1\}\cup(\mathsf{u}+1))\smallsetminus\{j\},j]}| \\ &= (K - l - |\mathsf{u}| + 1)|Y_N^{[\{1\}\cup(\mathsf{u}+1),\emptyset]}| \\ &\quad + \sum_{j\in\{1\}\cup(\mathsf{u}+1)} |Y_N^{[(\{1\}\cup(\mathsf{u}+1))\smallsetminus\{j\},\emptyset]}|,\end{aligned}$$

where we have used the fact that $|Y_N^{[\mathsf{w}\cup\{j\},\emptyset]}| + |Y_N^{[\mathsf{w},j]}| = |Y_N^{[\mathsf{w},\emptyset]}|$ whenever $\{j\} \cup \mathsf{w} \subset \{1, \dots, K\}$ and $j \notin \mathsf{w}$.

If $K \in \mathsf{u}$, then

$$\begin{aligned}|Y_{N+1}^{[\mathsf{u},\emptyset]}| &= (K - l + 2)|Y_N^{[\{1\}\cup(\mathsf{u}+1),\emptyset]}| + \sum_{j\in\{1\}\cup(\mathsf{u}+1),\, j\leq K} |Y_N^{[(\{1\}\cup(\mathsf{u}+1))\smallsetminus\{j\},j]}| \\ &\quad + |Y_N^{[(\{1\}\cup(\mathsf{u}+1))\smallsetminus\{K+1\},\emptyset]}| = |Y_N^{[(\{1\}\cup(\mathsf{u}+1))\smallsetminus\{K+1\},\emptyset]}|,\end{aligned}$$

since (3.2)–(3.3) guarantee that all other expressions in the middle term of this equation vanish.

In either case, (3.6) is satisfied. ∎

In order to prove an analogous recursion formula for $Z = X_{K+1,l}^{[\emptyset,K]}$ we denote by $Z_N^{[\mathsf{v},\emptyset]} \subset Z_N$ the set of all paths of length N which *begin* with an element of $\mathsf{B}_{K+1,l}^{[\mathsf{v},K]}$ for some finite set $\mathsf{v} \subset \mathbb{Z}$. Note that $\mathsf{B}_{K+1,l}^{[\mathsf{v},K]} = \emptyset$ whenever $|\mathsf{v}| > l$ or $\mathsf{v} \not\subset \{0, \dots, K - 1\}$.

LEMMA 3.7. *For every $N \geq 1$ and every $\mathsf{v} \subset \{0, \ldots, K-1\}$ with $|\mathsf{v}| \leq l$,*

$$|Z_{N+1}^{[\emptyset,\mathsf{v}]}| = (l - |\mathsf{v}|)|Z_N^{[\emptyset,\{K-1\}\cup(\mathsf{v}-1)]}| + \sum_{j\in\{K-1\}\cup(\mathsf{v}-1)} |Z_N^{[\emptyset,(\{K-1\}\cup(\mathsf{v}-1))\setminus\{j\}]}|. \tag{3.7}$$

If $0 \in \mathsf{v}$, (3.7) reduces to

$$|Z_{N+1}^{[\emptyset,\mathsf{v}]}| = |Z_N^{[\emptyset,\{K-1\}\cup(\mathsf{v}'-1)]}|,$$

where $\mathsf{v}' = \mathsf{v} \setminus \{0\}$.

Proof. For every $\mathsf{a} \in \mathsf{B}_{K,l}$ we set $\bar{\mathsf{a}} = \{K - a : a \in \mathsf{a}\}$. We recall the definition of the isomorphism $\Psi_{K+1,l}\colon X_{K+1,l} \to X_{K+1,K+1-l}$ in (2.25) and note that $\Psi_{K+1,l}(Z_N^{[\emptyset,\mathsf{v}]}) = \overline{Y}_N^{[\bar{\mathsf{v}},\emptyset]}$, where $\overline{Y} = X_{K+1,K-l+1}^{[0,\emptyset]}$. In particular, $|Z_M^{[\emptyset,\mathsf{v}]}| = |\overline{Y}_M^{[\bar{\mathsf{v}},\emptyset]}|$ for every $M \geq 1$.

Assume for the moment that $0 \notin \mathsf{v}$. From Lemma 3.6 it follows that

$$\begin{aligned}
|Z_{N+1}^{[\emptyset,\mathsf{v}]}| &= |\overline{Y}_{N+1}^{[\bar{\mathsf{v}},\emptyset]}| \\
&= (l - |\mathsf{v}|)|\overline{Y}_N^{[\{1\}\cup(\bar{\mathsf{v}}+1),\emptyset]}| + \sum_{j\in\{1\}\cup(\bar{\mathsf{v}}+1)} |\overline{Y}_N^{[(\{1\}\cup(\bar{\mathsf{v}}+1))\setminus\{j\},\emptyset]}| \\
&= (l - |\mathsf{v}|)|Z_N^{[\emptyset,\overline{\{1\}\cup(\bar{\mathsf{v}}+1)}]}| + \sum_{j\in\{1\}\cup(\bar{\mathsf{v}}+1)} |Z_N^{[\emptyset,\overline{(\{1\}\cup(\bar{\mathsf{v}}+1))\setminus\{j\}}]}| \\
&= (l - |\mathsf{v}|)|Z_N^{[\emptyset,\overline{\{1\}}\cup\overline{\bar{\mathsf{v}}+1}]}| + \sum_{j\in\{1\}\cup(\bar{\mathsf{v}}+1)} |Z_N^{[\emptyset,(\overline{\{1\}}\cup\overline{\bar{\mathsf{v}}+1})\setminus\overline{\{j\}}]}| \\
&= (l - |\mathsf{v}|)|Z_N^{[\emptyset,\{K-1\}\cup(\mathsf{v}-1)]}| + \sum_{j\in\{K-1\}\cup(\mathsf{v}-1)} |Z_N^{[\emptyset,(\{K-1\}\cup(\mathsf{v}-1))\setminus\{j\}]}|,
\end{aligned}$$

where we have used the facts that $\bar{\bar{\mathsf{v}}} = \mathsf{v}$, $\overline{\bar{\mathsf{v}} + 1} = K - (K - \mathsf{v} + 1) = \mathsf{v} - 1$, $\overline{\mathsf{u} \cup \mathsf{v}} = \bar{\mathsf{u}} \cup \bar{\mathsf{v}}$, $\overline{\mathsf{u} \setminus \mathsf{v}} = \bar{\mathsf{u}} \setminus \bar{\mathsf{v}}$ and $|\mathsf{v}| = |\bar{\mathsf{v}}|$.

If $0 \in \mathsf{v}$, then $K \in \bar{\mathsf{v}}$, and (3.2)–(3.3) imply that

$$\begin{aligned}
|Z_{N+1}^{[\emptyset,\mathsf{v}]}| = |\overline{Y}_{N+1}^{[\bar{\mathsf{v}},\emptyset]}| = |\overline{Y}_N^{[(\{1\}\cup(\bar{\mathsf{v}}+1))\setminus\{K+1\},\emptyset]}| &= |Z_N^{[\emptyset,\overline{(\{1\}\cup(\bar{\mathsf{v}}+1))\setminus\{K+1\}}]}| \\
= |Z_N^{[\emptyset,\overline{\{1\}\cup((\bar{\mathsf{v}}+1)\setminus\{K+1\})}]}| &= |Z_N^{[\emptyset,\{K-1\}\cup(\mathsf{v}'-1)]}|.
\end{aligned}$$

This proves (3.7). ■

Finally, we investigate the relation between $Y_N^{[\cdot,\emptyset]}$ and $Z_N^{[\emptyset,\cdot]}$. We prove the following statement by induction on N.

LEMMA 3.8. *For all $N \geq 1$, $1 \leq l \leq K/2$, $0 \leq m \leq l$ and every $\mathsf{u} \subset \{1, \ldots, K\}$ with $|\mathsf{u}| = m$, we have*

$$[K - l + 1]_m |Y_N^{[\mathsf{u},\emptyset]}| \leq [l]_m |Z_N^{[\emptyset,\bar{\mathsf{u}}]}| \tag{3.8}$$

where

$$[x]_m = \begin{cases} x(x-1)\cdots(x-m+1) & \text{if } m \geq 1, \\ 1 & \text{if } m = 0. \end{cases}$$

Proof. Let $N = 1$. Then $|Y_1^{[\mathsf{u},\emptyset]}| = \binom{K-m}{l-m-1}$, since we are choosing $l-1-|\mathsf{u}|$ elements in the set $\{1,\dots,K\} \smallsetminus \mathsf{u}$. Similarly, $|Z_1^{[\emptyset,\bar{\mathsf{u}}]}| = \binom{K-m}{l}$. Then

$$\begin{aligned}
(3.9)\qquad [K-l+1]_m |Y_1^{[\mathsf{u},\emptyset]}| &= [K-l+1]_m \binom{K-m}{l-m-1} \\
&= \frac{(K-m)!}{(l-m-1)!(K-l-m+1)!} \\
&\leq \frac{(K-m)!}{(l-m)!(K-l-m)!} \\
&= [l]_m \binom{K-m}{l} = [l]_m |Z_1^{[\emptyset,\bar{\mathsf{u}}]}|,
\end{aligned}$$

where we have used the assumption $l \leq K - l$. By treating (3.9) as our induction hypothesis, applying Lemmas 3.6 and 3.7, and remembering that $|\mathsf{u}| = |\bar{\mathsf{u}}| = m$, we get

$$\begin{aligned}
[K-l+1]_m |Y_{N+1}^{[\mathsf{u},\emptyset]}| &= [K-l+1]_{m+1} |Y_N^{[\{1\}\cup(\mathsf{u}+1),\emptyset]}| \\
&\quad + \sum_{j\in(\{1\}\cup(\mathsf{u}+1))} [K-l+1]_m |Y_N^{[(\{1\}\cup(\mathsf{u}+1))\smallsetminus\{j\},\emptyset]}| \\
&\leq [l]_{m+1} |Z_N^{[\emptyset,\overline{\{1\}\cup(\mathsf{u}+1)}]}| + \sum_{j\in\{1\}\cup(\mathsf{u}+1)} [l]_m |Z_N^{[\emptyset,\overline{(\{1\}\cup(\mathsf{u}+1))\smallsetminus\{j\}}]}| \\
&= [l]_m \big\{ (l-|\bar{\mathsf{u}}|) |Z_N^{[\emptyset,\{K-1\}\cup(\bar{\mathsf{u}}-1)]}| \\
&\quad + \sum_{j\in(\{K-1\}\cup(\bar{\mathsf{u}}-1))} |Z_N^{[\emptyset,\{K-1\}\cup(\bar{\mathsf{u}}-1)\smallsetminus\{j\}]}| \big\} = [l]_m |Z_{N+1}^{[\emptyset,\bar{\mathsf{u}}]}|. \ \blacksquare
\end{aligned}$$

Proof of Proposition 3.5. By taking $m = 0$ (and hence $\mathsf{u} = \emptyset$) in Lemma 3.8 we obtain $|Y_N| = |Y_N^{[\emptyset,\emptyset]}| \leq |Z_N^{[\emptyset,\emptyset]}| = |Z_N|$ for every $N \geq 2$. As noted in the penultimate paragraph before Lemma 3.6, this guarantees that

$$\begin{aligned}
h(X_{K+1,l}^{[0,\emptyset]}) = h(Y) &= \lim_{N\to\infty} \frac{1}{N} \log |Y_N| \\
&\leq \lim_{N\to\infty} \frac{1}{N} \log |Z_N| = h(Z) = h(X_{K+1,l}^{[\emptyset,K]})
\end{aligned}$$

for every $K \geq 1$ and $l \leq (K+1)/2$. We have proved (3.5), or equivalently (3.1). $\blacksquare$

4. The parity cocycle. If $\omega = (\omega_k) \in \Omega_K$ is a periodic point with period p, say, then

$$\pi^{(\omega)}_{(0,p)} := (\tilde{\omega}_0 \ (\mathrm{mod}\ p), \dots, \tilde{\omega}_{0+p-1} \ (\mathrm{mod}\ p))$$

is a permutation of $(0,\dots,p-1)$ (cf. Lemma 2.1(3)). What is the parity (or sign) of this permutation? In this section we prove that these parities are determined by the function $a\colon \Omega_K \to \mathbb{Z}$ in (2.3) and a continuous cocycle $\mathsf{s}\colon \mathbb{Z} \times \Omega_K \to \{\pm 1\}$ for the shift σ on Ω_K.

THEOREM 4.1. *Let $K \geq 1$, define $c\colon \Omega_K \to \mathbb{Z}$ by*

$$c(\omega) = |\{k < 0 : \tilde{\omega}_k > \tilde{\omega}_0\}| = |\{k = -K+1, \dots, -1 : \tilde{\omega}_k > \tilde{\omega}_0\}|, \tag{4.1}$$

and let $\mathsf{c}\colon \mathbb{Z} \times \Omega_K \to \mathbb{Z}$ be given by

$$\mathsf{c}(n,\omega) = \begin{cases} \sum_{k=0}^{n-1} c(\sigma^k \omega) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\mathsf{c}(-n, \sigma^n \omega) & \text{if } n < 0. \end{cases} \tag{4.2}$$

Consider the multiplicative group $C_2 := \{\pm 1\} \subset \mathbb{R}$ and define $s\colon \Omega_K \longmapsto C_2$ and $\mathsf{s}\colon \mathbb{Z} \times \Omega_K \to C_2$ by setting $s(\omega) = (-1)^{c(\omega)+a(\omega)}$ and

$$\mathsf{s}(n,\omega) = (-1)^{\mathsf{c}(n,\omega)+na(\omega)} = \prod_{k=0}^{n-1} s(\sigma^k\omega). \tag{4.3}$$

Then s satisfies the cocycle equation

$$\mathsf{s}(m+n,\omega) = \mathsf{s}(m,\sigma^n\omega)\mathsf{s}(n,\omega) \tag{4.4}$$

for all $m,n \in \mathbb{Z}$ and $\omega \in \Omega_K$. Furthermore, if $\omega \in \Omega_K$ is periodic with period p, then the parity of the permutation $\pi^{(\omega)}_{(0,p)}$ defined in Lemma 2.1(3) is given by

$$\operatorname{sgn} \pi^{(\omega)}_{(0,p)} = (-1)^{a(\omega)}\,\mathsf{s}(p,\omega). \tag{4.5}$$

Proof. The only statement requiring verification is (4.5). We fix $\omega \in \Omega_K$, $p > K \geq 2$ and use the conventions of Figure 1. We call a pair $(\mathbf{a} = (a_1,a_2), \mathbf{b} = (b_1,b_2)) \in \mathbb{Z}^2$ an *inversion* if $a_1 < b_1$ and $a_2 > b_2$ (i.e., if $\mathbf{b}$ lies above and to the right of $\mathbf{a}$ in Figure 1). Let

$$\mathscr{I}(p) = \{(\mathbf{a},\mathbf{b}) \in (\tilde{Q} \cup \tilde{A}') \times S(\omega) : (\mathbf{a},\mathbf{b}) \text{ is an inversion}\}. \tag{4.6}$$

According to (4.2), $c(\omega) = |\{\mathbf{b} \in S(\omega) : ((\omega_0, 0), \mathbf{b}) \text{ is an inversion}\}|$. Hence

$$\mathsf{c}(p,\omega) = |\mathscr{I}(p)|.$$

Now suppose that ω is periodic with period $p > K$. Then $S(\omega)$ is invariant under translation by (p,p) (which in Figure 1 moves every point p steps down and to the right), and $\tilde{A}' = \tilde{A} + (p,p)$.

We populate A^* by translating all points in $\tilde{A}'$ into A^* by adding $(-p,0)$ (or equivalently by adding $(0,p)$ to all points in $\tilde{A}$), leaving the points in $\tilde{Q}$ unchanged: for $\mathbf{a} \in \tilde{A} \cup \tilde{Q} \cup \tilde{A}'$ we let

$$\mathbf{a}^* = \begin{cases} \mathbf{a} & \text{if } \mathbf{a} \in \tilde{Q}, \\ \mathbf{a} - (p,0) & \text{if } \mathbf{a} \in \tilde{A}', \\ \mathbf{a} + (0,p) & \text{if } \mathbf{a} \in \tilde{A}, \end{cases}$$

and we set $\tilde{A}^* = \{\mathbf{a}^* : \mathbf{a} \in \tilde{A}\} = \{\mathbf{a}^* : \mathbf{a} \in \tilde{A}'\}$. Under our assumption that ω has period p, the set $\tilde{S} := \tilde{Q} \cup \tilde{A}^*$ has exactly one element in each row and column of the square Q. Hence $\tilde{S}$ determines a permutation of $\{0, \dots, p-1\}$ which coincides with $\pi^{(\omega)}_{(0,p)}$. The parity of this permutation is the parity of the number of inversions occurring in $\pi^{(\omega)}_{(0,p)}$: if

$$\mathscr{J}(p) = \{(\mathbf{a}, \mathbf{b}) \in \tilde{S} \times \tilde{S} : (\mathbf{a}, \mathbf{b}) \text{ is an inversion}\}, \tag{4.7}$$

then

$$\operatorname{sgn} \pi^{(\omega)}_{(0,p)} = (-1)^{|\mathscr{J}(p)|}. \tag{4.8}$$

When comparing this with

$$\mathsf{s}(p, \omega) = (-1)^{|\mathscr{I}(p)| + pa(\omega)}, \tag{4.9}$$

we have to consider several cases:

(a) If $(\mathbf{a}, \mathbf{b}) \in \tilde{Q}$ or $(\mathbf{a}, \mathbf{b}) \in \tilde{A}'$, then $(\mathbf{a}, \mathbf{b}) \in \mathscr{I}(p)$ if and only if $(\mathbf{a}^*, \mathbf{b}^*) \in \mathscr{J}(p)$.

(b) If $\mathbf{a} = (a_1, a_2) \in \tilde{B}$ and $\mathbf{b} = (b_1, b_2) \in \tilde{A}$, then $(\mathbf{a}, \mathbf{b}) \in \mathscr{I}(p)$ if and only if $a_1 < b_1$, in which case $(\mathbf{b}^*, \mathbf{a}^*) \notin \mathscr{J}(p)$. If $a_1 > b_1$, then $(\mathbf{a}, \mathbf{b}) \notin \mathscr{I}(p)$, but $(\mathbf{b}^*, \mathbf{a}^*) \in \mathscr{J}(p)$. Since $|\tilde{B} \times \tilde{A}| = a(\omega)(K - a(\omega))$, we conclude that $|\mathscr{I}(p) \cap (\tilde{B} \times \tilde{A})| + |\mathscr{J}(p) \cap (\tilde{A}^* \times \tilde{B})| = a(\omega)(K - a(\omega))$.

(c) $(\tilde{A}' \times \tilde{C}) \cap \mathscr{I}(p) = (\tilde{C} \times \tilde{A}') \cap \mathscr{I}(p) = \emptyset$, but $(\tilde{A}^* \times \tilde{C}) \subset \mathscr{J}(p)$. Note that $|\tilde{A}' \times \tilde{C}| = |\tilde{C} \times \tilde{A}'| = |\tilde{A}^* \times \tilde{C}| = a(\omega)(p - 2K + a(\omega))$.

(d) If $\mathbf{a} = (a_1, a_2) \in \tilde{D}$ and $\mathbf{b} = (b_1, b_2) \in \tilde{A}'$, then $(\mathbf{a}, \mathbf{b}) \in \mathscr{I}(p)$ if and only if $a_2 < b_2$, in which case $(\mathbf{b}^*, \mathbf{a}^*) \notin \mathscr{J}(p)$. If $a_2 > b_2$, then $(\mathbf{a}, \mathbf{b}) \notin \mathscr{I}(p)$, but $(\mathbf{b}^*, \mathbf{a}^*) \in \mathscr{J}(p)$. Since $|\tilde{A}' \times \tilde{D}| = a(\omega)(K - a(\omega))$, we conclude that $|\mathscr{I}(p) \cap (\tilde{D} \times \tilde{A}')| + |\mathscr{J}(p) \cap (\tilde{A}^* \times \tilde{D})| = a(\omega)(K - a(\omega))$.

Clearly,

$$\begin{aligned} |\mathscr{I}(p)| &= |\mathscr{I}(p) \cap (\tilde{Q} \times \tilde{Q})| + |\mathscr{I}(p) \cap (\tilde{A}' \times \tilde{A}')| + |\mathscr{I}(p) \cap (\tilde{A}' \times \tilde{Q})| \\ &\quad + |\mathscr{I}(p) \cap (\tilde{Q} \times \tilde{A})| + |\mathscr{I}(p) \cap (\tilde{Q} \times \tilde{A}')| \\ &= |\mathscr{I}(p) \cap (\tilde{Q} \times \tilde{Q})| + |\mathscr{I}(p) \cap (\tilde{A}' \times \tilde{A}')| \\ &\quad + |\mathscr{I}(p) \cap (\tilde{B} \times \tilde{A})| + |\mathscr{I}(p) \cap (\tilde{D} \times \tilde{A}')|, \\ |\mathscr{J}(p)| &= |\mathscr{J}(p) \cap (\tilde{Q} \times \tilde{Q})| + |\mathscr{J}(p) \cap (\tilde{A}^* \times \tilde{A}^*)| \\ &\quad + |\mathscr{J}(p) \cap (\tilde{A}^* \times \tilde{Q})| + |\mathscr{J}(p) \cap (\tilde{Q} \times \tilde{A}^*)| \\ &= |\mathscr{J}(p) \cap (\tilde{Q} \times \tilde{Q})| + |\mathscr{J}(p) \cap (\tilde{A}^* \times \tilde{A}^*)| + |\mathscr{J}(p) \cap (\tilde{A}^* \times \tilde{B})| \\ &\quad + |\mathscr{J}(p) \cap (\tilde{A}^* \times \tilde{D})| + |\tilde{A}^* \times \tilde{C}|. \end{aligned}$$

By combining the cases (a)–(d) listed above and remembering that $|\tilde{C}| = p - 2K + a(\omega)$ we obtain

$$\begin{aligned}\mathscr{J}(p) &= |\mathscr{I}(p) \cap (\tilde{Q} \times \tilde{Q})| + |\mathscr{I}(p) \cap (\tilde{A}' \times \tilde{A}')| - |\mathscr{I}(p) \cap (\tilde{B} \times \tilde{A})| \\ &\quad - |\mathscr{I}(p) \cap (\tilde{D} \times \tilde{A}')| + 2a(\omega)(K - a(\omega)) + a(\omega)(p - 2K + a(\omega)).\end{aligned}$$

Hence

$$|\mathscr{J}(p)| = |\mathscr{I}(p)| + a(\omega)(p + a(\omega)) = |\mathscr{I}(p)| + a(\omega)p + a(\omega) \pmod 2.$$

If we recall (4.8)–(4.9) we obtain (4.5). ∎

Examples 4.2. (1) Let $K = 2$ (cf. Example 2.2). The map $c\colon \Omega_2 \to \mathbb{Z}$ in (4.1) is given by

$$c(\omega) = \begin{cases} 1 & \text{if } \omega_0 = 0 \text{ and } \omega_{-1} = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathsf{c}(n, \omega) = |\{k : 0 \le k < n : \omega_k = 0 \text{ and } \omega_{k-1} = 2\}|.$$

(2) More generally, if $K \ge 1$ and $\omega \in \Omega_{K,1}$, then

$$c(\omega) = \begin{cases} 1 & \text{if } \omega_0 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathsf{c}(n, \omega) = |\{k : 0 \le k < n : \omega_k = 0\}|$.

(3) For $\omega \in \Omega_{K,l}$ with $1 \le l \le K-1$, $c(\omega)$ can be calculated by using the isomorphism $\phi_{K,l}\colon \Omega_{K,l} \to X_{K,l}$ in (2.10): $c(\omega) = |\{a \in \phi_{K,l}(\omega)_0 : a > \omega_0\}|$, where ω_0 is determined by (2.11).

As one would expect, the parity cocycle $\mathsf{s}\colon \mathbb{Z} \times \Omega_K \to C_2$ in (4.3) is *nontrivial* in the sense that its group of essential values is equal to C_2. More precisely, the following is true.

Proposition 4.3. *Let $K \ge 2$, $1 \le l \le K-1$, and let $\mu_{K,l}$ be the unique shift-invariant probability measure with maximal entropy on $\Omega_{K,l}$. Then the skew-product transformation $\tilde{\sigma}\colon \Omega_{K,l} \times C_2 \to \Omega_{K,l} \times C_2$ defined by*

$$\tilde{\sigma}(\omega, j) = (\sigma\omega, s(\omega)j)$$

for every $(\omega, j) \in \Omega_{K,l} \times C_2$ is ergodic with respect to the product measure $\tilde{\mu}_{K,l} = \mu_{K,l} \times \lambda_{C_2}$, where $\lambda_{C_2}(\{1\}) = \lambda_{C_2}(\{-1\}) = 1/2$.

Proof. According to [12, Corollary 5.4], we have to show the following: for every Borel set $B \subset \Omega_{K,l}$ with $\mu_{K,l}(B) > 0$ and every $j \in C_2$,

$$B \cap \sigma^{-m} B \cap \{\omega \in \Omega_{K,l} : \mathsf{s}(m, \omega) = j\} \ne \emptyset \tag{4.10}$$

for some $m > 0$.

Consider the cylinder sets

$$E = \{\omega \in \Omega_{K,l} : \omega_i = l \text{ for } i = 0, \ldots, K+1\}$$

and

$F = \{\omega \in \Omega_{K,l} : \omega_i = l \text{ for } i = 0, \dots, K-2, K+1,\ \omega_{K-1} = l+1,\ \omega_K = l-1\}$.

Since $\Omega_{K,l}$ is irreducible and aperiodic, $\beta := \mu_{K,l}(D) > 0$, where $D = E \cup F$. We define a homeomorphism $V\colon \Omega_{K,l} \to \Omega_{K,l}$ by setting $V\omega = \omega$ if $\omega \notin D$, and

$$(V\omega)_i = \begin{cases} l & \text{if } \omega \in E \text{ and } i = 0, \dots, K-2, K+1, \\ l+1 & \text{if } \omega \in E \text{ and } i = K-1, \\ l-1 & \text{if } \omega \in E \text{ and } i = K, \\ l & \text{if } \omega \in F \text{ and } i = 0, \dots, K+1. \end{cases}$$

Then $V^2 = \mathrm{Id}_{\Omega_{K,l}}$ and $VE = F$.

We fix $B \subset \Omega_{K,l}$ with $\mu_{K,l}(B) > 0$. By approximating B with closed and open subsets of $\Omega_{K,l}$ we see that $\lim_{m\to\infty} \mu_{K,l}(B \cap \sigma^{-m}V\sigma^m B) = \mu_{K,l}(B)$ and $\lim_{m\to\infty} \mu_{K,l}(\sigma^{-2m}B \cap \sigma^{-m}V\sigma^{-m}B) = \mu_{K,l}(B)$. Furthermore, since $\mu_{K,l}$ is mixing of every order, $\lim_{m\to\infty} \mu_{K,l}(B[0] \cap \sigma^{-m}D \cap \sigma^{-2m}B) = \beta\mu_{K,l}(B)^2$. We conclude that

$$\begin{aligned} \beta\mu_{K,l}(B)^2 &= \lim_{m\to\infty} \mu_{K,l}(B \cap \sigma^{-m}D \cap \sigma^{-2m}B) \\ &= \lim_{m\to\infty} \mu_{K,l}(B \cap \sigma^{-m}V\sigma^m B \cap \sigma^{-m}D \cap \sigma^{-2m}B \cap \sigma^{-m}V\sigma^{-m}B). \end{aligned}$$

Let $m > K$ be sufficiently large so that the set $E = B \cap \sigma^{-m}V\sigma^m B \cap \sigma^{-m}D \cap \sigma^{-2m}B[0] \cap \sigma^{-m}V\sigma^{-m}B$ is nonempty. For every $\omega \in E$, we have

$$\begin{aligned} \omega &\in B, & \sigma^{2m}\omega &\in B, & \sigma^m\omega &\in D, \\ \omega' &\in B, & \sigma^{2m}\omega' &\in B, & \sigma^m\omega' &\in D, \end{aligned}$$

where $\omega' = \sigma^{-m}V\sigma^m\omega$. A glance at the definition of the cocycle s in (4.3) shows that, for every $\omega \in E$ and $k \in \mathbb{Z}$,

$$s(\sigma^k\omega') = \begin{cases} s(\sigma^k\omega) & \text{if } k \neq m+K, \\ -s(\sigma^k\omega) & \text{if } k = m+K. \end{cases}$$

In particular, $\{\omega, \omega'\} \subset B \cap \sigma^{-2m}B$ and $\mathsf{s}(2m, \omega) = -\mathsf{s}(2m, \omega')$. This proves (4.10). ∎

Example 4.4 (Generalized circulants). The study of the space Ω_K, its irreducible components $\Omega_{K,l}$, and the permutations corresponding to periodic elements of Ω_K was partly motivated by expressions occurring in the calculation of entropy of (expansive) algebraic actions of the discrete Heisenberg group (cf. [7, Section 8]).

Let $\phi_0, \dots, \phi_K$ be continuous complex-valued functions on $\mathbb{T}$. We are interested in bi-infinite 'generalized circulants' of the form

(4.11)

$$A_{t,\alpha}=\begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & \phi_0(t-\alpha) & \phi_1(t-\alpha) & \cdots & \phi_{K-1}(t-\alpha) & \phi_K(t-\alpha) & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \phi_0(t) & \phi_1(t) & \cdots & \phi_{K-1}(t) & \phi_K(t) & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \phi_0(t+\alpha) & \phi_1(t+\alpha) & \cdots & \phi_{K-1}(t+\alpha) & \phi_K(t+\alpha) & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

for $(t,\alpha)\in\mathbb{T}^2$. The matrix $A_{t,\alpha}$ acts by left multiplication on the space $\ell^\infty(\mathbb{Z},\mathbb{C})^\top$ of bounded column vectors with complex entries. If α is rational with $\alpha=p/q$ in lowest terms, then $A_{t,p/q}$ acts by left multiplication on the set of elements in $\ell^\infty(\mathbb{Z},\mathbb{C})^\top$ with period q, which we identify with $\mathbb{C}^q\simeq\ell^\infty(\mathbb{Z}/q\mathbb{Z},\mathbb{C})$. The determinant of this linear transformation of $\mathbb{C}^q$ can be expressed in terms of *SFT* Ω_K, using the parity cocycle (4.3)–(4.4): if $P_q(\Omega_K)$ and $P_q(\Omega_{K,l})$ are the sets of points of period q in Ω_K and $\Omega_{K,l}$, then

$$\det A_{t,p/q}=\sum_{\omega\in P_q(\Omega_K)}(-1)^{a(\omega)}\mathsf{s}(q,\omega)\prod_{j=0}^{q-1}\phi_{\omega_j}(t+jp/q) \tag{4.12}$$
$$=\sum_{l=0}^{K}(-1)^l\sum_{\omega\in P_q(\Omega_{K,l})}\prod_{j=0}^{q-1}s(\sigma^j\omega)\phi_{\omega_j}(t+jp/q).$$

As explained in [7, Section 8] one should normalize $\det A_{t,p/q}$ by setting

$$D(A_{t,p/q})=|\det A_{t,p/q}|^{1/q}. \tag{4.13}$$

In the context of algebraic actions of the discrete Heisenberg group the functions ϕ_i, $i=0,\dots,K$, are trigonometric polynomials arising from the element f in the integer group ring $\mathbb{Z}\Gamma$ which defines the action, and the quantity $\int_{\mathbb{T}}\log D(A_{t,p/q})\,dt$ measures the contribution to the entropy of this action associated with a *rational* rotation number $\alpha=p/q$ representing the central generator $z=\left(\begin{smallmatrix}1&0&1\\0&1&0\\0&0&1\end{smallmatrix}\right)$ of Γ. If this action is expansive, the asymptotic behaviour (as $q\to\infty$) of the expressions $D(A_{t,p/q})$ in (4.13) determines the entropy of the algebraic action (cf. [4] and [7, Section 8]). For nonexpansive actions of this form one might still expect that $\limsup_{q\to\infty}D(A_{t,p/q})=h(\alpha_f)$ for every $t\in\mathbb{T}$, but this statement is currently only conjectural.

5. Permutations of $\mathbb{Z}^d$ with restricted movement. We start by still assuming that $d=1$, but by allowing $\mathsf{A}\subset\mathbb{Z}$ to be an arbitrary nonempty finite set. Define Π_{A} and Ω_{A} as at the beginning of Section 1. Then Ω_{A} is an *SFT* by Lemma 2.1.

PROPOSITION 5.1.

(1) *If $|\mathsf{A}|\geq 2$, then Ω_{A} is not irreducible and hence not mixing.*

(2) *The SFT* Ω_{A} *is finite if* $|\mathsf{A}| \le 2$, *and has positive topological entropy if* $|\mathsf{A}| \ge 3$.

Proof. In view of Proposition 1.1 we assume that $0 \in \mathsf{A} \subset \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and choose $K \ge 1$ so that $\mathsf{A} \subset \mathsf{A}_K$ (cf. (1.2)). For the proof of (1) we note that, for every $a \in \mathsf{A}$, the fixed point $\underline{a} = (\dots, a, a, a, \dots)$ lies in $\Omega_{\mathsf{A}} \cap \Omega_{K,a}$. For $a, b \in \mathsf{A}$ with $a < b$, the fixed points $\underline{a}$ and $\underline{b}$ lie in the distinct irreducible components $\Omega_{K,a}$ and $\Omega_{K,b}$ of Ω_K. The sets $\Omega_{\mathsf{A}} \cap \Omega_{K,a}$ and $\Omega_{\mathsf{A}} \cap \Omega_{K,b}$ are disjoint, nonempty, shift-invariant, open subsets of Ω_{A}, so that Ω_{A} cannot be topologically mixing. This proves (1).

We turn to (2). If $\mathsf{A} = \{0\}$ then $\Omega_{\mathsf{A}} = \{\underline{0}\}$. If $\mathsf{A} = \{0, a\}$ for some $a \ge 1$, then $|\Omega_{\mathsf{A}}| = |\Pi_{\mathsf{A}}| = 2^a$: for every $u = (u_0, \dots, u_{a-1}) \in \{0,1\}^a$ there exists a unique $\pi^{(u)} \in \Pi_{\mathsf{A}}$ such that $\pi^{(u)}(i + ma) = i + (m + u_i)a$ for every $i = 0, \dots, a-1$ and $m \in \mathbb{Z}$. Furthermore, every $\pi \in \Pi_{\mathsf{A}}$ is of this form for some $u \in \{0,1\}^a$.

Finally, we assume that $\mathsf{A} \supset \{0, a, b\}$ with $0 < a < b$. For every $n \in \mathbb{Z}$ we consider the finite permutation of $\mathbb{Z}$ defined by

$$\tau_n = \begin{pmatrix} n & n+1 & \cdots & n+a-1 & n+a & n+a+1 & \cdots & n+b-1 \\ n+b-a & n+b-a+1 & \cdots & n+b-1 & n & n+1 & \cdots & n+b-a-1 \end{pmatrix}.$$

Clearly, the permutations τ_m and τ_n commute if $|m - n| \ge b$. This allows us to define, for every $v = (v_n) \in \{0,1\}^{\mathbb{Z}}$, a permutation $\tau^{(v)} \in S^\infty(\mathbb{Z})$ by setting

$$\tau^{(v)} = \prod_{n \in \mathbb{Z}} \tau_{bn}^{v_n},$$

where τ_m^0 is the identity permutation for every $m \in \mathbb{Z}$. Note that $\varsigma^a \circ \tau^{(v)} \in \Pi_{\mathsf{A}}$ for every $v \in \{0,1\}^{\mathbb{Z}}$ (cf. Proposition 1.1).

In order to understand what is going on here it may help to consider an element $v \in \{0,1\}^{\mathbb{Z}}$ of the form $v = (\dots, 1, \dot{0}, 1, \dots)$, where the dot marks the zero-th coordinate of v. Then $\tau^{(v)}$ is the permutation

$$\left(\begin{array}{c|ccccc|ccccc|ccccc|c} \cdots & -b & \cdots & -b+a-1 & -b+a & \cdots & -1 & 0 & \cdots & a-1 & a & \cdots & b-1 & b & \cdots & b+a-1 & b+a & \cdots & 2b-1 & \cdots \\ \cdots & -a & \cdots & -1 & -b & \cdots & -a-1 & 0 & \cdots & a-1 & a & \cdots & b-1 & 2b-a & \cdots & 2b-1 & b & \cdots & 2b-a-1 & \cdots \end{array}\right),$$

where we have separated the individual permutations $\dots, \tau_{-b}, \tau_0, \tau_b^0, \dots$ by vertical bars. The permutation $\varsigma^a \circ \tau^{(v)}$ is of the form

$$\left(\begin{array}{c|cccccc|cccccc|cccccc|c} \cdots & -b & \cdots & -b+a-1 & -b+a & \cdots & -1 & 0 & \cdots & a-1 & a & \cdots & b-1 & b & \cdots & b+a-1 & b+a & \cdots & 2b-1 & \cdots \\ \cdots & 0 & \cdots & a-1 & -b+a & \cdots & -1 & a & \cdots & 2a-1 & 2a & \cdots & a+b-1 & 2b & \cdots & a+2b-1 & a+b & \cdots & 2b-1 & \cdots \end{array}\right)$$

and obviously lies in Π_{A}. By doing so for every $v \in \{0,1\}^{\mathbb{Z}}$ we have, in effect, embedded a full two-shift in the coordinates $b\mathbb{Z}$ of Ω_{A}. This implies that Ω_{A} has entropy $\ge \frac{1}{b} \log 2$. ∎

Next we take a look at the case where $d \ge 2$ and consider dynamical properties of the $\mathbb{Z}^d$-*SFT* $\Omega_{\mathsf{A}} \subset \mathsf{A}^{\mathbb{Z}^d}$, such as entropy and topological mixing.

THEOREM 5.2. *If* A *is a finite subset of* $\mathbb{Z}^2$, *then* Ω_{A} *has positive entropy if and only if* $|\mathsf{A}| \geq 3$.

Proof. We assume without loss in generality that $\mathbf{0} = (0,0) \in \mathsf{A}$: otherwise we replace A by $\mathsf{A}' = \mathsf{A} - \mathbf{u}$ for some $\mathbf{u} \in \mathsf{A}$. Then $\mathbf{0} \in \mathsf{A}'$, and $\Omega_{\mathsf{A}'}$ is topologically conjugate to Ω_{A} with conjugating map $\phi_{\mathsf{u}}\colon \Omega_{\mathsf{A}'} \to \Omega_{\mathsf{A}}$ given by $\phi_{\mathbf{m}}(\omega)_{\mathbf{n}} = \omega_{\mathbf{n}} + \mathsf{u}$ for every $\omega \in \Omega_{\mathsf{A}}$ and $\mathbf{n} \in \mathbb{Z}^2$.

Suppose that A is not contained in a one-dimensional subspace of $\mathbb{R}^2$. Then we can find two elements $\mathbf{a}, \mathbf{b} \in \mathsf{A}$ which are linearly independent over $\mathbb{R}$. We denote by Δ, Γ, with $\Delta \subset \Gamma \subset \mathbb{Z}^2$, the subgroups generated by $\{\mathbf{a}+\mathbf{b}, 3\mathbf{a}\}$ and $\{\mathbf{a}, \mathbf{b}\}$, respectively. For every $x = (x_{\mathbf{m}})_{\mathbf{m}\in\Delta} \in \{0,1\}^{\Delta}$ we define a permutation $\pi_x \in \Pi_{\mathsf{A}}$ as follows:

(i) if $\mathbf{n} \in \mathbb{Z}^2 \smallsetminus \Gamma$ we set $\pi_x(\mathbf{n}) = \mathbf{n}$;
(ii) if $\mathbf{n} \in \Delta$ and $x_{\mathbf{n}} = 0$ we set $\pi_x(\mathbf{n}) = \mathbf{n}+\mathbf{a}$, $\pi_x(\mathbf{n}+\mathbf{a}) = \mathbf{n}+\mathbf{a}+\mathbf{b}$, $\pi_x(\mathbf{n}+2\mathbf{a}) = \mathbf{n}+2\mathbf{a}$, and $\pi_x(\mathbf{n}+\mathbf{b}) = \mathbf{n}+\mathbf{b}$;
(iii) if $\mathbf{n} \in \Delta$ and $x_{\mathbf{n}} = 1$ we set $\pi_x(\mathbf{n}) = \mathbf{n}+\mathbf{b}$, $\pi_x(\mathbf{n}+\mathbf{b}) = \mathbf{n}+\mathbf{a}+\mathbf{b}$, $\pi_x(\mathbf{n}+\mathbf{a}) = \mathbf{n}+\mathbf{a}$, and $\pi_x(\mathbf{n}+2\mathbf{a}) = \mathbf{n}+2\mathbf{a}$.

It is easy to check that π_x is a permutation of $\mathbb{Z}^2$ which lies in Π_{A}. By varying x in $\{0,1\}^{\Delta}$ we conclude that $h(\Omega_{\mathsf{A}}) \geq \log 2/|\mathbb{Z}^2/\Delta|$.

If A is contained in a one-dimensional subspace of $\mathbb{R}^2$, we can find a primitive element $\mathbf{v} \in \mathbb{Z}^2$ such that $\mathsf{A} \subset S := \mathbb{Z}\mathbf{v} = \{k\mathbf{v} : k \in \mathbb{Z}\} \cong \mathbb{Z}$. Choose a second primitive element $\mathbf{w} \in \mathbb{Z}^2$ so that $\{\mathbf{v}, \mathbf{w}\}$ forms a basis of $\mathbb{Z}^2$. The restriction (or projection) $\Omega_{\mathsf{A}}|_S$ of Ω_{A} to $S = \mathbb{Z}\mathbf{v} = \{k\mathbf{v} : k \in \mathbb{Z}\} \cong \mathbb{Z}$ is an *SFT* under the shift $\sigma^{\mathbf{v}}$, which has positive entropy if and only if $|\mathsf{A}| = |\mathsf{A} \cap S| \geq 3$ (Proposition 5.1). Since the restrictions of Ω_{A} to $S + k\mathbf{w}$, $k \in \mathbb{Z}$, are all isomorphic and independent of each other, they all have the same entropy $h(\Omega_{\mathsf{A}}|_S)$ under $\sigma^{\mathbf{v}}$, and $h(\Omega_{\mathsf{A}}) = h(\Omega_{\mathsf{A}}|_S) > 0$ if and only if $|\mathsf{A}| \geq 3$. ■

The proof of Theorem 5.2 yields two corollaries. For use in their proofs, we recall that a nonzero subgroup $\Gamma \subset \mathbb{Z}^d$ is called *primitive* if $\mathbb{Z}^d/\Gamma$ is torsion-free. Similarly, a nonzero element $\mathbf{n} \in \mathbb{Z}^d$ is *primitive* if the cyclic subgroup $\{k\mathbf{n} : k \in \mathbb{Z}\} \subset \mathbb{Z}^d$ is primitive.

COROLLARY 5.3. *If* A *is a finite subset of* $\mathbb{Z}^d$, $d \geq 2$, *then* Ω_{A} *has positive entropy if and only if* $|\mathsf{A}| \geq 3$.

Proof. Again we may assume that $\mathbf{0} \in \mathsf{A}$. If $|\mathsf{A}| = 2$, Proposition 5.1 and the last part of the proof of Theorem 5.2 can easily be adapted to show that $h(\Omega_{\mathsf{A}}) = 0$. If $|\mathsf{A}| \geq 3$, choose a primitive subgroup $\Gamma \subset \mathbb{Z}^d$ such that $\Gamma \cong \mathbb{Z}^2$ and $\Gamma \cap \mathsf{A}$ has at least three elements. A slight extension of the last part of the proof of Theorem 5.2 shows that $h(\Omega_{\mathsf{A}}) > 0$. ■

The announced second corollary concerns topological mixing of Ω_{A}, $\mathsf{A} \subset \mathbb{Z}^d$. Recall that the *SFT* $\Omega_{\mathsf{A}} \subset \mathsf{A}^{\mathbb{Z}^d}$ is *mixing* if there exists, for any pair of nonempty finite sets $V, V' \subset \mathbb{Z}^d$, an $N \in \mathbb{N}$ with the following property: for any $\omega, \omega' \in \Omega_{\mathsf{A}}$ and every $\mathbf{n} \in \mathbb{Z}^2$ with $\|\mathbf{n}\| \geq N$ there exists an $\omega'' \in \Omega_{\mathsf{A}}$ with $\omega''|_V = \omega|_V$ and $\omega''|_{V'+\mathbf{n}} = \omega'|_{V'+\mathbf{n}}$. Here $\|\cdot\|$ is the maximum norm on $\mathbb{Z}^d$, and $\omega|_W \in \mathsf{A}^W$ denotes the restriction of ω to its coordinates in a nonempty subset $W \subset \mathbb{Z}^d$.

Corollary 5.4. *If $\mathsf{D} = \mathsf{A} - \mathsf{A}$ is contained in a one-dimensional subspace of $\mathbb{R}^d$, then Ω_{A} is not mixing.*

Proof. If $\mathsf{D} = \mathsf{A} - \mathsf{A}$ is contained in a one-dimensional subspace $V \subset \mathbb{R}^d$, and if $\mathbf{m} \in \mathsf{A}$, then there exists a primitive element $\mathbf{v} \in \mathbb{Z}^d$ such that $\mathsf{A}' = \mathsf{A} - \mathbf{m} \subset S := \mathbb{Z}\mathbf{v}$. By Proposition 5.1, the *SFT* $\Omega_{\mathsf{A}'}|_S$ is not mixing under the action of $\sigma^{\mathbf{v}}$, and the last part of the proof of Theorem 5.2 shows that neither $\Omega_{\mathsf{A}'}$ nor Ω_{A} can be mixing. ∎

Proposition 5.1 shows that Ω_A is nonmixing for every finite set $\mathsf{A} \subset \mathbb{Z}$ with at least two elements. For finite subsets $\mathsf{A} \subset \mathbb{Z}^d$ with $d > 1$, the situation can be different, as the following examples show.

Example 5.5. Let $\mathsf{A} = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2$. Then the $\mathbb{Z}^2$-*SFT* Ω_{A} is topologically mixing.

Since the verification of this claim will reappear—in a slightly more complicated form—in the proof of Theorem 5.6, we shall describe it in detail.

We start with a bit of notation and terminology. Let $\mathsf{A} \subset \mathbb{Z}^d$, $d \geq 1$, be a finite set containing $\mathbf{0}$. A subset $p \subset \mathbb{Z}^d$ is *allowed* if it consists either of a single point or of a bi-infinite sequence $\{p_k\}_{k\in\mathbb{Z}}$ such that $p_{k+1} - p_k \in \mathsf{A} \smallsetminus \{\mathbf{0}\}$ for every $k \in \mathbb{Z}$. If $p \in \mathbb{Z}^d$ is allowed, its *future* p^+ is defined by

$$p^+ = \begin{cases} \{\mathbf{n}\} & \text{if } p = \{\mathbf{n}\} \text{ for some } \mathbf{n} \in \mathbb{Z}^d, \\ \{p_k\}_{k\geq 1} & \text{if } p = \{p_k\}_{k\in\mathbb{Z}} \text{ with } p_{k+1} - p_k \in \mathsf{A} \smallsetminus \{\mathbf{0}\} \text{ for all } k \in \mathbb{Z}. \end{cases}$$

The definition of the *past* p^- of p is analogous.

A collection p of disjoint subsets of $\mathbb{Z}^d$ is *allowed* if it consists of allowed sets. More generally, if $S \subset \mathbb{Z}^d$ is a subset, a collection q of disjoint subsets of S is *allowed* if there exists an allowed collection p of disjoint subsets of $\mathbb{Z}^d$ such that $\mathsf{q} = \mathsf{p} \cap S = \{p \cap S : p \in \mathsf{p}\}$.

Every permutation $\pi \in \Pi_{\mathsf{A}}$ can be represented by the allowed partition $\mathsf{p}^{(\pi)}$ of $\mathbb{Z}^d$ into orbits or 'paths' of π. Conversely, if p is an allowed collection of disjoint subsets of $\mathbb{Z}^d$, we denote by $\mathsf{U}_{\mathsf{p}} = \bigcup_{p\in\mathsf{p}} p$ the *union* of p and extend p to an allowed partition $\tilde{\mathsf{p}}$ of $\mathbb{Z}^2$ by adding to it the singletons $\{\mathbf{n}\}$, $\mathbf{n} \in \mathbb{Z}^2 \smallsetminus \mathsf{U}_{\mathsf{p}}$. The partition $\tilde{\mathsf{p}}$ is the set of orbits of a unique permutation $\pi_{\tilde{\mathsf{p}}} \in \Pi_{\mathsf{A}}$.

We return to the above set $\mathsf{A} = \{(0,0),(1,0),(0,1)\}$. Take a finite set $Q_1 \subset \mathbb{Z}^2$ and a permutation $\pi_1 \in \Pi_{\mathsf{A}}$, and consider the collection $\mathsf{p}^{(\pi_1)}(Q_1) = \{p \in \mathsf{p}^{(\pi_1)} : p \cap Q_1 \neq \emptyset\}$ of all orbits in the partition $\mathsf{p}^{(\pi_1)}$ which pass through Q_1. We modify the orbits $p \in \mathsf{p}^{(\pi_1)}(Q_1)$ outside Q_1 in such a way that they are still allowed and move vertically almost all the time. Denote this family of modified orbits by $\mathsf{p}'_1(Q_1)$. Then we do the same for another permutation $\pi_2 \in \Pi_{\mathsf{A}}$ and another finite set $Q_2 \subset \mathbb{Z}^2$ with sufficient horizontal distance from Q_1, and obtain a modified allowed collection $\mathsf{p}'_2(Q_2)$. Since Q_1 and Q_2 have sufficient horizontal distance, $\mathsf{U}_{\mathsf{p}'_1(Q_1)} \cap \mathsf{U}_{\mathsf{p}'_2(Q_2)} = \emptyset$ and the union $\mathsf{p}' = \mathsf{p}'_1(Q_1) \cup \mathsf{p}'_2(Q_2)$ is again an allowed collection of orbits. Finally, we extend p' to an allowed partition $\tilde{\mathsf{p}}'$ of $\mathbb{Z}^2$ by adding singletons and obtain a $\pi' = \pi_{\tilde{\mathsf{p}}'} \in \Pi_{\mathsf{A}}$ which coincides on Q_1 and Q_2 with π_1 and π_2, respectively.

If the sets Q_1 and Q_2 are separated vertically rather than horizontally, the modification process described above has to be changed accordingly.

Now the details: let $M \geq 0$, set $Q = Q^{(M)} = \{0, \dots, M\}^2 \subset \mathbb{Z}^2$, and define $\mathbb{E}_k = \{0, \dots, M+k\} \times \mathbb{N} \subseteq \mathbb{Z}^2$. Fix $\pi_1 \in \Pi_{\mathsf{A}}$. The collection $\mathsf{p} := \mathsf{p}^{(\pi_1)}(Q)$ of all π_1-orbits intersecting Q is finite: it has at most $|Q|$ elements.

We start the induction process by setting $\mathsf{p}^{(0)} = \mathsf{p}$ and $\mathsf{q}^{(0)} = \emptyset$. Suppose that $K \geq 0$, and that we have found allowed collections of disjoint sets $\mathsf{p}^{(k)} \subset \mathsf{p}$ and $\mathsf{q}^{(k)}$ which satisfy the following conditions for $k = 0, \dots, K$:

(i) $q^+ \subset \mathbb{E}_k$ for every $q \in \mathsf{q}^{(k)}$;
(ii) $E_{p,q} := (p \cap \tilde{Q}) \cap (q \cap \tilde{Q}) = \emptyset$ for every $p \in \mathsf{p}^{(k)}$ and $q \in \mathsf{q}^{(k)}$, where $\tilde{Q} = Q \cup \pi_1(Q)$;
(iii) the sets $\{p \cap \tilde{Q} : p \in \mathsf{p}^{(k)}\} \cup \{q \cap \tilde{Q} : q \in \mathsf{q}^{(k)}\}$ form a partition of $\tilde{Q}$ which coincides with $\{p \cap \tilde{Q} : p \in \mathsf{p}\}$.

For the induction step we suppose that $\mathsf{p}^{(K)} \neq \emptyset$ and set $\mathsf{p}_1^{(K)} = \{p \in \mathsf{p}^{(K)} : p^+ \subset \mathbb{E}_{K+1}\}$ and $\mathsf{p}_2^{(K)} = \mathsf{p}^{(K)} \smallsetminus \mathsf{p}_1^{(K)}$. If $\mathsf{p}_1^{(K)} \neq \emptyset$, let $\mathsf{p}^{(K+1)} = \mathsf{p}_2^{(K)}$. If $\mathsf{p}_1^{(K)} = \emptyset$, every $p \in \mathsf{p}^{(K)}$ will move into $\mathbb{E}_{K+2}$ after having passed through $\mathbb{E}_{K+1} \smallsetminus \mathbb{E}_K$ (this follows from the fact that every infinite orbit of π_1 can only move up or right in steps of size one). Since p is finite, and since $p^+ \cap \mathbb{E}_{K+1}$ is finite for every $p \in \mathsf{p}^{(K)}$ by assumption, the set $F = \bigcup_{p \in \mathsf{p}^{(K)}} p^+ \cap (\mathbb{E}_{K+1} \smallsetminus \mathbb{E}_K)$ is finite and contains at least one element $\mathbf{m}$ whose second coordinate is maximal (i.e., satisfies $\mathbf{m} + l\mathbf{e}^{(2)} \notin F$ for every $l > 0$). We denote by $p' = \{p'_k\}_{k \in \mathbb{Z}} \in \mathsf{p}^{(K)}$ the element containing $\mathbf{m}$ with $p'_{k_0} = \mathbf{m}$, say, and define another allowed set $p'' = \{p''_k\}_{k \in \mathbb{Z}}$ by setting

$$p''_k = \begin{cases} p'_k & \text{if } k \leq k_0, \\ p'_{k_0} + (k - k_0)\mathbf{e}^{(2)} & \text{if } k > k_0. \end{cases}$$

Let $\mathsf{p}^{(K+1)} = \mathsf{p}^{(K)} \smallsetminus \{p'\}$ and $\mathsf{q}^{(K+1)} = \mathsf{q}^{(K)} \cup \{p''\}$. By assumption, $\mathsf{q}^{(K+1)}$ is again allowed.

In either case, $\mathsf{p}^{(K+1)} \subsetneq \mathsf{p}^{(K)}$, and the sets $\mathsf{p}^{(k)}, \mathsf{q}^{(k)}$, $k = 0, \dots, K+1$, satisfy the conditions (i)–(iii) above with $K+1$ replacing K.

Since p is finite and $\mathsf{p}^{(k+1)} \subsetneq \mathsf{p}^{(k)}$ for every $k \geq 0$, there has to exist a $K \geq 1$ with $K \leq |Q|$ such that $\mathsf{p}^{(K)} = \emptyset$, and hence with $q^+ \subset \mathbb{E}_{K+1}$ for every $q \in \mathsf{q}^{(K)}$.

We have arrived at an allowed collection $\mathsf{q} = \mathsf{q}^{(K)}$ of disjoint subsets of $\mathbb{Z}^2$ such that $q^+ \subset \mathbb{E}_{K+1}$ for every $q \in \mathsf{q}$ and the partitions $\{q \cap \tilde{Q} : q \in \mathsf{q}\}$ and $\{p \cap \tilde{Q} : p \in \mathsf{q}\}$ coincide. We extend q to an allowed partition $\tilde{\mathsf{q}}$ of $\mathbb{Z}^2$ by adding singletons and obtain a permutation $\tilde{\pi}_1 := \pi_{\tilde{\mathsf{q}}} \in \Pi_{\mathsf{A}}$ with the properties that $\tilde{\pi}_1^k(\mathbf{n}) = \pi_1^k(\mathbf{n})$ for every $\mathbf{n} \in Q$ and $k \leq 1$, and that $\tilde{\pi}_1^k(\mathbf{n}) \in \mathbb{E}_{|Q|+1}$ for every $\mathbf{n} \in Q$ and $k \geq 0$.

Exactly the same argument, but with directions reversed, allows us to find a permutation $\tilde{\tilde{\pi}}_1 \in \Pi_{\mathsf{A}}$ such that $\tilde{\tilde{\pi}}_1^k(\mathbf{n}) = \tilde{\pi}_1^k(\mathbf{n})$ for every $\mathbf{n} \in Q$ and $k \geq -1$, and that $\tilde{\tilde{\pi}}_1^k(\mathbf{n}) \in \{-|Q|-1, \dots, M\} \times (-\mathbb{N})$ for every $\mathbf{n} \in Q$ and $k \leq 0$.

The permutation $\tilde{\tilde{\pi}}_1$ has the properties that $\tilde{\tilde{\pi}}_1|_Q = \tilde{\pi}_1|_Q = \pi_1|_Q$, and that each orbit of $\tilde{\tilde{\pi}}_1'$ lies in the vertical strip $\{-|Q|-1, \dots, M+|Q|+1\} \times \mathbb{Z}$.

An analogous argument yields a permutation $\tilde{\tilde{\pi}}_1' \in \Pi_{\mathsf{A}}$ such that $\tilde{\tilde{\pi}}_1'|_Q = \pi_1|_Q$, and that each orbit of $\tilde{\tilde{\pi}}_1$ lies in the horizontal strip $\mathbb{Z} \times \{-|Q|-1, \dots, M+|Q|+1\}$.

By translating this back to the shift space Ω_{A} we conclude that there exists, for every $M \geq 0$, every pair $\omega_1, \omega_2 \in \Omega_{\mathsf{A}}$, and every $\mathbf{m} \in \mathbb{Z}^2$ with $\|\mathbf{m}\| > 7|Q^{(M)}|$, an element $\omega_3 \in \Omega_{\mathsf{A}}$ with $\omega_3|_{Q^{(M)}} = \omega_1|_{Q^{(M)}}$ and $\omega_3|_{Q^{(M)}+\mathbf{m}} = \omega_2|_{Q^{(M)}+\mathbf{m}}$. Clearly this implies that Ω_{A} is mixing.

Example 5.5 illustrates the following general result.

THEOREM 5.6. *Let $d \geq 2$, and let $\mathsf{A} \subset \mathbb{Z}^d$ be a finite set. Then the $\mathbb{Z}^d$-SFT Ω_{A} is topologically mixing if and only if $\mathsf{D} = \mathsf{A} - \mathsf{A}$ does not lie in a one-dimensional subspace of $\mathbb{R}^d$.*

Proof. We begin with a simplification and a bit of notation. As we observed in the proof of Theorem 5.2, we may assume without loss in generality that $\mathbf{0} \in \mathsf{A}$; in fact, we shall assume that $\mathbf{0}$ is a vertex of the closed convex hull $\bar{\mathsf{A}}$ of A in $\mathbb{R}^d$.

Write $C(\mathsf{A}) = \{\sum_{\mathbf{n} \in \mathsf{A}} t_{\mathbf{n}} \mathbf{n} : t_{\mathbf{n}} \geq 0 \text{ for every } \mathbf{n} \in \mathsf{A}\} \subset \mathbb{R}^d$ for the *cone* of A. Let $\mathbf{e} \in \mathsf{A}$ be another vertex of $\bar{\mathsf{A}}$ such that the ray $\{t\mathbf{e} : t \geq 0\} \subset C(\mathsf{A})$ is extremal, and let $\mathsf{E} = \{t\mathbf{e} : t \in \mathbb{R}\}$. Define $\mathsf{B} = \mathsf{A} \smallsetminus \mathsf{E}$ and denote by $\bar{\mathsf{B}}$ the convex hull of B. Then E and $\bar{\mathsf{B}}$ are disjoint convex subsets of $\mathbb{R}^d$, one of which is compact, and the strict hyperplane separation theorem (cf. [1, Section 2.5.1] or [6]) implies that there exist a vector $\mathbf{w} \in \mathbb{R}^d$ and real numbers $c_1 < c_2$ such that $\langle \mathbf{w}, \mathbf{v} \rangle \leq c_1$ and $\langle \mathbf{w}, \mathbf{v}' \rangle \geq c_2$ for every $\mathbf{v} \in \mathsf{E}$ and $\mathbf{v}' \in \bar{\mathsf{B}}$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. Since the first inequality holds for every $\mathbf{v} \in \mathsf{E}$, we conclude that $\langle \mathbf{w}, \mathbf{e} \rangle = c_1 = 0$, whereas $\langle \mathbf{w}, \mathbf{n} \rangle \geq c_2 > 0$ for every $\mathbf{n} \in \mathsf{B}$.

Now we can imitate the argument in Example 5.5. Take a finite set $Q \subset \mathbb{Z}^d$ and set $L_1 = \max_{\mathbf{n}\in Q} |\langle \mathbf{w}, \mathbf{n}\rangle|$, $L_2 = \max_{\mathbf{n}\in \mathsf{B}}\langle \mathbf{w}, \mathbf{n}\rangle + 1$, and $\mathbb{E}_k = \{\mathbf{k} \in \mathbb{Z}^d : -L_1 \le \langle \mathbf{w}, \mathbf{k}\rangle \le L_1 + kL_2\}$, $k \ge 0$. Then the induction argument in Example 5.5 can be used essentially unchanged, apart from replacing $\mathbf{e}^{(2)}$ with $\mathbf{e}$.

As there, we start with an element $\pi_1 \in \varPi_{\mathsf{A}}$, consider the collection $\mathsf{p} := \mathsf{p}^{(\pi_1)}(Q)$ of all π_1-orbits intersecting Q, and end up with an allowed collection q of disjoint subsets of $\mathbb{Z}^d$ and a $K \ge 0$ such that $q^+ \subset \mathbb{E}_K$ for every $q \in \mathsf{q}$ and the partitions $\{q \cap \tilde{Q} : q \in \mathsf{q}\}$ and $\{p \cap \tilde{Q} : p \in \mathsf{p}\}$ coincide, where $\tilde{Q} = Q \cup \pi_1(Q)$. Again we extend q to an allowed partition $\tilde{\mathsf{q}}$ of $\mathbb{Z}^d$ by adding singletons and obtain a permutation $\tilde{\pi}_1 := \pi_{\tilde{\mathsf{q}}} \in \varPi_{\mathsf{A}}$ with the properties that $\tilde{\pi}_1^k(\mathbf{n}) = \pi_1^k(\mathbf{n})$ for every $\mathbf{n} \in Q$ and $k \le 1$, and that $\tilde{\pi}_1^k(\mathbf{n}) \in \mathbb{E}_K$ for every $\mathbf{n} \in Q$ and $k \ge 0$. This takes care of forward orbits; the backward orbits of π_1 are dealt with by reversing directions as in Example 5.5.

Finally, we note that we can replace the vertex $\mathbf{e} \in \mathsf{A}$ by any other vertex $\mathbf{e}' \in \mathsf{A}$ giving rise to an extremal ray of $C(\mathsf{A})$ (here we are using our assumption that $\mathsf{D} = \mathsf{A} - \mathsf{A}$ is not one-dimensional). This allows us to complete the proof of Theorem 5.6 as in Example 5.5. ∎

Remarks 5.7. (1) The dynamical properties of $\varOmega_{\mathsf{A}}$ discussed here, like topological mixing or topological entropy, are *affine* invariants: for every finite set $\mathsf{A} \subset \mathbb{Z}^d$ and every $\gamma \in \mathbb{Z}^d \rtimes \mathrm{GL}(d, \mathbb{Z})$, $\varOmega_{\gamma\mathsf{A}}$ is obtained from $\varOmega_{\mathsf{A}}$ through an affine re-parametrization of the coordinates. In particular, $\varOmega_{\gamma\mathsf{A}}$ is mixing if and only if the same is true for $\varOmega_{\mathsf{A}}$, and $h(\varOmega_{\gamma\mathsf{A}}) = h(\varOmega_{\mathsf{A}})$.

(2) Permutations with restricted movement can be considered in a more general context than we have done here. Let $\varGamma$ be a countable discrete group, and let $\mathsf{A} \subset \varGamma$ be a nonempty finite set. Consider the set $\varPi_{\mathsf{A}} \subset S^\infty(\varGamma)$ of all permutations $\pi\colon \gamma \mapsto \pi(\gamma)$ such that

$$\omega_\gamma^{(\pi)} := \pi(\gamma)\gamma^{-1} \in \mathsf{A} \quad \text{for every } \gamma \in \varGamma. \tag{5.1}$$

We set

$$\varOmega_{\mathsf{A}} = \{\omega^{(\pi)} : \pi \in \varPi_{\mathsf{A}}\} \tag{5.2}$$

and observe as for $\varGamma = \mathbb{Z}$ that $\varOmega_{\mathsf{A}} \subset \mathsf{A}^\varGamma$ is a shift of finite type for the right shift action σ of $\varGamma$ on $\mathsf{A}^\varGamma$, defined by

$$(\sigma^\delta \omega)_\gamma = \omega_{\gamma\delta} \tag{5.3}$$

for every $\omega = (\omega_\gamma)_{\gamma\in\varGamma} \in \mathsf{A}^\varGamma$ and $\delta \in \varGamma$. This construction gives rise to a natural class of examples of $\varGamma$-*SFT*'s for any countable discrete group $\varGamma$.

6. Example 5.5 revisited. As an illustration of properties of the multiparameter *SFT*'s appearing in Section 5 we return to the $\mathbb{Z}^2$-*SFT* $\varOmega_{\mathsf{A}}$, $\mathsf{A} = \{(0,0), (1,0), (0,1)\}$, of Example 5.5. As described there, every $\omega \in \varOmega_{\mathsf{A}}$ determines a permutation $\pi^{(\omega)}$ of $\mathbb{Z}^2$, each orbit of which consists either of a single point or of a bi-infinite sequence $(\mathbf{n}_k)_{k\in\mathbb{Z}}$ with $\mathbf{n}_{k+1}-\mathbf{n}_k \in \{(1,0),(0,1)\}$ for every $k\in\mathbb{Z}$. If we represent each infinite orbit of $\pi^{(\omega)}$ by a bi-infinite directed polygonal path in $\mathbb{Z}^2$, we obtain a collection $\mathsf{p}^{(\pi^{(\omega)})}$ of nonintersecting paths in $\mathbb{Z}^2$ moving either north or east at each step. Figure 2 shows the intersection of $\mathsf{p}^{(\pi^{(\omega)})}$ with a square $Q\subset\mathbb{Z}^2$. In the terminology of Example 5.5, $\mathsf{q}=\mathsf{p}^{(\pi^{(\omega)})}\cap Q$ is an *allowed collection* of disjoint subsets of Q or, for convenience, an *allowed configuration of paths* in Q. Conversely, every allowed configuration of paths in Q arises in this manner from some element of Ω_{A}.

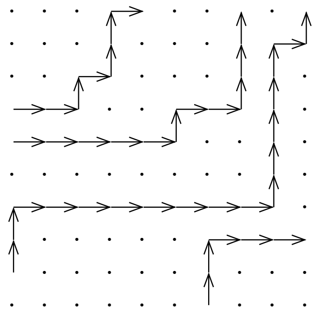


Fig. 2. An allowed configuration of paths in a square Q

The entropy of Ω_{A} is positive by Theorem 5.2. We are grateful to Christian Krattenthaler for pointing out and explaining to us [13, Theorem 3.1], which yields an explicit formula for the number of allowed configurations of k paths leading from the bottom and left edges to the top and right edges of the square Q in Figure 2. By using this formula and varying both k and the size Q one computes that the topological entropy of Ω_{A} is about $\log 1.38\ldots$.

According to Theorem 5.6, the *SFT* Ω_{A} is topologically mixing. However, it does not have the *uniform filling property* [10, Definition 3.1], nor is it *strongly irreducible* in the sense of [3, Definition 1.10]. The following proposition shows that Ω_A nevertheless has an abundance of periodic points, allowing us to express its entropy in terms of the logarithmic growth rate of the number of its periodic points.

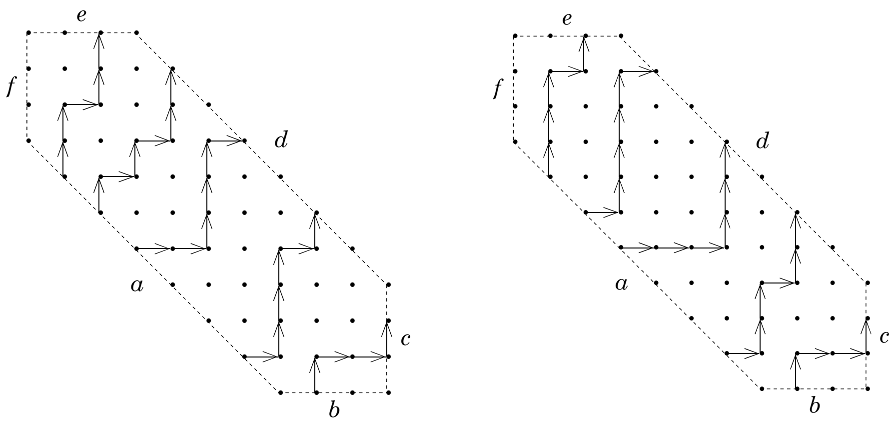
For every finite-index subgroup $\Gamma\subset\mathbb{Z}^2$ denote by $\mathrm{Fix}_\Gamma(\Omega_{\mathsf{A}})=\{\omega\in\Omega_{\mathsf{A}} : \sigma^{\mathbf{n}}\omega=\omega \text{ for every } \mathbf{n}\in\Gamma\}$ the set of Γ-periodic points in Ω_{A}. Then the following is true.

Proposition 6.1. *The set of periodic points is dense in Ω_{A}. Furthermore,*

$$h(\Omega_{\mathsf{A}})=\lim_{K\to\infty}\frac{1}{|\mathbb{Z}^2/\Delta_K|}\cdot\log|\mathrm{Fix}_{\Delta_K}(\Omega_{\mathsf{A}})|, \tag{6.1}$$

where $\Delta_K = \{2k(K^3 + 2K) \cdot (1,0) + 2lK \cdot (1,1) : k, l \in \mathbb{Z}\} \subset \mathbb{Z}^2$ *for every* $K \geq 1$.

Proof. Figure 3 shows two allowed configurations q and p of paths in a polygon $R \subset \mathbb{R}^2$ with edges a, b, c, d, e, f of lengths $|a| = |d|$ and $|b| = |c| = |e| = |f|$, whose intersections with the boundary ∂R of R (i.e., with the union of the edges of R) coincide.



(a) The configuration q (b) The configuration p

Fig. 3. Two allowed configurations of paths in a polygon R

If we rotate the configuration p in Figure 3 clockwise by 90°, flip the resulting pattern horizontally, and reverse the direction of all arrows, we obtain another allowed configuration p' of paths in R, shown in Figure 4.

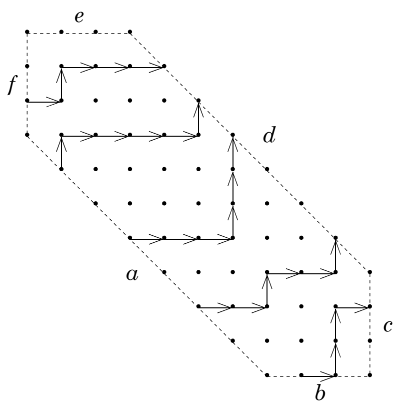


Fig. 4. The configuration p'

In Figure 5 we glue together the configurations q and p' along the edges labelled d and a, respectively, put in 'dotted' arrows to connect up loose ends, and obtain an allowed configuration (q, p) of paths in a bigger polygon $\tilde{R} \subset \mathbb{Z}^2$.

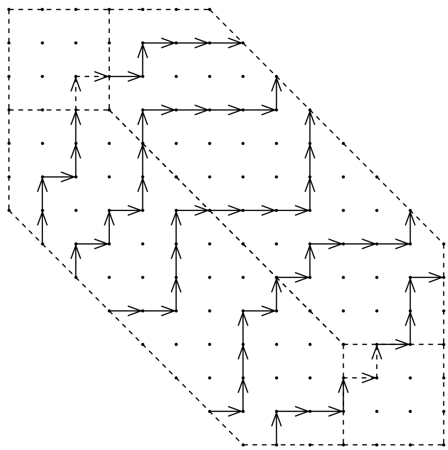


Fig. 5. The configuration (q, p) of paths in the bigger polygon $\tilde{R}$.

In Figure 6 we extend the configuration (q, p) in Figure 5 periodically, again putting in dotted arrows to connect loose ends.

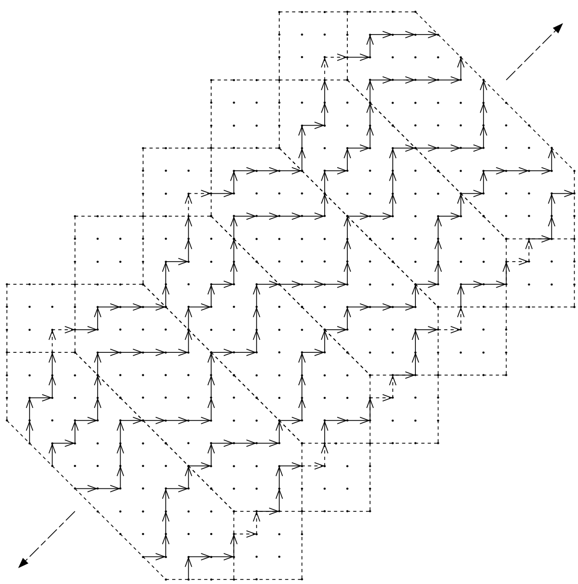


Fig. 6. The periodic extension of the configuration (q, p) in Figure 5.

Finally, we repeat the diagonal strip in Figure 6 periodically in the horizontal direction, allowing sufficient separation between the strips. Figure 7 shows the repetition of Figure 6 with a horizontal period of size $(|a| + 2|b|\sqrt{2})\sqrt{2}$ (the length of the hypotenuse of the triangle ABC).

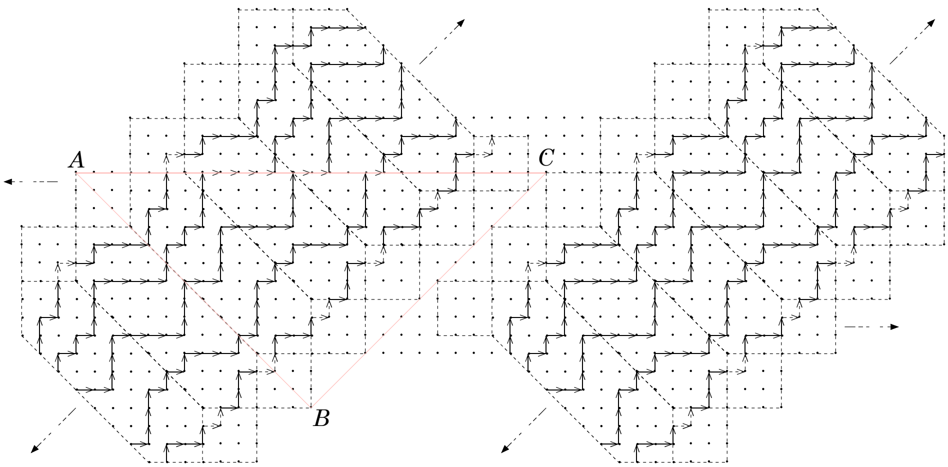


Fig. 7. The periodic repetition of the diagonal strip in Figure 6.

If we assume that the edges a and b of the polygon $R \subset \mathbb{Z}^2$ in Figure 3 have lengths $|a| = K^3\sqrt{2}$ and $|b| = K$ and write R_K instead of R to emphasise the dependence of this polygon on the integer K, then the periodic configuration of paths in Figure 7 corresponds to an element $\omega' \in \mathrm{Fix}_{\Delta_K}(\Omega_{\mathsf{A}})$, where

$$\begin{aligned} \Delta_K &= \{k(|a| + 2|b|\sqrt{2})\sqrt{2}(1,0) + 2l|b|(1,1) : k,l \in \mathbb{Z}\} \\ &= \{2k(K^3 + 2K)(1,0) + 2lK(1,1) : k,l \in \mathbb{Z}\} \subset \mathbb{Z}^2. \end{aligned} \tag{6.2}$$

If $\omega \in \Omega_{\mathsf{A}}$ is a point giving rise to the configuration q of paths in R_K, then the coordinates of ω and ω' coincide on R_K. Since K was arbitrary, this proves the density of periodic points in Ω_A.

For every $K \geq 2$, the intersection $[R_K] := R_K \cap \mathbb{Z}^2$ has cardinality $|[R_K]| = (2K+1)(K^3+1) + K^2$. The configuration q of paths in R_K arising from an element $\omega \in \Omega_{\mathsf{K}}$ is determined by the coordinates $\omega_{\mathbf{n}}$, $\mathbf{n} \in R_K$, of ω. Conversely, q determines all the coordinates $\omega_{\mathbf{n}}$, $\mathbf{n} \in R_K$, with the exception of (some of) the coordinates $\omega_{\mathbf{n}}$ with $\mathbf{n}$ lying on the boundary ∂R_K of R_K (cf. Figure 3). We write N_K for the number of allowed configurations of paths in R_K and conclude that

$$\lim_{K\to\infty} \frac{1}{|[R_K]|} \log N_K = \lim_{K\to\infty} \frac{1}{|[R_K]|} \log |\Pi_{[R_K]}(\Omega_{\mathsf{A}})| = h(\Omega_{\mathsf{A}}), \tag{6.3}$$

where we keep in mind that $([R_K])_{K\geq 1}$ is a Følner sequence in $\mathbb{Z}^2$, and where

$\varPi_F\colon \varOmega_{\mathsf{A}}[0] \to \mathsf{A}^F$ denotes the projection of every $\omega \in \varOmega_{\mathsf{A}}$ onto its coordinates in a set $F \subset \mathbb{Z}^2$.

Since ∂R_K intersects $\mathbb{Z}^2$ in $2K^3 + 4K \le 3K^3$ points, a set C of at least $M_K := N_K/|\mathsf{A}|^{3K^3} = N_K/3^{3K^3}$ of these configurations must coincide on ∂R_K. By taking all pairs of elements in C we obtain at least $M_K^2 = N_K^2/3^{6K^3}$ distinct allowed configurations (q,p), $\mathsf{q},\mathsf{p} \in \mathsf{C}$, of paths in the polygon $\tilde{R}_K$ in Figure 5, each of which determines an element of $\mathrm{Fix}_{\varDelta_K}(\varOmega_{\mathsf{A}})$, where $\varDelta_K \subset \mathbb{Z}^2$ is given in (6.2) (cf. Figure 7). Hence $|\mathrm{Fix}_{\varDelta_K}(\varOmega_{\mathsf{A}})| \ge N_K^2/3^{6K^3}$. We set $[\tilde{R}_K] = \tilde{R}_K \cap \mathbb{Z}^2$ and note that

$$|[\tilde{R}_K]| = (4K+1)(K^3+1) + 4K^2 = 2|[R_K]| + 2K^2 - K^3 - 1.$$

Since $|\mathbb{Z}^2/\varDelta_K| = 4K^4 + 8K^2$, we obtain

$$\begin{aligned}
(6.4)\qquad & \frac{1}{|\mathbb{Z}^2/\varDelta_K|}\log|\mathrm{Fix}_{\varDelta_K}(\varOmega_{\mathsf{A}})| \\
&= \frac{|[\tilde{R}_K]|}{4K^4+8K^2}\cdot\frac{2|[R_K]|}{|[\tilde{R}_K]|}\cdot\frac{1}{2|[R_K]|}\log|\mathrm{Fix}_{\varDelta_K}(\varOmega_{\mathsf{A}})| \\
&\ge \frac{4K^4}{4K^4+8K^2}\cdot\frac{2|[R_K]|}{|[\tilde{R}_K]|}\cdot\frac{1}{|[R_K]|}\log(N_K/3^{3K^3}) \\
&\ge \frac{4K^4}{4K^4+8K^2}\cdot\frac{2|[R_K]|}{|[\tilde{R}_K]|}\cdot\left(\frac{1}{|[R_K]|}\log N_K - \frac{3K^3}{2K^4}\log 3\right)
\end{aligned}$$

for every $K \ge 2$. By letting $K \to \infty$ and using (6.3) we find that

$$\liminf_{K\to\infty}\frac{1}{|\mathbb{Z}^2/\varDelta_K|}\log|\mathrm{Fix}_{\varDelta_K}(\varOmega_{\mathsf{A}})| \ge h(\varOmega_{\mathsf{A}}).$$

Since the opposite inequality $\limsup_{K\to\infty}\frac{1}{|\mathbb{Z}^2/\varDelta_K|}\log|\mathrm{Fix}_{\varDelta_K}(\varOmega_{\mathsf{A}})| \le h(\varOmega_{\mathsf{A}})$ is obvious, we have proved (6.1). ∎

Corollary 6.2.

$$\limsup_{K\to\infty}\frac{1}{K^2}\log|\mathrm{Fix}_{K\mathbb{Z}^2}(\varOmega_{\mathsf{A}})| = \lim_{K\to\infty}\frac{1}{16K^6}\log|\mathrm{Fix}_{4K^3\mathbb{Z}^2}(\varOmega_{\mathsf{A}})| = h(\varOmega_{\mathsf{A}}).$$

Proof. For every $K \ge 1$ we consider the polygon $\tilde{R}_K$ appearing in Figure 5. There we showed that there exists a set $\tilde{\mathsf{C}}$ of more than $N_K^2/3^{6K^3}$ distinct allowed configurations of paths in $\tilde{R}_K$ corresponding to points in $\mathrm{Fix}_{\varDelta_K}(\varOmega_{\mathsf{A}})$, all of which coincide on the boundary $\partial\tilde{R}_K$ of $\tilde{R}_K$.

We set $\varDelta'_K = 2(K^3+2K)\mathbb{Z}^2 \subset \varDelta_K$. Then $|\varDelta_K/\varDelta'_K| = K^2+2$, and $\varDelta_K$ is the disjoint union of the cosets $\varDelta'_K + \mathbf{v}$, $\mathbf{v} \in S_K$, with $|S_K| = K^2+2$. We choose, for every $\mathbf{v} \in S_K$, an arbitrary configuration $\mathsf{q}_{\mathbf{v}} \in \tilde{\mathsf{C}}$, fill the polygon $\tilde{R}_K + \mathbf{v}$ with the corresponding translate $\mathsf{q}_{\mathbf{v}} + \mathbf{v}$ of $\mathsf{q}_{\mathbf{v}}$, and obtain in this manner a family $\tilde{\mathsf{D}}$ of at least $(N_K^2/3^{6K^3})^{|\varDelta_K/\varDelta'_K|}$ distinct allowed configurations of paths in the set $F_K := \bigcup_{\mathbf{v}\in S_K}\tilde{R}_K + \mathbf{v}$, each of which has a unique Δ'_K-invariant extension to $\mathbb{Z}^2$ and determines an element of $\mathrm{Fix}_{\Delta'_K}(\Omega_{\mathsf{A}})$. From (6.4) we obtain

$$\frac{1}{|\mathbb{Z}^2/\Delta'_K|}\log|\mathrm{Fix}_{\Delta'_K}(\Omega_{\mathsf{A}})| \geq \frac{1}{|\mathbb{Z}^2/\Delta_K|}\log N_K/3^{3K^3}$$
$$\geq \frac{|[\tilde{R}_K]|}{4K^4+8K^2}\cdot\frac{2|[R_K]|}{|[\tilde{R}_K]|}\cdot\left(\frac{1}{|[R_K]|}\log N_K - \frac{3K^3}{2K^4}\log 3\right) \xrightarrow{K\to\infty} h(\Omega_{\mathsf{A}}).$$

As $\limsup_{K\to\infty}\frac{1}{|\mathbb{Z}^2/\Delta'_K|}\log|\mathrm{Fix}_{\Delta'_K}(\Omega_{\mathsf{A}})| \leq h(\Omega_{\mathsf{A}})$, this proves the corollary. ∎

PROBLEMS 6.3. (1) Is $\lim_{K\to\infty}\frac{1}{K^2}\log|\mathrm{Fix}_{K\mathbb{Z}^2}(\Omega_{\mathsf{A}})|$ equal to $h(\Omega_{\mathsf{A}})$?

(2) Does Ω_{A} have a unique shift-invariant probability measure of maximal entropy?

(3) Let μ be a (or *the*) shift-invariant probability measure of maximal entropy on Ω_{A}. For every $\omega \in \Omega_{\mathsf{A}}$ and every $n \geq 1$, consider the allowed configuration $\mathsf{p}^{(\omega)}(Q_n) := \mathsf{p}^{(\pi^{(\omega)})} \cap Q_n$ of paths in the square $Q_n = \{0,\dots,n\}^2 \subset \mathbb{Z}^2$ determined by ω (cf. Figure 2), and we write $N(\mathsf{p}^{(\omega)}(Q_n))$ for the number of paths (or connected components) of $\mathsf{p}^{(\omega)}(Q_n)$. Is it true that $\lim_{n\to\infty}\frac{1}{n}N(\mathsf{p}^{(\omega)}(Q_n)) = \frac{2}{3}$ for μ-a.e. $\omega \in \Omega_{\mathsf{A}}$, as numerical evidence suggests?

(4) How general are the results in this section? Are analogous statements true for every finite set $\mathsf{A} \subset \mathsf{Z}^2$ such that Ω_{A} is topologically mixing?

Acknowledgments. The first author would like to thank Christian Krattenthaler, Elon Lindenstrauss, and Benjy Weiss for helpful conversations, and the Banach Center (Simons Foundation and PMW) for hospitality and support both in Będlewo and Warszawa during the completion of this article.

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