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Topological radicals, VI. Scattered elements in Banach Jordan and associative algebras

by

Peng Cao (Beijing) and Yurii V. Turovskii (Baku)

Abstract. A Jordan or associative algebra is called scattered if it consists of elements with countable spectrum (so called scattered elements). It is proved that for sub-Banach, Jordan or associative, algebras there exists the largest scattered ideal and it is closed. Accordingly, this determines the scattered topological radical. The characterization of the scattered radical is given, and the perturbation class of scattered elements is considered.

1. Introduction. The paper continues the series of works [33, 34, 35, 22, 36], each of which can be read independently. Topological radicals were introduced by Dixon [13], for the classes $\mathfrak{U}_{na}$ and $\mathfrak{U}_{ba}$ of all normed associative and Banach associative algebras, respectively. He also suggested the axioms for algebraic radicals that were somewhat stronger than the classical Amitsur–Kurosh axioms. Later, especially in [35, 36], axioms for topological radicals for general classes of normed associative algebras were unified. In [22] some topological radicals on the class of Banach Lie algebras were considered.

The scattered radical $\mathcal{R}_s$ was introduced in [32] and studied mainly in [36] for the class $\mathfrak{U}_{ba}$. For each $A \in \mathfrak{U}_{ba}$, $\mathcal{R}_s(A)$ is the largest scattered ideal of A; an ideal is *scattered* if its elements all have countable spectrum. One of the main goals of this paper is to show that the scattered radical $\mathcal{R}_s(J)$ exists for each Banach Jordan algebra J, and that $\mathcal{R}_s$ is a topological radical on the class $\mathfrak{U}_{bj}$ of all Banach Jordan algebras, and even on the wider class $\mathfrak{U}_{sbj}$ of all sub-Banach Jordan algebras. An important fact about the latter class is that analytic functional calculus exists in every unital sub-Banach Jordan algebra (see Lemma 2.5).

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For Banach Jordan algebras J, there are well known examples of scattered ideals: the Jacobson radical, rad, and the socle, soc. Using analytic multifunctions, Aupetit gave nice spectral characterizations of them. Namely, let $x \in J$. Then x is in $\mathrm{rad}(J)$ if and only if the spectrum $\sigma(U_a x)$ is zero for each $a \in J$, and if and only if $\sup_{\lambda\in\mathbb{C}} \rho(a + \lambda x) < \infty$ for each $a \in J$, where U_a is the quadratic operator generated by a, and ρ is the spectral radius [2, Corollary 1 and Theorem 2]. Note that this is also valid for normed Jordan Q-algebras [2, Remark]. If J is semisimple then x is in $\mathrm{soc}(J)$ if and only if $U_a x$ has finite spectrum for each $a \in J$ [3, Theorem 3.11]. Wilkins [40] obtained a spectral characterization of inessential ideals (whose elements have at most 0 as a limit point of spectrum). Aupetit and Baribeau [4, Theorem 19] transferred the Barnes Theorem [6] on existence of the socle in semisimple scattered Banach associative algebras to the case of Banach Jordan algebras. They showed that each separable scattered Banach Jordan algebra J has an increasing transfinite chain $(I_\alpha)_{\alpha<\gamma}$ of closed ideals such that $I_0 = \mathrm{rad}(J)$, $I_\gamma = J$ and the quotients $I_{\alpha+1}/I_\alpha$ are modular annihilator.

Our paper is based on the technique of analytic multifunctions developed, in particular, by Aupetit [1]. One of the main results we apply to Banach Jordan algebras J is due to Aupetit and Zraïbi [5, Theorem 1]: If $f : D \to J$ is an analytic function on an open set $D \subset \mathbb{C}$, then $\lambda \mapsto \sigma(f(\lambda))$ is an analytic multifunction. We also use the Scarcity Theorem and the Aupetit–Zemánek Theorem [1, Theorems 7.2.8 and 7.2.13] that determine the behavior of scattered analytic multifunctions.

An ideal I of a normed (not necessarily associative) Q-algebra L is called *thin* if $\sigma(x) \setminus \widehat{\sigma}(x/I)$ is countable for each $x \in L$, where x/I is the coset of x in L/I and $\widehat{\sigma}(x)$ is the full spectrum of x. Every thin ideal is clearly scattered and, by [36, Theorem 8.14], each scattered ideal and its closure are thin for every sub-Banach associative algebra. In Corollary 3.4 we obtain a stronger result: the closure of a scattered ideal in the completion of a sub-Banach Jordan or associative algebra is thin. This result leans on Theorem 3.3 that is of independent interest, and improves the Gohberg–Kreĭn Theorem [16, Theorem 5.1 and Lemma 5.2] and the Aupetit Theorem on perturbation by inessential elements [1, Theorem 5.7.4].

As a consequence, we deduce that there is the largest scattered ideal $\mathcal{R}_s(L)$ in each sub-Banach Jordan or associative algebra L, it is closed (Corollary 3.5), and the map $\mathcal{R}_s : L \mapsto \mathcal{R}_s(L)$ is a topological radical on the class $\mathfrak{U}_{sbj}$ and on the class $\mathfrak{U}_{sba}$ of all sub-Banach associative algebras (Corollary 3.6). For associative algebras, this result was obtained earlier in [36, Section 8] in a different way.

We improve the results of Aupetit and Baribeau mentioned above. We prove that $\mathcal{R}_s(L)$ has a similar chain of closed ideals for any (not necessarily separable) sub-Banach Jordan or associative algebra L, and that every scattered semisimple sub-Banach Jordan or associative algebra has a non-zero socle (Corollary 3.7). Moreover, we show that $\mathcal{R}_s$ is the restriction of some algebraic radical (depending on rad and soc) to $\mathfrak{U}_{sbj}$ and $\mathfrak{U}_{sba}$ (Corollary 3.9).

The technique of analytic multifunctions allows us to give a characterization of $\mathcal{R}_s(L)$. We show that an element $x \in L$ is in $\mathcal{R}_s(L)$ if and only if $\widehat{\sigma}(x + a) \setminus \widehat{\sigma}(a)$ is countable for any $a \in L$, and if and only if $\{\lambda \in \mathbb{C} : 0 \in \sigma(a + \lambda x)\}$ is countable for every a in L^1 (the unitization of L) with $0 \notin \widehat{\sigma}(a)$ (Theorems 3.10 and 3.11). It follows from this that $\mathcal{R}_s(A)$ and $\mathcal{R}_s(A^+)$ coincide where A is a sub-Banach associative algebra and the Jordan algebra A^+ is the algebra A considered with the product $x \cdot y = (xy + yx)/2$ (Corollary 3.12).

Given a linear space X, the *perturbation class* of a subset $G \subset X$ is the set $\mathrm{Per}(G)$ of all $x \in X$ such that $x + y \in G$ for every $y \in G$. Perturbation classes were introduced by Lebow and Schechter [23] for the study of semi-Fredholm operators. For a Banach associative algebra A, the following result of Zemánek is well known [41, Theorem 2]: $a \in \mathrm{rad}(A)$ if and only if a belongs to the perturbation class of the set of all quasinilpotent elements of A.

We consider the perturbation class $Z(L) := \mathrm{Per}(S(L))$ of the set $S(L)$ of all scattered elements of L and show that, under some natural conditions on L, $Z(L)$ is a full subalgebra of L (Theorem 4.4). If $L = A$ is associative and complete normed, and D is a bounded derivation on A, then $D(A) \subset \mathcal{R}_s(A)$ whenever $D(A) \subset Z(A)$ (Theorem 4.6). If $L = J$ is a unital Banach Jordan algebra, then the associators of elements of $Z(J)$ are in $\mathcal{R}_s(J)$ (Theorem 4.8). At the end of paper we consider some conditions equivalent to the equality $Z(L) = S(L)$. In particular, we show in Corollary 4.10 that, for a Banach associative algebra A, $Z(A) = S(A)$ if and only if $S(A)$ lies in the center of A modulo the scattered radical.

Note that our results are formulated in full generality, for incomplete normed algebras. For simplicity, the reader can restrict attention to the case of complete normed algebras, but it seems that the use of incomplete normed algebras is more appropriate, especially for topological radical axioms. Note also that Theorem 3.3 answers the question for which scattered normed algebras their completion is also scattered. In Theorem 4.1 we present an example of a scattered commutative associative Q-algebra with $\mathcal{R}_s$-semisimple completion.

2. Preliminaries. In what follows, all spaces and algebras are taken over the field $\mathbb{C}$ of complex numbers. For $\lambda \in \mathbb{C}$ and $\delta > 0$, let $N(\lambda, \delta)$ denote the open disk $\{\mu \in \mathbb{C} : |\lambda - \mu| < \delta\}$. Let ∂G and $\widehat{G}$ denote the *boundary* and *polynomially convex hull* of any bounded set $G \subset \mathbb{C}$, respectively.

If X is a normed space and $Y \subset X$ then $\overline{Y}$ denotes the closure of Y in X, and $\widehat{X}$ the completion of X. If Y is a subspace of X, we identify $\widehat{Y}$ with the closed subspace of $\widehat{X}$, so $\overline{Y} \subset \widehat{Y}$. In general, $\overline{Y} \neq \widehat{Y}$.

First we recall some definitions and results on Jordan algebras. We also assume that the reader is familiar with the corresponding definitions and results in the associative setting.

2.1. Jordan algebras. A *Jordan algebra* J is a non-associative algebra whose product $\cdot$ satisfies $a \cdot b = b \cdot a$ and $(a \cdot b) \cdot a^2 = a \cdot (b \cdot a^2)$ for all $a, b \in J$, where $a^2 = a \cdot a$. The *associator* $[a, b, c]$ of elements of J is defined by

$$[a, b, c] = (a \cdot b) \cdot c - a \cdot (b \cdot c). \tag{2.1}$$

By a result of Jacobson (see [10, Lemma 3.1.23]), $b \mapsto [a, b, c]$ is a derivation of J. It is easy to check from the commutativity of the product that

$$[a, b, c] + [b, c, a] + [c, a, b] = 0 \quad \text{for all } a, b, c \in J. \tag{2.2}$$

For $x \in J$, the operator of "left" multiplication L_x is defined by $L_x a := x \cdot a$ for all $a \in J$. The operator of "right" multiplication obviously coincides with L_x. As usual, $U_{a,b}$ denotes the operator on J defined by

$$U_{a,b} x = \{a, x, b\} = (L_a L_b + L_b L_a - L_{a \cdot b}) x$$

and we set $U_a = U_{a,a}$ for all $a, b, x \in J$.

An associative algebra A endowed with the new product $a \cdot b = (ab + ba)/2$ is a well-known example of a Jordan algebra; it is denoted by A^+. Any Jordan algebra which is isomorphic to a Jordan subalgebra of A^+ is called *special.*

The *identity element* (or the *unit*) in J is an element 1 such that $a \cdot 1 = a$ for each $a \in J$. Let J^1 denote J if J is unital, and the Jordan algebra $J \oplus \mathbb{C}$ obtained from J by adjoining the unit 1 otherwise. Let $p \in J$ be a projection ($p^2 = p$) and $\mathfrak{J}_1(p) := \{x \in J : p \cdot x = x\}$. By [19, (3.1.6) and Lemma 3.1.1], $\mathfrak{J}_1(p) = U_p J$, $\mathfrak{J}_1(p)$ is a subalgebra of J, and p is the identity element of $\mathfrak{J}_1(p)$.

An element a in a unital Jordan algebra J is *invertible* if there is $b \in J$ such that $a \cdot b = 1$ and $a^2 \cdot b = a$. The inverse is unique and denoted by a^{-1}. The set of all invertible elements of J is denoted by $\mathrm{Inv}(J)$.

If J is not necessarily unital then $a \in J$ is *quasi-invertible* if $1 - a$ is invertible in J^1. For any Jordan algebra J, there is the largest ideal $\mathrm{rad}(J)$ consisting of quasi-invertible elements of J; it is called the *Jacobson radical* of J. By [25, (3)], for invertible $x, y \in J$,

$$x^{-1} - y^{-1} = U_{y^{-1}} U_{x-y} x^{-1} - U_{y^{-1}}(x - y). \tag{2.3}$$

An element a is invertible in J if and only if U_a is invertible, and in this case $U_a^{-1} = U_{a^{-1}}$. The equality

$$U_{U_a b} = U_a U_b U_a \quad \text{for all } a, b \in J \tag{2.4}$$

shows that

$$U_a b \text{ is invertible } \Leftrightarrow \ a \text{ and } b \text{ are invertible.} \tag{2.5}$$

Taking into account that $a^{-1} = U_{a^{-1}} a = U_a^{-1} a$ for each $a \in \mathrm{Inv}(J)$, we see from the above that

$$(U_a b)^{-1} = U_{U_a b}^{-1} U_a b = U_{a^{-1}} U_{b^{-1}} U_{a^{-1}} U_a b = U_{a^{-1}} b^{-1} \tag{2.6}$$

for all $a, b \in \mathrm{Inv}(J)$. It is easy to check that

$$U_{a-\lambda} = U_a - 2\lambda L_a + \lambda^2 \tag{2.7}$$

for all $a \in J$ and $\lambda \in \mathbb{C}$. In particular, $U_1 = U_{-1}$ is the identity operator on J.

For $x \in J$, the *spectrum* of x (denoted by $\sigma_J(x)$ or $\sigma(x)$) is the set of $\lambda \in \mathbb{C}$ for which $\lambda - x$ is not invertible in J^1. By the definition, $\sigma_J(x) = \sigma_{J^1}(x)$. Let $\mathrm{Res}(x) := \mathbb{C} \setminus \sigma(x)$ be the *resolvent set* of x.

A subspace M of J is called a *quadratic ideal* of J if $U_M J \subset M$ [42, Section 15.1], and an *inner ideal* of J if $U_M J^1 \subset M$ [26]. For example, $U_x J$ is an inner ideal called the *principal inner ideal* generated by $x \in J$. Any inner ideal of J is a subalgebra in J, and the intersection of inner (quadratic) ideals is an inner (quadratic) ideal. By [4, Proposition 4c)], if p is a projection in J and $x \in U_p J$, then $\sigma_{U_p J}(x) \subset \sigma_J(x)$.

For $e \in J$, an inner ideal M is called *e-modular* if $U_{1-e} J + U_{1-e,M} J^1 \subset M$ and $e^2 - e \in M$; if M is maximal among proper e-modular inner ideals, it is called a *maximal e-modular inner ideal.* Finally, M is a *maximal modular inner ideal* if it is maximal e-modular for some $e \in J$. The intersection of all maximal modular inner ideals is an ideal that coincides with $\mathrm{rad}(J)$ [18, Theorem 4.1]. If J is unital, $\mathrm{rad}(J)$ is the intersection of all maximal inner ideals of J [18, Theorem 1.1].

A largest ideal of J contained in a maximal modular inner ideal of J is called *primitive.* Let $\mathrm{Prim}(J)$ be the set of all primitive ideals of J; J is called *primitive* if $0 \in \mathrm{Prim}(J)$. The Jacobson radical is the intersection of all primitive ideals of J [26, Theorem III.5.3.1] and

$$\sigma_J(x) = \sigma_{J/\mathrm{rad}(J)}(x/\mathrm{rad}(J)) = \bigcup \{\sigma_{J/P}(x/P) : P \in \mathrm{Prim}(J^1)\}$$

for every $x \in J$ (see [39, Lemma 1]), where $x/\mathrm{rad}(J)$ and x/P are the cosets of x in $J/\mathrm{rad}(J)$ and J/P, respectively. (Usually one writes $x + I$ for x/I, but $x + I$ may be used for the set $\{x + y : y \in I\}$.)

Let I be an ideal of J. Then the *kernel-hull closure* $\mathrm{kh}(I)$ of I is an ideal of J defined by

$$\mathrm{kh}(I) = \bigcap \{P \in \mathrm{Prim}(J^1) : I \subset P\} = \bigcap \{P \in \mathrm{Prim}(J) : I \subset P\}. \tag{2.8}$$

As $\{P/I : P \in \mathrm{Prim}(J),\ I \subset P\} = \mathrm{Prim}(J/I)$, we have

$$\mathrm{kh}(I)/I = \mathrm{rad}(J/I) = \mathrm{rad}(J^1/I). \tag{2.9}$$

LEMMA 2.1. *Let J be a Jordan (or associative) algebra, I an ideal of J, and p a projection in J. If $p \in \mathrm{kh}(I)$ then $p \in I$.*

Proof. Let $\widetilde{p} = p/I$. As $\widetilde{p} \in \mathrm{rad}(J/I)$ by (2.9), $\widetilde{p}$ is quasi-invertible and $U_{1-\widetilde{p}}$ is invertible in $(J/I)^1$. As $U_{1-\widetilde{p}}\widetilde{p} = 0$, we have $\widetilde{p} = 0$ and $p \in I$. ∎

A Jordan algebra J is *non-degenerate* [26] if $U_x J^1 = 0$ implies $x = 0$ for $x \in J$. For instance, if $\mathrm{rad}(J) = 0$ then J is non-degenerate. It is easy to see that any ideal I of a non-degenerate Jordan algebra is non-degenerate: if $U_x I^1 = 0$ for $x \in I$ then $U_{U_x J^1} J^1 \subset U_x(U_{J^1} U_x J^1) \subset U_x I^1 = 0$ by (2.4), whence $U_x J^1 = 0$ and $x = 0$.

If a Jordan algebra J is non-degenerate, the *socle* $\mathrm{soc}(J)$ of J is defined as a sum of all minimal inner ideals [27]. Then $\mathrm{soc}(J)$ is the sum of the simple ideals of J generated by all completely primitive projections [27, Theorem 17], so $\mathrm{soc}(J)$ is an ideal of J. Recall that a projection p is *completely primitive* (or *division*) if the subalgebra $\mathfrak{J}_1(p)$ (equal to the image $U_p(J)$, see [19, (3.1.6) and Lemma 3.1.1]) is a division Jordan algebra. The following assertion is folklore.

LEMMA 2.2. *Let J, J_1, J_2 be non-degenerate Jordan algebras. Then:*

(1) *If θ is a homomorphism from J_1 onto J_2 then $\theta(\mathrm{soc}(J_1)) \subset \mathrm{soc}(J_2)$.*
(2) *If I is an ideal of J then $\mathrm{soc}(I) = I \cap \mathrm{soc}(J)$ and $\mathrm{soc}(\mathrm{soc}(I)) = \mathrm{soc}(I)$.*

Proof. (1) Let M be a minimal inner ideal of J_1. It is clear that $\theta(M)$ is an inner ideal of J_2. Assume that $\theta(M) \neq 0$. If K is a non-zero inner ideal of J_2 and $K \subset \theta(M)$ then $\theta^{-1}(K) \cap M$ is an inner ideal of J_1, whence $M \subset \theta^{-1}(K)$ and $\theta(M) \subset K$. Therefore $\theta(M)$ is a minimal inner ideal of J_2. This shows that $\theta(\mathrm{soc}(J_1)) \subset \mathrm{soc}(J_2)$.

(2) It is easy to see by using (2.4) that if K is an inner ideal of J then $U_{U_K J^1} J^1 \subset U_K J^1$, whence $U_K J^1$ is also an inner ideal J.

Let M be a minimal inner ideal of J. Then $U_M J^1 = M$ or $U_M J^1 = 0$, but the latter case is impossible by assumption. As $M \cap I$ is also an inner ideal of J, we have either $M \subset I$ or $M \cap I = 0$. Assume that K is a non-zero inner ideal of I and $K \subset M \subset I$. Then $U_K J^1$ is a non-zero inner ideal of J: $U_{U_K J^1} J^1 \subset U_K I \subset U_K J^1$. As M is minimal in J, $U_K J^1 = K = M$, i.e. M is a minimal inner ideal of I. This shows that $I \cap \mathrm{soc}(J) \subset \mathrm{soc}(I)$.

Let N be a minimal inner ideal of I. Then $U_N I^1 = N$ and $U_N J^1 = U_{U_N J^1} J^1 \subset U_N I^1 = N$, whence N is an inner ideal of J. Assume that K

is a non-zero inner ideal of J and $K \subset N$. Then $U_K J^1$ is a non-zero inner ideal of I, whence $U_K J^1 = K = N$, i.e. N is a minimal inner ideal of J. This shows that $\operatorname{soc}(I) \subset \operatorname{soc}(J)$.

To prove $\operatorname{soc}(\operatorname{soc}(I)) = \operatorname{soc}(I)$ it suffices to replace I by $\operatorname{soc}(I)$ in $\operatorname{soc}(I) = I \cap \operatorname{soc}(J)$. ■

A subalgebra M of J is called *strongly associative* if $[L_a, L_b] = 0$ for any $a, b \in M$, *spectral* if $\sigma_M(a) \cup \{0\} = \sigma_J(a) \cup \{0\}$ for each $a \in M$, and *full* (or *inverse closed*) if M contains the inverse of each element of M invertible in J. Each strongly associative subalgebra is associative. By [25, Corollary 2.3], for any $a \in J$ there is a maximal strongly associative subalgebra containing a which is inverse closed whenever J is unital; this subalgebra is obviously spectral. If J is a special Jordan algebra, i.e. $J \subset A^+$ for some associative algebra A, then $U_{a,b}x = (axb + bxa)/2$ for all $a, b, x \in J$, $x \cdot x = (xx + xx)/2 = xx$, $x \cdot x \cdot x = xxx$ and so on. So maximal strongly associative subalgebras of J are really subalgebras of A.

LEMMA 2.3. *Let J be a unital Jordan algebra, and let $x, a \in J$ be invertible. Then $\sigma(a^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(a)\}$ and $\sigma(U_x a^2) = \sigma(U_a x^2)$.*

Proof. Let $\mu \in \mathbb{C}$ be non-zero. As $a - \mu = U_a(a^{-1} - \mu a^{-2})$, we see that $a - \mu$ and $a^{-1} - \mu a^{-2}$ are simultaneously invertible or not. Let A be a maximal strongly associative subalgebra of J containing a. Then $a^{-1} - \mu a^{-2}$ is invertible in J if and only if it is so in A. As A is an associative algebra, $a^{-1} - \mu a^{-2}$ is invertible in A if and only if $1 - \mu a^{-1}$ is. As A is an inverse closed subalgebra of J, $1 - \mu a^{-1}$ is invertible in A if and only if it is so in J. Hence

$$\sigma(a^{-1}) = \{\mu^{-1} : \mu \in \sigma(a)\}. \tag{2.10}$$

It is clear that $U_x a^2$ and $U_a x^2$ are invertible. Let $\lambda \in \mathbb{C}$ be non-zero. Then it follows from

$$U_x a^2 - \lambda = U_x(a^2 - \lambda x^{-2}) = U_x U_a(1 - \lambda U_{a^{-1}} x^{-2})$$

that $U_x a^2 - \lambda$ and $1 - \lambda U_{a^{-1}} x^{-2}$ are simultaneously invertible or not. As $U_{a^{-1}} x^{-2}$ is the inverse of $U_a x^2$ (see (2.6)), we have $\sigma(U_x a^2) = \sigma(U_a x^2)$ by using (2.10). ■

An element x in J is called *scattered* in J if $\sigma_J(x)$ is finite or countable. In what follows, we omit "finite or" for brevity. The set of all scattered elements in J is denoted by $S(J)$. If J is not unital then $S(J) = S(J^1) \cap J$ and $S(J^1) = S(J) + \mathbb{C}$. A subalgebra or ideal of J is called *scattered* if it consists of scattered elements of J. All these notions can be transferred to associative algebras. So, if A is an associative algebra then $S(A)$ denotes the set of all scattered elements of A.

2.2. Banach Jordan algebras. A Jordan algebra J is called *normed* if it is a normed space with norm $\|\cdot\|$ and $\|x\cdot y\| \le \|x\|\,\|y\|$ for all $x, y \in J$. If J is unital, we assume that $\|1\| = 1$. If J is complete under the norm then J is called a *Banach* (or *complete normed*) *Jordan algebra.*

Let J be a unital normed Jordan algebra and $\mathcal{A}^{\mathrm{ic}}(x)$ (or $\mathcal{A}^{\mathrm{ic}}(x;J)$ if it is necessary to indicate J) be a closed full subalgebra of J generated by $x \in J$. It follows from [25] that $\mathcal{A}^{\mathrm{ic}}(x)$ is associative and is the closure of the algebra of rational functions $p(x)\cdot q(x)^{-1}$, where p, q are polynomials and $q(x)$ is invertible. If $J \subset A^{+}$ for a normed associative algebra A, then $\mathcal{A}^{\mathrm{ic}}(x, J^1)$ is a closed full commutative subalgebra of A^1.

By [25, Theorem 2.8], for a Banach Jordan algebra J, $\sigma_J(x) = \sigma_{\mathcal{A}^{\mathrm{ic}}(x)}(x)$ is a compact non-empty set in $\mathbb{C}$ and $\lambda \mapsto (\lambda - x)^{-1}$ is analytic on $\mathrm{Res}(x)$. Moreover, by analytic functional calculus (see [25, Theorem 2.9]), for any $\mathbb{C}$-valued function f analytic on some neighborhood V of $\sigma(x)$ there is an element $f(x) \in \mathcal{A}^{\mathrm{ic}}(x)$ defined by

$$f(x) = (2\pi i)^{-1} \int_{\Gamma} f(\xi)(\xi - x)^{-1}\, d\xi \tag{2.11}$$

where Γ is a suitable contour in V surrounding $\sigma(x)$, $f(x)$ does not depend on the choice of V and Γ, and the map $f \mapsto f(x)$ is a homomorphism from the algebra $\mathcal{O}(V)$ of functions analytic on V into J; this homomorphism is continuous with respect to convergence of functions on compact subsets of V (see [30, Theorem 10.27]). If $\Gamma \subset \mathrm{Res}(x)$ surrounds the clopen part σ of $\sigma(x)$ then

$$p_\sigma(x) = (2\pi i)^{-1} \int_{\Gamma} (\xi - x)^{-1}\, d\xi \tag{2.12}$$

is called the *spectral projection* of x *corresponding to* σ; if $\Gamma = \partial N(\lambda, \delta)$, it is convenient to write $p_{\lambda.\delta}(x)$ for $p_\sigma(x)$. One of the important properties of functional calculus is the *Spectral Mapping Theorem*:

$$f(\sigma(x)) = \sigma(f(x)) \tag{2.13}$$

for every function f analytic in some open neighborhood of $\sigma(x)$. In particular, $\sigma(p_\sigma(x))$ is equal to $\sigma \cup \{0\}$ if $\sigma(x) \setminus \sigma$ is not empty, and to σ otherwise. Hence $\sigma = \emptyset$ if and only if $p_\sigma(x) = 0$.

Let $x \in J$. If U_1 is an open set containing $\sigma(x)$ and f is holomorphic on U_1, and U_2 is an open set containing $f(\sigma(x))$ and g is holomorphic on U_2, then $(g \circ f)(x) = g(f(x))$ (see for instance [1, Exercise 3.13]).

Let J be a normed Jordan algebra. Then $\rho(x) = \lim \|x^n\|^{1/n}$ is the *spectral radius* of $x \in J$; x is called *quasinilpotent* if $\rho(x) = 0$. If J is complete normed then, by the Beurling–Gelfand formula (see also [25]),

$$\rho(x) = \rho_J(x) := \max\{|\lambda| : \lambda \in \sigma(x)\}, \tag{2.14}$$

and the map $x \mapsto \sigma(x)$ is upper semicontinuous on J [3, Theorems 2.1 and 2.2]. In particular, $x \mapsto \rho(x)$ is upper semicontinuous on J.

For a compact set $K \subset \mathbb{C}$, recall that $\widehat{K}$ is the set of all $\mu \in \mathbb{C}$ such that $|p(\mu)| \le \sup_{\lambda\in K} |p(\lambda)|$ for every polynomial p [15, Section 3.1]. In particular, $\widehat{K}$ is a compact set in $\mathbb{C}$,

$$\sup_{\mu\in\widehat{K}} |p(\mu)| = \sup_{\lambda\in K} |p(\lambda)| = \sup_{\mu\in\partial\widehat{K}} |p(\mu)| = \sup_{\lambda\in\partial K} |p(\lambda)| \tag{2.15}$$

for every polynomial p, and $\widehat{\partial K} = \widehat{K}$. We write $\widehat{\sigma}(x)$ for $\widehat{\sigma(x)}$. The set $\widehat{\sigma}(x)$ is called the *full spectrum* of x. Using upper semicontinuity of ρ and (2.15), it is easy to show that the map $x \mapsto \widehat{\sigma}(x)$ is upper semicontinuous on J.

By [10, Proposition 4.1.28], if J is a Banach Jordan algebra with unit 1 and if M is a closed subalgebra containing 1, then

$$\sigma_J(x) \subset \sigma_M(x) \quad \text{and} \quad \partial\sigma_M(x) \subset \partial\sigma_J(x) \quad \text{for any } x \in M,$$

whence $\widehat{\sigma}_M(x) = \widehat{\sigma}_J(x)$. Hence $S(L) = S(J) \cap L$ for any closed subalgebra L of J. In particular, each scattered closed subalgebra is a scattered Jordan algebra. If $J \subset A^+$ for a Banach associative algebra A then $S(J)$ coincides with $S(A) \cap J$ elementwise.

As the spectrum of any element in a normed Jordan algebra is not empty, the Gelfand–Mazur Theorem [14, 1.2] shows that every division normed Jordan algebra is isomorphic to $\mathbb{C}$. So the socle of a normed non-degenerate Jordan algebra J is the sum of the simple ideals generated by all minimal projections; a projection p in J is called *minimal* if $U_pJ = \mathbb{C}p$.

2.3. Sub-Banach Jordan algebras. A normed Jordan algebra J is called a *Q-algebra* if $\mathrm{Inv}(J^1)$ is open. By [29, Theorem 4], J is a Q-algebra if and only if $\rho_J(a) \le \|a\|$ for every $a \in J$, and if and only if J^1 is a Q-subalgebra of its completion $\widehat{J^1}$. It is easy to see (from $\rho_J(\cdot) \le \|\cdot\|$) that J is a Q-algebra if and only if the same is true of $\mathcal{A}^{\mathrm{ic}}(x; J^1)$ for every $x \in J$. As any normed associative algebra is a Q-algebra if and only if it is a spectral subalgebra of its completion (see [21, Lemma 20.9] and [28, Theorem 4.2.10]), J is a Q-algebra if and only if it is a spectral subalgebra of $\widehat{J}$, and if and only if $\rho_J(a) = \rho(a)$ for any $a \in J$ (see also [10, Theorem 4.6.11]).

Let J be a normed Jordan Q-algebra. Then any inverse closed subalgebra of J is spectral. Let I be an ideal of J and $x \in I$. As $\mathcal{A}^{\mathrm{ic}}(x; I^1) \subset \mathcal{A}^{\mathrm{ic}}(x; J^1) \cap I + \mathbb{C}$, we have

$$\sigma_I(x) \cup \{0\} = \sigma_J(x) \cup \{0\} \quad \text{for each } x \in I.$$

Hence I is a spectral subalgebra of J, $S(I) = S(J) \cap I$, and if I consists of scattered elements then I is a scattered normed Jordan algebra.

A normed Jordan algebra J is a normed Jordan Q-algebra if and only if all maximal modular inner ideals are closed [10, Theorem 4.4.72]. So all primitive ideals and the Jacobson radical of a normed Jordan Q-algebra are closed. In particular, if I is an ideal of a normed Jordan Q-algebra J then $\overline{I} \subset P$ for every primitive ideal of J such that $I \subset P$. Hence (also in the associative setting)

$$\overline{I} \subset \mathrm{kh}(I) = \mathrm{kh}(\overline{I}) = \overline{\mathrm{kh}(I)}. \tag{2.16}$$

Let L be a unital normed, Jordan or associative, Q-algebra. Then $\sigma_L(x)$ is equal to $\sigma_{\widehat{L}}(x)$ or to $\sigma_{\widehat{L}}(x) \cup \{0\}$ for each $x \in L$. Hence, for every function f analytic on some neighborhood of $\sigma_L(x)$, $f(x)$ given by (2.11) is an element of the completion $\widehat{L}$ by analytic functional calculus.

We say that *analytic functional calculus exists* in L if, for each $x \in L$ and each function f analytic on some neighborhood of $\sigma_L(x)$, the element $f(x)$ of the completion $\widehat{L}$ belongs to L. It should be noted that in this case $\sigma_L(x) = \sigma_{\widehat{L}}(x)$ for each $x \in L$. Indeed, if x is invertible in $\widehat{L}$ then the inverse x^{-1} in $\widehat{L}$ is the value of the integral of the form (2.11) and must belong to L by assumption.

LEMMA 2.4. *Let L be a normed Jordan or associative Q-algebra and I be a proper ideal of L. Then:*

(1) *If p is a projection in L and $p \in \overline{I}$ then $p \in I$.*
(2) *Assume that analytic functional calculus exists in L^1. Then:*
   (a) *If $x \in I$ then $f(x) - f(0) \in I$ for every function f analytic on some neighborhood V of $\sigma(x)$. That is, analytic functional calculus exists in I^1.*
   (b) *If M is a unital closed full subalgebra of L^1 then analytic functional calculus exists in M.*
   (c) *If $x, y \in L$ and $x - y \in I$ then $f(x) - f(y) \in \overline{I}$ for each function f analytic on some neighborhood of $\sigma(x) \cup \sigma(y)$. In particular, if $\partial N(\lambda, \delta) \subset \mathrm{Res}(x) \cap \mathrm{Res}(y)$ then $p_{\lambda,\delta}(x) - p_{\lambda,\delta}(y) \in \overline{I}$.*

Proof. (1) As $p \in \overline{I}$, we have $p \in \mathrm{kh}(I)$ by (2.16). By Lemma 2.1, $p \in I$.

(2a) As I is proper, x is not invertible. So f is defined at zero for every function f analytic on some neighborhood of $\sigma_L(x)$.

As $(\xi - x)^{-1} = \xi^{-1} + \xi^{-1}x(\xi - x)^{-1}$ for every non-zero $\xi \in \mathrm{Res}(x)$, we have

$$f(x) - f(0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi}\, d\xi + \frac{x}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi} (\xi - x)^{-1}\, d\xi - f(0).$$

From the Cauchy integral formula

$$f(0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi}\, d\xi,$$

so $f(x) - f(0) \in I$.

(2b) If $x \in M$ then $(\xi - x)^{-1} \in M$ for each $\xi \in \mathrm{Res}(x)$. So partial sums of (2.11) belong to M. As M is closed, $f(x) \in M$.

(2c) By (2.3), $(\xi - x)^{-1} - (\xi - y)^{-1} \in I$ for every $\xi \in \mathrm{Res}(x) \cap \mathrm{Res}(y)$. As $f(x) - f(y) = (2\pi i)^{-1} \int_\Gamma f(\xi)((\xi - x)^{-1} - (\xi - y)^{-1})\, d\xi$ (see (2.11)), the partial sums of this integral lie in I. So $f(x) - f(y)$ belongs to $\overline{I}$. ■

Let L be a Jordan or associative algebra. Recall that a subalgebra M of L is called a *subideal* of L if there is a finite chain $K_n \subset \cdots \subset K_1 \subset K_0$ of subalgebras of L such that K_{i+1} is an ideal of K_i for any i and $M = K_n$, $L = K_0$. A normed algebra is called a *Banach subideal* if it is a subideal of some Banach algebra; it is clear that such an algebra may not be complete.

We call an algebra M a *sub-Banach algebra* if there is a complete normed algebra L with a finite chain $K_n \subset \cdots \subset K_1 \subset K_0$ of subalgebras of L such that $M = K_n$, $L = K_0$, and for each i, either K_{i+1} is an ideal of K_i, or K^1_{i+1} is a closed full subalgebra of K^1_i. Identifying K^1_{i+1} with a subalgebra of K^1_i for each i, we obtain

$$M^1 = K^1_n \subset K^1_{n-1} \subset \cdots \subset K^1_1 \subset K^1_0 = L^1. \tag{2.17}$$

Clearly, each ideal and each closed full subalgebra of a unital sub-Banach algebra are sub-Banach algebras.

LEMMA 2.5. *Let M be a sub-Banach Jordan or associative algebra. Then:*

(1) *M is a normed Q-algebra.*
(2) *Analytic functional calculus exists in M^1.*
(3) *$M^1 = N^1$ for some normed algebra N that is a subideal of $\widehat{N}$.*
(4) *If I is a proper ideal of M then I is a Banach subideal.*
(5) *If I is a closed ideal of M then*
   (a) *M/I is (isometrically isomorphic to) a sub-Banach algebra;*
   (b) *if M is unital then $\bigcap_{y\in I} \sigma(x + y) \subset \widehat{\sigma}(x/I)$ for every $x \in M$.*

Proof. (1) follows from the fact that every Banach Jordan or associative algebra is a normed Q-algebra, and every ideal or closed full subalgebra of a unital normed Q-algebra is again a normed Q-algebra.

(2) follows from Lemma 2.4 by induction on n in (2.17).

(3) Let M^1 be taken from (2.17). As L^1 is complete normed, one may consider its completion $\widehat{M^1}$ as a closed subalgebra of L^1. Hence

$$M^1 = N^1_n \subset N^1_{n-1} \subset \cdots \subset N^1_1 \subset N^1_0 = \widehat{M^1}, \tag{2.18}$$

where $N_i = K_i \cap \widehat{M^1}$ for any i. If K_{i+1} is an ideal of K_i for some i then N_{i+1} is an ideal of N_i.

Suppose that K^1_{i+1} is a closed full subalgebra of K^1_i. Then N^1_{i+1} is a closed subalgebra of N^1_i. But it follows from (2.18) that N^1_{i+1} is dense in N^1_i.

Hence $N^1_{i+1} = N^1_i$. Let $N = N_n$. Then M is either N or N^1, and N is a subideal of $\widehat{N}$.

(4) follows from (3).

(5a) Assume that (2.18) holds where each N_{i+1} is an ideal of N_i. One may clearly consider $\widehat{I}$ as a closed ideal in $\widehat{M^1}$. Then consider the following chain of subalgebras of $\widehat{M^1}$:

$$M^1 + \widehat{I} = F^1_n \subset F^1_{n-1} \subset \cdots \subset F^1_1 \subset F^1_0 = \widehat{M^1},$$

where $F_i = N_i + \widehat{I}$ for any i. It is clear that each F_{i+1} is an ideal of F_i, and $\widehat{I}$ is a closed ideal of each algebra F^1_i. As a result, we obtain the following chain of subalgebras of $\widehat{M^1}/\widehat{I}$:

$$(M^1 + \widehat{I})/\widehat{I} = F^1_n/\widehat{I} \subset F^1_{n-1}/\widehat{I} \subset \cdots \subset F^1_1/\widehat{I} \subset F^1_0/\widehat{I} = \widehat{M^1}/\widehat{I}.$$

Here $\widehat{M^1}/\widehat{I}$ is complete normed, and it is easy to see that M/I and $F_n/\widehat{I}$ are isometrically isomorphic under $x/I \mapsto x/\widehat{I}$. Identifying them we conclude that M/I is a sub-Banach algebra.

(5b) Let $\lambda \notin \widehat{\sigma}(x/I)$. As $\widehat{\sigma}(x/I) = \widehat{\sigma}_{\widehat{M/I}}(x/I)$ and 0 is in an unbounded component of $\mathrm{Res}((\lambda - x)/I)$, [30, 10.30] shows that $(\lambda - x)/I$ has a logarithm in $\widehat{M/I}$. By (2), it has a logarithm in M/I. Thus there is $z \in M$ such that $(\lambda - x)/I = \exp(z/I) = \exp(z)/I$. So there is $y \in I$ such that $\lambda - x = \exp(z) + y$. Hence $\lambda - x - y$ is invertible and $\lambda \notin \sigma(x + y)$. ∎

Let J be a unital sub-Banach Jordan algebra, and define

$$\Omega_1(J) = \bigcup_{n\in\mathbb{N}} \{U_{\exp(a_1)} \cdots U_{\exp(a_n)} 1 : a_1, \ldots, a_n \in J\}.$$

It follows from (2.4) that $\Omega_1(J) \subset \mathrm{Inv}(J)$, and $\exp(a) = U_{\exp(a/2)}1$ lies in $\Omega_1(J)$ for any $a \in J$. Hence $\{x \in J : \|1 - x\| < 1\} \subset \Omega_1(J)$ (because if $\|1 - x\| < 1$ then $0 \notin \widehat{\sigma}(x)$, whence x has a logarithm in J). If $a = U_{\exp(a_1)} \cdots U_{\exp(a_n)} 1$ and $f(\lambda) = U_{\exp(\lambda a_1)} \cdots U_{\exp(\lambda a_n)} 1$ then $f(\lambda)$ lies in $\mathrm{Inv}(J)$ for every $\lambda \in \mathbb{C}$, the functions f and $\lambda \mapsto f(\lambda)^{-1}$ are analytic on $\mathbb{C}$, $f(1) = a^{-1}$ and $f(0) = f(0)^{-1} = 1$. Hence

$$a^{-1}, U_a b \in \Omega_1(J) \quad \text{if } a, b \in \Omega_1(J), \tag{2.19}$$

and $\Omega_1(J)$ is a connected component of $\mathrm{Inv}(J)$ containing 1. If J is complete normed, $\Omega_1(J)$ is a principal component of $\mathrm{Inv}(J)$ by [24].

Note that, for a unital sub-Banach associative algebra A, $\Omega_1(A)$ can be defined by

$$\Omega_1(A) = \bigcup_{n\in\mathbb{N}} \{\exp(a_1) \cdots \exp(a_n) : a_1, \ldots, a_n \in J\}.$$

Then $\Omega_1(A)$ is a subgroup of $\mathrm{Inv}(A)$ and a connected component of $\mathrm{Inv}(A)$ containing 1.

2.4. Topological radicals. Let $\mathfrak{U}$ be a class of normed (not necessarily associative) algebras. We may assume that $\mathfrak{U}$ is closed under images of *topological morphisms*: If $\theta : L_1 \to L_2$ is an open continuous homomorphism of L_1 onto L_2, and $\mathfrak{U}$ contains one of them, then it contains the other. It is convenient to identify two algebras if there is an isometric isomorphism between them. By default, by a *morphism* we mean a topological morphism.

A class $\mathfrak{U}$ of normed algebras is called *ground* if it contains all closed ideals and the corresponding quotients of all algebras from $\mathfrak{U}$, and *universal* if it is ground and contains arbitrary ideals of all algebras from $\mathfrak{U}$. The class $\mathfrak{U}_{bj}$ of all Banach Jordan algebras is ground, and the class $\mathfrak{U}_{nj}$ of all normed Jordan algebras is universal. By Lemma 2.5, the class of all Banach Jordan subideals and the class $\mathfrak{U}_{sbj}$ of all sub-Banach Jordan algebras are universal; the same is valid for the class of all Banach associative subideals and the class $\mathfrak{U}_{sba}$ of all sub-Banach associative algebras.

Let $\mathfrak{U}$ be a ground or universal class of normed algebras. The map P on $\mathfrak{U}$ that associates with each algebra L a closed ideal $P(L)$ of L is called a *topological radical on* $\mathfrak{U}$ if it satisfies the following conditions (called the *topological radical axioms*):

1. $\theta(P(L_1)) \subset P(L_2)$ for any morphism $\theta : L_1 \to L_2$ of algebras from $\mathfrak{U}$;
2. $P(L/P(L)) = 0$;
3. $P(P(L)) = P(L)$;
4. $P(I)$ is an ideal of L contained in $P(L)$, for any ideal $I \in \mathfrak{U}$ of L.

An algebra $L \in \mathfrak{U}$ is called *P-semisimple* if $P(L) = 0$, and *P-radical* if $L = P(L)$.

The most famous radical in the class $\mathfrak{U}_j$ of all Jordan algebras, as well as in the class $\mathfrak{U}_a$ of all associative algebras, is the Jacobson radical rad. An algebra L from $\mathfrak{U}_j$ or $\mathfrak{U}_a$ is called *semisimple* if $\mathrm{rad}(L) = 0$, and *radical* if $L = \mathrm{rad}(L)$. The Jacobson radical is a radical in the Amitsur–Kurosh sense and also an algebraic radical in Dixon's sense: it satisfies the *algebraic radical axioms* that repeat Axioms 1–4 with one exception: in Axiom 1 *algebraic morphisms* are assumed, i.e. onto homomorphisms.

Let us introduce now the following useful notions. A map P on $\mathfrak{U}$ such that $P(L)$ is an ideal of L, for each $L \in \mathfrak{U}$, is called an (*algebraic*) *preradical* [36] if it satisfies (algebraic) Axiom 1. For normed algebras, a preradical P is *topological* if $P(L)$ is a closed ideal of L for each L. A preradical P is *hereditary* if $P(I) = I \cap P(L)$ for any ideal $I \in \mathfrak{U}$ of $L \in \mathfrak{U}$. For any hereditary topological radical P on $\mathfrak{U}_{bj}$, the assignment $P \mapsto P^r$ defined by $P^r(J) = J \cap P(\widehat{J})$ maps P into a hereditary topological radical P^r on $\mathfrak{U}_{nj}$. (For normed associative algebras this was proved in [33, Theorem 2.21]; the same proof is valid for normed non-associative algebras, in particular for normed Jordan algebras.) This procedure is called *regular*. For more details of the theory of topological and algebraic radicals we refer to [36].

The Jacobson radical is not a topological radical in the class $\mathfrak{U}_{nj}$ or in the class of all normed associative algebras, but it is an algebraic hereditary preradical. In fact, Dixon [13, Example 10.1] constructed an algebra $A \in \mathfrak{U}_{na}$ for which $\operatorname{rad}(A)$ is not closed, whence $A^+ \in \mathfrak{U}_{nj}$ and $\operatorname{rad}(A^+) = \operatorname{rad}(A)^+$ is not closed in A^+. However, rad is a hereditary topological radical on the class of all normed Jordan Q-algebras because rad is a hereditary algebraic radical on $\mathfrak{U}_j$ and $\operatorname{rad}(J)$ is closed in J for any normed Jordan Q-algebra J.

Let Rad be the restriction of rad to the classes $\mathfrak{U}_{bj}$ of all Banach Jordan algebras and $\mathfrak{U}_{ba}$ of all Banach associative algebras. Then Rad is a hereditary topological radical on both of these classes. Then Rad^r is a hereditary topological radical on $\mathfrak{U}_{nj}$ and $\mathfrak{U}_{na}$; it is called the *regular Jacobson radical.*

One of the most useful tools for the theory of radicals of Jordan algebras is the Slin'ko Theorem [42, Theorem 14.12] that states that if J is a Jordan algebra, I is an ideal of J and K is an ideal of I, and if I/K has no nilpotent ideals, then K is an ideal of J.

2.5. Analytic multifunctions. By a *multifunction* we mean any map K from an open subset D of $\mathbb{C}$ into the set of all non-empty compacts in $\mathbb{C}$; K is *analytic* on D if K is upper semicontinuous on D and if for any open set G in D and for any function ψ plurisubharmonic on a neighborhood of $\{(\lambda, z) : \lambda \in G,\ z \in K(\lambda)\}$ the function $\varphi(\lambda) = \sup\{\psi(\lambda, z) : z \in K(\lambda)\}$ is subharmonic on G. The other names in the literature are *analytic multivalued* (or *set-valued*), *functions.*

Recall two facts from the theory of analytic multifunctions [1, Chapter 7]. Let K be an analytic multifunction from an open set $D \subset \mathbb{C}$ into $\mathbb{C}$. Then either $\{\lambda \in D : K(\lambda) \text{ is countable}\}$ has capacity zero, or $K(\lambda)$ is countable for all $\lambda \in D$, by the *Scarcity Theorem* [1, Theorem 7.2.8]. In the last case, for a fixed $\eta \in \mathbb{C}$, the set $\{\lambda \in D : \eta \in K(\lambda)\}$ is either countable or equal to D, by the *Aupetit–Zemánek Theorem* [1, Theorem 7.2.13]. We also note the important Localization Principle [1, Theorem 7.1.5].

It is important for us to underline that the Aupetit–Zraïbi Theorem [5, Theorem 1] is valid for a sub-Banach Jordan (or associative) algebra L: *if $f : D \to L$ is an analytic function then the map $\lambda \mapsto \sigma(f(\lambda))$ is an analytic multifunction.* Indeed, as L is a full subalgebra of its completion $\widehat{L}$ (see Lemma 2.5), it follows that $\sigma(f(\lambda)) = \sigma_{\widehat{L}}(f(\lambda))$ for every $\lambda \in D$. So we can refer to the Aupetit–Zraïbi Theorem for Banach Jordan (or associative) algebras.

Remark 2.6. If L is a normed Jordan (or associative) Q-algebra and if $f : D \to L$ is an analytic function and $K(\lambda) = \sigma(f(\lambda))$ for any $\lambda \in D$, then the Scarcity Theorem and the Aupetit–Zemánek Theorem are applicable to K. Indeed, let $K'(\lambda) = \sigma_{\widehat{L}}(f(\lambda))$ for $\lambda \in D$. Then K' is an analytic multifunction, and (the conclusions of) these theorems hold simultaneously for K and K', because L is a spectral subalgebra of $\widehat{L}$.

3. The scattered radical

3.1. Preparatory lemma. We start with the following important results.

LEMMA 3.1. *Let J be a unital sub-Banach Jordan algebra and $a, b \in J$. Then:*

(1) *For each $\lambda \notin \widehat{\sigma}(a)$ there exist a number $\delta(\lambda) > 0$ and an analytic function $g : N(\lambda, \delta(\lambda)) \to \mathcal{A}^{\mathrm{ic}}(a; J)$ such that, for every μ in $N(\lambda, \delta(\lambda))$,*
   (a) *$g(\mu)$ is invertible in J and $g^{-2}(\mu) = a - \mu$;*
   (b) *$g(\mu) = f_\mu(a)$ for some analytic $\mathbb{C}$-valued function f_μ defined on a suitable neighborhood of $\sigma(a)$.*

(2) *If $U_{g(\mu)}b$ is scattered for each $\mu \in N(\lambda, \delta(\lambda))$ and $\lambda \notin \widehat{\sigma}(a)$ then the set $\sigma(a + b) \setminus \widehat{\sigma}(a)$ is countable.*

(3) *If A is a normed associative Q-algebra, $a, b \in A$ and $(a - \lambda)^{-1}b$ is scattered for any $\lambda \notin \widehat{\sigma}(a)$ then $\sigma(a + b) \setminus \widehat{\sigma}(a)$ is countable.*

Proof. (1) Let $\lambda \notin \widehat{\sigma}(a)$. As 0 lies in an unbounded component of $\mathrm{Res}(a - \lambda)$, choose a simply connected open set Ω with $0 \notin \Omega$ and $\sigma(a - \lambda) \subset \Omega$. (For instance, take $\Omega = \mathbb{C} \setminus E$ where E is a simple continuous curve in $\mathrm{Res}(a - \lambda)$ joining 0 and ∞.) By [12, Theorem 7.2.2], there is an analytic function $f : \Omega \to \mathbb{C}$ such that $\exp(f(\eta)) = \eta$ for each $\eta \in \Omega$. As $x \mapsto \sigma(x)$ is upper semicontinuous, there is $\delta(\lambda) > 0$ such that $\sigma(b - \lambda) \subset \Omega$ for every $b \in J$ with $\|a - b\| < \delta(\lambda)$. By holomorphic functional calculus for such b, the elements $f(b - \lambda)$ are well defined; in particular, $f(a - \mu)$ exists for all $\mu \in N(\lambda, \delta(\lambda))$.

Set $g(\mu) = \exp(-f(a - \mu)/2)$ for $\mu \in N(\lambda, \delta(\lambda))$. It is easy to see that $g^{-2}(\mu) = a - \mu$. The function $g(\mu)$ is analytic on $N(\lambda, \delta(\lambda))$ as it is the composition of analytic functions $\exp(-f/2)$ and $\mu \mapsto a - \mu$.

If f_μ is the composition of the analytic functions $\exp(-f/2)$ and $\xi \mapsto \xi - \mu$ (defined on a suitable neighborhood of $\sigma(a)$) then f_μ is an analytic $\mathbb{C}$-valued function with $f_\mu(a) = g(\mu)$ for any $\mu \in N(\lambda, \delta(\lambda))$.

(2) Let $\lambda \notin \widehat{\sigma}(a)$, and let $\gamma : [0, 1] \to \mathbb{C} \setminus \widehat{\sigma}(a)$ be a continuous curve joining λ and some point η where $|\eta| > \rho(a+b)$. By (1), for every point $\gamma(t)$ there are $\delta(t) > 0$ and an analytic function $g_t : N(\gamma(t), \delta(t)) \to \mathcal{A}^{\mathrm{ic}}(a)$ satisfying $g_t^{-2}(\mu) = a - \mu$ for every $\mu \in N(\gamma(t), \delta(t))$. As $\gamma([0, 1])$ is a compact set covered by the open sets $N(\gamma(t), \delta(t))$, there are reals $t_0 = 0 < t_1 < \cdots < t_n = 1$

such that

$$\gamma([0,1]) \subset \bigcup_{j=0}^{n} N(\gamma(t_j), \delta(t_j)).$$

Let $D_j = N(\gamma(t_j), \delta(t_j))$ and $f_j(\mu) = U_{g_{t_j}(\mu)} b$ for any $\mu \in D_j$, for $j = 0, 1, \ldots, n$. As f_j is evidently analytic, the map $K_j : \mu \mapsto \sigma(f_j(\mu))$ is an analytic multifunction on D_j by [5, Theorem 1]. As $U_{g_{t_j}(\mu)}$ is invertible, we deduce (by using $a - \mu = g_{t_j}^{-2}(\mu) = U_{g_{t_j}^{-1}(\mu)} 1$ and $b = U^{-1}_{g_{t_j}(\mu)} U_{g_{t_j}(\mu)} b = U_{g_{t_j}^{-1}(\mu)} U_{g_{t_j}(\mu)} b$) that

$$a + b - \mu = U_{g_{t_j}^{-1}(\mu)} (1 + U_{g_{t_j}(\mu)} b) = U_{g_{t_j}^{-1}(\mu)} (1 + f_j(\mu))$$

for every $\mu \in D_j$, and $a + b - \mu$ is not invertible if and only if $1 + f_j(\mu)$ is not invertible. Hence

$$\sigma(a+b) \cap D_j = \{\mu \in D_j : -1 \in K_j(\mu)\}. \tag{3.1}$$

By assumption, $K_j(\mu)$ is countable for any $\mu \in D_j$. By the Aupetit–Zemánek Theorem, $G_j := \{\mu \in D_j : -1 \in K_j(\mu)\}$ is either countable or equal to D_j, for $j = 0, 1, \ldots, n$.

Assume that $G_0 = D_0$. Then $D_0 \subset \sigma(a+b)$ by (3.1). As $D_0 \cap D_1$ consists of an uncountable number of points of $\sigma(a+b)$, the set $G_1 = \sigma(a+b) \cap D_1$ cannot be countable. Therefore, by the Aupetit–Zemánek Theorem, $G_1 = D_1 \subset \sigma(a+b)$. Repeating this argument a finite number of times, we find that $G_n = D_n \subset \sigma(a+b)$. As $\eta \in D_n$ and $\rho(a+b) < |\eta|$, we obtain a contradiction.

Therefore, by the Aupetit–Zemánek Theorem, G_0 is at most countable, and so also is $\sigma(a+b) \cap D_0$ by (3.1). We have proved in fact that $\sigma(a+b) \cap N(\lambda, \delta(\lambda))$ is countable for any $\lambda \notin \widehat{\sigma}(a)$. As $\mathbb{C} \setminus \widehat{\sigma}(a)$ is a separable metric space, there is a countable covering by inscribed open sets each of which contains a countable number of points of $\sigma(a+b)$. So $\sigma(a+b) \setminus \widehat{\sigma}(a)$ is countable.

(3) Let $D = \mathbb{C} \setminus \widehat{\sigma}(a)$ and $K(\lambda) = \sigma((a-\lambda)^{-1} b)$ for $\lambda \in D$. Then K is a multifunction on D, and $K(\lambda)$ is countable for any $\lambda \in D$. Since $a + b - \lambda = (a - \lambda)(1 + (a - \lambda)^{-1} b)$ for $\lambda \in D$, we have $\sigma(a+b) \cap D = G$ where $G = \{\lambda \in D : -1 \in K(\lambda)\}$. By the Aupetit–Zemánek Theorem (see Remark 2.6), G is either countable or equal to D. The latter case is impossible because $\sigma(a+b)$ is bounded, but D is not. Therefore $\sigma(a+b) \setminus \widehat{\sigma}(a)$ is countable. ∎

Under the assumptions of Lemma 3.1, $\sigma(a+b) \setminus \widehat{\sigma}(a)$ *is countable if and only if* $\widehat{\sigma}(a+b) \setminus \widehat{\sigma}(a)$ *is countable, and in this case*

$$\sigma(a+b) \setminus \widehat{\sigma}(a) = \widehat{\sigma}(a+b) \setminus \widehat{\sigma}(a) \quad \textit{for all } a, b. \tag{3.2}$$

This follows from the fact that if a compact set in $\mathbb{C}$ is the union of a polynomially convex set K and a countable set Z then it is polynomially convex. Indeed, let $\lambda \in \widehat{K \cup Z}$. If $\lambda \in \widehat{K}$ then clearly $\lambda \in K$. Assume now that $\lambda \notin K$. As K is polynomially convex, there is a polynomial p such that

$$\max_{\mu \in K} |p(\mu)| < |p(\lambda)| \leq \max_{\mu \in Z} |p(\mu)| .$$

Let $Z' = \{\mu \in Z : |p(\lambda)| \leq |p(\mu)|\}$. It is easy to see that Z' is a compact set and $\lambda \in \widehat{Z'}$. But every countable compact set in $\mathbb{C}$ is polynomially convex. So $\lambda \in Z' \subset Z$.

3.2. Scattered ideals

3.2.1. First we introduce a useful technical notion. Let L be a unital sub-Banach Jordan or associative algebra, I an ideal of L, x an arbitrary element of L, and G a polynomially convex compact set containing $\widehat{\sigma}(x)$. An open disk $N(\lambda, \delta)$ with $\delta > 0$ is called *(x, G, I)-special* if $G \cap N(\lambda, \delta)$ is countable, the contour $\Gamma_{\lambda,\delta} := \partial N(\lambda, \delta)$ lies in $\mathbb{C} \setminus G$, and the spectral projection $p_{\lambda,\delta}(x) := (2\pi i)^{-1} \int_{\Gamma_{\lambda,\delta}} (\xi - x)^{-1} \, d\xi$ of x corresponding to the set $\sigma(x) \cap N(\lambda, \delta)$ belongs to I. We say that $\lambda \in \mathbb{C}$ *has an (x, G, I)-special disk* if there is an (x, G, I)-special disk $N(\lambda, \delta)$ for some $\delta > 0$.

Lemma 3.2. *Let L be a unital sub-Banach Jordan or associative algebra, let I be an ideal of L, let $x \in L$, and suppose $\widehat{\sigma}(x) \subset G = \widehat{G}$ for some compact set $G \subset \mathbb{C}$. Assume that $N(\lambda, \delta)$ is an (x, G, I)-special disk. Then:*

1. *In any interval $[t_1, t_2] \subset \mathbb{R}$ with $0 < t_1 < t_2 < \delta$ there is $t > 0$ such that $N(\lambda, t)$ is an (x, G, I)-special disk.*
2. *If $\mu \in N(\lambda, \delta)$ then μ has an (x, G, I)-special disk.*
3. *If M is a polynomially convex compact set in $\mathbb{C}$ and $M \setminus G$ is countable then λ has an $(x, G \cup M, I)$-special disk.*

Proof. Without loss of generality one can assume that L is complete normed.

(1) Assume, to the contrary, that there is an interval $[t_1, t_2]$ with $0 < t_1 < t_2 < \delta$ such that $N(\lambda, t)$ is not a (λ, G, I)-special disk. Then for every $t \in [t_1, t_2]$ either $\partial N(\lambda, t) \cap G \neq \emptyset$ or $\partial N(\lambda, t) \subset \mathbb{C} \setminus G$ and $p_{\lambda,t}(x)$ does not lie in I. In the latter case $p_{\lambda,t}(x) = p_{\lambda,t}(x) \cdot p_{\lambda,\delta}(x)$ belongs to I, a contradiction. So the former case must hold for all $t \in [t_1, t_2]$. But then $G \cap N(\lambda, \delta)$ is uncountable, also a contradiction.

(2) If $\mu \in N(\lambda, \delta)$ then there is $t' > 0$ such that $N(\mu, t) \subset N(\lambda, \delta)$ for any $t < t'$. Then $G \cap N(\mu, t)$ is countable. Arguing as in (1), we conclude that there is a positive $t < t'$ such that $\partial N(\mu, t) \subset \mathbb{C} \setminus G$ and $p_{\mu,t}(x) \in I$. Therefore $N(\mu, t)$ is an (x, G, I)-special disk.

(3) Let $F = G \cup M$. Then $\widehat{\sigma}(x) \subset F = \widehat{F}$ and $F \cap N(\lambda, \delta)$ is countable. If $\partial N(\lambda, \delta) \cap F \neq \emptyset$, there is a positive $t < \delta$ such that $\partial N(\lambda, t) \subset \mathbb{C} \setminus F$ and $p_{\lambda,t}(x) \in I$. Hence $N(\lambda, t)$ is an (x, F, I)-special disk. ∎

Let I be an ideal of L and $x \in L$. Let $\omega(x; I)$ be the set of all $\lambda \in \widehat{\sigma}(x)$ having no $(x, \widehat{\sigma}(x), I)$-special disks. Taking into account that $\widehat{\sigma}(x - \eta) = \widehat{\sigma}(x) - \eta$, it is easy to see that $\omega(x - \eta; I) = \omega(x; I) - \eta$ for every $\eta \in \mathbb{C}$ whenever $\omega(x; I)$ is not empty. The following theorem generalizes [1, Theorem 5.7.4].

THEOREM 3.3. *Let L be a unital sub-Banach Jordan or associative algebra, let I be an ideal of L and $x \in L$. Then:*

(1) *$\omega(x; I)$ is a polynomially convex compact set.*
(2) *$\widehat{\sigma}(x) \setminus \omega(x; I)$ is countable.*
(3) *If $\omega(x; I)$ is not empty then $\partial\omega(x; I) \subset \sigma(x)$.*
(4) *If I is scattered then*
   (a) *$\omega(x; I) = \omega(x + y; I)$ for every $y \in I$;*
   (b) *if $\omega(x; I)$ is empty then $\widehat{\sigma}(x/\widehat{I}) = \{0\}$ ($\widehat{I}$ is the closure of I in $\widehat{L}$);*
   (c) *if $\omega(x; I)$ is not empty then $\omega(x; I) = \widehat{\sigma}(x/\widehat{I})$.*

Proof. (1) The set $\mathbb{C} \setminus \omega(x; I)$ is covered by open disks each of which is inscribed in $\mathbb{C} \setminus \omega(x; I)$ and intersects $\widehat{\sigma}(x)$ in a countable set, whence $\mathbb{C} \setminus \omega(x; I)$ is open and $\omega(x; I)$ is closed. So $\omega(x; I)$ is a compact set. If $\lambda \in \mathbb{C} \setminus \omega(x; I)$ then there is an $(x, \widehat{\sigma}(x), I)$-special disk $N(\lambda, \delta)$. Let $E(\mu) = \{t\lambda + (1 - t)\mu : 0 < t < 1\}$ for $\mu \in \partial N(\lambda, \delta)$. As $\widehat{\sigma}(x) \cap N(\lambda, \delta)$ is countable, there is $\mu' \in \partial N(\lambda, \delta)$ such that $E(\mu') \subset \mathbb{C} \setminus \widehat{\sigma}(x)$. As $\mu' \in \partial N(\lambda, \delta) \subset \mathbb{C} \setminus \widehat{\sigma}(x)$, there is a simple continuous curve in $\mathbb{C} \setminus \widehat{\sigma}(x)$ joining μ' and ∞. Therefore λ lies in an unbounded component of $\mathbb{C} \setminus \omega(x; I)$. Hence $\omega(x; I)$ is polynomially convex by [15, Lemma 3.1.3].

(2) As $\mathbb{C} \setminus \omega(x; I)$ is a separable metric space which is covered by $(x, \widehat{\sigma}(x), I)$-special disks, it admits an inscribed countable covering, whence it intersects $\widehat{\sigma}(x)$ in a countable set. Thus $\widehat{\sigma}(x) \setminus \omega(x; I)$ is countable.

(3) Let us show that $\partial\omega(x; I) \subset \sigma(x)$. Assume, to the contrary, that $\lambda \in \partial\omega(x; I)$ but $\lambda \in \mathrm{Res}(x)$. Hence $V \subset \mathrm{Res}(x)$ for some neighborhood V of λ. As $\lambda \in \partial\omega(x; I)$, as we saw in (1), V contains a point μ that lies in $\mathbb{C} \setminus \widehat{\sigma}(x)$. Hence $\lambda \in \mathbb{C} \setminus \widehat{\sigma}(x)$, a contradiction. So $\partial\omega(x; I) \subset \sigma(x)$.

(4a) *The case when $L = J$, a unital sub-Banach Jordan algebra.* By Lemma 3.1(1), for any $\lambda \in \mathbb{C} \setminus \widehat{\sigma}(x)$ there exist an open disk $N(\lambda, \delta)$ and an analytic function $g : N(\lambda, \delta) \to \mathcal{A}^{\mathrm{ic}}(x)$ such that $g^{-2}(\mu) = x - \mu$ for every $\mu \in N(\lambda, \delta)$. As $y \in I$, the element $U_{g(\mu)}y$ is scattered for every $\mu \in N(\lambda, \delta)$. By Lemma 3.1(2), $\sigma(x + y) \setminus \widehat{\sigma}(x)$ is countable. By (3.2), $\widehat{\sigma}(x + y) \setminus \widehat{\sigma}(x)$ is countable.

Assume, to the contrary, that there is a point $\eta \in \omega(x+y;I) \setminus \omega(x;I)$. Then η has an $(x,\widehat{\sigma}(x),I)$-special disk $N(\eta,\delta)$. By Lemma 3.2(3), the point η also has an $(x,\widehat{\sigma}(x)\cup\widehat{\sigma}(x+y),I)$-special disk $N(\eta,t)$. Then $p_{\eta,t}(x+y)-p_{\eta,t}(x) \in \overline{I}$ by Lemma 2.4(2c). As $p_{\eta,t}(x) \in I$, we obtain $p_{\eta,t}(x+y) \in \overline{I}$. By Lemma 2.4(1), $p_{\eta,t}(x+y) \in I$, whence $\eta \in \mathbb{C} \setminus \omega(x+y;I)$, a contradiction.

Therefore $\omega(x+y;I) \subset \omega(x;I)$. Exchanging x and $x+y$, we also obtain the converse inclusion $\omega(x;I) \subset \omega(x+y;I)$.

(4a) *The case when $L = A$ is a unital sub-Banach associative algebra.* As $y \in I$, the element $(\lambda - x)^{-1}y$ is scattered for any $\lambda \in \mathbb{C} \setminus \widehat{\sigma}(x)$. By Lemma 3.1(3), $\sigma(x+y) \setminus \widehat{\sigma}(x)$ is countable. The rest of the proof is similar to the case when $L = J$.

(4b)&(4c) If $\lambda \in \sigma(x) \setminus \omega(x;I)$ then there is an $(x,\widehat{\sigma}(x),I)$-special disk $N(\lambda,\delta)$. So the spectral projection $p_{\lambda,\delta}(x)$ corresponding to $\sigma = \sigma(x) \cap N(\lambda,\delta)$ lies in I. Then $\lambda \in \sigma$ and

$$\begin{aligned}\sigma(x/\widehat{I}) &= \sigma\big((x\cdot(1-p_{\lambda,\delta}(x)))/\widehat{I}\big) \subset \sigma_{\widehat{L}}(x\cdot(1-p_{\lambda,\delta}(x)))\\ &= \sigma(x\cdot(1-p_{\lambda,\delta}(x))) \subset \{0\}\cup(\sigma(x)\setminus\sigma).\end{aligned}$$

So, if $\lambda \neq 0$ then $x/\widehat{I} - \lambda$ is invertible, whence $\sigma \setminus \{0\} \subset \mathrm{Res}(x/\widehat{I})$. Thus

$$\sigma(x/\widehat{I}) \subset \{0\} \cup \omega(x;I). \tag{3.3}$$

If $\omega(x;I)$ is empty then $\sigma(x/\widehat{I}) = \{0\}$.

Assume that $\omega(x;I)$ is not empty. By (3.3) applied to $a = x + \mu$ with $|\mu| > \rho(x)$, we see that $\sigma(a/\widehat{I}) \subset \omega(a;I)$, whence $\sigma(x/\widehat{I}) \subset \omega(x;I)$. As $\omega(x;I)$ is polynomially convex, $\widehat{\sigma}(x/\widehat{I}) \subset \omega(x;I)$.

By (3) and (4a), $\omega(x;I) = \omega(x+y;I)$ and $\partial\omega(x;I) \subset \sigma(x+y)$ for each $y \in I$. Hence $\partial\omega(x;I) \subset \sigma(x+y)$ for each $y \in \widehat{I}$ by upper semicontinuity of the map $z \mapsto \sigma(z)$ on $\widehat{L}$. By Lemma 2.5(5b),

$$\partial\omega(x;I) \subset \bigcap_{y\in\widehat{I}} \sigma(x+y) \subset \widehat{\sigma}(x/\widehat{I}) \subset \omega(x;I),$$

whence $\widehat{\sigma}(x/\widehat{I}) = \omega(x;I)$. ∎

3.2.2. *Thin ideals.* Let L be a sub-Banach Jordan or associative algebra. A closed ideal I of L is called *thin* [36] if $\sigma(a)\setminus \widehat{\sigma}(a/I)$ is countable for each $a \in L$; an arbitrary ideal is called *thin* if its closure is thin. Each thin ideal is scattered. The converse follows from Theorem 3.3(2)&(4).

Corollary 3.4. *Let L be a sub-Banach Jordan or associative algebra. If I is a scattered ideal of L then $\widehat{I}$ is a thin ideal of $\widehat{L}$. In particular, if L is scattered then $\widehat{L}$ is scattered.*

Corollary 3.4 cannot be extended to normed Jordan or associative Q-algebras. Indeed, by the example due to Dixon [13, Example 9.3], there is a radical normed associative algebra A such that $\widehat{A}$ is semisimple. It is obvious that A is a Q-algebra, and $\widehat{A}$ cannot be scattered.

Let L be a normed Jordan or associative Q-algebra, I an ideal of J and $q_I : J \to J/I$ the quotient map. In particular,

$$\overline{I} \subset \mathrm{kh}(I) \tag{3.4}$$

(for Jordan algebras, see (2.16); in normed associative Q-algebras all primitive ideals are closed (see, for instance, [33, Theorem 2.1]), which implies (3.4)).

Corollary 3.5. *Let L, L_1, L_2 be sub-Banach Jordan* (*or associative*) *algebras. Then:*

(1) *If I is a scattered ideal of L then $\mathrm{kh}(\widehat{I})$ is a scattered ideal of $\widehat{L}$ and $q_{\overline{I}}^{-1} S(L/\overline{I}) = S(L)$.*
(2) *If I_1 and I_2 are scattered ideals of L then $I_1 + I_2$ is a scattered ideal.*
(3) *There is the largest scattered ideal for L and it is closed.*
(4) *If $\mathcal{R}_s(L)$ denotes the largest scattered ideal of L then*

 (a) *$\theta(\mathcal{R}_s(L_1)) \subset \mathcal{R}_s(L_2)$ for any algebraic morphism θ from J_1 onto J_2;*
 (b) *$\mathcal{R}_s(L/\mathcal{R}_s(L)) = \{0\}$;*
 (c) *$\mathcal{R}_s(\mathcal{R}_s(L)) = \mathcal{R}_s(L)$;*
 (d) *if I is an ideal of L then $\mathcal{R}_s(I) = I \cap \mathcal{R}_s(L)$;*
 (e) *$\mathcal{R}_s(L) = L \cap \mathcal{R}_s(\widehat{L})$.*

Proof. (1) Let $x \in \mathrm{kh}(\widehat{I})$. By Theorem 3.3, $\sigma(x/\widehat{I}) = \{0\} = \widehat{\sigma}(x/\widehat{I})$, whence $\omega(x; I)$ is empty or $\{0\}$. Therefore $\widehat{\sigma}(x)$ is countable and x is scattered.

Now suppose $x/\overline{I}$ is scattered for some $x \in L$. By Theorem 3.3, $\omega(x; I)$ and $\widehat{\sigma}(x) \setminus \omega(x; I)$ are countable. So $\widehat{\sigma}(x)$ is countable and x is scattered. This proves $q_{\overline{I}}^{-1} S(L/\overline{I}) \subset S(L)$. The converse inclusion is obvious.

(2) By (1) and (3.4), one can assume that I_1 and I_2 are closed. Let $x = a+b$ with $a \in I_1$ and $b \in I_2$. It is easy to see that $\mathrm{Res}(a) \subset \mathrm{Res}(x/I_2)$, so x/I_2 is scattered. Then $\omega(x; I_2)$ is countable. By Theorem 3.3, $\widehat{\sigma}(x) \setminus \omega(x; I_2)$ is countable, whence x is scattered.

(3) If (I_α) is a chain of scattered ideals of L then the ideal $\bigcup I_\alpha$ is clearly scattered in L. So, by Zorn's Lemma, there are maximal scattered ideals of L. It follows from (2) that they all coincide, so there is only one maximal scattered ideal K, and this ideal is largest. By (1), K is closed.

(4a) Clearly, $\sigma(\theta(a)) \subset \sigma(a)$ for each $a \in L_1$, whence $\theta(\mathcal{R}_s(L_1))$ is a scattered ideal of L_2. Hence it is contained in $\mathcal{R}_s(L_2)$.

(4b) Let $I = \mathcal{R}_s(L)$ and $q_I : L \to L/I$ be the quotient map. Let $I' = q_I^{-1}(\mathcal{R}_s(L/I))$. By (1), $I' \subset S(L)$, i.e. I' is a scattered ideal of L, whence $I' \subset \mathcal{R}_s(L) = I$ and $\mathcal{R}_s(L/I) = \{0\}$.

(4c) is obvious.

(4d) *The case when I is an ideal of a sub-Banach Jordan algebra J.* It is clear that $I \cap \mathcal{R}_s(J) \subset \mathcal{R}_s(I)$. Since $\mathcal{R}_s(I/\mathcal{R}_s(I)) = \{0\}$ by (4b), $I/\mathcal{R}_s(I)$ is semisimple and has no non-zero nilpotent ideals. So $\mathcal{R}_s(I)$ is an ideal of J by Slin'ko's Theorem [42, Theorem 14.12], whence $\mathcal{R}_s(I) \subset \mathcal{R}_s(J)$.

The case when I is an ideal of a sub-Banach associative algebra A. Let M be an ideal of A generated by $\mathcal{R}_s(I)$. As $M^3 \subset \mathcal{R}_s(I) \subset M$, M^3 and hence $\overline{M^3}$ are scattered ideals of A. As $\overline{M}/\ \overline{M^3}$ is nilpotent, $\overline{M}$ is a scattered ideal of A by (1). Hence $\mathcal{R}_s(I) \subset \overline{M} \subset \mathcal{R}_s(A)$ and $\mathcal{R}_s(I) \subset I \cap \mathcal{R}_s(A)$. As $I \cap \mathcal{R}_s(A) \subset \mathcal{R}_s(I)$, we obtain equality.

(4e) As $L \cap \mathcal{R}_s(\widehat{L})$ is a scattered ideal of L, we get $L \cap \mathcal{R}_s(\widehat{L}) \subset \mathcal{R}_s(L)$. On the other hand, the completion $\widehat{\mathcal{R}_s(L)}$ can be identified with a scattered ideal of $\widehat{L}$ by Corollary 3.4. So $\mathcal{R}_s(L) \subset \widehat{\mathcal{R}_s(L)} \subset \mathcal{R}_s(\widehat{L})$ and $\mathcal{R}_s(L) \subset L \cap \mathcal{R}_s(\widehat{L})$. ∎

Corollary 3.5(4a)–(4d) yields

Corollary 3.6. *The map $\mathcal{R}_s : L \mapsto \mathcal{R}_s(L)$ is a hereditary topological radical on the class $\mathfrak{U}_{sbj}$ of all sub-Banach Jordan algebras and on the class $\mathfrak{U}_{sba}$ of all sub-Banach associative algebras.*

The map $\mathcal{R}_s$ is called the *scattered radical.* For every normed Jordan algebra J, the ideal $J \cap \mathcal{R}_s(\widehat{J})$ is scattered and closed in J. The map $\mathcal{R}_s^r : J \mapsto J \cap \mathcal{R}_s(\widehat{J})$ is a hereditary topological radical on the class $\mathfrak{U}_{nj}$ of all normed Jordan algebras (see Section 2.4). This radical is called the *regular scattered radical.* Proposition 2.28 of [33] shows that $\mathcal{R}_s^r(A)$ may be smaller than the largest scattered ideal for a normed associative Q-algebra A. Taking into account Corollary 3.12 below, we obtain such an example for the normed Jordan Q-algebra A^+.

3.2.3. *Structure theorem for the scattered radical*

Theorem 3.7. *Let L be a sub-Banach Jordan or associative algebra. Then:*

(1) *There is an increasing transfinite chain $(P_\alpha(L))_{\alpha \le \gamma}$ of closed ideals of L such that $P_0(L) = \mathrm{kh}(0)$, $P_{\alpha+1}(L) = \mathrm{kh}(q_{P_\alpha(L)}^{-1}(\mathrm{soc}(L/P_\alpha(L))))$ for any α, $P_\beta(L) = \mathrm{kh}(\bigcup_{\alpha<\beta} P_\alpha(L))$ for any limit ordinal β, and $P_\gamma(L) = \mathcal{R}_s(L)$.*

(2) *If L is scattered and semisimple then L has a non-zero socle.*

Proof. We will only give the proof for Jordan algebras; for associative algebras, the assertion is well known. Let $I_\alpha = P_\alpha(L)$ for all α.

(1) *The case when L is complete normed.* First we show that all ideals I_α are scattered. Let I be a scattered closed ideal of L such that L/I is semisimple, and let $I' = q_I^{-1}(\mathrm{soc}(L/I))$. By Theorem 3.3, $\omega(x;I) = \widehat{\sigma}(x/I)$ and $\widehat{\sigma}(x) \setminus \omega(x;I)$ is countable for every $x \in L$. As $x \in I'$ if and only if $x/I \in \mathrm{soc}(L/I)$, the set $\widehat{\sigma}(x/I)$ is finite for every $x \in I'$ by [3, Theorem 3.11]. Hence $\widehat{\sigma}(x)$ and $\sigma(x)$ are countable for every $x \in I'$. Therefore I' and $\overline{I'}$ are scattered ideals of L. Further, $\mathrm{kh}(I') = \mathrm{kh}(\overline{I'})$ is also a scattered ideal. As $\mathrm{kh}(I') = q_{\overline{I'}}^{-1}(\mathrm{rad}(L/\overline{I'}))$, $L/\mathrm{kh}(I')$ is semisimple. This proves that if I_α is scattered then so is $I_{\alpha+1}$. Taking into account that I_0 is scattered, we can assume that I_α are scattered for $\alpha < \beta$ where β is a limit ordinal. As such ideals lie in $\mathcal{R}_s(L)$, the ideals $\bigcup_{\alpha<\beta} I_\alpha$ and I_β are also scattered. By transfinite induction, all ideals I_α are scattered.

As L is a set, the transfinite sequence stabilizes at some ordinal γ: $I_\gamma = I_{\gamma+1}$. As I_γ is scattered, we have $I_\gamma \subset \mathcal{R}_s(L)$. Assume, to the contrary, that $I_\gamma \neq \mathcal{R}_s(L)$. Then $\mathcal{R}_s(L)/I_\gamma$ is non-zero, scattered and semisimple. By [4, Theorem 19], $\mathrm{soc}(\mathcal{R}_s(L)/I_\gamma)$ is non-zero. By Lemma 2.2, $\mathrm{soc}(\mathcal{R}_s(L)/I_\gamma) \subset \mathrm{soc}(L/I_\gamma)$. Hence we can add to $(I_\alpha)_{\alpha\le\gamma}$ an ideal $I_{\gamma+1} := \mathrm{kh}(q_{I_\gamma}^{-1}(\mathrm{soc}(L/I_\gamma)))$ $\neq I_\gamma$, a contradiction. So $I_\gamma = \mathcal{R}_s(L)$.

(2) By Lemma 2.5, $L^1 = N^1$ where N is a subideal of $\widehat{N}$. We can assume that $N \neq \widehat{N}$. Since N is scattered, it follows from Corollary 3.4 that $\widehat{N}$ is also scattered. Then for $\widehat{N}$ there is a chain $(I_\alpha)_{\alpha\le\gamma}$ of ideals described in (1). Let β be a smallest ordinal for which $N \cap I_\beta \neq \{0\}$. Clearly, $N \cap I_\beta$ is an ideal of N. As N is semisimple, N contains an element $a \in N \cap I_\beta$ with non-zero spectrum. As N is scattered, $\sigma(a)$ is countable and there is a non-zero $\lambda \in \sigma(a)$ such that $N(\lambda,\delta) \setminus \{\lambda\} \subset \mathrm{Res}(a) \setminus \{0\}$. As $N \cap I_\beta$ is a sub-Banach algebra, analytic functional calculus exists in $(N \cap I_\beta)^1$ by Lemma 2.5. Then the spectral projection $p := p_{\lambda,\delta}$ of a corresponding to $\sigma(a) \cap N(\lambda,\delta)$ belongs to $(N \cap I_\beta)^1$. As λ is a non-zero isolated point of $\sigma(a)$, p is a value of some analytic function $f(a)$ with $f(0) = 0$. Then $p \in N \cap I_\beta$ by Lemma 2.4(2a).

It is clear that $\beta > 0$. If β is a limit ordinal then, by Lemma 2.1, $p \in I_\alpha$ for some $\alpha < \beta$, but $N \cap I_\alpha = \{0\}$, a contradiction. Hence $\beta = \alpha + 1$ for some α and $p \in q_{I_\alpha}^{-1}(\mathrm{soc}(\widehat{N}/I_\alpha)) \subset I_{\alpha+1}$. By assumption, $\mathrm{soc}(\widehat{N}/I_\alpha) \subset I_{\alpha+1}/I_\alpha$, and by Lemma 2.2(2), $\mathrm{soc}(I_{\alpha+1}/I_\alpha) = \mathrm{soc}(\widehat{N}/I_\alpha)$. Hence p/I_α is in $\mathrm{soc}(I_{\alpha+1}/I_\alpha)$.

It is clear that $N \cap I_{\alpha+1}$ and $(N \cap I_{\alpha+1} + I_\alpha)$ are subideals of $I_{\alpha+1}$. Then $M := (N \cap I_{\alpha+1} + I_\alpha)/I_\alpha$ is a subideal of $I_{\alpha+1}/I_\alpha$, whence $\mathrm{soc}(M) = M \cap \mathrm{soc}(I_{\alpha+1}/I_\alpha)$ by Lemma 2.2(2). As $p/I_\alpha \in M$, we get $p/I_\alpha \in \mathrm{soc}(M)$. Let $\theta : x \mapsto x/I_\alpha$ be a homomorphism from $N \cap I_{\alpha+1}$ onto M. It follows that if $x/I_\alpha = y/I_\alpha$ for $x, y \in N \cap I_{\alpha+1}$ then $x - y \in N \cap I_\alpha = \{0\}$. So θ is injective and θ^{-1} is a homomorphism from M onto $N \cap I_{\alpha+1}$. By Lemma

2.2(1), we have $p \in \theta^{-1}(\mathrm{soc}(M)) \subset \mathrm{soc}(N \cap I_{\alpha+1})$. As $N \cap I_{\alpha+1}$ is an ideal of L, we conclude that $p \in \mathrm{soc}(L)$ by Lemma 2.2(1).

(1) *The general case* follows exactly as in the above case of complete normed algebras by using (2). ∎

The maps $P_\alpha : L \mapsto P_\alpha(L)$ in Theorem 3.7 may be applied to any Jordan or associative algebra. Recall that the algebraic radical axioms repeat Axioms 1–4 word for word, only in Axiom 1 it is necessary to take algebraic morphisms. Let P, T be maps on the class $\mathfrak{U}$ of algebras such that $P(L)$ and $T(L)$ are ideals of L for any L. Then P is called an (*algebraic*) *under radical* [13] if P satisfies all (algebraic) radical axioms apart from Axiom 2. Let us define the *convolution* $P * T$ of P and T by $(P*T)(L) = q_{T(L)}^{-1}(P(L/T(L)))$ for any L, where $q_{T(L)}$ is the quotient map $L \to L/T(L)$. For details on the convolution we refer to [36, Section 4].

Let $R(L) = \mathrm{kh}((\mathrm{soc} * \mathrm{rad})(L))$ for any Jordan or associative algebra L. Define the ideals $R_\alpha(L)$ by induction as follows: $R_1(L) = 0$ and $R_{\alpha+1}(L) = (R*R_\alpha)(L)$ for any ordinal α. If β is a limit ordinal, set $R_\beta(L) = \bigcup_{\alpha<\beta} R_\alpha(L)$.

LEMMA 3.8. *Let R and R_α be defined as above, and let P_α be defined as in Theorem* 3.7, *for any ordinal α. Then:*

(1) *If T and P are algebraic hereditary preradicals then $T * P$ is an algebraic hereditary preradical.*
(2) *$R : L \mapsto R(L)$ is an algebraic hereditary preradical on $\mathfrak{U}_j$ and $\mathfrak{U}_a$.*
(3) *All R_a and P_α are algebraic hereditary preradicals.*
(4) *$P_{\alpha+k} = R_{\alpha+k}$ for any α and $k > 0$.*

Proof. (1) By [36, Lemma 4.10], which also holds in the non-associative algebra context, the convolution of preradicals is a preradical. So $T * P$ is a preradical.

Let I be an ideal of L. As $P(I) = I \cap P(L)$, we infer that $I/P(I)$ is isomorphic to the ideal $I' := (I + P(L))/P(L)$ of $L/P(L)$, and that $T(I/P(I))$ is isomorphic to $T(I') = I' \cap T(L/P(L))$. Now if $x \in (T * P)(I)$ then $q_{P(I)}(x) \in T(I/P(I))$, whence $q_{P(L)}(x) \in T(L/P(L))$, and therefore $x \in I \cap (T * P)(L)$. Conversely, if $y \in I \cap (T * P)(L)$ then $q_{P(L)}(y) \in T(I')$, whence $q_{P(I)}(y) \in T(I/P(I))$. As $y \in I$, we have $y \in (T * P)(I)$. Therefore

$$(T * P)(I) = I \cap (T * P)(L). \tag{3.5}$$

(2) Let $T = \mathrm{soc}$ and $P = \mathrm{rad}$, and let $\mathrm{psoc}(L) = (T * P)(L)$, the *presocle* of L. Although soc has the properties of a hereditary preradical on semisimple algebras (see Lemma 2.2), an argument similar to one in (1) shows that (3.5) holds.

Let us check that psoc is a preradical. Let $f : L_1 \to L_2$ be an algebraic morphism. As $f(P(L_1)) \subset P(L_2)$, there is an algebraic morphism

$g : L_1/P(L_1) \to L_2/P(L_2)$ with $q_{P(L_2)} \circ f = g \circ q_{P(L_1)}$, and $g(T(L_1/P(L_1))) \subset T(L_2/P(L_2))$, whence

$$f((T * P)(L_1)) = f\big(q^{-1}_{P(L_1)}(T(L/P(L_1)))\big) \subset q^{-1}_{P(L_2)}(g(T(L_1/P(L_1))))$$
$$\subset q^{-1}_{P(L_2)}(T(L_2/P(L_2))) = (T * P)(L_2).$$

So psoc is a hereditary preradical. As $R = \mathrm{rad} * \mathrm{psoc}$, R is a hereditary preradical by (1).

(3) Assume that we have already proved that R_α and P_α are hereditary preradicals for all $\alpha < \beta$. If $\beta = \alpha + 1$, it follows from (1) that $R_{\alpha+1}$ and $P_{\alpha+1}$ are hereditary preradicals. The case of a limit ordinal β is evident.

(4) Clearly, $R_1 = R = \mathrm{rad} * (\mathrm{soc} * \mathrm{rad})$ and $P_1 = (\mathrm{rad} * \mathrm{soc}) * \mathrm{rad}$. By [36, Lemma 4.10], convolution is associative for preradicals. In our case it is easy to check that $P_1 = R_1$.

Further, $R_2 = \mathrm{rad} * \mathrm{soc} * \mathrm{rad} * R_1$ and $P_2 = \mathrm{rad} * \mathrm{soc} * P_1$. But as rad is a radical, it is evident that $\mathrm{rad} = \mathrm{rad} * \mathrm{rad}$, whence $P_2 = R_2$. Hence, if $P_\alpha = R_\alpha$ for some ordinal α then clearly $P_{\alpha+1} = R_{\alpha+1}$. If $P_\alpha = R_\alpha$ for all $\alpha < \beta$, where β is a limit ordinal, it is easy to see that $P_\beta = \mathrm{rad} * R_\beta$ and $P_{\beta+1} = R_{\beta+1}$. ∎

Corollary 3.9. *The scattered radical $\mathcal{R}_s$ is the restriction of some algebraic radical (denoted by $\mathrm{rad}^{\mathrm{soc}}$) from the classes $\mathfrak{U}_j$ and $\mathfrak{U}_a$ to the classes $\mathfrak{U}_{sbj}$ and $\mathfrak{U}_{sba}$, respectively.*

Proof. Let P_α and R_α be defined for any ordinal α as in Lemma 3.8. As P_α and R_α are hereditary preradicals by Lemma 3.8, it is easy to see that they are under radicals. Note that [13, Theorem 6.6] is also valid for non-associative algebras. By that result, for any Jordan or associative algebra L, the sequence $(R_\alpha(L))$ stabilizes at some γ and R_γ is an algebraic radical. We denote this radical by $\mathrm{rad}^{\mathrm{soc}}$. By using Lemma 3.8(4), we conclude that the sequence $(P_\alpha(L))$ stabilizes at $\gamma + 1$ and $P_{\gamma+1}(L) = R_{\gamma+1}(L) = R_\gamma(L) = \mathrm{rad}^{\mathrm{soc}}(L)$. It follows from Theorem 3.7 that $\mathcal{R}_s$ is the restriction of rad to the classes $\mathfrak{U}_{sbj}$ and $\mathfrak{U}_{sba}$. ∎

3.3. Characterization of the largest scattered ideal. First we consider the case of sub-Banach Jordan algebras.

Theorem 3.10. *Let J be a sub-Banach Jordan algebra and $x \in J$. The following are equivalent:*

(1) $x \in \mathcal{R}_s(J)$.
(2) *$U_a x$ is scattered for every $a \in J^1$.*
(3) *$\widehat{\sigma}(x + a) \setminus \widehat{\sigma}(a)$ is countable for every $a \in J$.*
(4) *$\{\lambda \in \mathbb{C} : 0 \in \sigma(a + \lambda x)\}$ is countable for every $a \in J^1$ with $0 \notin \widehat{\sigma}(a)$.*
(5) *$\{\lambda \in \mathbb{C} : 0 \in \sigma(a + \lambda x)\}$ is countable for every $a \in \Omega_1(J^1)$.*

Proof. (1)⇒(2) is obvious, and (2)⇒(3) follows from Lemma 3.1.

(3)⇒(4): Condition (3) can be written as follows: $\widehat{\sigma}(a+\lambda x)\setminus\widehat{\sigma}(a)$ is countable for all $a\in J^1$ and $\lambda\in\mathbb{C}$. Let $a\in J^1$ and $0\notin\widehat{\sigma}(a)$. Let G be the set of all $\eta\in\mathbb{C}$ such that $0\in\sigma(a+\eta x)$. Let $\lambda\in G$. As $\widehat{\sigma}(a+\lambda x)\setminus\widehat{\sigma}(a)$ is countable, there are disjoint open sets $V_1,V_2\subset\mathbb{C}$ such that $\sigma(a+\lambda x)\subset V_1\cup V_2$ and $0\in V_1\subset\mathbb{C}\setminus\widehat{\sigma}(a)$. As the map $y\mapsto\sigma(y)$ is upper semicontinuous, there is $\delta(\lambda)>0$ such that $\sigma(a+\mu x)\subset V_1\cup V_2$ for any $\mu\in N(\lambda,\delta(\lambda))$. Let $K(\mu):=\sigma(a+\mu x)\cap V_1$ for $\mu\in N(\lambda,\delta(\lambda))$. By the Localization Principle [1, Theorem 7.1.5], $K(\mu)$ is either empty or not empty, simultaneously for all $\mu\in N(\lambda,\delta(\lambda))$, and in the latter case $K(\mu)$ is an analytic multifunction on $N(\lambda,\delta(\lambda))$. Since $0\in K(\lambda)$, we conclude that K is an analytic multifunction on $N(\lambda,\delta(\lambda))$. As $K(\mu)\subset\mathbb{C}\setminus\widehat{\sigma}(a)$, $K(\mu)$ is countable for any $\mu\in N(\lambda,\delta(\lambda))$. By the Aupetit–Zemánek Theorem [1, Theorem 7.2.13], $G\cap N(\lambda,\delta(\lambda))$ is either countable or equal to $N(\lambda,\delta(\lambda))$.

Now let $Q=\{\lambda\in G: N(\lambda,\delta(\lambda))\subset G\}$. It is evident that Q is open in $\mathbb{C}$. On the other hand, if $\lambda_n\to\lambda$ as $n\to\infty$ for some $\{\lambda_n\}\subset Q$ then $\lambda\in G$ (because the limit $U_{a+\lambda x}$ of non-invertible operators $U_{a+\lambda_n x}$ is not invertible). As $N(\lambda,\delta(\lambda))\cap N(\lambda_n,\delta(\lambda_n))$ contains an uncountable number of points of G for all sufficiently large n, we have $N(\lambda,\delta(\lambda))\subset G$ and therefore $\lambda\in Q$. So Q is also closed in $\mathbb{C}$. Hence Q is equal to $\emptyset$ or $\mathbb{C}$. If $Q=\mathbb{C}$ then $0\in\widehat{\sigma}(a)$, a contradiction.

So $Q=\emptyset$, whence $G\cap N(\lambda,\delta(\lambda))$ is countable for every $\lambda\in G$. As G is a separable metric space, it admits a countable covering by disks $N(\lambda,\delta(\lambda))$, whence it is countable.

(2)⇒(5): Let $a\in\Omega_1(J^1)$ be such that $\|1-a\|<1$. Then $0\notin\widehat{\sigma}(a)$ and a has a logarithm in $\widehat{J^1}$ by [30, Theorem 10.30], whence a has a logarithm in J^1 since analytic functional calculus exists in J^1 by Lemma 2.5. Then $a=\exp(b)$ for some $b\in J^1$. Let $c=\exp(-b/2)$. Then c is invertible and lies in J^1, and also $a=c^{-2}$. As $a+\lambda x=U_{c^{-1}}(1+\lambda U_c x)$, for $\lambda\neq 0$ this element is not invertible if and only if $-1/\lambda\in\sigma(U_c x)$. As $U_c x$ is scattered, $\{\lambda\in\mathbb{C}:0\in\sigma(a+\lambda x)\}$ is countable.

Let now $a\in\Omega_1(J^1)$ be arbitrary. Then $a=U_{\exp(b_n)}\cdots U_{\exp(b_1)}1$ for some $b_1,\dots,b_n\in J^1$. Let $f(\mu)=U_{\exp(\mu b_n)}\cdots U_{\exp(\mu b_1)}1$ for $\mu\in\mathbb{C}$. Then f is an analytic function, $f(0)=1$, $f(1)=a$, and there is $\delta>0$ such that $\|1-f(\mu)\|<1$ for each $\mu\in N(1,\delta)$. By the above, $\{\lambda\in\mathbb{C}: 0\in\sigma(f(\mu)+\lambda x)\}$ is countable for any $\mu\in N(1,\delta)$. Let $F(\mu)$ be the operator $U_{\exp(-\mu b_1)}\cdots U_{\exp(-\mu b_n)}$ on J for $\mu\in\mathbb{C}$. As

$$F(\mu)=U^{-1}_{\exp(\mu b_1)}\cdots U^{-1}_{\exp(\mu b_n)}=(U_{\exp(\mu b_n)}\cdots U_{\exp(\mu b_1)})^{-1},$$

we see that $F(\mu)f(\mu)=1$ and

$$F(\mu)(f(\mu)+\lambda x)=1+\lambda F(\mu)x. \tag{3.6}$$

It follows from (2.5) and (3.6) that, for $\lambda \neq 0$, the element $f(\mu) + \lambda x$ is not invertible if and only if $-1/\lambda \in \sigma(F(\mu)x)$. Therefore $\sigma(F(\mu)x)$ is countable for any $\mu \in N(1,\delta)$. As $\mu \mapsto \sigma(F(\mu)x)$ is an analytic multifunction and the capacity of $N(1,\delta)$ is not zero, $\sigma(F(\mu)x)$ is countable for any $\mu \in \mathbb{C}$ by the Scarcity Theorem [1, Theorem 7.2.8]. In particular, $\sigma(F(1)x)$ is countable, whence so is $\{\lambda \in \mathbb{C} : 0 \in \sigma(a + \lambda x)\}$.

(5)⇒(4): If $0 \notin \widehat{\sigma}(a)$ then $a = \exp(b)$ for some $b \in J^1$ as above. Hence $a \in \Omega_1(J^1)$, and by (5), the set $\{\lambda \in \mathbb{C} : 0 \in \sigma(a + \lambda x)\}$ is countable.

(4)⇒(2): Let $a \in J^1$. Define $g, f : \mathbb{C} \setminus \widehat{\sigma}(a) \to J$ by $g(\mu) = a - \mu$ and $f(\mu) = g^{-2}(\mu)$ for any $\mu \in \mathbb{C} \setminus \widehat{\sigma}(a)$. It is easy to check that $0 \notin \widehat{\sigma}(f(\mu))$ for any $\mu \in \mathbb{C} \setminus \widehat{\sigma}(a)$. Since

$$U_{g(\mu)}(f(\mu) + \lambda x) = U_{g(\mu)} U_{g(\mu)^{-1}} 1 + \lambda U_{g(\mu)} x = 1 + \lambda U_{g(\mu)} x,$$

for $\lambda \neq 0$, $f(\mu) + \lambda x$ is not invertible if and only if $-1/\lambda \in \sigma(U_{g(\mu)} x)$. As $\{\lambda \in \mathbb{C} : 0 \in \sigma(f(\mu) + \lambda x)\}$ is countable, so is $K(\mu) := \sigma(U_{g(\mu)} x)$ for any $\mu \in \mathbb{C} \setminus \widehat{\sigma}(a)$. As the multifunction $\mu \mapsto K(\mu)$ is analytic on $\mathbb{C}$ and $K(\mu)$ is countable on the sets of positive capacity in $\mathbb{C} \setminus \widehat{\sigma}(a)$, we conclude that $K(\mu)$ is countable for any $\mu \in \mathbb{C}$ by the Scarcity Theorem. In particular, $U_a x$ is scattered.

(2)⇒(1): Let $M = \{z \in J : U_a z \text{ is scattered for any } a \in J^1\}$ and let $x, y \in M$. We have just proved that (2) is equivalent to (3) and (5). By (3) applied to x and y, the sets $\widehat{\sigma}(x + y + a) \setminus \widehat{\sigma}(a + y)$ and $\widehat{\sigma}(a + y) \setminus \widehat{\sigma}(a)$ are countable for any $a \in J$, whence so is $\widehat{\sigma}(x + y + a) \setminus \widehat{\sigma}(a)$. Hence $x + y \in M$. So M is a linear subspace of J.

Let $b, c \in \Omega_1(J)$. By (2.19), $U_{b^{-1}} c \in \Omega_1(J)$. As $c + \lambda U_b x = U_b(U_{b^{-1}} c + \lambda x)$ for any $\lambda \in \mathbb{C}$, we infer from (5) that $\{\lambda \in \mathbb{C} : 0 \in \sigma(c + \lambda U_b x)\}$ is countable. By (2), $U_d U_b x$ is scattered for any $b \in \Omega_1(J^1)$ and $d \in J^1$.

Let $a \in J^1$. Then $a - \lambda \in \Omega_1(J^1)$ for every $\lambda \in \mathbb{C} \setminus \widehat{\sigma}(a)$. So $U_d U_{a-\lambda} x$ is scattered for any λ taken from a set of positive capacity. By the Scarcity Theorem, $U_d U_{a-\lambda} x$ is scattered for every $\lambda \in \mathbb{C}$ and $d \in J^1$, whence $U_{a-\lambda} x \in M$ for every $\lambda \in \mathbb{C}$. As $M \subset J$, by (2.7) we have $L_a x = (U_a x - U_{a-1} x + x)/2 \in M$ for every $a \in J^1$. Therefore M is a scattered ideal of J^1. Thus $M \subset \mathcal{R}_s(J^1) \cap J = \mathcal{R}_s(J)$, whence $x \in \mathcal{R}_s(J)$. ∎

THEOREM 3.11. *Let A be a sub-Banach associative algebra and $x \in A$. The following are equivalent*:

(1) *$x \in \mathcal{R}_s(A)$.*
(2) *$(a - \lambda)^{-1} x$ is scattered for every $a \in A$ and $\lambda \notin \widehat{\sigma}(a)$.*
(3) *$\widehat{\sigma}(x + a) \setminus \widehat{\sigma}(a)$ is countable for every $a \in A$.*
(4) *$\{\lambda \in \mathbb{C} : 0 \in \sigma(a + \lambda x)\}$ is countable for every $a \in A^1$ with $0 \notin \widehat{\sigma}(a)$.*

Proof. It is clear that (2) can be written as follows:

(2′) $a^{-1} x$ is scattered for every $a \in A^1$ with $0 \notin \widehat{\sigma}(a)$.

Further, (1)⇒(2) is obvious, (2)⇒(3) follows from Lemma 3.1(3), and (3)⇒(4) can be proved word for word as the corresponding conditions in Theorem 3.10.

(4)⇒(2): Let $a \in A$ and $D = \mathbb{C} \setminus \widehat{\sigma}(a)$. As

$$a - \mu + \lambda x = (a - \mu)(1 + \lambda(a - \mu)^{-1}x)$$

for every $\mu \in D$, it follows that, for $\lambda \neq 0$, $\mu \in \sigma(a + \lambda x)$ if and if $\lambda^{-1} \in \sigma((a - \mu)^{-1}x)$. Since $\{\lambda \in \mathbb{C} : 0 \in \sigma(a - \mu + \lambda x)\}$ is countable for every $\mu \in D$, so is $\sigma((a - \mu)^{-1}x)$ for every $\mu \in D$.

So (2′), (2), (3) and (4) are mutually equivalent.

(3)⇒(1): Let M be the set of all $z \in A$ for which $\widehat{\sigma}(z + a) \setminus \widehat{\sigma}(a)$ is countable for any $a \in A^1$. Let $x, y \in M$. Then $\widehat{\sigma}(x + y + a) \setminus \widehat{\sigma}(y + a)$ and $\widehat{\sigma}(y+a)\setminus\widehat{\sigma}(a)$ are countable for every $a \in A^1$, whence so is $\widehat{\sigma}(x+y+a)\setminus\widehat{\sigma}(a)$. This means that $x + y \in M$. So M is a subspace of A.

Let $a \in A^1$ and $D = \mathbb{C} \setminus \widehat{\sigma}(a)$. Then $0 \notin \widehat{\sigma}((a - \lambda)^{-1})$ and $a - \lambda$ is the inverse element to $(a - \lambda)^{-1}$ for any $\lambda \in D$, whence $(a - \lambda)x$ is scattered for any $\lambda \in D$ by (2′). As D contains sets of positive capacity, $(a - \lambda)x$ is scattered for any $\lambda \in \mathbb{C}$ by the Scarcity Theorem. Hence ax is scattered for any $a \in A^1$. As $\sigma(uv)\setminus\{0\} = \sigma(vu)\setminus\{0\}$ for all $u, v \in A$, in particular, $b^{-1}ax$ and $b^{-1}xa$ are scattered for every $b \in A^1$ with $0 \notin \widehat{\sigma}(b)$. As (2′) and (3) are equivalent, $\widehat{\sigma}(ax+c)\setminus\widehat{\sigma}(c)$ and $\widehat{\sigma}(xa+c)\setminus\widehat{\sigma}(c)$ are countable for any $c \in A^1$. Therefore $ax, xa \in M$ for any $a \in A^1$, whence M is an ideal of A. As M is scattered, $M \subset \mathcal{R}_s(A)$ and $x \in \mathcal{R}_s(A)$. ∎

In general, the condition $x \in \mathcal{R}_s(A)$ does not imply that $\sigma(x + a) \setminus \sigma(a)$ is countable for every $a \in A$. Indeed, let A be the algebra of all bounded operators on the Hilbert space $H = l^2(\mathbb{Z})$, U the shift $e_k \mapsto e_{k+1}$ for any e_k from the standard orthonormal basis of $l^2(\mathbb{Z})$, and $K = -e_0^* \otimes e_1$ where e_0^* is the functional $e_0^*(x) = (x, e_0)$ for any $x \in H$. As $\mathcal{R}_s(A)$ is the ideal of compact operators on H, we have $K \in \mathcal{R}_s(A)$. It is easy to see that the spectrum of U is the unit circle, but the spectrum of $K+U$ is the closed unit disk (see also the observation after [7, Theorem 12]). So $\sigma(K + U) \setminus \sigma(U)$ is not countable.

Corollary 3.12. *Let A be a sub-Banach associative algebra and $x \in A$. Then:*

1. *If I is a one-sided scattered ideal of A then $I \subset \mathcal{R}_s(A)$.*
2. *$x \in \mathcal{R}_s(A)$ if and only if $x \in \mathcal{R}_s(A^+)$, and if and only if axa is scattered for every $a \in A^1$.*

Proof. (1) follows from the implication (2)⇒(1) of Theorem 3.11.

(2) follows from comparing Theorems 3.10 and 3.11. Indeed, the spectra of the same element in A and A^+ coincide, and conditions (3) and (4) are the same for both of these theorems. ∎

4. Perturbation class of scattered elements

4.1. If X is a linear space and $Y \subset X$, the *perturbation class* $\mathrm{Per}(Y)$ of Y is the set of all $x \in X$ such that $x + y \in Y$ for all $y \in Y$. Clearly, $\mathrm{Per}(Y) + \mathrm{Per}(Y) \subset \mathrm{Per}(Y)$; if $\mathbb{C}Y \subset Y$ then $\mathrm{Per}(Y)$ is a subspace of X.

Let L be either a normed Jordan Q-algebra J or a normed associative Q-algebra A, and let $Z(L) = \mathrm{Per}(S(L))$ be the perturbation class of the set $S(L)$ of all scattered elements of L. It is a subspace of L. As $\sigma_A(x) = \sigma_{A^+}(x)$ for each $x \in A$, it follows that $x \in Z(A)$ if and only if $x \in Z(A^+)$. By [11], for a unital Banach associative algebra A, $Z(A)$ is a full subalgebra of A and a Lie ideal of A. Moreover, $[Z(A), A] \subset \mathcal{R}_s(A)$, and for $x \in A$, the element ax is scattered for each scattered $a \in A$ if and only if $x \in Z(A)$.

It is well known that the set of all scattered elements is not closed in general. For instance, in the famous Kakutani example of discontinuity of spectrum (see the solution of [17, Problem 104]) some sequence of nilpotent operators tends to a non-scattered operator. Now we are going to show that, for a Banach associative algebra A, $Z(A)$ may also be non-closed even if it is commutative.

Theorem 4.1. *Let X be a Cantor set, and let A be the algebra $C(X)$ of all continuous complex-valued functions on X. Then $\mathcal{R}_s(A) = \{0\}$, and $Z(A)$ coincides with $S(A)$ and is dense but not closed in A.*

Proof. In the proof we use the fact that $\sigma(a) = a(X)$ for every $a \in C(X)$.

Assume that $\mathcal{R}_s(A)$ is not zero. Then, by Theorem 3.7, A has a non-zero socle, so there is a minimal projection p of A. As Ap is one-dimensional, there is only one $\xi \in X$ such that $p(\xi) = 1$. As $\sigma(p) = \{0, 1\}$, we have $p(\varsigma) = 0$ for every $\varsigma \in X \setminus \{\xi\}$. It is easy to see that ξ is an isolated point in X. But X is a perfect compact set, a contradiction. Thus $\mathcal{R}_s(A) = \{0\}$.

Let E be the set of all functions in A having finite image. Then it is a symmetric subalgebra of A. For all distinct $\xi, \varsigma \in X$ there is a decomposition of X into two disjoint closed sets G and F such that $\xi \in G$ and $\varsigma \in F$. So E separates the points of X. By Stone's Theorem, E is dense in A.

Let $a, b \in A$. It is clear that if $a(X)$ and $b(X)$ are countable then so is $(a + b)(X)$. Hence $E \subset Z(A) = S(A)$. As the function $x : \xi \mapsto \xi$ has uncountable spectrum X, $Z(A)$ is not closed in A. ∎

Note that $Z(A)$ from Theorem 4.1 is an example of a commutative scattered Q-algebra having an $\mathcal{R}_s$-semisimple completion.

Theorem 4.2. *Let L be either a normed Jordan Q-algebra J or an associative Q-algebra A. Then:*

(1) *If L is not unital then $Z(L) = Z(L^1) \cap L$ and $Z(L^1) = Z(L) + \mathbb{C}$, and if L is sub-Banach and I is a closed scattered ideal of L, then $Z(L) = q_I^{-1}(Z(L/I))$.*

(2) *If $L = J$ and $x \in J$ then*

(a) *if $x \in Z(J)$ then $U_a x$ is scattered for all scattered $a \in J^1$;*

(b) *if J is sub-Banach and $U_a x$ is scattered for all scattered $a \in J^1$ then $x \in Z(J)$.*

(3) *If $L = A$ and $x \in A$, then $x \in Z(A)$ if and only if ax is scattered for any scattered $a \in A$.*

Proof. (1) The first assertion follows from the fact that $1 \in Z(J^1)$.

Let us prove the second one. By Corollary 3.5(1), since L is a sub-Banach algebra, $S(L) = q_I^{-1}(S(L/I))$. If $x \in Z(L)$ then $x + a$ is scattered for any $a \in S(L)$, whence $q_I(x) + q_I(a)$ is scattered. Since $S(L/I) = q_I(S(L))$, we have $q_I(x) \in Z(L/I)$.

Conversely, let $x \in q_I^{-1}(Z(L/I))$. As $q_I(x + a)$ is scattered for any $a \in S(L) = q_I^{-1}(S(L/I))$, $x + a$ is also scattered, whence $x \in Z(L)$.

(2a) Let $x \in Z(J)$ and $a \in S(J)$. Assume that a is invertible. As $U_a x - \lambda = U_a(x - \lambda a^{-2})$, we have

$$\sigma(U_a x) = \{\lambda \in \mathbb{C} : 0 \in \sigma(x - \lambda a^{-2})\}. \tag{4.1}$$

Let $f(\lambda) = x - \lambda a^{-2}$ and $K(\lambda) = \sigma(f(\lambda))$ for $\lambda \in \mathbb{C}$. Then $f(\lambda)$ is an entire function, and K is a multifunction. As $a^{-2} \in S(J)$, we have $f(\lambda) \in S(J)$ and $K(\lambda)$ is countable, for any $\lambda \in \mathbb{C}$. By the Aupetit–Zemánek Theorem and Remark 2.6, $\{\lambda \in \mathbb{C} : 0 \in K(\lambda)\}$ is either countable or equal to $\mathbb{C}$. By (4.1), the latter case is impossible since $\sigma(U_a x)$ is bounded, so $\sigma(U_a x)$ is countable.

Let now $a \in S(J)$ be arbitrary. Then the multifunction $L(\lambda) := \sigma(U_{a-\lambda} x)$ (defined on $\mathbb{C}$) is countable for any $\lambda \in \mathrm{Res}(a)$, and $\mathrm{Res}(a)$ contains sets of positive capacity. By the Scarcity Theorem and Remark 2.6, $L(\lambda)$ is countable for any $\lambda \in \mathbb{C}$, whence $U_a x$ is scattered.

(2b) Let $x \in J$ be such that $U_a x \in S(J)$ for every $a \in S(J)$. Let $a \in J$ be scattered. Then $\sigma(a) = \widehat{\sigma}(a)$, whence, by Lemma 3.1(1), for every $\lambda \in \mathrm{Res}(a)$ there are an open disk $N(\lambda, \delta)$ and an analytic function $g : N(\lambda, \delta) \to A^{\mathrm{ic}}(a)$ such that $g(\mu) = f_\mu(a)$ for some analytic $\mathbb{C}$-valued function f_μ defined on a suitable neighborhood of $\sigma(a)$, and $g^{-2}(\mu) = a - \mu$ for any $\mu \in N(\lambda, \delta)$. As $\sigma(f_\mu(a)) = f_\mu(\sigma(a))$ by the Spectral Mapping Theorem, $g(\mu)$ is scattered for any $\mu \in N(\lambda, \delta)$, whence so is $U_{g(\mu)} x$ by assumption. By Lemma 3.1, $\sigma(x + a) \cap \mathrm{Res}(a)$ is countable. As a is scattered, $\sigma(x + a) \cap \sigma(a)$ is also countable. Hence $x + a$ is scattered.

(3) Let $x \in Z(A)$ and $a \in S(A)$. Then we have $x - \mu(a - \lambda)^{-1} = (a - \lambda)^{-1}((a - \lambda)x - \mu)$ for any $\lambda \in \mathrm{Res}(a)$ and $\mu \in \mathbb{C}$, whence $\sigma((a - \lambda)x) = \{\mu \in \mathbb{C} : 0 \in K(\mu)\}$ where $K(\mu) = \sigma(x - \mu(a - \lambda)^{-1})$. As $K(\mu)$ is countable for each $\mu \in \mathbb{C}$, $\{\mu \in \mathbb{C} : 0 \in K(\mu)\}$ is either countable or equal to $\mathbb{C}$ by the Aupetit–Zemánek Theorem and Remark 2.6. In the latter case, since

$\eta x - (a-\lambda)^{-1} \to -(a-\lambda)^{-1}$ as $\eta \to 0$, we infer that $(a-\lambda)^{-1}$ is not invertible, a contradiction. Hence $(a-\lambda)x$ is scattered for each $\lambda \in \mathrm{Res}(a)$. By the Scarcity Theorem and Remark 2.6, $(a-\lambda)x$ is scattered for any $\lambda \in \mathbb{C}$, whence ax is scattered.

Now let $x \in A$ be such that ax is scattered for any $a \in S(A)$. Then $(a-\lambda)^{-1}$ is also a scattered element of A for every $\lambda \notin \widehat{\sigma}(a)$. By Lemma 3.1(3), $\sigma(x+a) \setminus \widehat{\sigma}(a)$ is countable. As $\widehat{\sigma}(a)$ is countable, $x+a$ is scattered for any $a \in S(A)$. Therefore $x \in Z(A)$. ∎

Theorem 4.3. *Let J be a unital sub-Banach Jordan algebra. The following statements are equivalent:*

(1) $x \in Z(J)$.
(2) $\{\lambda : 0 \in \sigma(y+\lambda x)\}$ *is countable for every* $y \in S(J) \cap \mathrm{Inv}(J)$.

Proof. (1)⇒(2): Since $x \in Z(J)$, it follows that $y+\lambda x$ is scattered for every $y \in S(J) \cap \mathrm{Inv}(J)$ and $\lambda \in \mathbb{C}$. Let $f(\lambda) = \sigma(y+\lambda x)$ for $\lambda \in \mathbb{C}$. Then $\lambda \mapsto f(\lambda)$ is an analytic multifunction. As $f(\lambda)$ countable for any $\lambda \in \mathbb{C}$, the set $E := \{\lambda : 0 \in \sigma(y+\lambda x)\}$ is either countable or equal to $\mathbb{C}$ by the Aupetit–Zemánek Theorem. As y is invertible, E is not equal to $\mathbb{C}$, whence it is countable.

(2)⇒(1): For $a \in S(J)$ and $\mu \in \mathrm{Res}(a)$, let $y = (a-\mu)^{-2}$. Then $y \in S(J) \cap \mathrm{Inv}(J)$ by analytic functional calculus (see Lemma 2.5). As $U_{a-\mu}(y+\lambda x) = 1+\lambda U_{a-\mu}x$, $y+\lambda x$ is not invertible if and only if $1+\lambda U_{a-\mu}x$ is not invertible, i.e. $-1/\lambda \in \sigma(U_{a-\mu}x)$. By (2), $\{\lambda : 0 \in \sigma(y+\lambda x)\}$ is countable, whence so is $\sigma(U_{a-\mu}x)$. Let $h(\mu) = \sigma(U_{a-\mu}(x))$ for $\mu \in \mathbb{C}$. Then $h(\cdot)$ is an analytic multifunction on $\mathbb{C}$. As $h(\mu)$ is countable for every $\mu \in \mathrm{Res}(a)$, $h(\mu)$ is countable for every $\mu \in \mathbb{C}$ by the Scarcity Theorem. For $\mu = 0$ we find that $U_a x$ is scattered. So $x \in Z(J)$ by Theorem 4.2(2b). ∎

Theorem 4.4. *Let L be a unital sub-Banach Jordan algebra or a unital normed associative Q-algebra. Then $Z(L)$ is a full subalgebra of L.*

Proof. The case when $L = J$, a sub-Banach Jordan algebra. Let $u \in Z(J)$ be invertible. If $a \in J$ is scattered and invertible then a^{-1} is scattered by Lemma 2.3, and $U_{a^{-1}}u$ is scattered by Theorem 4.2(2a). As $U_a u^{-1} = (U_{a^{-1}}u)^{-1}$ by (2.6), $U_a u^{-1}$ is scattered.

If a is not invertible then it follows from the above that $U_{a-\lambda}u^{-1}$ is scattered for every $\lambda \in \mathrm{Res}(a)$. Define $G(\lambda) = \sigma(U_{a-\lambda}u^{-1})$ for $\lambda \in \mathbb{C}$. As G is an analytic multifunction, $G(\lambda)$ is countable for every $\lambda \in \mathbb{C}$ by the Scarcity Theorem. In particular, $U_a u^{-1}$ is scattered. As $a \in S(J)$ is arbitrary, $u^{-1} \in Z(J)$ for every invertible $u \in Z(J)$ by Theorem 4.2(2b).

Let now $x, y \in Z(J)$. Assume first that y is invertible. For a scattered and invertible element $a \in J$, using analytic functional calculus, one can find scattered and invertible elements $b, z \in J$ such that $b^2 = a$ and $z^2 = y$.

As $U_b y$ is scattered by Theorem 4.2(2a) and $\sigma(U_z a) = \sigma(U_b y)$ by Lemma 2.3, $U_z a$ is scattered. Also $U_z a$ is invertible, whence $\{\lambda : 0 \in \sigma(U_z a + \lambda x)\}$ is countable by Theorem 4.3. It is clear that $U_z a + \lambda x$ is invertible if and only if $a + \lambda U_{z^{-1}} x$ is. As a is an arbitrary element of $S(J) \cap \mathrm{Inv}(J)$, we see that $U_{z^{-1}} x \in Z(J)$ by Theorem 4.3. As $y^{-1} \in Z(J)$ by the above, we have similarly $U_{z^{-1}} y^{-1} \in Z(J)$, but $U_{z^{-1}} y^{-1} = U_{z^{-1}} (z^{-1})^2 = (z^{-1})^4 = (y^2)^{-1}$. So we conclude that $y^2 \in Z(J)$.

By Lemma 2.3, $\sigma(U_{y^{-1}} a) = \sigma(U_b y^{-2})$, and $U_b y^{-2}$ is scattered by Theorem 4.2(2a). Thus $U_{y^{-1}} a$ is scattered. Also $U_{y^{-1}} a$ is invertible, whence $\{\lambda : 0 \in \sigma(U_{y^{-1}} a + \lambda x)\}$ is countable by Theorem 4.3. As $a + \lambda U_y x = U_y (U_{y^{-1}} a + \lambda x)$, $U_{y^{-1}} a + \lambda x$ is invertible if and only if $a + \lambda U_y x$ is. As a is an arbitrary element of $S(J) \cap \mathrm{Inv}(J)$, we obtain $U_y x \in Z(J)$ by Theorem 4.3.

Assume now that $y \in Z(J)$ is arbitrary and $K(\lambda) = \sigma(U_{y-\lambda} x + a)$ for $\lambda \in \mathbb{C}$. As usual, $K(\lambda)$ is countable for any $\lambda \in \mathrm{Res}(y)$, and $\mathrm{Res}(y)$ contains sets of positive capacity. By the Scarcity Theorem, $K(\lambda)$ is countable for any $\lambda \in \mathbb{C}$. Therefore $U_y x \in Z(J)$ for all $x, y \in Z(J)$.

Now it is easy to show that $Z(J)$ is a subalgebra of J. Indeed, if $x, y \in Z(J)$ then $U_x y, U_{x-1} y \in Z(J)$, so that

$$x \cdot y = L_x y = (U_x y - U_{x-1} y + y)/2 \in Z(J).$$

As $Z(J)$ contains the inverses of its elements, it is a full subalgebra of J.

The case when $L = A$ is a normed associative Q-algebra. Let $x, y \in Z(A)$. By Theorem 4.2(3), $xyS(A) \subset xS(A) \subset S(A)$, whence $xy \in Z(A)$. So $Z(A)$ is a subalgebra of A. Assume that x is invertible. Then $x(a - \lambda)$ is scattered for any $a \in S(A)$ and $\lambda \in \mathbb{C}$. Hence $(a - \lambda)^{-1} x^{-1}$ is scattered for any $\lambda \in \mathrm{Res}(a)$. By Lemma 3.1(3), $x^{-1} \in Z(A)$. Therefore $Z(A)$ is a full subalgebra of A. ∎

The proof of Theorem 4.4 yields

Corollary 4.5. *Let J be a unital sub-Banach Jordan algebra. If $x \in Z(J)$ then $U_x a \in S(J)$ for every $a \in S(J)$.*

Proof. We know that $x^2 \in Z(J)$ by Theorem 4.4. If a is invertible then there is an invertible and scattered element b with $b^2 = a$. As $\sigma(U_x a) = \sigma(U_b x^2)$ by Lemma 2.3, and $U_b x^2$ is scattered by Theorem 4.2(2a), it follows that $U_x a$ is scattered.

If a is not invertible, $U_x(a - \lambda)$ is scattered for each $\lambda \in \mathrm{Res}(a)$. As $\lambda \mapsto \sigma(U_x(a - \lambda))$ is an analytic multifunction, $\sigma(U_x(a - \lambda))$ is countable for each $\lambda \in \mathbb{C}$ by the Scarcity Theorem. So $U_x a$ is scattered. ∎

The converse is not always true. For example, it is easy to find an operator x in $B(H)$ such that $x^2 = 0$ and $x \notin K(H) + \mathbb{C}$. Then $(U_x a)^2 = 0$ for each $a \in B(H)$ but $x \notin Z(B(H))$ (see Corollary 4.7).

4.2. Derivations. Let L be complete normed, and let D be a bounded derivation of L. Then $\exp(\lambda D)$ is a bounded automorphism of L [10, Lemma 2.2.1]. *If $D^2x = 0$ for some $x \in L$ then Dx is a quasinilpotent element of L.* The last statement is a well-known variant of the Kleinecke–Shirokov Theorem (and holds even if D is unbounded: see [38, Theorem 2.9] and [39, Theorem]). One can argue in the spirit of [1] as follows. If $D^2x = 0$ then $\exp(\lambda D)x = x + \lambda Dx$ and $\rho(\mu x + Dx) = |\mu|\,\rho(\exp(\mu^{-1}D)x) = |\mu|\,\rho(x)$ for any non-zero $\mu \in \mathbb{C}$. As the function $\mu \mapsto \rho(\mu x + Dx)$ is subharmonic, by [1, Theorem A.1.2] we have

$$\rho(Dx) = \limsup_{0\neq\mu\to 0} \rho(\mu x + Dx) = \limsup_{0\neq\mu\to 0} |\mu|\,\rho(x) = 0. \tag{4.2}$$

For a normed Jordan algebra the spectral radius does not change under passing to the completion of the algebra, so the statement itself and (4.2) also hold for normed Jordan algebras.

THEOREM 4.6. *Let L be either a Banach Jordan algebra J or a Banach associative algebra A, and let D be a bounded derivation of L. Then:*

(1) *$\mathcal{R}_s(L)$ and $Z(L)$ are invariant for D. In particular, if $L = J$ then $[a, x, b] \in Z(J)$ for any $a, b \in J$ and $x \in Z(J)$.*
(2) *If $L = A$ and $D(A) \subset Z(A)$ then $D(A) \subset \mathcal{R}_s(A)$.*

Proof. (1) As $\exp(\lambda D)$ is an automorphism of L, it preserves spectrum. So, for $L = J$, it is easy to see from Theorem 3.10(2) that $U_{\exp(\lambda D)a} \exp(\lambda D)x$ is scattered for any $a \in J$ and $x \in \mathcal{R}_s(J)$. Hence $\exp(\lambda D)x \in \mathcal{R}_s(J)$. As $x \in \mathcal{R}_s(J)$ and $\mathcal{R}_s(J)$ is closed, it follows that $Dx \in \mathcal{R}_s(J)$. For $L = A$, the corresponding result is proved similarly.

Further, it is clear that $\exp(\lambda D)S(L) = S(L)$ and $\exp(\lambda D)Z(L) = Z(L)$. Let $x \in Z(L)$, $a \in S(L)$,

$$f(\lambda) := (\exp(\lambda D)x - x)/\lambda + a \quad \text{for } \lambda \neq 0 \quad \text{and} \quad f(0) := D(x) + a.$$

Then f is an entire function, and the multifunction $\lambda \mapsto \sigma(f(\lambda))$ is analytic on $\mathbb{C}$ and countable for any $\lambda \neq 0$. By the Aupetit–Zemánek Theorem, $\sigma(f(0))$ is also countable. Therefore $D(x) + a$ is scattered for any $a \in S(L)$, i.e. $D(x) \in Z(L)$ for any $x \in Z(L)$.

In particular, for $L = J$, as $x \mapsto [a, x, b] = (a \cdot x) \cdot b - a \cdot (x \cdot b)$ is an inner derivation for any $a, b \in J$, we have $[a, x, b] \in Z(J)$ for any $x \in Z(J)$.

(2) Let $D(A) \subset Z(A)$ and $x \in S(A)$. Then $D(xa), Da \in Z(A)$ and $xDa \in S(A)$ for any $a \in A$. Hence

$$(Dx)a = D(xa) - xDa \in Z(A) + S(A) \subset S(A).$$

Therefore $Dx \in \mathcal{R}_s(A)$.

As D preserves $\mathcal{R}_s(A)$, in view of Theorem 4.2(1) one can assume that $\mathcal{R}_s(A) = \{0\}$. Thus we need only show that $D = 0$. For any $y \in A$, we have $Dy \in Z(A) \subset S(A)$, whence $D^2y \in \mathcal{R}_s(A) = \{0\}$. Thus Dy is quasinilpotent for any $y \in A$ by (4.2). By [31, Proposition 2.2], $D(A) \subset \operatorname{rad}(A) = \{0\}$. ∎

COROLLARY 4.7. *Let A be a Banach associative algebra, and let x be a scattered element of A. Then $x \in Z(A)$ if and only if $xa - ax \in \mathcal{R}_s(A)$ for every $a \in A$. In particular, $Z(A)$ is the set of scattered elements that lie in the center $\mathcal{C}_{\mathcal{R}_s(A)}(A)$ of A modulo the scattered radical.*

Proof. Passing to cosets modulo $\mathcal{R}_s(A)$, one may assume that the scattered radical of A is zero by Theorem 4.2(1).

For every $a \in A$, the inner derivation $z \mapsto az - za$ preserves $Z(A)$ by Theorem 4.6(1). So if $x \in Z(A)$ then the inner derivation $a \mapsto xa - ax$ maps A into $Z(A)$. By Theorem 4.6(2), $xa - ax \in \mathcal{R}_s(A) = \{0\}$ for every $a \in A$.

Conversely, let $x \in S(A)$ and $xa - ax \in \mathcal{R}_s(A) = \{0\}$ for each $a \in A$. Then x commutes with any scattered element b of A. As $\sigma(x + b) \subset \sigma(x) + \sigma(b)$, $x + b$ is a scattered element of A. Hence $x \in Z(A)$.

Returning to the initial assumptions of the corollary, we conclude from above that $Z(A) = S(A) \cap \mathcal{C}_{\mathcal{R}_s(A)}(A)$. ∎

It is not clear whether statement (2) of Theorem 4.6 is valid for Banach Jordan algebras.

THEOREM 4.8. *Let J be a Banach Jordan algebra. Then:*

(1) $[Z(J), Z(J), Z(J)] \subset \mathcal{R}_s(J)$, $\big[\overline{Z(J)}, J, \overline{Z(J)}\big] \subset S(J)$ *and* $\big[J, \overline{Z(J)}, J\big] \subset S(J)$.
(2) *If* $\mathcal{R}_s(J) = \{0\}$ *then* $\big[\overline{Z(J)}, J, \overline{Z(J)}\big]$ *and* $\big[J, \overline{Z(J)}, J\big]$ *consist of quasinilpotents of* J.

Proof. (1) Without loss of generality we may assume that J is unital. Let $a, b, c \in Z(J)$ and $x \in J$. Then $[x, a, b], [a, b, x] \in Z(J)$ by Theorem 4.6(1). By (2.2), we have $[b, x, a] \in Z(J)$. Thus

$$[Z(J), J, Z(J)] \subset Z(J). \tag{4.3}$$

As $T_{(a,c)} : y \mapsto [a, y, c]$ is a derivation on J, and as $T_{(a,c)}(b \cdot x), T_{(a,c)}x \in Z(J)$ by (4.3) and $b \cdot T_{(a,c)}x \in Z(J)$ by Theorem 4.4, we obtain

$$(T_{(a,c)}b) \cdot x = T_{(a,c)}(b \cdot x) - b \cdot T_{(a,c)}x \in Z(J).$$

This immediately shows that

$$[Z(J), Z(J), Z(J)] \cdot J \subset Z(J). \tag{4.4}$$

Further, $T_{(a,c)}$ is also a *derivation* in the *associator Lie triple system* of J. Recall that the latter is a vector space inherited from J considered with the map $(u,v,w) \mapsto [u,v,w]$ (see [19, Section 8.1]). So it is easy to see that

$$T_{(a,c)}[x,x,b] = [T_{(a,c)}x,x,b] + [x,T_{(a,c)}x,b] + [x,x,T_{(a,c)}b].$$

As $T_{(a,c)}x \in Z(J)$ by (4.3), we have $[T_{(a,c)}x,x,b] \in Z(J)$ again by (4.3), and $[x,T_{(a,c)}x,b] \in Z(J)$ by Theorem 4.6(1). As also $T_{(a,c)}[x,x,b] \in Z(J)$ by (4.3), we have

$$[x,x,T_{(a,c)}b] = T_{(a,c)}[x,x,b] - [T_{(a,c)}x,x,b] - [x,T_{(a,c)}x,b] \in Z(J). \tag{4.5}$$

As $L_{x^2}T_{(a,c)}b \in Z(J)$ by (4.4), we obtain

$$\begin{aligned} U_xT_{(a,c)}b &= 2(L_x^2 - L_{x^2})T_{(a,c)}b + L_{x^2}T_{(a,c)}b \\ &= -2[x,x,T_{(a,c)}b] + L_{x^2}T_{(a,c)}b \in Z(J). \end{aligned}$$

In other words, $U_xT_{(a,c)}b \in S(J)$ for every $x \in J$. By Theorem 3.10, we have $T_{(a,c)}b \in \mathcal{R}_s(J)$.

As $\mathcal{R}_s(J)$ is a thin ideal of J, to complete the proof of (1) it suffices to prove (2).

(2) Assume that $\mathcal{R}_s(J) = \{0\}$. It follows from the above that

$$[\overline{Z(J)},\overline{Z(J)},\overline{Z(J)}] \subset \overline{[Z(J),Z(J),Z(J)]} = \{0\}, \tag{4.6}$$

whence $T_{(a,c)}b \in \overline{Z(J)}$ for every $b \in J$ and $a,c \in \overline{Z(J)}$ by (4.3). By (4.6), $T_{(d,e)}T_{(a,c)} = 0$ for all $d,e \in \overline{Z(J)}$. In particular, $T_{(a,c)}^2 b = 0$ and $T_{(a,c)}b$ is a quasinilpotent (see (4.2)). This shows that $[\overline{Z(J)},J,\overline{Z(J)}]$ consists of quasinilpotents of J.

Let $x,y \in J$. Then $T_{(x,y)}(\overline{Z(J)}\ ) \subset \overline{Z(J)}$ by Theorem 4.6. As $\overline{Z(J)}$ is a Banach commutative associative algebra, $T_{(x,y)}(\overline{Z(J)}) \subset \operatorname{rad}(\overline{Z(J)})$ by the Singer–Wermer Theorem [37]. This shows that $[J,\overline{Z(J)},J]$ consists of quasinilpotents of J. ∎

Let J be a Banach Jordan algebra, and let D be a bounded derivation on J. Assume that $D(J)$ consists of quasinilpotents or even $D^2 = 0$. Then [9, Examples 4.7 and 4.8] shows that D does not necessarily map J into $\operatorname{rad}(J)$.

Let L be complete normed. We consider the conditions that guarantee equality of $S(L)$ and $Z(L)$.

Corollary 4.9. *Let J be a Banach Jordan algebra. Then the following conditions are equivalent:*

(1) *$a+b$ is scattered for any scattered $a,b \in J$.*
(2) *U_ab is scattered for any scattered $a,b \in J^1$.*

Proof. Follows from Theorem 4.2(2). ∎

COROLLARY 4.10. *Let A be a Banach associative algebra. Then the following conditions are equivalent:*

(1) *$a+b$ is scattered for any scattered $a, b \in A$.*
(2) *ab is scattered for any scattered $a, b \in A$.*
(3) *$S(A)$ is a Lie ideal (a subalgebra) of A.*
(4) *$[S(A^+), S(A^+), S(A^+)] \subset \mathcal{R}_s(A^+)$.*
(5) *$ax - xa$ has finite spectrum modulo $\mathcal{R}_s(A)$ for every $x \in A$ and $a \in S(A)$.*
(6) *$S(A)$ is commutative modulo $\mathcal{R}_s(A)$.*
(7) *$S(A)$ lies in the center of A modulo $\mathcal{R}_s(A)$.*

Proof. By Corollary 3.5(1) and Theorem 4.2(1), it suffices to show the required implications under the condition $\mathcal{R}_s(A) = \{0\}$. So we assume in what follows that A is $\mathcal{R}_s$-semisimple. By Corollary 3.12, so is A^+.

(1)$\Leftrightarrow$(7) follows from Corollary 4.7, and (1)$\Leftrightarrow$(2) from Theorem 4.2(3). The implications (3)$\Rightarrow$(1), (7)$\Rightarrow$(3), (7)$\Rightarrow$(5) and (7)$\Rightarrow$(6) are obvious, and (6)$\Rightarrow$(1) follows from the inclusion $\sigma(a+b) \subset \sigma(a)+\sigma(b)$ for any commuting $a, b \in A$.

(7)$\Rightarrow$(4): It is clear that $S(A)$ is a commutative subalgebra of A. Hence the Jordan product induced in $S(A)$ from A^+ coincides with the usual product in $S(A)$. Therefore $S(A)^+$ is an associative algebra. But $S(A)^+ = S(A^+)$.

(4)$\Rightarrow$(6): Note that $[a, b, b] = 0$ implies $[b, [b, a]] = 0$. Then, by the Kleinecke–Shirokov Theorem, $[b, a]$ is quasinilpotent for any scattered (in particular, quasinilpotent) elements $a, b \in A$. Hence the set of quasinilpotent elements of A coincides with $\mathrm{rad}(A)$ by [20, Theorem 2]. As A is semisimple, $S(A)$ is commutative.

(5)$\Rightarrow$(7): Assume that $ax - xa$ has finite spectrum for each $x \in A$ and $a \in S(A)$. Let D_a be the derivation $x \mapsto ax - xa$ for $a \in S(A)$. Then $D_a(A)$ consists of elements with finite spectrum. As A is semisimple, D_a maps A into $\mathrm{soc}(A)$ by [8, Theorem 3.2]. But $\mathrm{soc}(A) \subset \mathcal{R}_s(A) = \{0\}$. So $ax = xa$ for each $x \in A$ and $a \in S(A)$. ∎

COROLLARY 4.11. *Let A be an $\mathcal{R}_s$-semisimple associative Banach algebra. Then for any non-zero quasinilpotent $a \in A$ there is a quasinilpotent $b \in A$ such that $\sigma(a+b)$ is infinite.*

Proof. Assume, to the contrary, that there is a non-zero quasinilpotent $a \in A$ such that $a+b$ has finite spectrum for any quasinilpotent $b \in A$. For every $c \in A$, define $f(\lambda) = (a - \exp(-\lambda c) a \exp(\lambda c))/\lambda$ for $0 \neq \lambda \in \mathbb{C}$ and set $f(0) = ca - ac$. Then f is analytic on $\mathbb{C}$, and $f(\lambda) - a$ is quasinilpotent for any $\lambda \in \mathbb{C}$. By the scarcity of elements with finite spectrum (see [1, Theorem 3.4.25]), the set of $\lambda \in \mathbb{C}$ with finite $\sigma(f(\lambda))$ either has zero capacity or is the whole $\mathbb{C}$. Hence $ca - ac$ has finite spectrum for any $c \in A$. Repeating the proof of (5)⇒(7) in Corollary 4.10, we see that a is in the center of A. So ax is a quasinilpotent for any $x \in A$, whence $a \in \mathrm{rad}(J) \subset \mathcal{R}_s(A) = \{0\}$, a contradiction. ■

Corollary 4.12. *Let H be a Hilbert space and $A = B(H)/K(H)$. Then for any non-zero quasinilpotent element of A there is a quasinilpotent element of A such that their sum has infinite spectrum.*

Proof. Note that A is $\mathcal{R}_s$-semisimple and apply Corollary 4.11. ■

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Peng Cao
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing, P.R. China
E-mail: cpeng@bit.edu.cn

Yurii V. Turovskii
Institute of Mathematics and Mechanics
National Academy of Sciences of Azerbaijan
9 Vahabzade Street
Baku AZ1141, Azerbaijan
E-mail: yuri.turovskii@gmail.com