COHOMOLOGY OF KOSZUL-VINBERG ALGEBROIDS AND POISSON MANIFOLDS, I

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Abstract. We introduce a cohomology theory of Koszul-Vinberg algebroids. The relationships between that cohomology and Poisson manifolds are investigated. We focus on the complex of chains of superorders [KJL1]. We prove that symbols of some sort of cycles give rise to so called bundlelike Poisson structures. In particular we show that if $E \to M$ is a transitive Koszul-Vinberg algebroid whose anchor is injective then a Koszul-Vinberg cocycle $\theta$ whose symbol has non-zero skew symmetric component defines a transversally Poissonian symplectic foliation in $M$.

1. Background material. Let $A$ be a real algebra whose multiplication map is denoted by

$$(a, b) \to ab.$$

Given three elements $a, b, c$ of $A$ their associator in $A$ is the quantity

$$(a, b, c) = a(bc) - (ab)c. \tag{1}$$

**Definition 1.1.** A real algebra $A$ is called a Koszul-Vinberg algebra if its associator map satisfies the identity

$$(a, b, c) = (b, a, c).$$

N.B. Koszul-Vinberg algebras are also called left symmetric algebras [NB1], [PA].

Let $A$ be a Koszul-Vinberg algebra and let $W$ be a real vector space with two bilinear maps

$$A \times W \to W : (a, w) \to aw;$$

$$W \times A \to W : (w, a) \to wa. \tag{2}$$

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The paper is in final form and no version of it will be published elsewhere.
We will set the following: given \( a \in A \) and \( w \in W \)
\[
(a, b, w) = a(bw) - (ab)w,
\]
\[
(a, w, b) = a(wb) - (aw)b,
\]
\[
(w, a, b) = w(ab) - (wa)b.
\]

**Definition 1.2.** A vector space equipped with two bilinear maps (2) is called a Koszul-Vinberg module of \( A \) if the following identities hold for any \( a, b \in A \) and \( w \in W \):
\[
(a, b, w) = (b, a, w),
\]
\[
(a, w, b) = (w, a, b).
\]

Given a Koszul-Vinberg algebra \( A \) and a Koszul-Vinberg module \( W \) of \( A \), one of the following spaces:
\[
J(A) = \{ c \in A/(a, b, c) = 0, \forall a \in A, \forall b \in A \};
\]
\[
J(W) = \{ w \in W/(a, b, w) = 0, \forall a \in A, \forall b \in A \}.
\]

The subspace \( J(A) \subset A \) is a subalgebra of \( A \) and the induced multiplication map is associative. In general the vector subspace \( J(W) \) is not invariant under the actions (2).

**Examples of Koszul-Vinberg algebras and their modules**

(\( e_1 \)) Every associative algebra is a Koszul-Vinberg algebra.

(\( e_2 \)) Let \((M, D)\) be a locally flat manifold, [KJL3]; then the vector space \( \Gamma(TM) \) of smooth vector fields on \( M \) is a Koszul-Vinberg algebra; its multiplication map is defined by
\[
(X, Y) \rightarrow XY = DXY.
\]

(\( e_3 \)) Given a locally flat manifold \((M, D)\) let \( W \) be the vector space of real valued smooth functions on \( M \). For any \( f \in W \) and \( X \in \Gamma(TM) \) we define \( Xf \in W \) and \( fX \in W \) by putting
\[
(Xf)(x) = < df, X > (x), \quad (fX)(x) = 0 \in \mathbb{R}.
\]

With the above operations \( W \) becomes a Koszul-Vinberg module of \( A = \Gamma(TM) \).

Given a Koszul-Vinberg algebra \( A \) and two Koszul-Vinberg modules of \( A \), called \( V \) and \( W \), let \( Hom(W, V) \) be the vector space of linear maps from \( W \) to \( V \). We consider the following actions of \( A \) in \( Hom(W, V) \): let \( \theta \in Hom(W, V) \), \( a \in A \), \( w \in W \) then we set
\[
(a\theta)(w) = a(\theta(w)) - \theta(aw), \quad (\theta a)(w) = (\theta(w))a.
\]

Under the actions defined in (4) the vector space \( Hom(W, V) \) becomes a Koszul-Vinberg module of \( A \). More generally the vector space \( Hom(\oplus^q W, V) \) of \( q \)-linear mappings from \( W \) to \( V \) is a Koszul-Vinberg module of \( A \) under the following actions: let \( \theta \in Hom(\oplus^q W, V) \), \( a \in A \) and \( w_1, ..., w_q \in W \), we set
\[
(a\theta)(w_1, ..., w_q) = a(\theta(w_1, ..., w_q)) - \sum_{1 \leq j \leq q} \theta(...aw_j, ..., w_q),
\]
\[
(\theta a)(w_1, ..., w_q) = (\theta(w_1, ..., w_q))a.
\]

Let \( q \) be a positive integer every pair \((j, w_0)\) where \( j \) is a non-negative integer with \( j \leq q \) and \( w_0 \in W \) will define a linear map from \( Hom(\otimes^q V, V) \) to \( Hom(\otimes^{q-1} W, V) \), called \( e_j(w_0) \). Let \( \theta \in Hom(\otimes^q W, V) \) then \( e_j(w_0)\theta \in Hom(\otimes^{q-1} W, V) \) is defined by
\[
(e_j(w_0)\theta)(w_1, ..., w_{q-1}) = \theta(w_1, ..., w_{j-1}, w_0, w_j, w_{j+1}, ..., w_q).
\]
The linear map $e_j(w_0)$ commutes with the right action of $A$, viz

$$(e_j(w_0)\theta)a = e_j(w_0)(\theta a).$$

Thus the notation $e_j(w_0)\theta a$ will be well defined.

We are now in a position to recall the definition of the complex

$$\ldots \rightarrow C^q(A, W) \xrightarrow{\delta_q} C^{q+1}(A, W) \rightarrow \ldots$$

Let $A$ be a Koszul-Vinberg algebra and let $W$ be a Koszul-Vinberg module of $A$. For each positive integer $q$ we set

$$C^q(A, W) = \text{Hom}(\otimes^q A, W)$$

and for $q = 0$ we set

$$C^0(A, W) = J(W).$$

Then the graded vector space

$$C(A, W) = \bigoplus_{q \geq 0} C^q(A, W)$$

is a cochain complex whose boundary operator is defined by

$$(\delta \theta)(a_1, \ldots, a_{q+1}) = \sum_{j \leq q} (-1)^j \{(a_j \theta)(\hat{a}_j \ldots a_{q+1}) + (e_q(a_j)\theta a_{q+1}(\ldots \hat{a}_j \ldots, \hat{a}_{q+1})\}.$$

The family $(\delta_q)_q$ satisfies the following identity

$$\delta_{q+1}\delta_q = 0.$$

The $q^{th}$ cohomology space of the cochain complex $C(A, W)$ is denoted by $H^q(A, W)$. We have

$$H^q(A, W) = \ker(\delta_q)/\text{im}(\delta_{q-1})$$

for $q > 0$ and

$$H^0(A, W) = \ker(\delta_0).$$

**Example.** Let $(M, D)$ be a locally flat manifold and let $A = \Gamma(TM)$ be the corresponding Koszul-Vinberg algebra. Regarding $A$ as a Koszul-Vinberg module of itself the subspace $J(A)$ consists of affine vector fields. Thus $\ker(\delta_0)$ is the subspace of locally linear vector fields. One sees that in general $H^0(A, A)$ will be non-trivial; e.g. if $(M, D)$ is the real flat torus then $\dim H^0(A, A) = \dim M$. On the other hand we have

$$H^1(A, A) = 0, \quad [\text{NB}_3].$$

**2. Koszul-Vinberg algebroids and coalgebroids.** Let $M$ be a smooth manifold and $E$ a vector bundle over $M$. The space of smooth sections of $E$ is denoted by $\Gamma(E)$.

**Definition 2.1.** A Koszul-Vinberg algebroid over $M$ is a vector bundle $E$ over $M$ with a bundle map $a : E \rightarrow TM$, called the anchor map, such that

(P1) $\Gamma(E)$ is a Koszul-Vinberg algebra;

(P2) The anchor $a : \Gamma(E) \rightarrow \Gamma(TM)$ satisfies the following identities: $\forall f \in C^\infty(M, \mathbb{R})$, $\forall s \in \Gamma(E)$, $\forall s' \in \Gamma(E)$

$$(fs)s' = f(ss'), \quad s(fs') = f(ss') + <df, a(s)> s'.$$
Remark. It follows from conditions (P1) and (P2) that the anchor map is a homomorphism of the associated Lie algebras.

Examples of Koszul-Vinberg algebroids

$$(e_1)$$ The tangent bundle of a locally flat manifold $(M, D)$ is a Koszul-Vinberg algebroid. Its anchor is Identity map; given two sections of $TM$, called $X, Y$ then

$$XY = D_XY.$$  

$$(e_2)$$ Let $\mathcal{F}$ be an affine foliation in a smooth manifold $M$ and let $E_\mathcal{F}$ be the tangent bundle of $\mathcal{F}$ in $TM$. Since each leaf of $\mathcal{F}$ is a locally flat manifold $E_\mathcal{F}$ is a Koszul-Vinberg algebroid over $M$.

$$(e_3)$$ Each completely integrable system in an $m$-dimensional symplectic manifold $(M, \omega)$ gives rise to an action of $\mathbb{R}^m$ in $M$. The orbits of that action are locally flat manifolds; thus every completely integrable system will generate a Koszul-Vinberg algebroid.

$$(e_4)$$ Given a lagrangian foliation $\mathcal{F}$ in a symplectic manifold $(M, \omega)$ one defines a Koszul-Vinberg algebroid $E$ as in $(e_2)$. If $s, s' \in \Gamma(E)$ then $ss'$ is defined by the relation

$$\iota(ss')\omega = L_s\iota(s')\omega$$

where $\iota(s')$ is the inner product by $s'$ and $L_s$ is the Lie derivation w.r.t. $s$. The multiplication in $\Gamma(E)$ given by (6) induces a locally flat structure in each leaf of $\mathcal{F}$.

Now given a Koszul-Vinberg algebroid $E$ whose anchor map is injective, it is natural to ask whether the locally flat structure of leaves of $E$ extends to a locally flat structure in $M$. The notion of Koszul-Vinberg co-algebroid together with cochain complex (5) help to study the extension that we just raised, [NBW] (see also [KI] for the notion of partial connection).

**Definition 2.2.** Given a Koszul-Vinberg algebroid $E \rightarrow M$, a Koszul-Vinberg coalgebroid of $E$ is a vector bundle $N \rightarrow M$ together with a bundle map $\alpha : N \rightarrow TM$ satisfying the following conditions:

$$(c_1) \Gamma(N)$$ is a Koszul-Vinberg algebra.

$$(c_2)$$ There exists a linear map $j : \Gamma(TM) \rightarrow \Gamma(N)$ such that the sequence

$$\Gamma(E) \xrightarrow{\alpha} \Gamma(TM) \xrightarrow{j} \Gamma(N) \rightarrow 0$$

is exact and $j \circ \alpha(s) = s, \forall s \in \Gamma(N)$.

$$(c_3)$$ Let $s, s'$ be elements of $\Gamma(N)$ and $f \in C^\infty(M, \mathbb{R})$; then

$$(fs)s' = f(ss')$$

and if $< df, \alpha(\sigma) > = 0$ for every $\sigma \in \Gamma(E)$ then

$$s(fs') = f(ss') + < df, \alpha(s) > s'.$$

**Example.** Let $\mathcal{F}$ be a locally flat foliation which is a transversally affine foliation at the same time. Then the Koszul-Vinberg algebroid $E_\mathcal{F}$ corresponding to $\mathcal{F}$ admits a Koszul-Vinberg coalgebroid, [NBW].

Indeed let $\mathcal{L}$ be the sheaf of locally linear sections of $E_\mathcal{F}$, i.e. $s \in \mathcal{L}$ iff $s's = 0$, $\forall s' \in \Gamma(E_\mathcal{F})$. We consider the quotient vector bundle $TM/E_\mathcal{F}$. Since $\mathcal{F}$ is transversally
affine the space of smooth sections of $N = TM/E_F$ admits a structure of Koszul-Vinberg algebra (every germ of submanifold which is transverse to $F$ is a germ of affine manifold). Thus $\Gamma(N)$ admits a Koszul-Vinberg algebra structure. Let us write $J(N) = J(\Gamma(N))$.

Then $C^\infty(M, \mathbb{R})J(N) = \Gamma(N)$. Using a riemannian metric on $M$ one constructs a section $\alpha : N \to TM$ of the exact sequence

$$0 \to E_F \xrightarrow{\alpha} TM \xrightarrow{j} N \to 0$$

where $j$ is the canonical projection.

In [NBW] we have used the Lie algebra $A = \text{norm}(L) \cap j^{-1}(J(N))$ to study the extension problem of the locally flat structure of $F$; $\text{norm}(L)$ is the normalizer of $L$ in the Lie algebra $\Gamma(TM)$.

Remark that every Koszul-Vinberg algebroid $E$ gives rise to a Lie algebroid $E_L$; the total space of $E_L$ is $E$; for $s$ and $s'$ in $\Gamma(E_L)$ the bracket is defined by

$$[s, s'] = ss' - s's.$$  

The anchor map of $E$ satisfies the identity

$$a([s, s']) = [a(s), a(s')].$$

Indeed let $s, s', s''$ be elements of $\Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$, then

$$[s, s'](fs'') = (ss')(fs'') - (s's)(fs'') = s(s'(fs'')) - s'(s(fs''))$$

and property $(P_2)$ of definition 2.1 implies that

$$< df, a([s, s']) > = a(s)(a(s')f) - a(s')(a(s)f),$$

where $a(s)f = < df, a(s) >$.

3. Real cohomology of Koszul-Vinberg algebroids. Let $E \to M$ be a Koszul-Vinberg algebroid. The vector space $W = C^\infty(M, \mathbb{R})$ is a Koszul-Vinberg module of $A = \Gamma(E)$. The left action and the right action are defined by

$$(sf)(x) = < df, a(s) >, \quad (f.s)(x) = 0,$$

where $a$ is the anchor map of $E$.

We will focus on the cochain complex

$$\cdots \to C^q(A, W) \xrightarrow{\delta_q} C^{q+1}(A, W) \to \cdots$$

The $q^{th}$ cohomology space of $(8)$ is denoted by $H^q(E, \mathbb{R})$, i.e. $H^q(E, \mathbb{R}) = H^q(A, W)$.

Definition 2.3. The vector space $H^q(E, \mathbb{R})$ is called the $q^{th}$ cohomology space of the Koszul-Vinberg algebroid $E \to M$.

Example. Let $E$ be a regular Koszul-Vinberg algebroid whose anchor map is denoted by $a$. Then $a(E)$ defines a foliation on $M$. A function $f$ belongs to $J(W)$ iff $L_{a(s)} \circ$
$L_{a(s')}(f) = 0$ for arbitrary sections $s, s'$ of $E$. Thus if the anchor is injective then $J(W)$ consists of smooth functions which are affine along each leaf of $a(E)$. Since $H^0(E, \mathbb{R}) = ker(\delta_0)$ we see that $H^0(E, \mathbb{R})$ is just the vector space of first integrals of $a(E)$.

**Theorem 3.1.** If a regular Koszul-Vinberg algebroid $E \to M$ admits a dense leaf then $\dim H^0(E, \mathbb{R}) = 1$.

**4. Koszul-Vinberg algebroids and Poisson manifolds.** To every Koszul-Vinberg algebroid $E \to M$ we attach the following new Koszul-Vinberg algebroid $\mathcal{E} = E \times \mathcal{R}$ where $\mathcal{R}$ is the trivial vector bundle $\mathcal{R} = M \times \mathbb{R}$. We identify $\Gamma(\mathcal{R})$ with the associative algebra $C^\infty(M, \mathbb{R})$ of smooth real valued functions on $M$. Thus we will identify $\Gamma(\mathcal{E})$ with $\Gamma(\mathcal{E}) \times C^\infty(M, \mathbb{R})$ as well.

Henceforth $\Gamma(\mathcal{E})$ is an algebra whose multiplication is

$$ (s, f)(s', f') = (ss', ff' + < df', a(s)>). $$

It is easy to see that (9) endows $\Gamma(\mathcal{E})$ with a structure of Koszul-Vinberg algebra. Moreover if $g \in C^\infty(M, \mathbb{R})$ then we have

$$(g(s, f))(s', f') = g((s, f)(s', f'))$$

and

$$(s, f)(g(s', f')) = g((s, f)(s', f')) + < dg, a(s)>(s', f').$$

Naturally the anchor map of $\mathcal{E}$ is defined by

$$a_\epsilon(s, f) = a(s)$$

where $a$ is the anchor of $E \to M$. The Koszul-Vinberg algebra $\Gamma(\mathcal{E})$ is the semi-product $\Gamma(E) \times C^\infty(M, \mathbb{R})$.

Now let $V$ be a vector space, let $r$ be a non-negative integer; we will put $T^r(V) = \otimes^r V$.

Henceforth we are concerned with the cochain complex

$$\ldots \to C^q(\mathcal{G}, W) \overset{\delta_q}{\to} C^q(\mathcal{G}, W) \to \ldots$$

where $\mathcal{G}$ is the Koszul-Vinberg algebra (9) and $W = C^\infty(M, \mathbb{R})$. For each non-negative integer $q$ the vector space $C^q(\mathcal{G}, W)$ is bigraded

$$C^q(\mathcal{G}, W) = \oplus_{r+s=q} C^{r,s}(\mathcal{G}, W)$$

with

$$C^{r,s}(\mathcal{G}, W) = Hom(T^r A \otimes T^s W, W).$$

$r$ and $s$ being non-negative integers.

The boundary operator $\delta_q$ goes from $C^{r,s}(\mathcal{G}, W)$ to the direct sum $C^{r+1,s}(\mathcal{G}, W) \oplus C^{r,s+1}(\mathcal{G}, W)$. Thus we will equip the cohomology space $H^q(\mathcal{G}, W)$ with the bigradation

$$H^q(\mathcal{G}, W) = \oplus_{r+s=q} H^{r,s}(\mathcal{G}, W)$$
with
\[ H^{r,s}(\mathcal{G}, W) = \frac{\ker(\delta_q : C^{r,s}(\mathcal{G}, W) \to C^{r+1,s}(\mathcal{G}, W) \oplus C^{r,s+1}(\mathcal{G}, W))}{\delta_{q-1}(C^{q-1}(\mathcal{G}, W)) \cap C^{r,s}(\mathcal{G}, W)} \].

Naturally one sees that
\[ \delta_{q-1}(C^{q-1}(\mathcal{G}, W)) \cap C^{r,s}(\mathcal{G}, W) = \delta_{q-1}(C^{r-1,s}(\mathcal{G}, W) + C^{r,s-1}(\mathcal{G}, W)) \cap C^{r,s}(\mathcal{G}, W). \]

We will develop the analogue of the complex of differential forms of superorder introduced by Jean-Louis Koszul, [KJL2].

To begin with, let \( \xi \in \mathcal{G} \), for a non-negative integer \( k \) and \( x \in M \) \( j_x^k \xi \) is the \( k \)-th jet at \( x \) of \( \xi \in \mathcal{G} \). We will present \( j_x^k \xi \) by
\[ j_x^k \xi = (d_x^1 \xi, ..., d_x^k \xi) \]
where \( d_x^l \xi \) is the \( l \)-th differential at \( x \) of the section \( \xi \in \Gamma(\mathcal{E}) \).

**Definition [KJL2].** A cochain \( \theta \in C^q(\mathcal{G}, W) \) is of order \( \leq k \) if at every \( x \in M \) and for \( \xi_1, ..., \xi_q \in \mathcal{G} \) the value at \( x \) of \( \theta(\xi_1, ..., \xi_q) \) depends on \( j_x^k \xi_1, ..., j_x^k \xi_q \).

Let \( I = (i_1, ..., i_q) \) be a \( q \)-tuple of non-negative integers such that \( i_l \leq k \). Given a \( q \)-cochain \( \theta \in C^q(\mathcal{G}, W) \) of order \( \leq k \), we set
\[ \theta^I(\xi_1, ..., \xi_q)(x) = \theta(d_x^{i_1} \xi_1, ..., d_x^{i_q} \xi_q). \]

Since \( \theta \) is \( q \)-multilinear (11) makes sense.

Thus every \( \theta \in C^q(\mathcal{G}, W) \) which is of order \( \leq k \) will be decomposed as follows
\[ \theta(\xi_1, ..., \xi_q) = \sum I \theta^I(\xi_1, ..., \xi_q) \]
where \( I = (i_1, ..., i_q) \) with \( 0 \leq i_1, ..., i_q \leq k \).

We call \( \theta^I \) the component of type \( I \) of \( \theta \).

The following definition is crucial for the forthcoming applications.

**Definition 4.1** Given a cochain of order \( \leq k \), say \( \theta \in C^q(\mathcal{G}, W) \), then its component of type \((k, ..., k)\) is called the symbol of \( \theta \).

Notice that the symbol of \( \theta \) may be zero.

**Proposition [NB4].** The symbol \( \sigma(\theta) \) of every \( q \)-cocycle \( \theta \in C^{0,q}(\mathcal{G}, W) \) is \( \delta_q \)-closed and satisfies the identity
\[ s \sigma(\theta) = 0 \]
for any arbitrary element \( s \in \Gamma(E) \).

We recall that
\[ (s(\sigma(\theta)))(\xi_1, ..., \xi_q) = a(s)(\sigma(\theta)(\xi_1, ..., \xi_q)) - \sum_{j \leq q} \sigma(\theta)(...s\xi_j, ..., \xi_q). \]

For every non-negative integer \( r \), \( H^{r,0}(\mathcal{G}, W) = 0 \). (That phenomenon may be explained by using an appropriate spectral sequence.)

We are going now to relate symbols of so called Koszul-Vinberg cocycle to Poisson manifolds structures.

We will deal with the vector spaces \( C^{r,s}(\mathcal{G}, W) \) such that \( rs = 0 \). For instance \( C^{0,2}(\mathcal{G}, W) \) may contain Poisson tensors as well as Jacobi tensors.
On the other hand let us suppose that the Koszul-Vinberg algebroid $E \to M$ has an injective anchor map. Then Riemannian metrics or symplectic structures on the vector bundle $E \to M$ give rise to elements of $C^{2,0}(G, W)$.

Definition 4.2. (i) A cochain $\theta \in C^2(G, W)$ is called a Koszul-Vinberg cochain if for arbitrary elements $\xi_1, \xi_2, \xi_3$ of $G$ one has

$$(\xi_1, \xi_2, \xi_3)_\theta = (\xi_2, \xi_1, \xi_3)_\theta$$

where

$$(\xi_1, \xi_2, \xi_3)_\theta = \theta(\xi_1, \theta(\xi_2, \xi_3)) - \theta(\theta(\xi_1, \xi_2), \xi_3).$$

(ii) $\theta \in C^2(G, W)$ is a Koszul-Vinberg cocycle if $\delta \Pi_\theta = \delta.\theta = 0$ and $(\xi_1, \xi_2, \xi_3)_\theta = (\xi_2, \xi_1, \xi_3)_\theta$.

Definition 4.2 makes sense because $W$ may be regarded as a subspace of $G$.

Every Koszul-Vinberg cochain $\theta \in C^2(G, W)$ defines a Koszul-Vinberg algebra structure whose multiplication is given by

$$\xi_1 \xi_2 = \theta(\xi_1, \xi_2).$$

Therefore we define in $G$ a new Lie algebra structure called $G_{\theta}$, whose bracket is given by

$$[\xi_1, \xi_2]_{\theta} = \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1).$$

Before continuing we will recall some differential geometry structures related to the cohomology of Koszul-Vinberg algebroids.

Definition 4.3. (i) A Poisson foliation in a manifold $M$ is a foliation $\mathcal{F}$ whose leaves are Poisson manifolds.

(ii) A transversally Poisson foliation in $M$ is a foliation whose sheaf of basic functions is a sheaf of Poisson algebra.

Part (ii) in definition 4.3 has the following meaning: the sheaf of local first integrals of $\mathcal{F}$ admits a Lie algebra bracket

$$(f, g) \to \{f, g\}$$

such that

$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$
is a derivation of the algebra $A$. This elementary result has deep consequences; for example given a smooth manifold $M$ with a star product in $C^\infty(M, \mathbb{R})$, say
\[ f \ast g = fg + \sum_{k>0} h^k B_k(f, g) \]
the bilinear map $B_1 : C^\infty(M, \mathbb{R})^2 \to C^\infty(M, \mathbb{R})$ is a cocycle of the Hochschild complex of $C^\infty(M, \mathbb{R})$. One deduces that $B_1$ is a bidifferential operator of order 1 whose skew symmetric component defines a Poisson manifold structure on $M$, [KM]. The same claim doesn’t hold in the cohomology theory of Koszul-Vinberg algebras. For instance in a Koszul-Vinberg algebra $A$ the multiplication map
\[ (a, b) \to ab \]
is an exact cocycle of $C(A, A)$, but the linear map
\[ b \to ab - ba \]
for a fixed $a$ is a derivation of $A$ iff $a \in J(A)$. That makes relevant the theorem which is stated below.

Let $E \to M$ be a Koszul-Vinberg algebroid and let $C(G, W)$ be the complex associated to $G = \Gamma(E)$.

**Theorem I [NB_4].** Let $\theta \in C^{0,2}(G, W)$ be a cocycle of order $\leq k$. If the skew symmetric component of the symbol $\sigma(\theta)$ is non-zero, then $k = 1$.

An important consequence of theorem is the following statement:

**Theorem II [NB_4].** The skew symmetric component of the symbol $\sigma(\theta)$ of every Koszul-Vinberg cocycle $\theta \in C^{0,2}(G, W)$ is a Poisson tensor.

Now let us assume the Koszul-Vinberg algebroid $E \to M$ to be regular. Then $E$ defines a foliation $E_F$ in $M$. Given any Koszul-Vinberg cocycle $\theta \in C^{0,2}(G, W)$ of order $\leq k$ we denote by $\Pi_\theta$ the skew symmetric component of $\sigma(\theta)$. The following corollary follows directly from theorem II.

**Corollary 4.4.** Every germ of submanifold in $M$ which is normal to $F_E$ is a germ of Poisson submanifold of $(M, \Pi_\theta)$. In particular if $F_E$ is simple then the quotient manifold $M/F_E$ admits a Poisson manifold structure $(M/F_E, \hat{\Pi}_\theta)$ such that the canonical projection from $M$ to $M/F_E$ is a Poisson morphism from $(M, \Pi_\theta)$ to $(M/F_E, \hat{\Pi}_\theta)$.

Considering the case of Koszul-Vinberg algebroids with injective anchor maps, we see that such algebroids define locally flat foliations in their base manifolds. Thus we can state the following

**Theorem III [NB_4].** Let $E \to M$ be a Koszul-Vinberg algebroid whose anchor map is injective. If $E$ is transitive, then every Koszul-Vinberg cocycle $\theta \in C^{0,2}(G, W)$ defines a regular Poisson structure on $M$.

Remark that $W$ being a Koszul-Vinberg submodule of $G$ every Koszul-Vinberg cochain $\hat{\theta} \in C^2(G, G)$ induces a Koszul-Vinberg cochain $\theta \in C^{0,2}(G, W)$. 
5. The Koszul-Vinberg analogues of star product. Let \( M \) be a smooth manifold and let \( W \) be the vector space \( C^\infty(M, \mathbb{R}) \) endowed with its natural structure of associative and commutative algebra.

Given a start product in \( W \), say 
\[
    f \ast f' = ff' + \sum_{k>0} h^k B_k(f, f')
\]
it is well known that the skew symmetric component of \( B_1 \) is a Poisson tensor on \( M \), [KM]. Regarding theorem II a natural question arises: does the same phenomenon persist in Koszul-Vinberg algebra structures.

Henceforth we will consider a Koszul-Vinberg algebroid \( E \rightarrow M \). As before we denote by \( G \) the vector space of smooth sections of the Wihtney sum \( E \oplus \mathbb{R} \). We consider the multiplication already defined by (9), i.e. for \( \xi = (s, f), \xi' = (s', f') \)
\[
    \xi \xi' = (ss', ff' + < df', a(s) >)
\]
where \( a \) is the anchor map of \( E \). Let \( h \) be some parameter; we will focus on the family of multiplication in \( G \)
\[
    \xi_\ast_h \xi' = \xi \xi' + \sum_{k>0} h^k \theta_k(f, f')
\]
with \( \theta_k \in C^2(G, G) \). We suppose the multiplication (12) to satisfy Definition 1.1, viz
\[
    (\xi_1, \xi_2, \xi_3) \ast_h = (\xi_2, \xi_1, \xi_3) \ast_h
\]
for elements \( \xi_1, \xi_2, \xi_3 \) of \( G \). Thus we obtain a family \( G_h \) of Koszul-Vinberg algebras. The coefficient \( \theta_1 \) is a cocycle of the complex \( C(G, G) \).

Each Koszul-Vinberg algebra \( G_h \) give rise to a Lie algebra whose bracket is given by 
\[
    [\xi, \xi']_h = [\xi, \xi']_h - \xi' \ast_h \xi = [\xi, \xi'] + \sum_{k>0} h^k \Lambda_k(\xi, \xi')
\]
with \( \Lambda_k(\xi, \xi') = \theta_k(\xi, \xi') - \theta_k(\xi', \xi) \).

In order that the pair \( (E \oplus \mathbb{R}, G_h) \) define a Koszul-Vinberg algebroid with the same anchor map \( a \) as the pair \( (E \oplus \mathbb{R}, G) \) it is necessary that
\[
    a([\xi, \xi']_h) = a([\xi, \xi]).
\]
Thus we must have
\[
    a(\sum_{k>0} h^k \Lambda_k(\xi, \xi')) = 0.
\]
Therefore we see that for every positive integer \( k \) one has \( a(\Lambda_k(\xi, \xi')) = 0 \). On the other hand recall that \( W \) is a two-sided ideal of the Koszul-Vinberg algebra \( G \) whose multiplication is (9). Then the \( W \)-component of the cocycle \( \theta_1 \) is a \( W \)-valued 2-cocycle of the cochain complex \( C(G, W) \). By assuming that the map \( a \) is also the anchor map of the pair
\[
    (E \oplus \mathbb{R}, G_h)
\]
we deduce from the condition
\[
    \xi(f \ast_h \xi') = f(\xi \ast \xi') + < df, a(\xi) > \xi'
\]
that the chains $\theta_k$ are of order zero, that is to say that each $\theta_k$ is tensorial. This phenomenon is in contrast to the case of star products in the associative and commutative algebra $C^\infty(M, \mathbb{R})$.

To end the present paper we deduce from (13) the following statement.

**Proposition 5.1.** Let $E \to M$ be a Koszul-Vinberg algebroid whose anchor map is injective. Suppose that the associated algebroid $E \oplus \mathbb{R}$ admits a one parameter family of deformations $(E \oplus \mathbb{R}, G_h)$ whose multiplication is

$$\xi \ast_h \xi' = \xi \xi' + \sum_{k>0} h^k \theta_k(\xi, \xi').$$

Then the coefficients $\theta_k$ are symmetric chains of the cochain complex $C(G, G)$.

**References**


