ACTA ARITHMETICA
176.3 (2016)

An explicit hybrid estimate for $L(1/2+it,\chi)$

by

Ghaith A. Hiary (Columbus, OH)

1. Introduction. Let χ be a Dirichlet character modulo q, and let $L(1/2+it,\chi)$ be the corresponding Dirichlet L-function on the critical line. Let $\tau(q)$ be the number of divisors of q. If $|t|\ge 3$, say, we define the analytic conductor of $L(1/2+it,\chi)$ to be $\mathfrak{q} := q|t|$.

We are interested in finding an explicit hybrid estimate for $L(1/2+it,\chi)$ in terms of $\mathfrak{q}$ and $\tau(q)$. Specifically, we would like to find constants c, κ_1, κ_2, κ_3, and $t_0\ge 3$ as small as possible, such that

$$|L(1/2+it,\chi)| \le c\tau(q)^{\kappa_1}\mathfrak{q}^{\kappa_2}\log^{\kappa_3}\mathfrak{q} \qquad (|t|\ge t_0). \tag{1}$$

If $|t|\le t_0$, then estimating $L(1/2+it,\chi)$ reduces, essentially, to bounding pure character sums. Barban, Linnik, and Tshudakov [1] gave Big-O bounds for such sums, as well as some applications.

The convexity bound in our context is $L(1/2+it,\chi)\ll \mathfrak{q}^{1/4}$. This can be derived using the standard method of the approximate functional equation. Habsieger derived such an approximate equation in [4]. And we use this in §7 to prove that if χ is a primitive character (1) modulo $q>1$, then we have the convexity bound

$$|L(1/2+it,\chi)| \le 124.46\mathfrak{q}^{1/4} \qquad (\mathfrak{q}\ge 10^9,\ |t|\ge \sqrt{q}). \tag{2}$$

Previously, Rademacher [11] derived the explicit bound

$$|L(\sigma+it,\chi)| \le \left(\frac{q|1+\sigma+it|}{2\pi}\right)^{(1+\eta-\sigma)/2}\zeta(1+\eta),$$

valid if $0<\eta\le 1/2$, $\sigma\le 1+\eta$, and $\chi \pmod q$ is primitive. This is nearly a convexity bound except for an additional $\eta>0$ in the exponent.

2010 *Mathematics Subject Classification*: Primary 11M06, 11L40.
Key words and phrases: van der Corput method, Weyl method, exponential sums, Dirichlet L-functions, powerfull modulus, hybrid estimates, explicit estimates.
Received 3 February 2016; revised 7 July 2016.
Published online 17 October 2016.

(1) We consider the principal character as neither primitive nor imprimitive.

DOI: 10.4064/aa8433-7-2016

Using partial summation, we obtain an explicit bound applicable for any t. Specifically, if χ is primitive modulo $q > 1$, then we derive in §7 that

$$|L(1/2+it,\chi)| \le 4q^{1/4}\sqrt{(|t|+1)\log q}. \tag{3}$$

The bound (3) is weaker than the convexity bound in general, but it can be useful in the limited region where t is small.

Our main result is Theorem 1.2. This theorem supplies the first example of an explicit hybrid Weyl bound (i.e. with $\kappa_2 = 1/6$ in (1)) for an infinite set of Dirichlet L-functions, namely, the set of Dirichlet L-functions corresponding to powerfull moduli. Theorem 1.2 takes a particularly simple form if q is a sixth power and χ is primitive, yielding Corollary 1.1 below.

Corollary 1.1. *Let χ be a primitive Dirichlet character modulo q. If q is a sixth power, then*

$$|L(1/2+it,\chi)| \le 9.05\tau(q)\mathfrak{q}^{1/6}\log^{3/2}\mathfrak{q} \qquad (|t| \ge 200).$$

In the notation of (1), Corollary 1.1 asserts that if q is a sixth power and χ is primitive, then the choice $c = 9.05$, $\kappa_1 = 1$, $\kappa_2 = 1/6$, $\kappa_3 = 3/2$, and $t_0 = 200$ is admissible. The constant $\kappa_3 = 3/2$ arises from two sources: a dyadic division that contributes 1, and the Weyl differencing method (see [13, §5.4]) which contributes 1/2. The constant $\kappa_1 = 1$ arises, in part, when counting the number of solutions to quadratic congruence equations in the Weyl differencing method. The $\kappa_2 = 1/6$ arises from proving that, on average, square-root cancellation occurs in certain short segments of the dyadic pieces $\sum_{V\le n<2V}\chi(n)/n^{1/2+it}$. The constant $c = 9.05$ is largely contributed by the part of the main sum over $\mathfrak{q}^{1/3} \ll n \ll \mathfrak{q}^{2/3}$. Last, the constant $t_0 = 200$ is due to technical reasons, and can be lowered with some work.

We state the main theorem below. See §2 for the definitions of $\mathrm{sqf}(q)$, $\mathrm{cbf}(q)$, $\mathrm{spf}(q)$, B, B_1, D, and $\Lambda(D)$. For now we remark that if χ is primitive, then $B = B_1 = 1$. And if q is a sixth power, then $\mathrm{sqf}(q) = \mathrm{cbf}(q) = \mathrm{spf}(q) = 1$. The number $\Lambda(D)$ is bounded by $\tau(D)$, and D is usually of size about $q^{1/3}$.

Theorem 1.2. *Let χ be a Dirichlet character modulo q. If $|t| \ge 200$, then*

$$|L(1/2+it,\chi)| \le \mathfrak{q}^{1/6}Z(\log\mathfrak{q}) + W(\log\mathfrak{q})$$

where

$$\begin{aligned}Z(X) :=\ & 6.6668\sqrt{\mathrm{cbf}(q)} - 16.0834\,\mathrm{spf}(q) + 15.6004\,\mathrm{spf}(q)X\\ &+ 1.7364\sqrt{\Lambda(D)\,\mathrm{cbf}(q)(65.5619 - 17.1704X - 2.4781X^2 + 0.6807X^3)}\\ &+ 1.7364\sqrt{\Lambda(D)\,\mathrm{cbf}(q)B\tau(D/B)(-1732.5-817.82X+71.68X^2+47.57X^3)}\end{aligned}$$

and

$$W(X) := -101.152 - 195.696B_1\,\mathrm{sqf}(q) + 19.092X + 94.978B_1\,\mathrm{sqf}(q)X.$$

For many applications, it suffices to focus on the case where χ is primitive. For if not, then letting $\chi_1 \pmod{q_1}$ be the primitive character inducing χ and using the Euler product and analytic continuation of $L(s,\chi)$, we have

$$|L(1/2+it,\chi)| \le |L(1/2+it,\chi_1)| \prod_{\substack{p\mid q\\ p\nmid q_1}} (1+1/\sqrt{p}). \tag{4}$$

Thus, we obtain an explicit bound on $L(1/2+it,\chi)$ by bounding $L(1/2+it,\chi_1)$ and using the inequality (4). In our proof of Theorem 1.2, though, we bound general sums of the form (5), and keep track of the dependence on B and B_1.

The main devices in our proofs are the hybrid van der Corput–Weyl Lemmas 4.1 and 4.2. These lemmas provide explicit bounds for sums of the form

$$\sum_{n=N+1}^{N+L} \chi(n) e^{2\pi i f(n)}, \tag{5}$$

where we take $f(x) = -\frac{t}{2\pi}\log x$ in our application. A pleasant feature of the resulting bounds is that they naturally split into two main terms, one originating from $\chi(n)$ and the other from n^{-iqt}. In particular, we can detect cancellation in the q and t aspects separately, then combine the savings routinely using the well-spacing Lemma 3.1.

The starting point in our proof of Theorem 1.2 is the Dirichlet series

$$L(1/2+it,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{1/2+it}}, \tag{6}$$

valid for χ nonprincipal. (If χ is principal, we use a bound for the Riemann zeta function.) We partition the sum in (6) into four parts: $1 \ll n \ll \mathfrak{q}^{1/3}$ which is bounded trivially, $\mathfrak{q}^{1/3} \ll n \ll \mathfrak{q}^{2/3}$ for which Lemma 4.1 is used, $\mathfrak{q}^{2/3} \ll n \ll \mathfrak{q}$ for which Lemma 4.2 is used, and the tail $\mathfrak{q} \ll n$ which is bounded using the Pólya–Vinogradov inequality.

We remark that the restriction in Corollary 1.1 that q is a sixth power may be relaxed to q being a cube provided that one starts with a main sum of length about $\sqrt{\mathfrak{q}}$ (as in the approximate functional equation) instead of the main sum (6). One then applies van der Corput lemmas analogous to those in [9], but for the twisted sums (5).

Interest in powerfull modulus L-functions has grown recently, both from the theoretical and computational perspectives. Milićević [10] has recently derived sub-Weyl bounds for pure character sums to prime-power modulus. And the present author [7] had derived an algorithm to compute hybrid sums to powerfull modulus in $\mathfrak{q}^{1/3+o(1)}$ time. If q is smooth (but not necessarily powerfull) or prime, then one can obtain explicit hybrid subconvexity bounds by deriving an explicit version of Heath-Brown's $\mathfrak{q}$-analogue of the van der Corput method in [5], and an explicit version of Heath-Brown's hybrid Burgess method in [6].

2. Notation. Let χ be a Dirichlet character modulo q. We factorize the modulus

$$q := p_1^{a_1} \cdots p_\omega^{a_\omega},$$

where the p_j are distinct primes and $a_j \geq 1$. For each prime power p^a, we define

$$C_1(p^a) := p^{\lceil a/2 \rceil}, \qquad D_1(p^a) := p^{a - \lceil a/2 \rceil},$$

then extend the definitions multiplicatively, i.e. $C_1(q) = C_1(p_1^{a_1}) \cdots C_1(p_\omega^{a_\omega})$. In addition, we define

$$C(p^a) := p^{\lceil a/3 \rceil}, \qquad D(p^a) := \begin{cases} 1, & a = 1, \\ p^{a - 2\lceil a/3 \rceil + 1}, & p = 2 \text{ and } a > 1, \\ p^{a - 2\lceil a/3 \rceil}, & p \neq 2 \text{ and } a > 1, \end{cases}$$

then extend the definitions multiplicatively. Since the quantities $C_1(q)$, $D_1(q)$, $C(q)$, and $D(q)$ will appear often, it is useful to introduce the short-hand notation $C_1 := C_1(q)$, $D_1 := D_1(q)$, $C := C(q)$, and $D := D(q)$. For example, $C_1 D_1 = q$.

Some additional arithmetic factors will appear in our estimates: $(m, n) \geq 0$ is the greatest common divisor of m and n, $\omega(m)$ is the number of distinct prime divisors of m, and $\Lambda(m)$ is the number of solutions of the congruence $x^2 \equiv 1 \pmod{m}$ with $0 \leq x < m$. Explicitly,

$$\Lambda(m) = \begin{cases} 2^{\omega(m)-1}, & m \equiv 2 \pmod{4}, \\ 2^{\omega(m)}, & m \not\equiv 2 \pmod{4} \text{ and } m \not\equiv 0 \pmod{8}, \\ 2^{\omega(m)+1}, & m \equiv 0 \pmod{8}. \end{cases}$$

We define $\Lambda := \Lambda(D)$, and

$$\operatorname{sqf}(p^a) := p^{\lceil a/2 \rceil - a/2}, \qquad \operatorname{cbf}(p^a) := p^{\lceil a/3 \rceil - a/3},$$
$$\operatorname{spf}(p^a) := \frac{p^{\lceil a/2 \rceil - \lceil a/3 \rceil/2 - a/6}}{\sqrt{D(p^a)}},$$

then extend the definitions multiplicatively. Note that $\operatorname{sqf}(q)$ is determined by the primes $p_j \mid q$ such that $a_j \not\equiv 0 \pmod{2}$, and $\operatorname{cbf}(q)$ by the primes p_j such that $a_j \not\equiv 0 \pmod{3}$. If q is a square, then $\operatorname{sqf}(q) = 1$. If q is a cube, then $\operatorname{cbf}(q) = 1$. And if q is a sixth power, then $\operatorname{sqf}(q) = \operatorname{cbf}(q) = 1$ and $\operatorname{spf}(q) \leq 1$.

The numbers B and B_1 that appear in Theorem 1.2 are defined in Lemma 3.3.

In the remainder of the paper, we use the following notation: $\exp(x) = e^x$ is the usual exponential function, $[x]$ is the closest integer to x, $\|x\|$ is the distance to the closest integer to x, $\bar{\ell} \pmod C$ is the modular inverse of $\ell \pmod C$ if it exists, and

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

3. Preliminary lemmas

Lemma 3.1. *Let $\{y_r : r = 0, 1, \ldots\}$ be a set of real numbers. Suppose that there is a number $\delta > 0$ such that $\min_{r \neq r'} |y_r - y_{r'}| \geq \delta$. If $P \geq 2$ and $y \geq x$ then*

$$\sum_{y_r \in [x,y]} \min(P, \|y_r\|^{-1}) \leq 2(y - x + 1)\bigl(2P + \delta^{-1} \log(eP/2)\bigr). \tag{7}$$

If $P < 2$, then replace the r.h.s. by $2(y - x + 1)(P + \delta^{-1})$.

Proof. We may assume that $\delta \leq 1/2$, otherwise the bounds follow on trivially estimating the number of points y_r in $[x, y]$ by $2(y - x) + 1$ and using the trivial bound $\min(P, \|y_r\|^{-1}) \leq P$.

For each integer $k \in [x, y]$, we consider the interval $[k - 1/2, k + 1/2]$. There are at most two points y_r in $[k - \delta, k + \delta)$, say $y_k^+ \in [k, k + \delta)$ and $y_k^- \in [k - \delta, k)$. If no such points exist, then we insert one (or both) of them subject to the condition $|y_k^+ - y_k^-| \geq \delta$. To preserve the δ-spacing condition, we slide the remaining points in $(y_k^+, k + 1/2]$ (resp. $[k - 1/2, y_k^-)$) to the right of y_k^+ (resp. left of y_k^-) in the obvious way. It is possible that a point falls off each edge, in which case we may discard it. This is permissible since the overall procedure that we described can only increase the magnitude of the sum in (7).

We have $y_k^+ = k + \rho_k \delta$ for some $\rho_k \in [0, 1)$, and so $y_k^- \leq k + (\rho_k - 1)\delta$. Hence, using the inequality

$$\min(P, \|y_r\|^{-1}) + \min(P, \|y_{r'}\|^{-1}) \leq \min(2P, \|y_r\|^{-1} + \|y_{r'}\|^{-1}),$$

and the formula $\|y_r\| = |y_r - k|$ if $|y_r - k| \leq 1/2$, we obtain

$$\sum_{|y_r - k| \leq 1/2} \min(P, \|y_r\|^{-1}) \leq \sum_{0 \leq r \leq 1/(2\delta)} \min\biggl(2P, \frac{1}{\delta(r + \rho_k)} + \frac{1}{\delta(r + 1 - \rho_k)}\biggr).$$

We observe that

$$\frac{1}{\delta(r + \rho_k)} + \frac{1}{\delta(r + 1 - \rho_k)} = \frac{1}{\delta} \frac{2r + 1}{r^2 + r + \rho_k - \rho_k^2} \leq \frac{2}{\delta r}.$$

Combining this with the observation $\frac{2}{r\delta} \ge 2P$ if $r \le \frac{1}{\delta P}$, we conclude that

$$(8)\qquad \sum_{|y_r-k|\le 1/2} \min(P, \|y_r\|^{-1}) \le 2P\left\lceil \frac{1}{\delta P}\right\rceil + \sum_{\lceil 1/(\delta P)\rceil \le r\le 1/(2\delta)} \frac{2}{r\delta}.$$

To bound the sum over r, we isolate the first term and estimate the remainder by an integral. If $P \ge 2$ (so that the integral below makes sense), then this gives the bound

$$\sum_{\lceil 1/(\delta P)\rceil \le r\le 1/(2\delta)} \frac{2}{r\delta} \le 2P + 2\delta^{-1} \int_{1/(\delta P)}^{1/(2\delta)} \frac{1}{x}\,dx.$$

The integral evaluates to $\log(P/2)$. Therefore, the r.h.s. in (8) is bounded by $2\delta^{-1} + 2P + 2P + 2\delta^{-1}\log(P/2)$. So the lemma follows if $P \ge 2$ as the cardinality of $\{k : x \le k \le y\}$ is $\le y - x + 1$. Finally, if $P < 2$, then the sum on the r.h.s. in (8) is empty, and so the bound is $2\delta^{-1} + 2P$. ∎

Lemma 3.2. *Let f be an analytic function on a disk of radius $\lambda(L-1)$ centered at $N+1$, where $\lambda > 1$ and $1 \le L \in \mathbb{Z}$. If there is a number η and an integer $J \ge 0$ such that $|f^{(j)}(N+1)|\lambda^j(L-1)^j/j! \le \eta/\lambda^j$ for $j > J$, then*

$$\Big|\sum_{n=N+1}^{N+L} \chi(n)e^{2\pi i f(n)}\Big| \le \nu_J(\lambda,\eta) \max_{0\le\Delta<L}\Big|\sum_{n=N+1+\Delta}^{N+L} \chi(n)e^{2\pi i P_J(n-N-1)}\Big|,$$

where

$$P_J(x) := \sum_{j=0}^{J} \frac{f^{(j)}(N+1)x^j}{j!},$$

$$\nu_J(\lambda,\eta) := \left(1 + \frac{\lambda^{-J}}{\lambda-1}\right)\exp\left(\frac{2\pi\eta\lambda^{-J}}{\lambda-1}\right).$$

Proof. If $L = 1$, the lemma is trivial. So assume that $L > 1$. We apply the Taylor expansion to obtain

$$f(N+1+z) = P_J(z) + \sum_{j>J} \frac{f^{(j)}(N+1)}{j!} z^j \qquad (|z| \le \lambda(L-1)).$$

Using the Taylor expansion once more, we find that

$$e^{2\pi i(f(N+1+z)-P_J(z))} = \sum_{j=0}^{\infty} c_j(J,N)z^j \qquad (|z| \le \lambda(L-1)).$$

So if we define $\nu_J^* := \sum_{j=0}^{\infty} |c_j(J,N)(L-1)^j|$, then partial summation gives

$$\Big|\sum_{n=N+1}^{N+L} \chi(n)e^{2\pi i f(n)}\Big| \le \nu_J^* \max_{0\le\Delta<L}\Big|\sum_{n=N+1+\Delta}^{N+L} \chi(n)e^{2\pi i P_J(n-N-1)}\Big|.$$

To estimate the coefficients $c_j(J,N)$, we use the Cauchy theorem applied with a circle of radius $\lambda(L-1)$ around the origin. In view of the growth condition on the derivatives of f, this yields

$$|c_j(J,N)| \le \frac{1}{2\pi}\left|\oint \frac{e^{2\pi i(f(N+1+z)-P_J(z))}}{z^{j+1}}\,dz\right| \le \frac{\exp\left(\frac{2\pi\eta\lambda^{-J}}{\lambda-1}\right)}{\lambda^j(L-1)^j}.$$

Noting that $c_j(J,N)=0$ for $1\le j\le J$, we therefore deduce that

$$\nu_J^* \le \exp\left(\frac{2\pi\eta\lambda^{-J}}{\lambda-1}\right)\left[1+\sum_{j>J}\frac{(L-1)^j}{\lambda^j(L-1)^j}\right] = \nu_J(\lambda,\eta). \ \blacksquare$$

LEMMA 3.3. *There exists an integer $\tilde L$ such that $\chi(1+C_1x)=e^{2\pi i\tilde L x/D_1}$ for all $x\in\mathbb{Z}$. If χ is primitive, then $B_1:=(\tilde L,D_1)=1$. Furthermore, there exist integers L_0 and L such that $\chi(1+Cx)=e^{4\pi i L_0x/(CD)+2\pi i Lx^2/D}$ for all $x\in\mathbb{Z}$. If χ is primitive, then L can be chosen so that $B:=(L,D)=1$.*

Proof. We start with the decomposition $\chi=\chi_1\cdots\chi_\omega$, where χ_j is a Dirichlet character modulo $p_j^{a_j}$. By [8, Lemma 3.4], there exists an integer $\tilde L_j$ such that

$$\chi_j(1+C_1(p_j^{a_j})x)=\exp(2\pi i\tilde L_j x/D_1(p_j^{a_j}))$$

for all $x\in\mathbb{Z}$. Hence,

$$\chi(1+C_1x)=\chi_1(1+C_1x)\cdots\chi_\omega(1+C_1x)=e^{2\pi i\tilde L x/D_1},$$

where, because of $C_1D_1=q$, we have

$$\tilde L = q\sum_{j=1}^{\omega}\tilde L_j/p_j^{a_j}.$$

Let $B_1=(\tilde L,D_1)$. It is clear that $\chi(1+qx/B_1)=1$ for all x. So q/B_1 is an induced modulus for χ. In particular, if $B>1$ then χ is imprimitive. This completes the proof of the first part of the lemma.

For the second part, we use [7, Lemma 4.2]. Consider first the case $p_j^{a_j}\notin\{4,8\}$ and $a_j>1$. Then there are integers $L_{0,j}$ and L_j such that

$$\chi_j(1+C(p_j^{a_j})x)=\exp\left(\frac{4\pi i L_{0,j}x}{C(p_j^{a_j})D(p_j^{a_j})}+\frac{2\pi i L_jx^2}{D(p_j^{a_j})}\right) \tag{9}$$

for all $x\in\mathbb{Z}$, and moreover we can take $L_{0,j}=-L_j$. If $a_j=1$, then $C(p_j^{a^j})=p_j^{a_j}$. So $\chi_j(1+C(p_j^{a_j})x)=1$ and we can take $L_{0,j}=L_j=0$. If $p_j^{a_j}=4$, then either $L_{0,j}=0$ and $L_j=1$, or χ is principal. If $p_j^{a_j}=8$, then either $L_{0,j}=L_j=1$, or $L_{0,j}=2$ and $L_j=0$ (an imprimitive character), or $L_{0,j}=-1$ and $L_j=1$, or χ is principal. Put together, this gives

$$\chi(1+Cx)=\chi_1(1+Cx)\cdots\chi_\omega(1+Cx)=\exp\left(\frac{4\pi i L_0x}{CD}+\frac{2\pi i Lx^2}{D}\right)$$

where

$$L_0 = C^2D\sum_{j=1}^{\omega}\frac{L_{0,j}}{C(p_j^{a_j})^2D(p_j^{a_j})},$$

$$L = C^2D\sum_{j=1}^{\omega}\frac{L_j}{C(p_j^{a_j})^2D(p_j^{a_j})}.$$

It remains to prove that if χ is primitive then $B=1$. To this end, we note that $Lq^2/(B^2C^2D)$ is an integer. So if we show that $2L_0q/(BC^2D)$ is an integer too, then $\chi(1+qx/B)=1$ for all $x\in\mathbb{Z}$. In particular, if $B>1$, then q/B is a nontrivial induced modulus and χ is imprimitive, which completes the proof of the second part of the lemma.

Now, to show that $2L_0q/(BC^2D)$ is an integer, we first note that L_0q/C^2 is always an integer. (Recall that $L_0=0$ if $a_j=1$.) Furthermore, if $a_j=1$ then $(D,p_j)=1$ and so $(B,p_j)=1$. In light of this, we may assume that $a_j>1$ for all j.

We consider two possibilities. If $p_j^{a_j}\notin\{4,8\}$ for any j, then $C^2D=q$ (if q is odd) or $2q$ (if q is even), and in any case $L_{0,j}=-L_j$ for all j. The last fact implies in turn that $L_0=-L$, hence $B=(L_0,D)$. In particular, B divides L_0 and we conclude that $2L_0q/(BC^2D)=L_0/B$ or $2L_0/B$, and so this is an integer in either case.

On the other hand, if $p_j^{a_j}\in\{4,8\}$ for some j, then $C^2D=2q$ and we appeal to the remark following (9). Accordingly, if χ is primitive and $p_j^{a_j}\in\{4,8\}$ then $L_j=1$ and so L must be odd. This shows that $B=(L,D/2)$. In addition, we have

$$L_0 = L - \begin{cases}\dfrac{C^2(L_j-L_{0,j})}{4}\,\dfrac{D}{2}, & p_1^{a_1}=4,\\[2ex] \dfrac{C^2(L_j-L_{0,j})}{8}\,\dfrac{D}{2}, & p_1^{a_1}=8.\end{cases}$$

Therefore, given the possibilities for $L_{0,j}$ and L_j stated after (9), we see that if χ is primitive then $L_0\equiv L \pmod{D/2}$, and so $B=(L_0,D/2)$. This shows that B is a divisor of L_0, hence $2L_0q/(BC^2D)=L_0/B$ is an integer. ∎

LEMMA 3.4. *Let* $M,N\in\mathbb{Z}_{\ge 1}$, $W_M(m):=1-m/M$, *and* $d_m(N):=(2m,N)$. *Then*

$$\sum_{m=1}^{M}W_M(m)\frac{d_m(N)}{m}\le\tau(N)\log M, \tag{10}$$

$$\sum_{m=1}^{M}W_M(m)d_m(N)\le\tau(N)M.$$

Proof. We prove the first bound, the second one being analogous. Let us write $N = 2^a N'$ with N' odd. We induct on a. If $a = 0$, then $d_m(N) = (m, N)$ and the result follows because

$$\sum_{m=1}^{M} W_M(m)\frac{d_m(N)}{m} \le \sum_{\substack{r|N\\ r\le 2M}} \sum_{1\le m'\le M/r} W_M(rm')\frac{1}{m'}$$
$$\le \tau(N) \sum_{1\le m'\le M} W_M(m')\frac{1}{m'} \le \tau(N)\log M.$$

If $a = 1$, then $d_m(N) = 2d_m(N')$. So using the previous calculation and observing that $2\tau(N') = \tau(N)$ yields the desired bound.

Henceforth, we assume that $a \ge 2$. We may further assume that $M > 2$, for if $M = 1$ or 2 then the lemma is trivial.

Since N is even by hypothesis, we have $d_m(N) = 2(m, N/2)$. Using this, and dividing the sum over m into even and odd terms, we obtain

$$\sum_{m=1}^{M} W_M(m)\frac{d_m(N)}{m} = 2 \sum_{1\le m'\le \lfloor M/2\rfloor} W_M(2m')\frac{(2m', N/2)}{2m'} \tag{11}$$
$$+ 2 \sum_{0\le m'\le \lfloor (M-1)/2\rfloor} W_M(2m'+1)\frac{(2m'+1, N/2)}{2m'+1}.$$

We have $W_M(2m') \le W_{\lceil M/2\rceil}(m')$ and, by definition, $(2m', N/2) = d_{m'}(N/2)$. It follows by induction that the first sum on the r.h.s. of (11) is bounded by $\tau(N/2)\log\lceil M/2\rceil$. Furthermore, the second sum is clearly bounded by

$$\sum_{0\le m'\le \lfloor (M-1)/2\rfloor} W_M(2m'+1)\frac{(2m'+1, N')}{m'+1/2}$$
$$\le 2(1-1/M) + \sum_{1\le m\le M} W_M(m)\frac{d_m(N')}{m},$$

which, by induction, is $\le 2(1-1/M) + \tau(N')\log M$. Therefore, using the bound $\log\lceil M/2\rceil \le \log M + 1/M - \log 2$ and the formula $\tau(N/2) + \tau(N') = \tau(N)$, we arrive at

$$\sum_{m=1}^{M} W_M(m)\frac{d_m(N)}{m}$$
$$\le \tau(N)\log M + \big(2 - 2/M + \tau(N/2)/M - \tau(N/2)\log 2\big).$$

We conclude that the bound (10) holds provided that $\tau(N/2) \ge 4$. This is always fulfilled if $a \ge 2$ unless $N = 4$ or 8. But the lemma follows in these cases also by direct calculation. ∎

4. Hybrid van der Corput–Weyl lemmas

Lemma 4.1. *Suppose that f is a function satisfying the hypothesis of Lemma 3.2 for some $\lambda > 1$, $\eta \geq 0$, and with $J = 1$. If $f(x)$ is real for real x, then*

$$\Big|\sum_{n=N+1}^{N+L} \chi(n)e^{2\pi i f(n)}\Big| \leq \frac{2\nu_1(\lambda,\eta)C_1}{\pi}\left(\log\frac{D_1}{2B_1} + \frac{7}{4} + \frac{\pi}{2}\right) + \frac{\nu_1(\lambda,\eta)C_1}{\pi}\min\left(\frac{\pi B_1 L}{q}, \|qf'(N+1)/B_1\|^{-1}\right).$$

Proof. Applying Lemma 3.2 with $J = 1$ gives

$$\Big|\sum_{n=N+1}^{N+L} \chi(n)e^{2\pi i f(n)}\Big| \leq \nu_1(\lambda,\eta)\max_{0\leq \Delta < L}\Big|\sum_{n=N+1+\Delta}^{N+L} \chi(n)e^{2\pi i P_1(n-N-1)}\Big|, \tag{12}$$

where $P_1(x) = f(N+1) + f'(N+1)x$. Let Δ^* be where the maximum is achieved on the r.h.s. of (12). Let $N^* := N + \Delta^*$ and $L^* = L - \Delta^*$. So we have

$$\Big|\sum_{n=N+1}^{N+L} \chi(n)e^{2\pi i f(n)}\Big| \leq \nu_1(\lambda,\eta)\Big|\sum_{n=N^*+1}^{N^*+L^*} \chi(n)e^{2\pi i P_1(n-N-1)}\Big|. \tag{13}$$

We split the range of summation $N^* + 1 \leq n \leq N^* + L^*$ into arithmetic progressions along the residue classes $\ell \pmod{C_1}$. For each residue class $0 \leq \ell < C_1$, the terms in the progression $n \equiv \ell \pmod{C_1}$ are indexed by the integers k that verify $N^* + 1 \leq \ell + C_1 k \leq N^* + L^*$. So we have

$$\lceil (N^* + 1 - \ell)/C_1\rceil \leq k \leq \lfloor (N^* + L^* - \ell)/C_1\rfloor.$$

Using the formula $\lceil x + \delta\rceil - \lfloor x\rfloor = 1$, valid for any x and $\delta \in (0,1)$, we deduce that $\lceil (N^* + 1 - \ell)/C_1\rceil - \lfloor (N^* - \ell)/C_1\rfloor = 1$. Therefore, if we define $H_\ell := \lfloor (N^* - \ell)/C_1\rfloor$, then each ℓ determines an integer $\Omega_\ell \leq \lceil L^*/C_1\rceil$ such that (we use the triangle inequality below)

$$\Big|\sum_{n=N^*+1}^{N^*+L^*} \chi(n)e^{2\pi i P_1(n-N-1)}\Big| \leq \sum_{\ell=0}^{C_1-1}\Big|\sum_{k=H_\ell+1}^{H_\ell+\Omega_\ell} \chi(\ell + C_1 k)e^{2\pi i P_1(\ell + C_1 k - N - 1)}\Big|.$$

From Lemma 3.3, and the formula $\chi(\ell + C_1 k) = \chi(\ell)\chi(1 + C_1\bar{\ell}k)$, valid for $(\ell, q) = 1$, we deduce that there are integers γ_1 and B_1 such that $(\gamma_1, D_1) = 1$, $B_1 \mid D_1$, and

$$\chi(\ell + C_1 k) = \chi(\ell)e^{2\pi i B_1\gamma_1\bar{\ell}k/D_1}, \quad (\ell, q) = 1.$$

If $(\ell,q)>1$, then $\chi(\ell+C_1k)=0$. Therefore,

$$(14)\qquad \Big|\sum_{n=N^*+1}^{N^*+L^*}\chi(n)e^{2\pi iP_1(n-N-1)}\Big| \le \sum_{\substack{\ell=0\\(\ell,q)=1}}^{C_1-1}\Big|\sum_{k=H_\ell+1}^{H_\ell+\Omega_\ell} e^{2\pi i(B_1\gamma_1\bar{\ell}/D_1+C_1f'(N+1))k}\Big|.$$

Let us define

$$z_f := \left[\frac{qf'(N+1)}{B_1}\right],\qquad \delta_f := \pm\left\|\frac{qf'(N+1)}{B_1}\right\|,$$

where δ_f is positive if z_f is obtained by rounding down, and negative if z_f is obtained by rounding up. In either case, since $D_1C_1=q$ by construction, we have $C_1f'(N+1)=(z_f+\delta_f)B_1/D_1$. Therefore,

$$(15)\qquad \left\|\frac{B_1\gamma_1\bar{\ell}}{D_1}+C_1f'(N+1)\right\| = \left\|\frac{\gamma_1\bar{\ell}+z_f+\delta_f}{D_1/B_1}\right\| =: U_{\gamma_1\bar{\ell}+z_f+\delta_f}.$$

In view of this, it follows by the Kuz'min–Landau Lemma [2, Lemma 2] that the inner sum in (14) satisfies

$$\Big|\sum_{k=H_\ell+1}^{H_\ell+\Omega_\ell} e^{2\pi i(B_1\gamma_1\bar{\ell}/D_1+C_1f'(N+1))k}\Big| \le \min\Big(\Omega_\ell, \frac{1}{\pi}U^{-1}_{\gamma_1\bar{\ell}+z_f+\delta_f}+1\Big).$$

Given this, we divide the sum over ℓ in (14) into segments of length D_1/B_1.

$$\Big[\frac{uD_1}{B_1},\frac{(u+1)D_1}{B_1}\Big),\qquad u\in\mathbb{Z},\ 0\le u<\frac{B_1C_1}{D_1}.$$

Over each segment, we can get an easy handle on $U_{\gamma_1\bar{\ell}+z_f+\delta_f}$. Indeed, as ℓ runs over the reduced residue classes modulo q (hence reduced modulo D_1/B_1) in a given segment, $\gamma_1\bar{\ell}+z_f$ runs over a subset of the residue classes modulo D_1/B_1, hitting each class at most once. Therefore, summing over the B_1C_1/D_1 segments, and recalling that $\Omega_\ell\le\lceil L/C_1\rceil$ by construction, we obtain

$$(16)\qquad \Big|\sum_{n=N^*+1}^{N^*+L^*}\chi(n)e^{2\pi iP_1(n-N-1)}\Big| \le \frac{B_1C_1}{D_1}\sum_{\ell\,(\mathrm{mod}\,D_1/B_1)}\min\Big(\lceil L/C_1\rceil, \frac{1}{\pi}U^{-1}_{\ell+\delta_f}+1\Big).$$

We choose the residue class representatives modulo D_1/B_1 to be in $[-D_1/2B_1, D_1/2B_1)$ if $\delta_f\ge 0$, and in $(-D_1/2B_1, D_1/2B_1]$ if $\delta_f<0$. In either case, let $\tilde{\ell}$ denote the representative of ℓ. Since $0\le|\delta_f|\le 1/2$, we deduce the formula

$$U_{\ell+\delta_f} = \begin{cases} \dfrac{|\tilde{\ell}| + \operatorname{sgn}(\tilde{\ell})\delta_f}{D_1/B_1}, & \tilde{\ell} \neq 0, \\ \dfrac{|\delta_f|}{D_1/B_1}, & \tilde{\ell} = 0. \end{cases}$$

Now, if $\delta_f \geq 0$, we isolate the terms corresponding to $\tilde{\ell} = 0$ and $\tilde{\ell} = -1$ (if they exist) on the r.h.s. of (16). And if $\delta_f < 0$, we isolate the terms for $\tilde{\ell} = 0$ and $\tilde{\ell} = 1$. Moreover, we use the lower bound $U_{\pm 1+\delta_f} \geq B_1/(2D_1)$ to control the term $\tilde{\ell} = \pm 1$. Then we sum over the remaining $\tilde{\ell}$, pairing the terms for $\tilde{\ell}$ and $-\tilde{\ell} - 1$ if $\delta_f \geq 0$, and the terms for $\tilde{\ell} + 1$ and $-\tilde{\ell}$ if $\delta_f < 0$. In summary, assuming that $D_1/B_1 \geq 2$ (so there are at least two residue classes modulo D_1/B_1), we obtain

$$(17) \quad \sum_{\ell \,(\mathrm{mod}\, D_1/B_1)} \min\left(\lceil L/C_1 \rceil, \frac{1}{\pi} U_{\ell+\delta_f}^{-1} + 1\right) \leq \min\left(\lceil L/C_1 \rceil, \frac{1}{\pi} U_{\delta_f}^{-1} + 1\right) + \left(\frac{2D_1}{\pi B_1} + 1\right) + \left(\frac{D_1}{B_1} - 2\right) + \frac{D_1}{\pi B_1} \sum_{1 \leq \ell < D_1/(2B_1)} \left(\frac{1}{\ell + |\delta_f|} + \frac{1}{\ell + 1 - |\delta_f|}\right).$$

The second sum over ℓ on the r.h.s. of (17) is bounded by

$$(18) \quad \sum_{1 \leq \ell < D_1/(2B_1)} \frac{2\ell + 1}{\ell^2 + \ell + |\delta_f| - \delta_f^2} \leq \frac{3}{2} + \sum_{2 \leq \ell < D_1/(2B_1)} \frac{2}{\ell} \leq \frac{3}{2} + 2 \log \frac{D_1}{2B_1}.$$

It is easy to check that the last two estimates still hold if $D_1/B_1 = 1$. Hence, substituting (18) into (17) we obtain, on noting that $\lceil L/C_1 \rceil \leq L/C_1 + 1$,

$$\sum_{\ell \,(\mathrm{mod}\, D_1/B_1)} \min\left(\lceil L/C_1 \rceil, \frac{1}{\pi} U_{\ell+\delta_f}^{-1} + 1\right) \leq \min\left(\frac{L}{C_1}, \frac{1}{\pi} U_{\delta_f}^{-1}\right) + \frac{2D_1}{\pi B_1}\left(1 + \frac{\pi}{2} + \frac{3}{4}\right) + \frac{2D_1}{\pi B_1} \log \frac{D_1}{2B_1}.$$

We multiply the last estimate by the outer factor B_1C_1/D_1 in (16). This gives

$$(19) \quad \Big|\sum_{n=N^*+1}^{N^*+L^*} \chi(n) e^{2\pi i P_1(n-N-1)}\Big| \leq \frac{B_1C_1}{D_1}\left(\min\left(\frac{L}{C_1}, \frac{1}{\pi} U_{\delta_f}^{-1}\right) + \frac{2C_1}{\pi}\left(\log \frac{D_1}{2B_1} + \frac{7}{4} + \frac{\pi}{2}\right)\right).$$

Finally, we use the formula $U_{\delta_f}^{-1} = \|qf'(N+1)/B_1\|^{-1} D_1/B_1$, and substitute (19) back into (13). After straightforward rearrangements, we obtain the lemma. ∎

LEMMA 4.2. *Suppose that f is a function satisfying the hypothesis of Lemma 3.2 for some $\lambda > 1$, $\eta \geq 0$, and with $J = 2$. Let $d_m := (2m, D/B)$. If $f(x)$ is real for real x, then*

$$(20)\qquad \Big|\sum_{n=N+1}^{N+L} \chi(n)e^{2\pi i f(n)}\Big|^2 \leq \frac{4\nu_2(\lambda,\eta)^2 \Lambda C L}{\pi}\left(\log\frac{D}{2B}+\frac{7}{4}+\frac{3\pi}{2\Lambda}\right) + \frac{4\nu_2(\lambda,\eta)^2\Lambda C^2}{\pi}\sum_{m=1}^{\lceil L/C\rceil}\left(1-\frac{m}{\lceil L/C\rceil}\right) \times \min\left(\frac{\pi d_m BL}{CD}, \left\|\frac{mC^2Df''(N+1)}{Bd_m}\right\|^{-1}\right).$$

Proof. We apply Lemma 3.2 with $J = 2$ to the sum. This shows (similarly to the beginning of the proof of Lemma 4.1) that

$$(21)\qquad \Big|\sum_{n=N+1}^{N+L} \chi(n)e^{2\pi i f(n)}\Big| \leq \nu_2(\lambda,\eta)\Big|\sum_{n=N^*+1}^{N^*+L^*} \chi(n)e^{2\pi i P_2(n-N-1)}\Big|,$$

where $P_2(x) = f(N+1)+f'(N+1)x+f''(N+1)x^2/2$ and $[N^*+1, N^*+L^*] \subset [N+1, N+L]$. We split the range of summation on the r.h.s. of (21) into arithmetic progressions along the residue classes ℓ modulo C. Letting $K_\ell := \lfloor (N^*-\ell)/C\rfloor$ and $\Delta_\ell := \lfloor (N^*+L^*-\ell)/C\rfloor - K_\ell$, we have

$$(22)\qquad \sum_{n=N^*+1}^{N^*+L^*} \chi(n)e^{2\pi i P_2(n-N-1)} = \sum_{\ell=0}^{C-1}\sum_{k=K_\ell+1}^{K_\ell+\Delta_\ell} \chi(\ell+Ck)e^{2\pi i P_2(\ell+Ck-N-1)}.$$

We make use of the following properties of Δ_ℓ. First, by construction, we have

$$(23)\qquad \sum_{\ell=0}^{C-1} \Delta_\ell = L^* \leq L.$$

Second, using the periodicity of Δ_ℓ as a function of $\ell \pmod C$, and the change of variable $r \equiv N^* - \ell \pmod C$, we obtain

$$(24)\qquad \sum_{\ell=0}^{C-1} \sqrt{\Delta_\ell} = \sum_{r=0}^{C-1} \sqrt{\lfloor (L^*+r)/C\rfloor} \leq \sum_{r=0}^{C-1} \sqrt{\lfloor (L+r)/C\rfloor}.$$

Furthermore, supposing that $L \equiv \ell_0 \pmod C$, where $0 \leq \ell_0 < C$, on considering the summation ranges $0 \leq r \leq C-\ell_0-1$ and $C-\ell_0 \leq r \leq C-1$ in (24) separately, we obtain

$$\sum_{r=0}^{C-1} \sqrt{\lfloor (L+r)/C\rfloor} = (C-\ell_0)\sqrt{(L-\ell_0)/C} + \ell_0\sqrt{(L-\ell_0)/C+1}.$$

If we view the r.h.s. above as a function of $0 \le \ell_0 < C$, say $p(\ell_0)$, then its maximum is achieved when $\ell_0 = 0$. Thus,

$$\sum_{\ell=0}^{C-1} \sqrt{\Delta_\ell} \le p(0) = \sqrt{CL}.$$

Also, we have the bound

$$\sum_{\ell=0}^{C-1} \Delta_\ell^2 \le \frac{L^2}{C} + (\tilde{\rho} - \tilde{\rho}^2)C, \qquad \tilde{\rho} := \ell_0/C. \tag{25}$$

We are now ready to return to (22). Lemma 3.3 asserts that there is a polynomial $g_\ell(x)$ of degree 2 in x such that

$$\chi(\ell + Ck) = \chi(\ell) e^{2\pi i g_\ell(k)}, \qquad (\ell, q) = 1,$$

where $g_\ell(x) = \alpha_\ell x + B\gamma\bar{\ell}^2 x^2/D$, $(\gamma, q) = 1$, and $B \mid D$. Therefore, applying the Cauchy–Schwarz inequality to the r.h.s. in (22), we obtain

$$\Big|\sum_{n=N^*+1}^{N^*+L^*} \chi(n) e^{2\pi i P_2(n-N-1)}\Big|^2 \le C \sum_{\substack{\ell=0 \\ (\ell,q)=1}}^{C-1} \Big|\sum_{k=K_\ell+1}^{K_\ell+\Delta_\ell} e^{2\pi i Q_\ell(k)}\Big|^2, \tag{26}$$

where $Q_\ell(x) := g_\ell(x) + P_2(\ell + Cx - N - 1)$. We bound the inner sum using the van der Corput–Weyl Lemma [2, Lemma 5]. In fact, we use the more precise form of the lemma at the bottom of page 1273 in [2]. This form implies that if M is a positive integer then

$$\Big|\sum_{k=K_\ell+1}^{K_\ell+\Delta_\ell} e^{2\pi i Q_\ell(k)}\Big|^2 \le (\Delta_\ell + M)\bigg(\frac{\Delta_\ell}{M} + \frac{2}{M}\sum_{m=1}^{M}\Big(1 - \frac{m}{M}\Big)|S'_m(\ell)|\bigg), \tag{27}$$

where

$$S'_m(\ell) := \sum_{r=K_\ell+1}^{K_\ell+\Delta_\ell-m} e^{2\pi i (Q_\ell(r+m) - Q_\ell(r))}.$$

Substituting (27) into (26), and using the properties (23) and (25) together with the upper bound $\Delta_\ell \le \lceil L/C \rceil$, we obtain

$$\begin{aligned} &\Big|\sum_{n=N^*+1}^{N^*+L^*} \chi(n) e^{2\pi i P_2(n-N-1)}\Big|^2 \\ &\le CL + \frac{L^2 + \tilde{\rho}(1-\tilde{\rho})C^2}{M} + 2C\Big(1 + \frac{\lceil L/C \rceil}{M}\Big)\sum_{m=1}^{M}\Big(1 - \frac{m}{M}\Big)\sum_{\substack{\ell=0 \\ (\ell,q)=1}}^{C-1} |S'_m(\ell)|. \end{aligned} \tag{28}$$

Since $Q_\ell(x)$ is a quadratic polynomial, we have the simpler expression

$$|S'_m(\ell)| = \Big|\sum_{r=K_\ell+1}^{K_\ell+\Delta_\ell-m} e^{2\pi i(2mB\gamma\bar{\ell}^2/D+mC^2f''(N+1))r}\Big|.$$

We plan to bound $S'_m(\ell)$ using the Kuz'min–Landau Lemma [2, Lemma 2]. With this in mind, recall the definition $d_m = (2m, D/B)$. Let

$$m' =: \frac{2m}{d_m}, \qquad P_m := \frac{D}{Bd_m}.$$

Let (2)

$$w_m := [P_m mC^2 f''(N+1)] = \left[\frac{m'C^2Df''(N+1)}{2B}\right],$$
$$\epsilon_m := \|P_m mC^2 f''(N+1)\| = \pm\left\|\frac{m'C^2Df''(N+1)}{2B}\right\|.$$

Here, ϵ_m is positive if w_m is obtained by rounding down, and negative if w_m is obtained by rounding up. Hence,

$$\left\|\frac{2mB\gamma\bar{\ell}^2}{D} + mC^2f''(N+1)\right\| = \left\|\frac{m'\gamma\bar{\ell}^2 + w_m + \epsilon_m}{P_m}\right\|.$$

With $U_{z,m} := \|z/P_m\|$, the Kuz'min–Landau Lemma furnishes the estimate

$$|S'_m(\ell)| \le \min\left(\Delta_\ell - m, \frac{1}{\pi}U^{-1}_{m'\gamma\bar{\ell}^2+w_m+\epsilon_m,m} + 1\right).$$

Therefore, using the inequality $\Delta_\ell \le \lceil L/C\rceil$ yields

$$\mathcal{S}_m = \sum_{\substack{\ell=0\\(\ell,q)=1}}^{C-1} |S'_m(\ell)| \le \sum_{\substack{\ell=0\\(\ell,q)=1}}^{C-1} \min\left(\lceil L/C\rceil - m, \frac{1}{\pi}U^{-1}_{m'\gamma\bar{\ell}^2+w_m+\epsilon_m,m} + 1\right).$$

To get an explicit expression for $U_{z,m}$, we consider subsums of $\mathcal{S}_m$ over the segments

$$[uP_m, (u+1)P_m), \qquad u \in \mathbb{Z},\ 0 \le u < C/P_m.$$

To this end, let

$$\Lambda_m := \#\{0 \le x < P_m : x^2 \equiv 1 \pmod{P_m}\}.$$

As ℓ runs over the reduced residue classes in each segment, we see that since $(m'\gamma, P_m) = 1$ and $\bar{\ell}$ is squared, if $m'\gamma\bar{\ell}^2 + w_m$ hits a residue class modulo P_m, it does so Λ_m times. Let $\mathcal{R}_m$ denote the classes that are hit. We find that the cardinality of $\mathcal{R}_m$ is $\le P_m/\Lambda_m$. If $\epsilon_m \ge 0$ we take $\mathcal{R}_m \subset [-P_m/2, P_m/2)$, while if $\epsilon_m < 0$ we take $\mathcal{R}_m \subset (-P_m/2, P_m/2]$. Furthermore, given $m'\gamma\bar{\ell}^2 + w_m$,

(2) If each prime factor of q occurs with multiplicity > 1, then $C^2D = q$ if q is odd, and $C^2D = 2q$ if q is even. So the expressions that follow can be simplified in this case.

let $\tilde{\ell} \in \mathcal{R}_m$ denote the class representative that it hits. Then, summing over the C/P_m segments, we obtain

$$\mathcal{S}_m \leq \frac{C\Lambda_m}{P_m} \sum_{\tilde{\ell} \in \mathcal{R}_m} \min\Big(\lceil L/C \rceil - m, \frac{1}{\pi} U^{-1}_{\tilde{\ell}+\epsilon_m, m} + 1\Big),$$

and we have the formula

$$U_{\tilde{\ell}+\epsilon_m, m} = \begin{cases} \dfrac{|\tilde{\ell}| + \operatorname{sgn}(\tilde{\ell})\epsilon_m}{P_m}, & \tilde{\ell} \neq 0, \\ \dfrac{|\epsilon_m|}{P_m}, & \tilde{\ell} = 0. \end{cases}$$

At worst, the classes that are hit concentrate in $[-P_m/2\Lambda_m, P_m/2\Lambda_m]$. If $\epsilon_m \geq 0$, we isolate the terms corresponding to $\tilde{\ell} = 0$ and $\tilde{\ell} = -1$ (if they exist), and pair the remaining terms for $\tilde{\ell}$ and $-\tilde{\ell} - 1$. On the other hand, if $\epsilon_m < 0$, we isolate the terms for $\tilde{\ell} = 0$ and $\tilde{\ell} = 1$, and pair the remaining terms for $\tilde{\ell} + 1$ and $-\tilde{\ell}$. Since $0 \leq |\epsilon_m| \leq 1/2$ and $P_m/\Lambda_m \geq 1$, this gives

$$\begin{aligned} \mathcal{S}_m \leq \frac{C\Lambda_m}{P_m} \min\Big(\lceil L/C \rceil - m, \frac{P_m}{\pi|\epsilon_m|} + 1\Big) + \frac{C\Lambda_m}{P_m}\Big(\frac{2P_m}{\pi} + 1\Big) \\ + \frac{C\Lambda_m}{P_m}\Big(\frac{P_m}{\Lambda_m} - 1\Big) + \frac{C\Lambda_m}{\pi} \sum_{1 \leq \ell < P_m/(2\Lambda_m)} \Big(\frac{1}{\ell + |\epsilon_m|} + \frac{1}{\ell + 1 - |\epsilon_m|}\Big). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\sum_{1 \leq \ell < P_m/(2\Lambda_m)} \Big(\frac{1}{\ell + |\epsilon_m|} + \frac{1}{\ell + 1 - |\epsilon_m|}\Big) \\ &\qquad \leq \sum_{1 \leq \ell < P_m/(2\Lambda_m)} \frac{2\ell + 1}{\ell^2 + \ell} \leq \frac{3}{2} + 2\log\frac{P_m}{2\Lambda_m}. \end{aligned}$$

Hence, using the trivial inequalities $\lceil L/C \rceil < L/C + 1$ and $1 \leq \Lambda_m \leq P_m$, together with the observation $P_m \,|\, D$ so that $\Lambda_m = \Lambda(P_m) \leq \Lambda(D) = \Lambda$, we obtain

$$\mathcal{S}_m \leq \frac{C\Lambda_m}{P_m} \min\Big(L/C - m, \frac{P_m}{\pi|\epsilon_m|}\Big) + \frac{2C\Lambda}{\pi} + 2C + \frac{C\Lambda}{\pi}\Big(\frac{3}{2} + 2\log\frac{D}{2B}\Big).$$

Now, we have $\sum_{1 \leq m \leq M}(1 - m/M) = (M-1)/2$. So summing over m we arrive at

$$\begin{aligned} (29) \qquad \sum_{m=1}^{M}\Big(1 - \frac{m}{M}\Big)\mathcal{S}_m \leq C \sum_{m=1}^{M}\Big(1 - \frac{m}{M}\Big)\frac{\Lambda_m}{P_m} \min\Big(L/C - m, \frac{P_m}{\pi|\epsilon_m|}\Big) \\ + C(M-1) + \frac{C\Lambda(M-1)}{\pi}\Big(\frac{7}{4} + \log\frac{D}{2B}\Big). \end{aligned}$$

In (29), we choose $M = \lceil L/C \rceil$, so that $M = L/C + 1 - \tilde{\rho}$. Then we substitute the resulting expression into (28), which gives

$$\Big| \sum_{n=N^*+1}^{N^*+L^*} \chi(n) e^{2\pi i P_2(n-N-1)} \Big|^2 \le CL + \frac{L^2 + \tilde{\rho}(1-\tilde{\rho})C^2}{L/C + 1 - \tilde{\rho}} + 4C^2 \sum_{m=1}^{\lceil L/C \rceil} \left(1 - \frac{m}{\lceil L/C \rceil}\right) \frac{\Lambda_m}{P_m} \min\left(L/C - m, \frac{P_m}{\pi |\epsilon_m|}\right) + 4CL + \frac{4\Lambda CL}{\pi}\left(\frac{7}{4} + \log\frac{D}{2B}\right). \tag{30}$$

At this point, we may assume that $L \ge C$, otherwise the lemma is trivial due to the first term in (20). Given this assumption, it is easy to check that the second term in (30), viewed as a function of $\tilde{\rho}$, has no critical points in the interval $[0,1)$, and so it is monotonic over that interval. Comparing the values at $\tilde{\rho} = 0$ and $\tilde{\rho} = 1$, we deduce that the maximum is at $\tilde{\rho} = 1$. Using this in (30) and substituting the result into (21) (after squaring both sides there) yields the lemma. ■

5. Proof of Theorem 1.2. If $\chi = \chi_0$ is the principal character, then

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} (1 - p^{-s}).$$

Bounding the product above trivially, we obtain

$$|L(1/2+it, \chi_0)| \le |\zeta(1/2+it)| \prod_{p|q} (1 + 1/\sqrt{p}) \le |\zeta(1/2+it)| \tau(q).$$

(Note that this is a large overestimate, but it is still fine since the difficult part of the proof is $\chi \ne \chi_0$.) Combining this with the bound for the Riemann zeta function in [9], we arrive at

$$|L(1/2+it, \chi_0)| \le 0.63 \tau(q) \mathfrak{q}^{1/6} \log \mathfrak{q} \qquad (|t| \ge 3). \tag{31}$$

So the theorem follows in this case. Henceforth, we assume that χ is non-principal, and so $q > 2$.

Let $\rho = 1.3$, which is a parameter that will control the size of the segments in our dyadic subdivision. The starting point of the dyadic subdivision is

$$v_0 = \left\lceil \frac{C|t|^{1/3}}{(\rho-1)^2} \right\rceil.$$

We assume that $|t| \ge t_0 \ge \rho^3/(\rho-1)^3$ where $t_0 := 200$. Since $q > 2$ by assumption, we observe that $\mathfrak{q} \ge \mathfrak{q}_0 := 3t_0$.

From the Dirichlet series definition of $L(s,\chi)$, we have

$$|L(1/2+it,\chi)| \le \Big|\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{1/2+it}}\Big|. \tag{32}$$

We divide the summation range on the r.h.s. of (32) into an initial sum followed by dyadic segments $[\rho^\ell v_0, \rho^{\ell+1} v_0)$. This gives

$$|L(1/2+it,\chi)| \le \Big|\sum_{n=1}^{v_0-1} \frac{\chi(n)}{n^{1/2+it}}\Big| + \sum_{\ell=0}^{\infty}\Big|\sum_{\rho^\ell v_0 \le n < \rho^{\ell+1} v_0} \frac{\chi(n)}{n^{1/2+it}}\Big|.$$

The ℓth dyadic segment is subdivided into blocks of length L_ℓ where

$$L_\ell = \begin{cases} \lceil (\rho-1)\rho^\ell v_0/|t|^{1/3}\rceil, & 0 \le \ell < \ell_0 := \log(CD|t|^{2/3}/v_0)/\log\rho \\ \lceil (\rho-1)\rho^\ell v_0/|t|^{1/2}\rceil, & \ell_0 \le \ell < \ell_1 := \log(q|t|/5v_0)/\log\rho, \\ \lceil (\rho-1)\rho^\ell v_0/|t|\rceil, & \ell_1 \le \ell, \end{cases}$$

plus a (possibly empty) boundary block. (Note that our assumption $|t| \ge t_0$ and the fact that $CD \le q$ imply $\ell_0 < \ell_1$.) So there are $R_\ell = \lceil (\rho-1)\rho^\ell v_0/L_\ell\rceil$ blocks in the ℓth segment. The rth block in the ℓth segment begins at

$$N_{r,\ell} + 1 = \lceil \rho^\ell v_0\rceil + rL_\ell \quad (0 \le r < R_\ell).$$

We first bound the initial sum, then we bound the sum over each range of ℓ separately.

5.1. Initial sum. The initial sum is bounded trivially by using the triangle inequality and the fact that $|\chi(n)/n^{1/2+it}| \le 1/\sqrt{n}$, and comparing with the integral $\int_0^{v_0-1} \frac{1}{\sqrt{x}}\,dx$. Recalling that $C/q^{1/3} = \mathrm{cbf}(q)$, this gives

$$\Big|\sum_{n=1}^{v_0-1} \frac{\chi(n)}{n^{1/2+it}}\Big| \le 2\sqrt{v_0-1} \le \mathfrak{v}_0\sqrt{\mathrm{cbf}(q)}\,\mathfrak{q}^{1/6}, \qquad \mathfrak{v}_0 := \frac{2}{\rho-1}. \tag{33}$$

5.2. Sum over $0 \le \ell < \ell_0$. Using the Cauchy–Schwarz inequality we obtain

$$\Big|\sum_{0\le\ell<\ell_0}\ \sum_{\rho^\ell v_0 \le n < \rho^{\ell+1} v_0} \frac{\chi(n)}{n^{1/2+it}}\Big|^2 \le (\ell_0+1) \sum_{0\le\ell<\ell_0}\Big|\sum_{\rho^\ell v_0 \le n < \rho^{\ell+1} v_0} \frac{\chi(n)}{n^{1/2+it}}\Big|^2. \tag{34}$$

We partition the ℓth dyadic segment in (34) into blocks of length L_ℓ. Then we apply partial summation to each segment to remove the weighting factor $1/\sqrt{n}$. Finally, we apply the Cauchy–Schwarz inequality to the sum of the blocks. This yields

$$\Big|\sum_{\rho^\ell v_0 \le n < \rho^{\ell+1} v_0} \frac{\chi(n)}{n^{1/2+it}}\Big|^2 \le \frac{R_\ell}{\rho^\ell v_0} \sum_{r=0}^{R_\ell-1} \max_{0\le\Delta\le L_\ell}\Big|\sum_{n=N_{r,\ell}+1}^{N_{r,\ell}+\Delta} \frac{\chi(n)}{n^{it}}\Big|^2. \tag{35}$$

We estimate the inner sum in (35) via Lemma 4.2. To this end, let

$$\lambda = \frac{1}{\sqrt{\rho-1}}, \qquad f(x) = -\frac{t}{2\pi}\log x,$$

and $1 \le L = \Delta \le L_\ell$. (Note that $\lambda > 1$, as required by the lemma.) We have

$$\frac{\lambda(L_\ell - 1)}{N_{r,\ell}+1} < \frac{1}{\lambda|t|^{1/3}} < 1. \tag{36}$$

So $f(N_{r,\ell}+1+z)$ is analytic on a disk of radius $|z| \le \lambda(L_\ell - 1)$. Moreover, as a consequence of (36),

$$\frac{|f^{(j)}(N_{r,\ell}+1)|}{j!}\lambda^j(L-1)^j = \frac{|t|\lambda^j(L-1)^j}{2\pi j(N_{r,\ell}+1)^j} \le \frac{1}{2\pi j\lambda^j} \qquad (j \ge 3).$$

In particular, the required bound on $|f^{(j)}(N_{r,\ell}+1)|/j!$ in Lemma 4.2 holds with $\eta = 1/(6\pi)$. Therefore, if we let $\nu_2 = \nu_2(1/\sqrt{\rho-1}, 1/(6\pi))$ and

$$y_{r,m,\ell} := \frac{mC^2Df''(N_{\ell,r}+1)}{Bd_m} = \frac{m}{d_m}\,\frac{C^2Dt}{2\pi B}\,\frac{1}{(N_{r,\ell}+1)^2},$$

Lemma 4.2 shows that the r.h.s. in (34) is bounded by

$$\frac{4\nu_2^2\Lambda}{\pi}(\ell_0+1)\sum_{0\le\ell<\ell_0}(*_\ell + **_\ell) \tag{37}$$

with

$$*_\ell := \frac{CL_\ell R_\ell^2}{\rho^\ell v_0}\left(\log\frac{D}{2B} + \frac{7}{4} + \frac{3\pi}{2\Lambda}\right),$$

$$**_\ell := \frac{C^2R_\ell}{\rho^\ell v_0}\sum_{m=1}^{\lceil L_\ell/C\rceil} W(m)\sum_{r=0}^{R_\ell-1}\min\left(\frac{\pi d_m BL_\ell}{CD}, \frac{1}{\|y_{r,m,\ell}\|}\right),$$

where for brevity we write

$$W(m) = 1 - \frac{m}{\lceil L_\ell/C\rceil}.$$

We consider the easier term $*_\ell$ first. Since $(\rho-1)^2v_0/|t|^{1/3} \ge C$, we obtain

$$\frac{(\rho-1)\rho^\ell v_0}{|t|^{1/3}} \le L_\ell \le \frac{(\rho-1)\rho^{\ell+1}v_0}{|t|^{1/3}}. \tag{38}$$

And the upper bound in (38) gives $(\rho-1)^2v_0/L_\ell \ge 1$. Hence,

$$\frac{(\rho-1)\rho^\ell v_0}{L_\ell} \le R_\ell \le \frac{(\rho-1)\rho^{\ell+1}v_0}{L_\ell}. \tag{39}$$

As can be seen from (39) and the definition of L_ℓ, we have

$$R_\ell \le \rho|t|^{1/3} \qquad (0 \le \ell < \ell_0). \tag{40}$$

Using this bound together with (39) and the inequality $\Lambda \ge 2$ (valid since $q > 2$ by assumption), we arrive at

$$\sum_{0\le \ell<\ell_0} *_\ell \le (\ell_0+1)\rho^2(\rho-1)C|t|^{1/3}\left(\log\frac{D}{2}+\frac{7}{4}+\frac{3\pi}{4}\right).$$

Furthermore, by our choice of v_0, we have

$$\ell_0+1 \le \frac{\log(\rho(\rho-1)^2 D|t|^{1/3})}{\log\rho}. \tag{41}$$

Therefore, the inequality $\log(D|t|^{1/3}/2) \le \log \mathfrak{q}^{1/3}$, the formula $C/q^{1/3} = \operatorname{cbf}(q)$, and incorporating the additional factor ℓ_0+1 from (37) into our estimate, give

$$(\ell_0+1)\sum_{0\le\ell<\ell_0} *_\ell \le \mathfrak{v}_1 \operatorname{cbf}(q)\mathfrak{q}^{1/3} Z_1(\log\mathfrak{q}), \qquad \mathfrak{v}_1 := \frac{\rho^2(\rho-1)}{27\log^2\rho}, \tag{42}$$

where

$$Z_1(X) := (X-\log t_0 + 21/4 + 9\pi/4)\big(X+3\log(2\rho(\rho-1)^2)\big)^2.$$

The term $**_\ell$ in (37) is more complicated to handle. First, we apply Lemma 3.1 to estimate the sum over r there. To this end, note that

$$(N_{r+1,\ell}+1)^2-(N_{r,\ell}+1)^2 \ge 2\lceil\rho^\ell v_0\rceil L_\ell \quad (0\le r<R_\ell-1).$$

Moreover, by construction, $N_{R_\ell-1,\ell}+1 \le \lfloor\rho^{\ell+1}v_0\rfloor$ and $N_{0,\ell}+1 \ge \lceil\rho^\ell v_0\rceil$. Hence,

$$\begin{aligned} |y_{r+1,m,\ell}-y_{r,m,\ell}| &\ge \frac{m}{d_m}\,\frac{C^2D|t|}{2\pi B}\,\frac{2\lceil\rho^\ell v_0\rceil L_\ell}{\lfloor\rho^{\ell+1}v_0\rfloor^4} \quad (0\le r<R_\ell-1),\\ |y_{R_\ell-1,m,\ell}-y_{0,m,\ell}| &\le \frac{m}{d_m}\,\frac{C^2D|t|}{2\pi B}\,\frac{\rho^2-1}{\lfloor\rho^{\ell+1}v_0\rfloor^2}. \end{aligned} \tag{43}$$

We apply Lemma 3.1 to the sequence $\{y_{r,m,\ell}\}_r$ with $y = y_{R_\ell-1,m,\ell}$, $x = y_{0,m,\ell}$, $P = \pi d_m B L_\ell/CD$, and (since $y_{r,m,\ell}$ is monotonic in r) with δ equal to the lower bound for $|y_{r+1,m,\ell}-y_{r,m,\ell}|$ in (43). With these parameter choices, Lemma 3.1 gives

$$\sum_{r=0}^{R_\ell-1}\min\left(\frac{\pi d_m B L_\ell}{CD}, \frac{1}{\|y_{r,m,\ell}\|}\right) \le 2(y-x+1)\big(2P+\delta^{-1}\log(e\max(P,2)/2)\big).$$

Multiplying out the brackets, we obtain four terms: $2(y-x)\delta^{-1}\log(e\max(P,2)/2))$, $2\delta^{-1}\log(e\max(P,2)/2))$, $4(y-x)P$, and $4P$. We estimate the sum of each term over m with the aid of the following inequalities, which are either straightforward to prove (the left two inequalities) or a consequence of Lemma 3.4:

$$\begin{aligned}\sum_{m=1}^{\lceil L_\ell/C\rceil} W(m) &\le \frac{L_\ell}{2C}, & \sum_{m=1}^{\lceil L_\ell/C\rceil} W(m)\frac{d_m}{m} &\le \tau(D/B)\log\lceil L_\ell/C\rceil,\\ \sum_{m=1}^{\lceil L_\ell/C\rceil} W(m)m &\le \frac{L_\ell^2}{2C^2}, & \sum_{m=1}^{\lceil L_\ell/C\rceil} W(m)d_m &\le \tau(D/B)\lceil L_\ell/C\rceil.\end{aligned} \tag{44}$$

Combining (38), (40), (43), and (44), together with the bound (here we use $L_\ell \ge C$)

$$\log\bigl(e\max(P,2)/2\bigr) \le \log\frac{e\pi L_\ell}{2C},$$

we routinely deduce the estimates

$$\begin{aligned}
&\frac{C^2R_\ell}{\rho^\ell v_0}\sum_{m=1}^{\lceil L_\ell/C\rceil} W(m)\frac{(\rho^2-1)\lfloor\rho^{\ell+1}v_0\rfloor^2}{\lceil\rho^\ell v_0\rceil L_\ell}\log\frac{e\pi L_\ell}{2C} \le \mathcal{B}_1(\ell),\\
&\frac{C^2R_\ell}{\rho^\ell v_0}\sum_{m=1}^{\lceil L_\ell/C\rceil} W(m)\frac{d_m}{m}\,\frac{2\pi B}{C^2D|t|}\,\frac{\lfloor\rho^{\ell+1}v_0\rfloor^4}{\lceil\rho^\ell v_0\rceil L_\ell}\log\frac{e\pi L_\ell}{2C} \le \mathcal{B}_2(\ell),\\
&\frac{C^2R_\ell}{\rho^\ell v_0}\sum_{m=1}^{\lceil L_\ell/C\rceil} W(m)\frac{4\pi d_m BL_\ell}{CD}\,\frac{m}{d_m}\,\frac{C^2D|t|}{2\pi B}\,\frac{\rho^2-1}{\lfloor\rho^{\ell+1}v_0\rfloor^2} \le \mathcal{B}_3(\ell),\\
&\frac{C^2R_\ell}{\rho^\ell v_0}\sum_{m=1}^{\lceil L_\ell/C\rceil} W(m)d_m\frac{4\pi BL_\ell}{CD} \le \mathcal{B}_4(\ell),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}_1(\ell) &:= \frac{\rho^3(\rho^2-1)}{2}C|t|^{1/3}\log\frac{e\pi L_\ell}{2C},\\
\mathcal{B}_2(\ell) &:= \frac{2\pi\rho^5}{(\rho-1)}\,\frac{\rho^\ell v_0}{D|t|^{1/3}}B\tau(D/B)\left(\log\frac{eL_\ell}{C}\right)\log\frac{e\pi L_\ell}{2C},\\
\mathcal{B}_3(\ell) &:= \rho^3(\rho-1)^3(\rho^2-1)C|t|^{1/3},\\
\mathcal{B}_4(\ell) &:= 4\pi\rho^2(\rho-1)^2\frac{\rho^\ell v_0}{D|t|^{1/3}}B\tau(D/B).
\end{aligned}$$

Incorporating the additional factor ℓ_0+1 from (34) into our estimate, we therefore conclude that

$$(\ell_0+1)\sum_{0\le\ell<\ell_0} **_\ell \le \sum_{j=1}^{4}(\ell_0+1)\sum_{0\le\ell<\ell_0}\mathcal{B}_j(\ell). \tag{45}$$

We estimate the sum over ℓ in (45) as a geometric progression. To this end, we use the bound on ℓ_0 in (41), the bound

$$L_\ell \le \rho(\rho-1)CD|t|^{1/3} \qquad (0 \le \ell < \ell_0),$$

which follows directly from the definitions of L_ℓ and ℓ_0, and consequently the bound

$$\log \frac{e\pi L_\ell}{2C} \le \log \frac{e\pi\rho(\rho-1)D|t|^{1/3}}{2} \qquad (0 \le \ell < \ell_0).$$

After some rearrangements, this yields

$$(\ell_0+1) \sum_{0\le\ell<\ell_0} B_1(\ell) \le \mathfrak{v}_2 \operatorname{cbf}(q)\mathfrak{q}^{1/3} Z_2(\log\mathfrak{q}), \qquad \mathfrak{v}_2 := \frac{\rho^3(\rho^2-1)}{54\log^2\rho},$$

$$(\ell_0+1) \sum_{0\le\ell<\ell_0} B_2(\ell) \le \mathfrak{v}_3 \operatorname{cbf}(q)B\tau(D/B)\mathfrak{q}^{1/3} Z_3(\log\mathfrak{q}),$$

$$\mathfrak{v}_3 := \frac{2\pi\rho^6}{27(\rho-1)^2\log\rho},$$

$$(\ell_0+1) \sum_{0\le\ell<\ell_0} B_3(\ell) \le \mathfrak{v}_4 \operatorname{cbf}(q)\mathfrak{q}^{1/3} Z_4(\log\mathfrak{q}), \qquad \mathfrak{v}_4 := \frac{\rho^3(\rho-1)^3(\rho^2-1)}{9\log^2\rho},$$

$$(\ell_0+1) \sum_{0\le\ell<\ell_0} B_4(\ell) \le \mathfrak{v}_5 \operatorname{cbf}(q)B\tau(D/B)\mathfrak{q}^{1/3} Z_5(\log\mathfrak{q}),$$

$$\mathfrak{v}_5 := \frac{4\pi\rho^3(\rho-1)}{3\log\rho},$$

where

$$\begin{aligned}
Z_2(X) &:= \big(X + 3\log(e\pi\rho(\rho-1))\big)\big(X + 3\log(2\rho(\rho-1)^2)\big)^2,\\
Z_3(X) &:= \big(X + 3\log(2e\rho(\rho-1))\big)\big(X + 3\log(e\pi\rho(\rho-1))\big)\\
&\qquad \times \big(X + 3\log(2\rho(\rho-1)^2)\big),\\
Z_4(X) &:= \big(X + 3\log(2\rho(\rho-1)^2)\big)^2,\\
Z_5(X) &:= X + 3\log(2\rho(\rho-1)^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(46) \qquad (\ell_0+1) \sum_{0\le\ell<\ell_0} {**}_\ell
&\le \mathfrak{v}_2 \operatorname{cbf}(q)\mathfrak{q}^{1/3} Z_2(\log\mathfrak{q}) + \mathfrak{v}_3 B\tau(D/B)\operatorname{cbf}(q)\mathfrak{q}^{1/3} Z_3(\log\mathfrak{q})\\
&\quad + \mathfrak{v}_4 \operatorname{cbf}(q)\mathfrak{q}^{1/3} Z_4(\log\mathfrak{q}) + \mathfrak{v}_5 B\tau(D/B)\operatorname{cbf}(q)\mathfrak{q}^{1/3} Z_5(\log\mathfrak{q}).
\end{aligned}$$

We combine (46) and (42), and use the inequality $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$ with $x, y \ge 0$. This gives

$$\sqrt{(\ell_0+1) \sum_{0\le\ell<\ell_0} ({*}_\ell + {**}_\ell)} \le \sqrt{\operatorname{cbf}(q)}\Big(\sqrt{Z_6(\log\mathfrak{q})} + \sqrt{B\tau(D/B) Z_7(\log\mathfrak{q})}\Big)\mathfrak{q}^{1/6}$$

where

$$Z_6(X) := \mathfrak{v}_1 Z_1(X) + \mathfrak{v}_2 Z_2(X) + \mathfrak{v}_4 Z_4(X),$$
$$Z_7(X) := \mathfrak{v}_3 Z_3(X) + \mathfrak{v}_5 Z_5(X).$$

Finally, we substitute this back into (34) to conclude that

$$\Big|\sum_{0\le\ell<\ell_0}\sum_{\rho^\ell v_0\le n<\rho^{\ell+1}v_0}\frac{\chi(n)}{n^{1/2+it}}\Big| \le \frac{2\nu_2}{\sqrt{\pi}}\sqrt{\Lambda\,\mathrm{cbf}(q)Z_6(\log\mathfrak{q})}\,\mathfrak{q}^{1/6} + \frac{2\nu_2}{\sqrt{\pi}}\sqrt{\Lambda\,\mathrm{cbf}(q)B\tau(D/B)Z_7(\log\mathfrak{q})}\,\mathfrak{q}^{1/6}. \tag{47}$$

5.3. Sum over $\ell_0 \le \ell < \ell_1$. Applying the triangle inequality and partial summation gives

$$\sum_{\ell_0\le\ell<\ell_1}\Big|\sum_{\rho^\ell v_0\le n<\rho^{\ell+1}v_0}\frac{\chi(n)}{n^{1/2+it}}\Big| \le \sum_{\ell_0\le\ell<\ell_1}\frac{1}{(\rho^\ell v_0)^{1/2}}\sum_{r=0}^{R_\ell-1}\max_{0\le\Delta\le L_\ell}\Big|\sum_{n=N_{r,\ell}+1}^{N_{r,\ell}+\Delta}\frac{\chi(n)}{n^{it}}\Big|. \tag{48}$$

We bound the inner sum in (48) via Lemma 4.1. Using a similar analysis to the beginning of Section 5.2, one verifies that the required analyticity conditions on $f(x) = -\frac{t}{2\pi}\log x$ in Lemma 4.1 hold with $J=1$, $\lambda = 1/\sqrt{\rho-1}$, and $\eta = 1/(4\pi)$. Therefore, if we let $\nu_1 = \nu_1(1/\sqrt{\rho-1}, 1/(4\pi))$ and

$$x_{r,\ell} := \frac{qf'(N_{\ell,r}+1)}{B_1} = -\frac{qt}{2\pi B_1}\,\frac{1}{N_{r,\ell}+1},$$

then by Lemma 4.1 the inner double sum in (48) is bounded by

$$\frac{2\nu_1}{\pi}\,\frac{C_1R_\ell}{(\rho^\ell v_0)^{1/2}}\left(\log\frac{D_1}{2B_1}+\frac{7}{4}+\frac{\pi}{2}\right) + \frac{\nu_1}{\pi}\,\frac{C_1}{(\rho^\ell v_0)^{1/2}}\sum_{r=0}^{R_\ell-1}\min\left(\frac{\pi B_1L_\ell}{q}, \frac{1}{\|x_{r,\ell}\|}\right). \tag{49}$$

We bound the sum over r in (49) using Lemma 3.1. To this end, note that

$$|x_{r+1,\ell}-x_{r,\ell}| \ge \frac{q|t|}{2\pi B_1}\,\frac{L_\ell}{\lfloor\rho^{\ell+1}v_0\rfloor^2},\quad |x_{R_\ell-1,\ell}-x_{0,\ell}| \le \frac{q|t|}{2\pi B_1}\,\frac{\rho-1}{\lfloor\rho^{\ell+1}v_0\rfloor}. \tag{50}$$

Furthermore, since the sequence $x_{r,\ell}$ is monotonic in r, we may set δ in Lemma 3.1 to be the lower bound for $|x_{r+1,\ell}-x_{r,\ell}|$ in (50), set $y-x$ as the upper bound for $|x_{R_\ell-1,\ell}-x_{0,\ell}|$ in (50), and set $P = \pi B_1L_\ell/q$. (Note that $P\ge 2$.) Therefore, applying the lemma, and multiplying out the brackets in the resulting bound $2(y-x+1)(2P+\delta^{-1}\log(eP/2))$, gives

$$\sum_{r=0}^{R_\ell-1}\min\left(\frac{\pi B_1 L_\ell}{q},\frac{1}{\|x_{r,\ell}\|}\right) \le \left(\frac{2(\rho-1)\lfloor\rho^{\ell+1}v_0\rfloor}{L_\ell}+\frac{4\pi B_1}{q|t|}\frac{\lfloor\rho^{\ell+1}v_0\rfloor^2}{L_\ell}\right)\log\frac{e\pi B_1 L_\ell}{2q} + \frac{2(\rho-1)|t|L_\ell}{\lfloor\rho^{\ell+1}v_0\rfloor}+\frac{4\pi B_1 L_\ell}{q}.$$

Using similar inequalities to those in Section 5.2, we deduce that

$$\frac{(\rho-1)\rho^\ell v_0}{|t|^{1/2}} \le L_\ell \le \frac{(\rho-1)\rho^{\ell+1}v_0}{|t|^{1/2}} \le \frac{1}{5}\rho(\rho-1)q|t|^{1/2} \qquad (\ell_0\le\ell<\ell_1).$$

Consequently, since $B_1 \le D_1$, we have

$$\log\frac{e\pi B_1 L_\ell}{2q} \le \log\frac{e\pi\rho(\rho-1)D_1|t|^{1/2}}{10} \qquad (\ell_0\le\ell<\ell_1).$$

Using these inequalities, the formulas $\sqrt{q}/D_1=\operatorname{sqf}(q)$, $C_1/(\sqrt{CD}\,q^{1/6})=\operatorname{spf}(q)$, and executing the geometric sum over ℓ, we therefore conclude that

$$\sum_{\ell_0\le\ell<\ell_1}\frac{C_1}{(\rho^\ell v_0)^{1/2}}\sum_{r=0}^{R_\ell-1}\min\left(\frac{\pi B_1 L_\ell}{q},\frac{1}{\|x_{r,\ell}\|}\right) \le \mathfrak{v}_8\operatorname{spf}(q)\mathfrak{q}^{1/6}Z_8(\log\mathfrak{q}) + \mathfrak{v}_9\operatorname{sqf}(q)B_1Z_8(\log\mathfrak{q})+\mathfrak{v}_{10}\operatorname{spf}(q)\mathfrak{q}^{1/6}+\mathfrak{v}_{11}\operatorname{sqf}(q)B_1, \tag{51}$$

where

$$Z_8(X):=X+2\log(e\pi\rho(\rho-1)/10)$$

and

$$\mathfrak{v}_8:=\frac{\rho^{3/2}}{\sqrt{\rho}-1},\qquad \mathfrak{v}_9:=\frac{2\pi\rho^{5/2}}{\sqrt{5}\,(\rho-1)(\sqrt{\rho}-1)},$$
$$\mathfrak{v}_{10}:=\frac{2\rho^{3/2}(\rho-1)^2}{\sqrt{\rho}-1},\qquad \mathfrak{v}_{11}:=\frac{4\pi\rho^{3/2}(\rho-1)}{\sqrt{5}\,(\sqrt{\rho}-1)}.$$

Furthermore, using the bound

$$R_\ell\le\rho|t|^{1/2}\qquad(\ell_0\le\ell<\ell_1),$$

we obtain

$$\sum_{\ell_0\le\ell<\ell_1}\frac{C_1R_\ell}{(\rho^\ell v_0)^{1/2}}\left(\log\frac{D_1}{2B_1}+\frac{7}{4}+\frac{\pi}{2}\right)\le\mathfrak{v}_{12}\operatorname{spf}(q)\mathfrak{q}^{1/6}Z_9(\log\mathfrak{q}), \tag{52}$$

where

$$Z_9(X)=X-2\log(2\sqrt{t_0})+\frac{7}{2}+\pi,\qquad \mathfrak{v}_{12}:=\frac{\rho^{3/2}}{2(\sqrt{\rho}-1)}.$$

We substitute (52) and (51) into (49) and (48), which gives (after some simplification)

$$(53)\qquad \sum_{\ell_0\le\ell<\ell_1}\Big|\sum_{\rho^\ell v_0\le n<\rho^{\ell+1}v_0}\frac{\chi(n)}{n^{1/2+it}}\Big| \le \frac{\nu_1}{\pi}(\mathfrak{v}_8\,\mathrm{spf}(q)\mathfrak{q}^{1/6}Z_8(\log\mathfrak{q}) + \mathfrak{v}_9\,\mathrm{sqf}(q)B_1Z_8(\log\mathfrak{q}) + \mathfrak{v}_{10}\,\mathrm{spf}(q)\mathfrak{q}^{1/6} + \mathfrak{v}_{11}\,\mathrm{sqf}(q)B_1) + \frac{2\nu_1}{\pi}\mathfrak{v}_{12}\,\mathrm{spf}(q)\mathfrak{q}^{1/6}Z_9(\log\mathfrak{q}).$$

5.4. Sum over $\ell_1\le\ell$. As before, we apply the triangle inequality, partial summation, and Lemma 3.2 with $J=0$, to obtain

$$(54)\qquad \sum_{\ell_1\le\ell}\Big|\sum_{\rho^\ell v_0\le n<\rho^{\ell+1}v_0}\frac{\chi(n)}{n^{1/2+it}}\Big| \le \sum_{\ell_1\le\ell}\frac{\nu_0}{(\rho^\ell v_0)^{1/2}}\sum_{r=0}^{R_\ell-1}\max_{0\le\Delta\le L_\ell}\Big|\sum_{n=N_{r,\ell}+1}^{N_{r,\ell}+\Delta}\chi(n)\Big|.$$

Here, $\nu_0 := \nu_0(1/\sqrt{\rho-1},1/(2\pi))$. We use the bound for nonprincipal characters at the bottom of p. 139 of [7]. Specifically, if $\chi \pmod q$ is nonprincipal, then $|\sum_n\chi(n)|\le 2\sqrt{q}\log q$. Using this, we deduce that the r.h.s. of (54) is

$$(55)\qquad \le 2\nu_0\sqrt{q}\log q\sum_{\ell_1\le\ell}\frac{1}{(\rho^\ell v_0)^{1/2}} \le \frac{2\sqrt{5}\,\nu_0\sqrt{\rho}}{\sqrt{\rho}-1}\,\frac{\log q}{\sqrt{|t|}} \le \nu_0\mathfrak{v}_{13}Z_{10}(\log\mathfrak{q}),$$

where

$$Z_{10}(X) := X-\log t_0,\qquad \mathfrak{v}_{13} := \frac{2\sqrt{5}\,\sqrt{\rho}}{(\sqrt{\rho}-1)\sqrt{t_0}}.$$

5.5. Summary. We combine (33), (47), (53), and (55), then evaluate the resulting numerical constants with $\rho=1.3$. This yields the theorem.

6. Proof of Corollary 1.1. We may assume that $q>1$, otherwise the corollary follows from the bound (31) for principal characters. By Lemma 3.3, if χ is primitive, then $B=B_1=1$. Also, since q is a sixth power, $\mathrm{sqf}(q)=\mathrm{cbf}(q)=1$ and $\mathrm{spf}(q)\le 1$. Therefore, the functions $Z(X)$ and $W(X)$ in Theorem 1.2 satisfy

$$\begin{aligned}Z(X) &\le -9.416+15.6004X+1.4327\sqrt{\Lambda(D)}X^{3/2}\\ &\quad + 12.1673\sqrt{\Lambda(D)\tau(D)}X^{3/2},\\ W(X) &\le -296.84+114.07X,\end{aligned}$$

where we have used the fact that for $X\ge\log(2t_0)$ we have

$$65.5619-17.1704X-2.4781X^2+0.6807X^3\le 0.6807X^3,$$
$$-1732-817.82X+71.68X^2+47.57X^3\le 49.1X^3.$$

These inequalities are verified using `Mathematica`.

It is easy to see that $\Lambda(p^a) \le \tau(p^a)$, which, by multiplicativity, implies that $\sqrt{\Lambda(D)\tau(D)} \le \tau(D)$. Furthermore, since q is a sixth power and $q > 1$, it follows that $\tau(D) \le 0.572\tau(q)$ (as can be seen by considering the case $q = 2^{6a}$), $\tau(q) \ge 7$, and $q \ge 2^6$. Substituting these bounds into the expressions for $Z(X)$ and $W(X)$, we verify via `Mathematica` that

$$\begin{aligned} Z(X) &\le \tau(q)(-1.3451 + 2.2287X + 7.2695X^{3/2}) \le 7.95\tau(q)X^{3/2}, \\ W(X) &\le \tau(q)(-42.4056 + 16.2958X) \le 16.30\tau(q)X \end{aligned}$$

for $X \ge \log(2^6 t_0)$. Therefore,

$$|L(1/2+it,\chi)| \le 7.95\tau(q)\mathfrak{q}^{1/6}\log^{3/2}\mathfrak{q} + 16.30\tau(q)\log\mathfrak{q}.$$

Finally, using the bound $\mathfrak{q} \ge 2^6 t_0$, we deduce that

$$|L(1/2+it,\chi)| \le 9.05\tau(q)\mathfrak{q}^{1/6}\log^{3/2}\mathfrak{q},$$

proving the corollary.

7. Proofs of bounds (3) and (2)

Proof of bound (3). Since χ is nonprincipal, we have

$$L(1/2+it,\chi) = \sum_{n\le M} \frac{\chi(n)}{n^{1/2+it}} + \mathcal{R}_M(t,\chi),$$

where the remainder $\mathcal{R}_M(t,\chi) := \sum_{n>M}^{\infty} \chi(n)n^{-1/2-it}$ is just the tail of the Dirichlet series. (We do not require that $M > 0$ be an integer.) To estimate the tail, we use partial summation [12, formula (1)]:

$$\tag{56} \begin{aligned} \Big|\sum_{M<n\le M_2} \frac{\chi(n)}{n^{1/2+it}}\Big| &\le \frac{1}{\sqrt{M_2}}\Big|\sum_{n\le M_2}\chi(n)\Big| + \frac{1}{\sqrt{M}}\Big|\sum_{n\le M}\chi(n)\Big| \\ &\quad + (1/2+|t|)\int_M^{M_2}\Big|\sum_{1\le n\le u}\chi(n)\Big|u^{-3/2}\,du. \end{aligned}$$

We bound the character sums on the r.h.s. of (56) using the Pólya–Vinogradov inequality in [3, §23]. This asserts that if χ is a primitive character modulo $q > 1$ then $|\sum_{N_1\le n<N_2}\chi(n)| \le \sqrt{q}\log q$. Substituting this in (56), taking the limit as $M_2 \to \infty$, and executing the integral gives

$$|L(1/2+it,\chi)| \le 2\sqrt{M} + \frac{2\sqrt{q}\log q}{\sqrt{M}}(|t|+1).$$

The claimed bound follows on choosing $M = (|t|+1)\sqrt{q}\log q$. ∎

Remark. If χ is merely assumed to be nonprincipal, then the bound (3) still holds but with an extra factor of $\sqrt{2}$ in front. One simply uses the Pólya–Vinogradov inequality stated in [7, p. 139] in the proof.

Proof of bound (2). Since $|L(1/2+it,\chi)| = |L(1/2-it,\overline{\chi})|$ and the proof will apply symmetrically to $L(1/2+it,\chi)$ and $L(1/2+it,\overline{\chi})$, we may assume that $t \geq 0$. Let $n_1 = \lfloor\sqrt{qt/(2\pi)}\rfloor$. Since $qt \geq 2\pi$, [4, Theorem 5.3] implies that

$$|L(1/2+it,\chi)| \leq (2+\delta_t)\left|\sum_{n=1}^{n_1}\frac{\chi(n)}{n^{1/2+it}}\right| + |\mathcal{R}(t,\chi)|, \tag{57}$$

where (3) $\delta_t := \exp\big(\frac{\pi}{24t}+\frac{1}{12t^2}\big)-1$ and

$$|\mathcal{R}(t,\chi)| \leq \frac{264.72q^{1/4}\log q}{t^{1/4}} + \frac{11.39q^{3/4}}{t^{3/4}}e^{-0.78\sqrt{t/q}}. \tag{58}$$

To prove this, we specialize [4, Theorem 5.3] to the critical line, taking $X=Y$ with $2\pi X^2 = qt$, then appeal to well-known properties of Gauss sums. Put together, this yields

$$L(1/2+it,\chi) = \sum_{n\leq X}\frac{\chi(n)}{n^{1/2+it}} + F(t,\chi)\sum_{n\leq X}\frac{\overline{\chi(n)}}{n^{1/2-it}} + \mathcal{R}(t,\chi), \tag{59}$$

where $G(\chi,-1)$ is a Gauss sum and

$$F(t,\chi) := \frac{(2\pi i)^{1/2+it}q^{-1/2-it}G(\chi,-1)}{\Gamma(1/2+it)}. \tag{60}$$

(Here, $(2\pi i)^{1/2+it}$ is defined using the principal branch of the logarithm.) We estimate $\mathcal{R}(t,\chi)$ in (59) using the case "$X \leq Y$" in [4, Theorem 5.3]. Since we specialized $X=\sqrt{qt/(2\pi)}$, we obtain

$$|\mathcal{R}(t,\chi)| \leq \left(167.2(2\pi)^{1/4}\log q + \frac{2.87(2\pi)^{3/4}\sqrt{q}}{\sqrt{t}}e^{-\sqrt{\pi^3/50}\sqrt{t/q}}\right)\frac{q^{1/4}}{t^{1/4}}.$$

The claimed estimate (58) for $\mathcal{R}(t,\chi)$ follows on noting that $167.2(2\pi)^{1/4} < 264.72$, $2.87(2\pi)^{3/4} < 11.39$, and $\pi^{3/2}/\sqrt{50} > 0.78$.

To bound the factor $1/\Gamma(1/2+it)$ appearing in the definition (60) of $F(t,\chi)$, we mimic the proof of [4, Lemma 2.1] with minor adjustments. This gives

$$\frac{1}{|\Gamma(1/2+it)|} \leq \frac{\exp\big(\frac{\pi t}{2}+\frac{\pi}{24t}+\frac{1}{12t^2}\big)}{\sqrt{2\pi}} \quad (t>0). \tag{61}$$

Combining (61) with the facts that $|G(\chi,-1)| = \sqrt{q}$ and $|(2\pi i)^{1/2+it}| = \sqrt{2\pi}\,e^{-\pi t/2}$ gives $|F(t,\chi)| \leq \exp\big(\frac{\pi}{24t}+\frac{1}{12t^2}\big) = 1+\delta_t$. Since the second sum in the approximate functional equation (59) is just the complex conjugate of the first sum there, this proves (57).

(3) The appearance of δ_t in (57) is due to a slight imperfection in the form of the approximate functional equation proved in [4], and is not significant otherwise.

Last, we trivially estimate the sum in (57), then use the assumption $t \geq \sqrt{q} \geq \sqrt{2}$ and monotonicity to bound $\mathcal{R}(t,\chi)$ and δ_t. This gives (on noting that $\log q \leq \log \mathfrak{q}^{2/3}$)

$$|L(1/2+it,\chi)| \leq \left(\frac{2(1+\delta_{\sqrt{2}})}{(2\pi)^{1/4}} + 11.39\exp\left(-\frac{0.78}{2^{1/4}}\right)\right)\mathfrak{q}^{1/4} + 264.72\mathfrak{q}^{1/12}\log\mathfrak{q}^{2/3}.$$

Denote the r.h.s. above by $(*)$. Using `Mathematica` we verify that the equation $(*) = 124.46\mathfrak{q}^{1/4}$ has no real solution if $\mathfrak{q} \geq 10^9$. Furthermore, $(*)$ is smaller than $124.46\mathfrak{q}^{1/4}$ when $\mathfrak{q} = 10^9$. Hence, $(*) \leq 124.46\mathfrak{q}^{1/4}$ for all $\mathfrak{q} \geq 10^9$, as claimed. ∎

Acknowledgments. I thank Tim Trudgian for pointing out the reference [11].

Preparation of this material is partially supported by the National Science Foundation under agreement No. DMS-1406190.

References

[1] M. B. Barban, Yu. V. Linnik, and N. G. Tshudakov, *On prime numbers in an arithmetic progression with a prime-power difference*, Acta Arith. 9 (1964), 375–390.

[2] Y. F. Cheng and S. W. Graham, *Explicit estimates for the Riemann zeta function*, Rocky Mountain J. Math. 34 (2004), 1261–1280.

[3] H. Davenport, *Multiplicative Number Theory*, 3rd ed., Grad. Texts in Math. 74, Springer, New York, 2000.

[4] L. Habsieger, *Explicit approximate functional equations for various classes of Dirichlet series*, Ramanujan J. 9 (2005), 93–110.

[5] D. R. Heath-Brown, *Hybrid bounds for Dirichlet L-functions*, Invent. Math. 47 (1978), 149–170.

[6] D. R. Heath-Brown, *Hybrid bounds for Dirichlet L-functions. II*, Quart. J. Math. Oxford Ser. (2) 31 (1980), 157–167.

[7] G. A. Hiary, *Computing Dirichlet character sums to a power-full modulus*, J. Number Theory 140 (2014), 122–146.

[8] G. A. Hiary, *An alternative to Riemann–Siegel type formulas*, Math. Comp. 85 (2016), 1017–1032.

[9] G. A. Hiary, *An explicit van der Corput estimate for $\zeta(1/2+it)$*, Indag. Math. (N.S.) 27 (2016), 524–533.

[10] D. Milićević, *Sub-Weyl subconvexity for Dirichlet L-functions to prime power moduli*, Compos. Math. 152 (2016), 825–875.

[11] H. Rademacher, *On the Phragmén–Lindelöf theorem and some applications*, Math. Z. 72 (1959/1960), 192–204.

[12] M. Rubinstein, *Computational methods and experiments in analytic number theory*, in: Recent Perspectives in Random Matrix Theory and Number Theory, London Math. Soc. Lecture Note Ser. 322, Cambridge Univ. Press, Cambridge, 2005, 425–506.

[13] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Clarendon Press, Oxford, 1986.

Ghaith A. Hiary
Department of Mathematics
The Ohio State University
231 West 18th Ave.
Columbus, OH 43210, U.S.A.
E-mail: hiary.1@osu.edu