On Intersections of Generic Perturbations of Definable Sets

by

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Summary. Consider an o-minimal expansion $\mathcal{R}$ of a real closed field $R$ and two definable sets $E$ and $M$. We introduce concepts of a locally transitive (abbreviated to l.t.) and a strongly locally transitive (abbreviated to s.l.t.) action of $E$ on $M$. In the former case, $M$ is supposed to be of pure dimension $m$; in the latter, both $M$ and $E$ are supposed to be of pure dimension. We treat the elements of $E$ as perturbations of the set $M$. We prove that if $E$ acts l.t. on $M$, and $A$ and $B$ are two non-empty definable subsets of $M$ of dimension $\dim A \leq \dim B < \dim M$, then

$$\dim(\sigma(A) \cap B) < \dim A$$

for a generic $\sigma$ in $E$; here $\dim \emptyset = -1$. And if $E$ acts s.l.t. on $M$ and $A$ and $B$ are two definable subsets of $M$, then

$$\dim(\sigma(A) \cap B) \leq \max\{\dim A + \dim B - m, -1\}$$

for a generic $\sigma$ in $E$. We give an example of an l.t. action $E$ on $M$ for which the latter conclusion of the intersection theorem fails. We also prove a theorem on the intersections of generic perturbations in terms of the exceptional set $T \subset M$ of points at which $E$ is not l.t. Finally, we provide some natural conditions which imply that $T$ is a nowhere dense subset of $M$.

1. Introduction and main results. Consider an o-minimal expansion $\mathcal{R}$ of a real closed field $R$ and two definable (with parameters) subsets $E$ and $M$ of $R^n$. Let $\dim E = e$ and assume that $M$ is of pure dimension $m$, i.e. the dimension of $M$ at each point $x \in M$ is $m$. Examples of such sets are, for instance, definable topological manifolds (possibly with boundary). In this paper we set $\dim \emptyset = -1$. We shall investigate continuous definable

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maps
\[ \alpha : E \times M \to M, \quad \alpha(\sigma, x) = \sigma \cdot x = \sigma(x), \]
which we call actions of \( E \) on \( M \). We treat the elements of \( E \) as perturbations of the set \( M \). For subsets \( X \subset E \) and \( Y \subset M \), write
\[ X \cdot Y := \{ x \cdot y \in M : x \in X, y \in Y \}. \]
It will be often convenient to think of the points \( x \in M \) as maps from \( E \) into \( M \). Thus we set
\[ \alpha^x : E \ni \sigma \mapsto \sigma \cdot x \in M. \]
We say that \( \alpha \) or \( E \) is locally transitive (abbreviated to l.t.) at a point \( x \in M \) if \( \Omega \cdot x = \alpha^x(\Omega) \) is a subset of \( M \) of dimension \( m \) for every non-empty, open subset \( \Omega \) of \( E \). Of course, if \( E \) is l.t. at \( x \), then \( e \geq m \).

**Remark 1.1.** It is not difficult to prove that \( E \) is l.t. at \( x \in M \) iff the map \( \alpha^x \) is generically a submersion, i.e. there exists a nowhere dense subset \( F \) of \( E \) such that the restriction of \( \alpha^x \) to \( E \setminus F \) is a submersion of class \( C^1 \).

The set \( E \) is called l.t. on a subset \( A \) of \( M \) if it is l.t. at every point \( x \in A \). Locally transitive actions can be characterized in terms of the rank of definable maps defined below.

Let \( f : V \to W \) be a definable map between definable subsets of \( R^n \). For any \( x \in R^n \) and \( r > 0 \), denote by \( B(x, r) \) the ball with center \( x \) and radius \( r \).

The function \( V \times (0, \infty) \ni (x, r) \mapsto \dim f(V \cap B(x, r)) \in \mathbb{N} \)
is definable, because the dimension of fibers from a definable family depends definably on the parameters (cf. [1, Chap. 4]). Consequently, for a fixed \( x \in V \), the function \( \dim f(V \cap B(x, r)) \) is constant for \( r > 0 \) small enough.

Its common value \( r_xf \) near zero will be called the rank of \( f \) at \( x \); \( r_xf \) is a definable function of the variable \( x \). It is clear that \( E \) is l.t. at \( x \) iff the map \( \alpha^x \) is of constant rank \( m \).

Suppose now that the definable set \( E \) is also of pure dimension \( e \). We say that \( E \) is strongly locally transitive (abbreviated to s.l.t.) at a point \( x \in M \) if the fibers of the map \( \alpha^x \) are of dimension \( \leq e - m \). An easy dimension calculus, based on the following proposition (see e.g. [1] Chap. 4, Prop. 1.5]), shows that \( E \) is l.t. at \( x \) if \( E \) is s.l.t. at \( x \).

**Proposition 1.2.** Let \( f : V \to W \) be a definable map between non-empty definable sets. Then
\[
\dim f^{-1}f(v) \leq k \text{ for all } v \in V \Rightarrow \dim V \leq k + \dim f(V), \\
\dim f^{-1}f(v) \geq k \text{ for all } v \in V \Rightarrow \dim V \geq k + \dim f(V).
\]
The set $E$ is called s.l.t. on a subset $A$ of $M$ if it is s.l.t. at every point $x \in A$.

The main purpose of this paper is to prove the following two theorems on intersections of generic perturbations.

**Theorem 1.3.** Suppose $M$ is a definable set of pure dimension $m$. Let $A$ and $B$ be non-empty, definable subsets of $M$. If $E$ is l.t. on $A$ and $\dim A \leq \dim B < m$, then there exists a definable, nowhere dense subset $Z$ of $E$ such that

$$\dim(\sigma(A) \cap B) < \dim A$$

for all $\sigma \in E \setminus Z$.

**Theorem 1.4.** Suppose $E$ and $M$ are definable sets of pure dimension $e$ and $m$, respectively. Let $A$ and $B$ be definable subsets of $M$. If $E$ is s.l.t. on $A$, then there exists a definable subset $Z$ of $E$ such that $\dim Z < e$ and

$$\dim(\sigma(A) \cap B) \leq d := \max\{\dim A + \dim B - m, -1\}$$

for all $\sigma \in E \setminus Z$.

The above results generalize the theorem on generic intersections from [6], which treated only the case where $E$ is a definable group. Their proofs will be given in the next two sections. Section 4 gives an example of a l.t. action $E$ for which Theorem 1.4 fails. In Section 5, we define an exceptional set $T \subset M$ of points at which $E$ is not l.t. and prove a theorem on intersections of generic perturbations in terms of $T$ (Corollary 5.1). Finally, we provide some natural conditions which imply that $T$ is a nowhere dense subset of $M$ (Theorem 5.2).

Some natural examples of manifolds which act l.t. (but not s.l.t.) are the following: the set of all reflections of $\mathbb{R}^m$ in affine hyperplanes, the set of all rotations in $\mathbb{R}^m$ around affine subspaces of dimension $m - 2$, the set of such rotations by a fixed non-zero angle, the counterparts of these sets in the sphere $S^m$ and in the hyperbolic space $\mathbb{H}^m$, as well as non-empty open subsets of the above-mentioned sets. The paper [8] provides a study of the l.t. action of the set of rotations in $\mathbb{R}^m$, $S^m$ and $\mathbb{H}^m$, and its results are applied in [4], devoted to a concept of a small set which refines the concept of a Tarski nullset.

**Remark 1.5.** If $\mathcal{R}$ is a polynomially bounded, o-minimal expansion of the field $\mathbb{R}$, then smooth (i.e. of class $C^\infty$) definable functions constitute a quasianalytic class, i.e. the identity principle holds: two quasianalytic functions on a connected open subset $U \subset \mathbb{R}^m$ coincide if so do their germs at a point $a \in U$. Hence the following characterization of local transitivity. Suppose $E$ and $M$ are connected, smooth manifolds definable in $\mathcal{R}$ and $\alpha : E \times M \to M$ is a smooth, definable map. Then $\alpha$ is l.t. at a point $x \in M$ iff the set $E \cdot x$ is a subset of $M$ of dimension $m$. For example, $\mathcal{R}$ may be an
analytic structure $\mathbb{R}_{an}$ (i.e. the expansion of the field $\mathbb{R}$ by restricted analytic functions) or, more generally, a quasianalytic structure (i.e. the expansion of the field $\mathbb{R}$ by restricted quasianalytic functions; see e.g. [9, 5, 7]).

**Remark 1.6.** Strong local transitivity can be expressed in terms of Remmert rank. Denote by $\dim A_x$ the dimension of a definable set $A$ at a point $x$. By the Remmert rank of a definable map $f : V \to W$ at a point $x \in V$ (cf. [3, Chap. V]) we mean the number

$$\varrho_x f := \dim V_x - \dim f^{-1}(f(x))_x.$$  

Clearly, $E$ is s.l.t. at $x$ iff the map $\alpha^x$ is of Remmert rank $\geq m$ everywhere on $E$.

2. **Proof of Theorem 1.3.** We first prove that the definable set

$$Z := \{ \sigma \in E : \exists a \in A \exists r > 0 \ [\sigma \cdot (A \cap B(a, r)) \subset B] \}$$

is a nowhere dense subset of $E$. Otherwise it would contain an open definable subset $U$ of $E$. By definable choice, there exist definable functions $a : U \to A$ and $r : U \to (0, 1)$ such that

$$\sigma \cdot (A \cap B(a(\sigma), r(\sigma))) \subset B.$$ 

After shrinking the open subset $U$, we may assume that the maps $a(\sigma)$ and $r(\sigma)$ are continuous. Take any point $\sigma_0 \in U$, $\varepsilon := r(\sigma_0)/3$ and a neighbourhood $U_0 \subset U$ of $\sigma_0$ such that

$$d(a(\sigma), a(\sigma_0)) < \varepsilon \quad \text{and} \quad r(\sigma) > 2\varepsilon \quad \text{for all } \sigma \in U_0;$$

here $d$ stands for the Euclidean distance in $\mathbb{R}^n$. Then

$$B(a(\sigma_0), \varepsilon) \subset B(a(\sigma), 2\varepsilon) \subset B(a(\sigma), r(\sigma))$$

for all $\sigma \in U_0$, and thus

$$\sigma \cdot (A \cap B(a(\sigma_0), \varepsilon)) \subset B \quad \text{for all } \sigma \in U_0.$$ 

In particular, we get

$$\dim(U_0 \cdot a(\sigma_0)) < m,$$

which contradicts the fact that $E$ is l.t. on the set $A$. Therefore the definable set $Z$ is a nowhere dense subset of $E$, as asserted.

Consequently, we get

$$\sigma \cdot (A \cap B(a, r)) \not\subset B$$

for all $a \in A$, $r > 0$ and $\sigma \in E \setminus Z$. Hence

$$\dim(\sigma(A) \cap B) < \dim A$$

for all $\sigma \in E \setminus Z$. Indeed, this is an immediate consequence of
Lemma 2.1. Given definable subsets $C$ and $D$, if $\dim(C \cap D) = \dim C$, then $C \cap B(a,r) \subset D$ for some $a \in C$ and $r > 0$.

This, in turn, follows directly from the existence of a finite definable cell decomposition compatible with the sets $C$ and $D$. Thus the proof of Theorem 1.3 is complete. ■

3. Proof of Theorem 1.4. The proof relies on dimension calculus similar to that applied in the proof of the theorem on generic intersections in [6]. We adopt the notation from that paper with the definable group $G$ replaced by the definable topological manifold $E$. Let

$$\Delta = \Delta_M := \{(x,x) : x \in M\} \text{ and } \pi : \Delta \to M$$

be the diagonal and the projection onto the first factor. Then

$$\sigma(A) \cap B = \pi((\sigma(A) \times B) \cap \Delta)$$

$$= \pi \circ (\sigma \times \text{Id}_M)((A \times B) \cap \{(x,\sigma(x)) : x \in M\}).$$

Hence the sets $\sigma(A) \cap B$ and $(A \times B) \cap \{(x,\sigma(x)) : x \in M\}$ are definably homeomorphic, and thus we have to find a definable, nowhere dense subset $Z$ of $E$ such that

$$\dim(A \times B) \cap \{(x,\sigma(x)) : x \in M\} \leq d \text{ for all } \sigma \in G \setminus Z.$$ 

Therefore Theorem 1.4 follows immediately from the lemma below (cf. [6, p. 23]).

Lemma 3.1. The subset $Z$ of all $\sigma \in E$ such that

$$\dim(A \times B) \cap \{(x,\sigma(x)) : x \in M\} > d$$

is definable and nowhere dense in $E$.

Its proof can be repeated verbatim because it requires only that the fibers

$$\{\sigma \in E : \sigma(x) = y\} = (\alpha^x)^{-1}(y), \quad x \in A, \; y \in B,$$

be of dimension $\leq e - m$ (in the present setting). But this is just the assumption that $E$ is s.l.t. on $A$.

4. Examples. We will show that Theorem 1.3 is asymmetric in the sense that the assumption $\dim A \leq \dim B$ is essential, even in the case where $E$ is a submanifold of a definable Lie group $G$ which acts transitively on a definable manifold $M$. In particular, in this example the action of $E$ on $M$ will be l.t. but the action of

$$E^{-1} := \{\sigma^{-1} \in G : \sigma \in E\}$$

will not be l.t. on $M$. 
Let $M := R^2 \setminus \{(0,0)\}$ and
\[
E := \left\{ \begin{bmatrix} 1 & -x \\ 1-xy & 1-y \end{bmatrix} \in \text{GL}(2, R) : x, y \in R \right\}.
\]
of course, the general linear group $\text{GL}(2, R)$ acts transitively on $M$ and $E$ is an algebraic submanifold of $\text{GL}(2, R)$. Then, by an easy calculation, for every $(a, b) \in M$ the set $E \cdot (a, b)$ contains all points $(c, d)$ with $c \neq 0$ and $d \neq 0$. Hence $E$ is l.t. on $M$. Clearly,
\[
E^{-1} := \left\{ \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \in \text{GL}(2, R) : x, y \in R \right\}.
\]
Further,
\[
E^{-1} \cdot (u, 0) = \{(u, v) : v \in R\}
\]
for all $u \neq 0$. Since the above set is of dimension 1, $E^{-1}$ is not l.t. at $(u, 0)$.

Now let us show that the assumption $\text{dim} A \leq \text{dim} B$ is needed. Keep the notation of the above example. Let $A$ be the line $\{(1, t) : t \in R\}$ and $B := \{(1, v)\}$. Thus $E$ is l.t. but $\text{dim} A > \text{dim} B$. For all $\sigma \in E$, we have $B \subset \sigma(A)$ (the equivalent relation $\sigma^{-1}(B) \subset A$ is obvious). Hence $\text{dim}(\sigma(A) \cap B) = 0$, while the conclusion of Theorem 1.3 would require it to be $-1$, i.e. the intersection to be empty.

**Remark 4.1.** Suppose that a definable group $G$ acts transitively on a definable manifold $M$ and that $E$ is a definable subset of $G$ of pure dimension. It is not difficult to check that if $E$ is s.l.t. on $M$, then $E^{-1}$ is l.t. on $M$.

When $\alpha : E \times M \to M$ is a definable map of class $C^1$ between two definable manifolds of class $C^1$, we call $\alpha$ a submersive action at a point $x \in M$ if the map $\alpha^x$ is a submersion. $E$ is called submersive on a subset $A$ of $M$ if it is submersive at every point $x \in A$. Obviously, every submersive action is s.l.t. We immediately obtain the following corollary to Theorem 1.4.

**Corollary 4.2.** Let $A$ and $B$ be definable subsets of $M$. If $E$ is submersive on $A$, then there exists a definable subset $Z$ of $E$ such that $\text{dim} Z < e$ and
\[
\text{dim}(\sigma(A) \cap B) \leq d := \max\{\text{dim} A + \text{dim} B - m, -1\}
\]
for all $\sigma \in E \setminus Z$. $\square$

**Remark 4.3.** This corollary follows also from an o-minimal version of the Thom transversality theorem (see e.g. [2, Chap. 3, Theorem 2.7] for the classical version) and the existence of a definable stratification.

**Proposition 4.4.** The weaker assumptions of Theorem 1.1 do not imply the inequality $\text{dim}(\sigma(A) \cap B) \leq d$ of Theorem 2.2.
Proof. Let $M := (\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}$ and

$$E := \left\{ \frac{1}{x_1^2 - x_2^2} \begin{bmatrix} x_1 & -x_2 & 0 \\ -x_2 & x_1 & 0 \\ x_2x_3 & -x_1x_3 & x_1^2 - x_2^2 \end{bmatrix} \in \text{GL}(3, \mathbb{R}) \right\};$$

of course, $E$ is an algebraic submanifold of $\text{GL}(3, \mathbb{R})$. Then, by an easy calculation, for every $a = (a_1, a_2, a_3) \in M$ the set $E \cdot a$ contains all points $b = (b_1, b_2, b_3)$ with $b_1 \neq 0$, $b_2 \neq 0$, $b_1^2 - b_2^2 \neq 0$ and $a_2b_1 - a_1b_2 \neq 0$. Hence $E$ is l.t. on $M$. Clearly,

$$E^{-1} := \left\{ \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & x_3 & 1 \end{bmatrix} \in \text{GL}(3, \mathbb{R}) \right\},$$

$E^{-1}$ is not l.t. on $M$ and thus $E$ is not s.l.t. on $M$.

Let $A$ be the circle

$$A := \{ a \in M : a_1^2 + a_2^2 = 1, \ a_3 = 1 \}$$

and let $B$ be the line

$$B := \{ b \in M : b_2 = 0, \ b_3 = 1 \}.$$ 

Then $d = \max\{ \dim A + \dim B - m, -1 \} = -1$.

Observe now that if $b = (b_1, 0, 1) \in B$ and

$$\sigma^{-1} := \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & x_3 & 1 \end{bmatrix} \in E^{-1},$$

then $\sigma^{-1}(b) = (b_1x_1, b_1x_2, 1)$. Hence $\sigma^{-1}(B)$ is the line through the points $(0,0,1)$ and $(x_1, x_2, 1)$. Therefore $A \cap \sigma^{-1}(B)$ is a two-point set. Hence $\dim(\sigma(A) \cap B) = 0 > d$ for every $\sigma \in E$, contrary to the inequality of Theorem 1.2.

5. Exceptional set of the action. Consider further an action of a definable set $E$ on a definable set $M$ of pure dimension $m$. By the exceptional subset $T$ of $M$ with respect to the action $E$ we mean the definable set of all points $x \in M$ at which $E$ is not l.t., i.e. of those $x \in M$ for which $U \cdot x$ is of dimension $< m$ for some non-empty, open subset $U$ of $E$. Thus the assumption of Theorem 1.3 says that $A \cap T = \emptyset$. But this can be replaced by the weaker assumption $\dim(A \cap T) < \dim A$:

Corollary 5.1. If $A$ and $B$ are non-empty, definable subsets of $M$ such that

$$\dim A \leq \dim B < m \quad \text{and} \quad \dim(A \cap T) < \dim A,$$
then there exists a definable, nowhere dense subset $Z$ of $E$ such that

$$\dim(\sigma(A) \cap B) < \dim A$$

for all $\sigma \in E \setminus Z$.

Proof. Subtract $T$ from $A$ and apply Theorem 1.3.

The smaller the dimension of $T$, the larger the class of pairs $A, B$ for which Theorem 1.3 and Corollary 5.1 apply. Now we are going to provide some conditions under which $T$ is of dimension $< m$. For a subset $F$ of $E$ set

$$M(F) := \{ x \in M : \dim(F \cdot x) < m \}.$$  

Then

(5.1) \( F_1 \subset F_2 \Rightarrow M(F_1) \supset M(F_2) \)

and

(5.2) \( M(F_1 \cup F_2) = M(F_1) \cap M(F_2) \).

Obviously, if $F$ is a definable set, so is $M(F)$. Further, we get

$$T = \bigcup_{U \subset E \text{ open}} M(U) = \bigcup_{\sigma \in E, r > 0} M(E \cap B(\sigma, r)).$$

This gives rise to the following definition. We call an element $\sigma \in E$ a singular perturbation with respect to $E$ if

$$\dim M(E \cap B(\sigma, r)) = m$$

for some $r > 0$.

The set of all singular perturbations with respect to $E$ is an open definable subset of $E$. We call its closure $E_s$ the singular locus of $E$. The complement $E_t := E \setminus E_s$ is an open definable subset of $E$, called the tame locus of $E$. It is not difficult to check that $E_t$ has no singular perturbation with respect to $E_t$.

**Theorem 5.2.** If $E$ has no singular perturbation with respect to $E$, then the exceptional set $T$ is a subset of $M$ of dimension $< m$.

Proof. Towards a contradiction, suppose that $T$ is of dimension $m$ and thus contains an open definable subset $\Omega$ of $M$. By definable choice, there are definable functions

$$c : \Omega \to E \quad \text{and} \quad r : \Omega \to (0, \infty) \subset R$$

such that $x \in M(E \cap B(c(x), r(x)))$ for all $x \in \Omega$. We may assume, after shrinking $\Omega$, that the functions $c$ and $r$ are continuous. Take a point $x_0 \in \Omega$ and a neighbourhood $\Omega_0$ of $x_0$ such that

$$d(c(x), c(x_0)) < r_0/3 \quad \text{and} \quad r(x) > 2r_0/3$$

for all $x \in \Omega_0$,

where $r_0 := r(x_0)$. Then

$$B(c(x), r(x)) \supset B(c(x_0), r_0/3)$$

for all $x \in \Omega_0$,
and
\[ x \in M(E \cap B(c(x), r(x))) \subset M(E \cap B(c(x_0), r_0/3)) \text{ for all } x \in \Omega_0. \]
Hence
\[ \Omega_0 \subset M(E \cap B(c(x_0), r_0/3)), \]
and thus \( c(x_0) \in E \) is a singular perturbation with respect to \( E \), contrary to the assumption. This finishes the proof. ■

**Remark 5.3.** Assume that \( \mathcal{R} \) is a polynomially bounded, o-minimal expansion of the field \( \mathbb{R} \), and that \( E \) and \( M \) are connected, smooth, definable manifolds. It follows from the identity principle for quasianalytic functions (cf. Remark 1.5) that \( M(U) = M(E) \) for any non-empty, open subset \( U \) of \( E \). Further, \( M(E) \) is a definable quasianalytic subset of \( M \), and hence so is the exceptional set \( T = M(E) \). Thus, unlike in the general settings, \( T \) is either a closed, nowhere dense subset of \( M \) or the whole manifold \( M \).

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