ANNALES
POLONICI MATHEMATICI
118.1 (2016)

A logarithmically improved regularity criterion for the 3D MHD system involving the velocity field in homogeneous Besov spaces

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Abstract. We consider a regularity criterion for the 3D MHD equations. It is proved that if

$$\int_0^T \frac{\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}}^{2/(1+r)}}{1+\ln(e+\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}})}\,d\tau<\infty$$

for some $0<r<1$, then the solution is actually smooth on $(0,T)$.

1. Introduction. In this paper, we consider the following three-dimensional (3D) magnetohydrodynamic (MHD) equations:

$$\begin{cases} \boldsymbol{u}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{0},\\ \boldsymbol{b}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}-\Delta\boldsymbol{b}=\boldsymbol{0},\\ \nabla\cdot\boldsymbol{u}=0,\\ \nabla\cdot\boldsymbol{b}=0,\\ \boldsymbol{u}(0)=\boldsymbol{u}_0,\ \boldsymbol{b}(0)=\boldsymbol{b}_0, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}=(u_1,u_2,u_3)$ is the fluid velocity field, $\boldsymbol{b}=(b_1,b_2,b_3)$ is the magnetic field, π is a scalar pressure, and $(\boldsymbol{u}_0,\boldsymbol{b}_0)$ are the prescribed initial data satisfying $\nabla\cdot\boldsymbol{u}_0=\nabla\cdot\boldsymbol{b}_0=0$ in the sense of distributions. Physically, (1.1) governs the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas, liquid metals, and salt water. Moreover, $(1.1)_1$ reflects the conservation of momentum, $(1.1)_2$ is the induction equation, and $(1.1)_3$ specifies the conservation of mass.

Besides its physical background, the MHD system (1.1) is also mathematically significant. Duvaut and Lions [6] constructed a global weak solution

2010 *Mathematics Subject Classification*: Primary 35B65; Secondary 35Q35, 76D03.
Key words and phrases: MHD equations, regularity criteria, regularity of solutions.
Received 25 February 2016; revised 13 September 2016.
Published online 21 October 2016.

DOI: 10.4064/ap3952-9-2016

to (1.1) for initial data with finite energy. However, the issue of regularity and uniqueness of such a given weak solution remains a challenging open problem. Many sufficient conditions (see e.g., [3, 4, 5, 8, 7, 9, 10, 12, 15, 16, 17, 18, 19, 20, 21] and the references therein) were derived to guarantee the regularity of the weak solution. In particular, It was shown in [4] that if

$$\boldsymbol{u} \in L^q(0,T; B^r_{p,\infty}(\mathbb{R}^3)) \tag{1.2}$$

with $2/q + 3/p = 1 + r$, $3/(1+r) < p \le \infty$, $-1 < r \le 1$ and $(p, r) \neq (\infty, 1)$, then the solution is regular on $(0, T)$. Here, $B^r_{p,\infty}$ is the inhomogeneous Besov spaces (see [1, Chapter 2] for example). The middle case $r = 0$ of (1.2) was improved (from inhomogeneous Besov spaces to homogeneous ones) as

$$\boldsymbol{u} \in L^2(0,T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)) \tag{1.3}$$

in [14].

The aim of the present paper is to make a further contribution in this direction.

Theorem 1.1. *Let $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$, and $T > 0$. Assume that $(\boldsymbol{u}, \boldsymbol{b})$ is the unique strong solution pair of the MHD system* (1.1) *with initial data $(\boldsymbol{u}_0, \boldsymbol{b}_0)$ on $(0, T)$. If*

$$\int_0^T \frac{\|\boldsymbol{u}(\tau)\|^{2/(1+r)}_{\dot{B}^r_{\infty,\infty}}}{1 + \ln(e + \|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}})}\, d\tau < \infty \tag{1.4}$$

for some $0 < r < 1$, then the solution can be smoothly extended past T.

Remark 1.2. An immediate consequence of Theorem 1.1 is the following regularity criterion:

$$\boldsymbol{u} \in L^{2/(1+r)}(0,T; \dot{B}^r_{\infty,\infty}(\mathbb{R}^3)) \quad (0 < r < 1). \tag{1.5}$$

This extends (1.2). In fact, we have $B^s_{p,q}(\mathbb{R}^3) = L^p(\mathbb{R}^3) \cap \dot{B}^s_{p,q}(\mathbb{R}^3)$ for any $s > 0$ and $1 \le p, q \le \infty$ (see [2, Theorem 6.3.2]).

Remark 1.3. Our result (1.4) is a logarithmically improved version, with the limiting cases $r = 0$ and $r = 1$ out of reach.

The main idea in proving Theorem 1.1 is the following lemma.

Lemma 1.4. *For $f \in \dot{B}^r_{\infty,\infty}(\mathbb{R}^3)$, $g, h \in H^1(\mathbb{R}^3)$ and any $\varepsilon > 0$, $0 < r < 1$, $k \in \{1, 2, 3\}$, we have*

$$\int_{\mathbb{R}^3} \partial_k f \cdot gh\, dx \le C\|f\|^{2/(1+r)}_{\dot{B}^r_{\infty,\infty}} \|(g,h)\|^2_{L^2} + \varepsilon\|\nabla(g,h)\|^2_{L^2}. \tag{1.6}$$

Proof. We have

$$
\begin{aligned}
\int_{\mathbb{R}^3} \partial_k f \cdot gh\,dx &= -\int_{\mathbb{R}^3} f \cdot \partial_k(gh)\,dx \\
&= -\int_{\mathbb{R}^3} \Lambda^r f \cdot \Lambda^{-r}\partial_k(gh)\,dx \qquad (\Lambda = (-\Delta)^{1/2}) \\
&\le C\|\Lambda^r f\|_{\dot B^0_{\infty,\infty}} \|\Lambda^{-r}\partial_k(gh)\|_{\dot B^0_{1,1}} \qquad \text{(by [1, Proposition 2.29])} \\
&\le C\|f\|_{\dot B^r_{\infty,\infty}} \|gh\|_{\dot B^{1-r}_{1,1}} \qquad \text{(by [1, Lemma 2.1])} \\
&\le C\|f\|_{\dot B^r_{\infty,\infty}} (\|g\|_{L^2}\|h\|_{\dot B^{1-r}_{2,1}} + \|g\|_{\dot B^{1-r}_{2,1}}\|h\|_{L^2}) \\
&\qquad \text{(by analogues of [1, Corollary 2.54])} \\
&\le C\|f\|_{\dot B^r_{\infty,\infty}} (\|g\|_{L^2}\|h\|^r_{\dot B^0_{2,\infty}} \|h\|^{1-r}_{\dot B^1_{2,\infty}} + \|g\|^r_{\dot B^0_{2,\infty}} \|g\|^{1-r}_{\dot B^1_{2,\infty}} \|h\|_{L^2}) \\
&\qquad \text{(by [1, Proposition 2.22])} \\
&\le C\|f\|_{\dot B^r_{\infty,\infty}} (\|g\|_{L^2}\|h\|^r_{L^2}\|\nabla h\|^{1-r}_{L^2} + \|g\|^r_{L^2}\|\nabla g\|^{1-r}_{L^2}\|h\|_{L^2}) \\
&\qquad \text{(by [1, Proposition 2.39])} \\
&\le C\|f\|_{\dot B^r_{\infty,\infty}} \|(g,h)\|^{1+r}_{L^2}\|\nabla(g,h)\|^{1-r}_{L^2} \\
&\le C\|f\|^{2/(1+r)}_{\dot B^r_{\infty,\infty}} \|(g,h)\|^2_{L^2} + \varepsilon\|\nabla(g,h)\|^2_{L^2}. \ \blacksquare
\end{aligned}
$$

2. Proof of Theorem 1.1. To prove Theorem 1.1, we only need to show that the H^3 norm of the solution is uniformly bounded on $(0,T)$ under the assumption (1.4).

Taking the inner product of $(1.1)_1$ with $-\Delta \boldsymbol{u}$, and of $(1.1)_2$ with $-\Delta \boldsymbol{b}$ in $L^2(\mathbb{R}^3)$, and adding the resulting equations, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|^2_{L^2} + \|\Delta(\boldsymbol{u},\boldsymbol{b})\|^2_{L^2} \\
&\quad = \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{u}\,dx - \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\Delta\boldsymbol{u}\,dx \\
&\qquad + \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\Delta\boldsymbol{b}\,dx - \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{b}\,dx \\
&\quad = -\sum_{k=1}^3 \int_{\mathbb{R}^3} [(\partial_k\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_k\boldsymbol{u}\,dx + \sum_{k=1}^3 \int_{\mathbb{R}^3} [(\partial_k\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\partial_k\boldsymbol{u}\,dx \\
&\qquad - \sum_{k=1}^3 \int_{\mathbb{R}^3} [(\partial_k\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\partial_k\boldsymbol{b}\,dx + \sum_{k=1}^3 \int_{\mathbb{R}^3} [(\partial_k\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\partial_k\boldsymbol{b}\,dx,
\end{aligned}
$$

where we use integration by parts, the fact that $\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = 0$ and its consequence

$$\int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\partial_k \boldsymbol{b}]\cdot\partial_k \boldsymbol{u} + [(\boldsymbol{b}\cdot\nabla)\partial_k \boldsymbol{u}]\cdot\partial_k \boldsymbol{b}\, dx = 0 \qquad (k=1,2,3).$$

By Lemma 1.4, we deduce

$$\frac{1}{2}\frac{d}{dt}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 \le C\|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}}^{2/(1+r)}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \tfrac{1}{2}\|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.$$

Consequently,

$$\begin{aligned}(2.1)\qquad \frac{d}{dt}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \|\Delta(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 &\le C\|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}}^{2/(1+r)}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\ &\le C\frac{\|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}}^{2/(1+r)}}{1+\ln(e+\|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}})}[1+\ln(e+\|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}})]\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2\\ &\le C\frac{\|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}}^{2/(1+r)}}{1+\ln(e+\|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}})}[1+\ln(e+\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2})]\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2,\end{aligned}$$

where in the last estimate we use the Sobolev embedding theorem $H^3(\mathbb{R}^3)\subset \dot{B}^r_{\infty,\infty}(\mathbb{R}^3)$ and the fact that $\boldsymbol{u}\in L^\infty(0,T;L^2(\mathbb{R}^3))$.

Due to (1.4), for any $0<\varepsilon\ll 1$ there exists $0<T_0<T$ such that

$$\int_{T_0}^{T}\frac{\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}}^{2/(1+r)}}{1+\ln(e+\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}})}\, d\tau < \varepsilon.$$

For any $T_0<t<T$, we set

$$y(t) = \sup_{\tau\in[T_0,t)}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2}.$$

By the monotonicity of $y(t)$, applying the Gronwall inequality to (2.1) yields

$$\begin{aligned}(2.2)\qquad &\|\nabla(\boldsymbol{u},\boldsymbol{b})(t)\|_{L^2}^2 + \int_{T_0}^{t}\|\Delta(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\, d\tau\\ &\le \|\nabla(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}\cdot\Big\{C(1+\ln(e+y(t)))\int_{T_0}^{t}\frac{\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}}^{2/(1+r)}}{1+\ln(e+\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}})}\, d\tau\\ &\qquad\cdot\exp[C(1+\ln(e+y(t)))\int_{T_0}^{t}\frac{\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}}^{2/(1+r)}}{1+\ln(e+\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}})}\, d\tau]\Big\}\\ &\le C_0\exp[2C\varepsilon(1+\ln(e+y(t)))] \qquad (\text{since } fe^f\le e^{2f})\\ &\le C_0(e+y(t))^{2C\varepsilon},\end{aligned}$$

where C_0 is a positive constant depending on T_0.

To close the estimate, we apply ∇^3 to $(1.1)_{1,2}$, multiply the resulting equations by $\nabla^3 \boldsymbol{u}$ and $\nabla^3 \boldsymbol{b}$ respectively, and sum them up to obtain

$$
\begin{aligned}
(2.3)\qquad & \frac{1}{2}\frac{d}{dt}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \|\nabla^4(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \nabla^3[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\nabla^3\boldsymbol{u}\,dx - \int_{\mathbb{R}^3} \nabla^3[(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\nabla^3\boldsymbol{b}\,dx \\
&\quad + \int_{\mathbb{R}^3} \{\nabla^3[(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\nabla^3\boldsymbol{u} + \nabla^3[(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\nabla^3\boldsymbol{b}\}\,dx \\
&= -\int_{\mathbb{R}^3} [\nabla^3,\boldsymbol{u}\cdot\nabla]\boldsymbol{u}\cdot\nabla^3\boldsymbol{u}\,dx - \int_{\mathbb{R}^3} [\nabla^3,\boldsymbol{u}\cdot\nabla]\boldsymbol{b}\cdot\nabla^3\boldsymbol{b}\,dx \\
&\quad + \int_{\mathbb{R}^3} \{[\nabla^3,\boldsymbol{b}\cdot\nabla]\boldsymbol{b}\cdot\nabla^3\boldsymbol{u} + [\nabla^3,\boldsymbol{b}\cdot\nabla]\boldsymbol{u}\cdot\nabla^3\boldsymbol{b}\}\,dx \\
&\quad ([f,g] = fg - gf, \text{ and we use the incompressibility condition}) \\
&\equiv J.
\end{aligned}
$$

To proceed further, we recall the following commutator estimate due to Kato–Ponce [11]:

$$
(2.4)\qquad \|[\Lambda^s, f]g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}})
$$

with

$$
(2.5)\quad s>0, \quad p_2,p_3\in(1,\infty), \quad p_2,p_4\in[1,\infty], \quad \frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{p_3}+\frac{1}{p_4}.
$$

Consequently,

$$
\begin{aligned}
J &\le \|[\nabla^3,\boldsymbol{u}\cdot\nabla]\boldsymbol{u}\|_{L^{\frac{4}{3}}}\|\nabla^3\boldsymbol{u}\|_{L^4} + \|[\nabla^3,\boldsymbol{u}\cdot\nabla]\boldsymbol{b}\|_{L^{\frac{4}{3}}}\|\nabla^3\boldsymbol{b}\|_{L^4} \\
&\quad + \|[\nabla^3,\boldsymbol{b}\cdot\nabla]\boldsymbol{b}\|_{L^{\frac{4}{3}}}\|\nabla^3\boldsymbol{u}\|_{L^4} + \|[\nabla^3,\boldsymbol{b}\cdot\nabla]\boldsymbol{u}\|_{L^{\frac{4}{3}}}\|\nabla^3\boldsymbol{b}\|_{L^4} \\
&\le C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^4}\cdot\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^4} \qquad (\text{by } (2.4)) \\
&\le C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/4}\|\nabla^4(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{7/4} \\
&\quad (\text{by the Gagliardo–Nirenberg inequality } \|\nabla^3 f\|_{L^4} \le C\|\nabla^2 f\|_{L^2}^{1/8}\|\nabla^4 f\|_{L^2}^{7/8}) \\
&\le C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^8\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \frac{1}{2}\|\nabla^4(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.
\end{aligned}
$$

Plugging this into (2.3), and absorbing the diffusion term, we get

$$
\frac{d}{dt}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 \le C\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^8\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.
$$

Integrating the above inequality over (T_0, t), we find

$$\begin{aligned}\|\nabla^3(\boldsymbol{u},\boldsymbol{b})(t)\|_{L^2} &\le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C\int_{T_0}^{t} \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^8 \|\nabla^2(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,d\tau \\ &\le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C\sup_{T_0<\tau<t} \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^8 \cdot \int_{T_0}^{t} \|\nabla^2(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,d\tau \\ &\le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C[e+y(t)]^{8C\varepsilon}\cdot[e+y(t)]^{2C\varepsilon} \qquad \text{(by (2.2))} \\ &\le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C[e+y(t)]^{10C\varepsilon}.\end{aligned}$$

Thus,

$$e+y(t) \le \|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}^2 + C[e+y(t)]^{10C\varepsilon}.$$

Choosing $\varepsilon = 1/(20C)$, we deduce

$$y(t) \le C(\|\nabla^3(\boldsymbol{u},\boldsymbol{b})(T_0)\|_{L^2}, T_0, T) < \infty,$$

as desired. The proof of Theorem 1.1 is thus complete.

Acknowledgements. Zujin Zhang is partially supported by the Natural Science Foundation of Jiangxi Province (grant no. 20151BAB201010) and the National Natural Science Foundation of China (grant nos. 11501125, 11361004).

References

[1] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren Math. Wiss. 343, Springer, Heidelberg, 2011.

[2] J. Bergh and J. Löfström, *Interpolation Spaces: an Introduction*, Grundlehren Math. Wiss. 223, Springer, Berlin, 1976.

[3] C. S. Cao and J. H. Wu, *Two regularity criteria for the 3D MHD equations*, J. Differential Equations 248 (2010), 2263–2274.

[4] Q. L. Chen, C. X. Miao and Z. F. Zhang, *On the regularity criterion of weak solutions for the 3D viscous magneto-hydrodynamics equations*, Comm. Math. Phys. 284 (2008), 919–930.

[5] Q. L. Chen, C. X. Miao and Z. F. Zhang, *The Beale–Kato–Majda criterion for the 3D magneto-hydrodynamics equations*, Comm. Math. Phys. 275 (2007), 861–872.

[6] G. Duvaut et J.-L. Lions, *Inéquations en thermoélasticité et magnétohydrodynamique*, Arch. Ration. Mech. Anal. 46 (1972), 241–279.

[7] C. He and Y. Wang, *On the regularity criteria for weak solutions to the magnetohydrodynamic equations*, J. Differential Equations 238 (2007), 1–17.

[8] C. He and Z. P. Xin, *On the regularity of weak solutions to the magnetohydrodynamic equations*, J. Differential Equations 213 (2005), 235–254.

[9] X. J. Jia and Y. Zhou, *On regularity criteria for the 3D incompressible MHD equations involving one velocity component*, J. Math. Fluid Mech. 18 (2016), 187–206.

[10] X. J. Jia and Y. Zhou, *Ladyzhenskaya–Prodi–Serrin type regularity criteria for the 3D incompressible MHD equations in terms of 3×3 mixture matrices*, Nonlinearity 28 (2015), 3289–3307.

[11] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier–Stokes equations*, Comm. Pure Appl. Math. 41 (1988), 891–907.

[12] Y. L. Liu, *On the critical one-component velocity regularity criteria to 3-D incompressible MHD system*, J. Differential Equations 260 (2016), 6989–7019.

[13] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. 36 (1983), 635–664.

[14] X. J. Xu, Z. Ye and Z. J. Zhang, *Remark on an improved regularity criterion for the 3D MHD equations*, Appl. Math. Lett. 42 (2015), 41–46.

[15] K. Yamazaki, *Regularity criteria of MHD system involving one velocity and one current density component*, J. Math. Fluid Mech. 16 (2014), 551–570.

[16] K. Yamazaki, *Regularity criteria of the three-dimensional MHD system involving one velocity and one vorticity component*, Nonlinear Anal. 135 (2016), 73–83.

[17] Z. J. Zhang, *Regularity criteria for the 3D MHD equations involving one current density and the gradient of one velocity component*, Nonlinear Anal. 115 (2015), 41–49.

[18] Z. J. Zhang, *Remarks on the global regularity criteria for the 3D MHD equations via two components*, Z. Angew. Math. Phys. 66 (2015), 977–987.

[19] Z. J. Zhang, *Refined regularity criteria for the MHD system involving only two components of the solution*, Appl. Anal. (2016), doi: 10.1080/00036811.2016.1207245.

[20] Y. Zhou, *Remarks on regularities for the 3D MHD equations*, Discrete Contin. Dynam. Systems 12 (2005), 881–886.

[21] Y. Zhou and J. S. Fan, *Logarithmically improved regularity criteria for the 3D viscous MHD equations*, Forum Math. 24 (2012), 691–708.

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