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Totally irreducible representations of algebras

by

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Abstract. The principal subject of the paper is the density in the strong operator topology of subalgebras of the algebra $L(X)$ of continuous operators in a locally convex space X. Besides a survey of known results we present refinements and generalizations of theorems related to the, still unsolved, problem of Fell and Doran.

1. Introduction. Although the Jacobson Density Theorem (JDT) is purely of algebraic character, it inspires a number of questions of topological nature in the theory of bounded operators on topological vector spaces.

Let $\mathcal{A}$ be an associative algebra over the field $\mathbb{R}$ or $\mathbb{C}$ (denoted as $\mathbb{K}$) and let X be a vector space over $\mathbb{K}$. A *representation* of $\mathcal{A}$ on X is a homomorphism T of $\mathcal{A}$ into the algebra $\mathcal{L}(X)$ of all linear endomorphisms of X. A representation T is *algebraically irreducible* if for every $0 \neq x \in X$ the orbit $\mathcal{O}(T,x) = \{T_a x : a \in \mathcal{A}\}$ coincides with X.

For a given representation T of $\mathcal{A}$ we denote $\mathcal{T} = \{T_a : a \in \mathcal{A}\}$.

If $\mathcal{S} \subset \mathcal{L}(X)$, the *algebraic commutant* of $\mathcal{S}$ is the set

$$\mathcal{S}_{\mathrm{a}}^{\mathrm{com}} = \{A \in \mathcal{L}(X) : AS = SA \text{ for all } S \in \mathcal{S}\}.$$

Let I denote the identity operator on X.

Theorem 1.1 (Jacobson Density Theorem). *Let X be a vector space over $\mathbb{K}$ and let $\mathcal{A}$ be an associative algebra over the same field. If (T,X) is an algebraically irreducible representation of $\mathcal{A}$ in X and $\mathcal{T}_{\mathrm{a}}^{\mathrm{com}} = \mathbb{K} I$ then for every linearly independent system $x_1,\dots,x_k \in X$ and for every system $y_1,\dots,y_k \in X$ there is $a \in \mathcal{A}$ such that $y_i = T_a x_i$, $1 \le i \le k$.*

In particular, if $A \in \mathcal{L}(X)$ and $V \subset X$ is a finite-dimensional space with a base $x_1,\dots,x_k$, there exists $a \in \mathcal{A}$ such that $Ax_i = T_a x_i$, $1 \le i \le k$, hence $A|V = T_a$.

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The property which appears in the conclusion of the JDT is called the algebraic k-irreducibility or, in other papers, strict k-transitivity of the representation. In explicit form, a representation T of $\mathcal{A}$ on X is *algebraically k-irreducible* if for every linearly independent system $x_1, \ldots, x_k \in X$ the orbit of the action of $\mathcal{A}$ on the cartesian product X^k given by the formula $T_a^{(k)}(x_1, \ldots, x_k) = (T_a x_1, \ldots, T_a x_k)$ coincides with X^k. A representation is *algebraically totally irreducible* if it is k-irreducible for every $k \in \mathbb{N}$. So, the Jacobson Density Theorem states that every algebraically irreducible representation with trivial commutant is algebraically totally irreducible.

If X is a topological vector space, the algebra $L(X)$ of continuous endomorphisms of X is a subalgebra of $\mathcal{L}(X)$ whose structure can be investigated by providing it with a topology and studying representations of other algebras in $L(X)$.

In this paper we consider the case of a locally convex space X.

Throughout what follows, by a *representation* of an algebra $\mathcal{A}$ on X we mean a homomorphism T of $\mathcal{A}$ into $L(X)$. A representation T is *irreducible* if for every $0 \neq x \in X$ the orbit $\mathcal{O}(T, x)$ is dense in X.

The strong operator topology on $L(X)$ is the topology of pointwise convergense of nets. A base of neighbourhoods of zero in this topology can be parametrized by systems $(\mathcal{V}, x_1, \ldots, x_k)$, where $\mathcal{V}$ is a neighbourhood of zero in X and $x_1, \ldots, x_k \in X$, and is given by

$$\mathcal{U}(\mathcal{V}, x_1, \ldots, x_k) = \{A \in L(X) : Ax_1, \ldots, Ax_k \in \mathcal{V}\}.$$

An operator $A \in L(X)$ belongs to the closure of a subset $\mathcal{S} \subset L(X)$ in the strong operator theory if and only if it can be approximated by elements of $\mathcal{S}$ on every finite-dimensional subspace of X.

In particular, if a representation T of an algebra $\mathcal{A}$ on a locally convex space X is algebraically irreducible and the algebraic commutant $\mathcal{T}_{\mathrm{a}}^{\mathrm{com}}$ is trivial, then by the JDT the space $\mathcal{T}$ is dense in $L(X)$.

The question which interests us is how to obtain analogous results with these strong algebraic assumptions replaced by others of topological character.

Formulating the problem in this way we can treat as a particular solution a theorem proved independently by C. E. Rickart [R] and B. Yood [Y] which states that for an algebraically irreducible representation T of $\mathcal{A}$ on a Banach space X the set $\mathcal{T}$ is dense in $L(X)$ in the weak operator topology.

Observe that nothing is assumed about the commutant of the representation. If X is a Banach space, then $L(X)$ has the structure of a Banach algebra, which is an exceptional situation.

When studying approximations of operators in the strong operator topology, it is convenient to use the notion of (topological) total irreducibility in the following form.

A representation T of an algebra $\mathcal{A}$ on a locally convex vector space X is *totally irreducible* if for every $k \in \mathbb{N}$ and every linearly independent system $x_1, \ldots, x_k \in X$ the orbit

$$\mathcal{O}(T^{(k)}, x_1, \ldots, x_k) = \{(T_a x_1, \ldots, T_a x_k) : a \in \mathcal{A}\}$$

is dense in X^k.

In the case of a locally convex space X this property is exactly equivalent to the density of $\mathcal{T}$ in $L(X)$ in the strong operator theory.

In the book [A] we find the following theorem.

THEOREM 1.2. *Let $\mathcal{A}$ be a Banach algebra and X a Banach space. If T is a continuous, algebraically irreducible representation of $\mathcal{A}$ then T is algebraically totally irreducible.*

In this theorem the "topologization" concerns the structure of the algebra $\mathcal{A}$, of the carrier space X and of the representation itself. Algebraic irreducibility is assumed and algebraic total irreducibility is deduced. Again no assumption about the commutant is necessary. This is explained by the fact that the structure of a Banach algebra provides an additional argument in the form of the Gelfand–Mazur theorem.

In Section 2 we prove a generalization of this theorem under the assumption that $\mathcal{A}$ is a Waelbroeck algebra and X is a locally convex space.

A quite different approach to the "topologization" of the JDT appears in the book [FD]. The conjecture of Fell and Doran proposes sufficient conditions for a subalgebra $\mathcal{A} \subset L(X)$ to be dense in $L(X)$ in the strong operator topology. Let $\mathcal{T}^{\mathrm{com}} = \{S \in L(X) : ST_a = T_a S \text{ for all } a \in \mathcal{A}\}$.

PROBLEM OF FELL AND DORAN. Let X be a topological vector space. Suppose that the representation T of an algebra $\mathcal{A}$ on X is irreducible and $\mathcal{T}^{\mathrm{com}}$ is trivial.

Is the representation T totally irreducible?

W. Żelazko dedicated a number of papers to the study of the problem of Fell and Doran treating it as a search of spaces X for which all irreducible representations with trivial commutant are totally irreducible. So far the conjecture has been confirmed completely only for the space (s) of real or complex sequences provided with the topology of pointwise convergence [Z3].

An important criterion of total irreducibility was proved in [Z2].

If (T, X) and (R, Y) are representations of an algebra $\mathcal{A}$, and S is a densely defined linear operator from X to Y, we say that S is (T, S)-*intertwining* if the domain $D(S)$ of S is $\mathcal{T}$-invariant and $ST_a x = R_a Sx$ for all $x \in D(S)$ and $a \in \mathcal{A}$.

Theorem 1.3 ([Z2]). *Let X be a real or complex topological vector space and let T be an irreducible representation of an algebra $\mathcal{A}$ on X such that $\mathcal{T}^{\mathrm{com}} = \mathbb{K} I$. Then T is totally irreducible if and only if all densely defined closed $(T, T^{(k)})$-intertwining operators are continuous for all $k \in \mathbb{N}$.*

The theorem provides a solution of the problem of Fell and Doran for algebraically irreducible representations in spaces X for which the Closed Graph Theorem is valid.

Corollary 1.4 ([Z2]). *Let X be a topological vector space for which every closed operator $S\colon X \to X$ is continuous. Let T be an algebraically irreducible representation of an algebra $\mathcal{A}$ on X such that $\mathcal{T}^{\mathrm{com}} = \mathbb{K} I$. Then T is totally irreducible and $\overline{\mathcal{T}} = L(X)$.*

The algebraic irreducibility assumption is a principal disadvantage of this theorem.

In [RT], in the case of a Banach space X, there is a result which can be treated as a possible approach to this matter. Instead of algebraic or topological irreducibility of the representation it is supposed that the algebra $\mathcal{T}$ has no non-trivial operator ranges, that is, no non-trival $\mathcal{T}$-invariant subspace of X can be represented as the image under a continuous intertwining operator $B\colon X^k \to X$ of a closed subspace $G \subset X^k$. This condition is weaker than algebraic irreducibility but obviously stronger than topological irreducibility. It is proved that an algebra $\mathcal{T} \subset L(X)$ which has no non-trivial injective operator ranges is dense in $L(X)$.

In Section 3 we prove a refinement of Theorem 1.3 in the case of a locally convex space X. Moreover, we show that k-irreducible representations occur in pairs: a representation (T, X) of the algebra $\mathcal{A}$ is k-irreducible if and only if the action of $\mathcal{A}$ on the dual space X' via the adjoint operators is k-irreducible in the weak* topology.

With these results we prove, in Section 4, that for a locally convex space X, a representation of $\mathcal{A}$ which has a trivial commutant $\mathcal{T}^{\mathrm{com}}$ and has no non-trivial injective operator ranges is totally irreducible.

We also obtain a criterion of total irreducibility of representations in a locally convex space X which involves only densely defined intertwining operators inside X, instead of $(T, T^{(k)})$-intertwining operators, which appear in Theorem 1.3:

We prove that an irreducible representation (T, X) of $\mathcal{A}$ in a locally convex vector space is totally irreducible if and only if every injective densely defined $\overline{\mathcal{T}}$-intertwining operator is scalar.

2. Algebraically irreducible representations of Waelbroeck algebras. A topological algebra W is called a *Waelbroeck algebra* if the subset G of elements invertible in W is open in W and the inverse mapping $G \ni a \mapsto a^{-1} \in G$ is continuous.

The spectral properties of Waelbroeck algebras are very close to those of Banach algebras. We recall the properties of Waelbroeck algebras which will be used further (see [Wa] or [W]).

LEMMA 2.1. *If W is a Waelbroeck algebra and $B \subset W$ is a subalgebra closed with respect to inversion, then B is a Waelbroeck algebra.*

If $J \subset W$ is a closed two-sided ideal in W then the quotient W/J is a Waelbroeck algebra.

GELFAND–MAZUR THEOREM. *Let W be a Waelbroeck algebra over $\mathbb{C}$ which admits a non-zero continuous linear functional. If W is a division algebra then $W \cong \mathbb{C}$.*

THEOREM 2.2. *Let W be a locally convex Waelbroeck algebra over $\mathbb{C}$ and let (T, X) be an algebraically irreducible representation of W on a locally convex space. Then (T, X) is algebraically totally irreducible.*

Proof. Let J be a maximal left ideal in W and let $E = W/J$. The left regular representation of W on E defined by the formula $L_a[u] = [au]$ is algebraically irreducible. We begin by proving the statement of the theorem for this representation.

Let S be a linear operator on E which commutes with all operators L_a, $a \in W$. For every $0 \neq v \in E$ the orbit $\mathcal{O}(L, v)$ coincides with E. Since $S(L_a v) = L_a S(v)$, the operator S is uniquely determined by the value $S(v)$. Let us choose $v = [e]$ and let $Sv = [u]$. Then for every $b \in J$ we have $0 = S([b]) = S(L_b[e]) = L_b S([e]) = L_b[u] = [bu]$. This proves that $Ju \subset J$.

Let $V = \{u \in W : Ju \subset J\}$. It is a closed subalgebra of W, and J is a two-sided ideal in V. The algebra V is closed with respect to inversion, because for $u \in V$ invertible the relation $Ju \subset J$ implies $J \subset Ju^{-1}$ and by the maximality of J we obtain $Ju^{-1} = J$.

For every $u \in V$ the formula $S_u[a] = [au]$ defines a linear operator commuting with the representation L. Moreover, $S_u = I$ if and only if $u \in J$. This permits us to identify $\mathcal{L}_{\mathrm{a}}^{\mathrm{com}}$, the algebraic commutant of the representation L, with the quotient algebra V/J, which is a Waelbroeck algebra according to Lemma 2.1.

The algebraic irreducibility of L implies that for every $S \in \mathcal{L}_{\mathrm{a}}^{\mathrm{com}}$ the spaces $\ker S$ and $\operatorname{im} S$ are trivial. It follows that $\mathcal{L}_{\mathrm{a}}^{\mathrm{com}}$ is a division algebra and V/J is a Waelbroeck division algebra. By the Gelfand–Mazur theorem, $V/J \cong \mathbb{C}$.

The representation (L, E) is algebraically totally irreducible by the Jacobson Density Theorem.

If (T, X) is an algebraically irreducible representation of W on a locally convex space X then for every $0 \neq x \in X$ the space $J_x = \{a \in W : T_a x = 0\}$ is a maximal ideal in W which is closed. The operator $R\colon W/J_x \ni [a] \mapsto T_a x$ is a well defined isomorphism intertwining the representations L and T. If S is a (T, T)-intertwining operator then $R^{-1}SR$ is in $\mathcal{L}_{\mathrm{a}}^{\mathrm{com}}$, hence it is scalar. By the JDT the representation (T, X) is algebraically totally irreducible. ∎

Notice that in Theorem 2.2, the continuity of the representation is not required. If we suppose the continuity of T, we can omit the assumption that W is locally convex.

Following the ideas of the last proof we can obtain a criterion of total irreducibility for representations of locally convex algebras on their quotient spaces.

Example 2.3. Let $\mathcal{A}$ be a locally convex algebra and let J be a maximal closed left ideal in $\mathcal{A}$. Then the natural action of $\mathcal{A}$ on $X = \mathcal{A}/J$ given by the formula $L_a[b] = [ab]$, $a \in \mathcal{A}$, is algebraically irreducible.

In this case every operator S in X which commutes with the action of $\mathcal{A}$ is continuous.

Indeed, if e denotes the unit in $\mathcal{A}$, we can represent S in the form $S[a] = T_a S[e]$. Denote $S[e] = [b]$. For $a \in J$ we have $0 = S[a] = L_a[b] = [ab]$. Thus $Jb \subset J$. The operator R_b of right multiplication by $b \in \mathcal{A}$ is continuous on $\mathcal{A}$ and it leaves the ideal J invariant. So, it defines a continuous operator on $X = \mathcal{A}/J$, which coincides with S. By the JDT the action of $\mathcal{A}$ on X is totally irreducible (equivalently, $\mathcal{A}$ is dense in $L(\mathcal{A}/J)$) if and only if the set $\tilde{J} = \{a \in \mathcal{A} : Ja \subset J\}$ coincides with J.

3. k-irreducibility for X locally convex. Theorem 1.3 of the Introduction provides a characterization of totally irreducible representations in a very general case.

According to this theorem a representation T of an algebra $\mathcal{A}$ on a topological vector space X, which is irreducible and has a trivial commutant $\mathcal{T}^{\mathrm{com}}$, is totally irreducible if and only if all closed densely defined $(T, T^{(k)})$-intertwining operators are continuous.

Note that if $R(x) = (R_1(x), \ldots, R_k(x))$ is a continuous $(T, T^{(k)})$-intertwining operator, then for every $1 \le i \le k$ the operator R_i belongs to $\mathcal{T}^{\mathrm{com}}$, so by assumption it is scalar. The total irreducibility of (T, X) is equivalent to all closed densely defined $(T, T^{(k)})$-intertwining operators being of the form $R(x) = (\lambda_1 x, \ldots, \lambda_k x)$, where $\lambda_i \in \mathbb{K}$.

We will prove a refinement of this theorem for X locally convex.

A representation (T, X) of the algebra $\mathcal{A}$ is said to be *k-irreducible* if for any $1 \le j \le k$ and any linearly independent $x_1, \ldots, x_j \in X$ the orbit $\mathcal{O}(T^{(j)}, x_1, \ldots, x_j)$ is dense in X^j.

Considering a representation T which is $(k-1)$-irreducible but not k-irreducible we obtain densely defined closed intertwining operators $S\colon X \supset D(S) \to X^k$ with additional properties. For X a Banach space, the analogous theorem proved in [RT] is called the Graph Theorem. We also generalize a result from [Ar] which states that for a representation with this property the orbit $\mathcal{O}(T^{(j)}, x_1, \ldots, x_j)$ consists of zero and of k-tuples of vectors which are linearly independent. This permits us to relate the problem of Fell and Doran to the problem of injective operator range. After proving a generalized Graph Theorem we obtain a generalization of Theorem 4.2 from [RT].

In this section, X is supposed to be a locally convex vector space. In the dual space X' we consider the weak* topology. We identify the dual space $(X^k)'$ with the space $(X')^k$ by associating to $\varphi \in (X^k)'$ the k-tuple $(\varphi_1, \ldots, \varphi_k) \in (X')^k$ such that $\varphi(x_1, \ldots, x_k) = \sum_{j=1}^k \varphi_j(x_j)$.

Denote by $\mathcal{A}_r$ the reversed algebra of $\mathcal{A}$, i.e. the vector space $\mathcal{A}$ provided with the product $a \cdot b = ba$. If (T, X) is a representation of $\mathcal{A}$ then the mapping $\mathcal{A} \ni a \mapsto T_a^*$ is a representation of $\mathcal{A}_r$ in the dual space X' which is denoted by (T^*, X').

THEOREM 3.1.

(i) *If a representation (T, X) is $(k-1)$-irreducible then it is not k-irreducible if and only if there exist a linearly independent k-tuple $x_1, \ldots, x_k$ in X and $\varphi_1, \ldots, \varphi_k$ in X' such that $\sum_{j=1}^k \varphi_j(T_a x_j) = 0$ for all $a \in \mathcal{A}$. Every nonzero k-tuple $(\varphi_1, \ldots, \varphi_k) \in (X')^k$ satisfying these equations consists of linearly independent elements.*

(ii) *The representation (T, X) of $\mathcal{A}$ is k-irreducible if and only if the representation (T^*, X') of $\mathcal{A}_r$ is k-irreducible.*

(iii) *If a representation (T, X) is $(k-1)$-irreducible but not k-irreducible then for every k-tuple $(x_1, \ldots, x_k) \in X^k$ of linearly independent coordinates and for every $a \in \mathcal{A}$ the system $(T_a x_1, \ldots, T_a x_k)$ is zero or consists of linearly independent elements.*

Proof. (i) Let (T, X) be a $(k-1)$-irreducible representation which is not k-irreducible. There is a linearly independent k-tuple $x_1, \ldots, x_k$ whose orbit in X^k is not dense. There is $\varphi \in (X^k)' \simeq (X')^k$ which vanishes on that orbit. Denoting $\varphi = (\varphi_1, \ldots, \varphi_k)$ we obtain, for every $a \in \mathcal{A}$,

$$\sum_{j=1}^{k} \varphi_j(T_a x_j) = \sum_{j=1}^{k} T_a^* \varphi_j(x_j) = 0. \tag{3.1}$$

It follows that the $T^{*(k)}$-orbit of φ in $(X')^k$ is not dense. We claim that the k-tuple $\varphi_1, \ldots, \varphi_k$ is linearly independent.

Suppose that, say $\varphi_k = \sum_{i=1}^{k-1} t_i\varphi_i$. Then for every $a \in \mathcal{A}$,

$$0 = \sum_{j=1}^{k} \varphi_j(T_a x_j) = \sum_{j=1}^{k-1} \varphi_j(T_a(x_j + t_j x_k)).$$

This means that the orbit $\mathcal{O}(T^{(k-1)}, x_1+t_1x_k, \ldots, x_{k-1}+t_{k-1}x_k)$ is not dense in X^{k-1}. As the representation (T, X) is $(k-1)$-irreducible, the $(k-1)$-tuple $\{x_j + t_j x_k\}_{j=1}^{k-1}$ is linearly dependent, a contradiction. This proves that the k-tuple $(\varphi_1, \ldots, \varphi_k)$ is linearly independent.

The converse implication is obvious, so assertion (i) proved.

We prove (ii) by induction. For $k = 1$ the assertion is valid, for if $V \subset X$ is a proper closed $\mathcal{T}$-invariant subspace, then $V^{\perp} = \{\varphi \in X' : \varphi(x) = 0,\ x \in V\}$ is a closed proper subspace invariant under all elements of $\mathcal{T}^*$. Similarly, if $U \subset X'$ is a closed non-trivial $\mathcal{T}^*$-invariant subspace, then $U_{\perp} = \{x \in X : \varphi(x) = 0,\ \varphi \in U\}$ is non-trivial, closed and $\mathcal{T}$-invariant.

Suppose that (ii) is valid for $j < k-1$. Let (T, X) be a $(k-1)$-irreducible representation which is not k-irreducible. By (i) there exist linearly independent k-tuples $x_1, \ldots, x_k$ in X and $\varphi_1, \ldots, \varphi_k$ in X' such that for all $a \in \mathcal{A}$,

$$\sum_{j=1}^{k} T_a^*\varphi_j(x_j) = \sum_{j=1}^{k} \varphi_j(T_a x_j) = 0.$$

Hence the $T^{*(k)}$-orbit of $(\varphi_1, \ldots, \varphi_k)$ in $(X')^k$ is not dense, so (T^*, X') is not k-irreducible.

In an analogous way one can prove that if (T^*, X') is not k-irreducible, then (T, X) is not k-irreducible.

(iii) In the notation of (i) we have $\sum_{j=1}^{k} \varphi_j(T_aT_bx_j) = \sum_{j=1}^{k} T_a^*\varphi_j(T_bx_j) = 0$ for all $a, b \in \mathcal{A}$. By (ii) the representation (T^*, X') is $(k-1)$-irreducible and is not k-irreducible. The system $\varphi_1, \ldots, \varphi_k$ is linearly independent, hence by the last statement of (i) the system $T_ax_1, \ldots, T_ax_k$ considered as a functional on $(X^k)'$ consists of linearly independent functionals. Obviously, this implies that it is linearly independent in X. ∎

THEOREM 3.2. *Let (T, X) be a representation which is $(k-1)$-irreducible but not k-irreducible. Then there exists a non-scalar densely defined closed $(T, T^{(k-1)})$-intertwining operator $S = (S_1, \ldots, S_{k-1})\colon X \to X^{k-1}$ such that each S_j is a densely defined injective operator commuting with all elements of $\mathcal{T}$.*

Proof. By Theorem 3.1(i) there exist linearly independent $x_1, \ldots, x_k$ in X and linearly independent $\varphi_1, \ldots, \varphi_k$ in X' such that

$$\sum_{j=1}^{k} T_a^*\varphi_j(x_j) = \sum_{j=1}^{k} \varphi_j(T_a x_j) = 0 \quad \text{for all } a \in A.$$

Let $G = \{(u_1, \ldots, u_k) \in X^k : \sum_{j=1}^k \varphi_j(T_a u_j) = 0,\, a \in \mathcal{A}\}$. Then G is a $T^{(k)}$-invariant closed subset of X^k which contains $(x_1, \ldots, x_k)$ and is not equal to X^k.

The set $D = \{u \in X : \exists u_2, \ldots, u_k \text{ such that } (u, u_2, \ldots, u_k) \in G\}$ is dense in X, because it contains the orbit of x_1.

Suppose that $(0, u_2, \ldots, u_k) \in G$. Then for all $a \in \mathcal{A}$,

$$\sum_{j=2}^{k} \varphi_j(T_a u_j) = \sum_{j=2}^{k} T_a^* \varphi_j(u_j) = 0.$$

This implies $u_2 = \cdots = u_k = 0$, because (T^*, X') is $(k-1)$-irreducible.

It follows that for every $u \in D$ and $(u, u_2, \ldots, u_k) \in G$ the operator defined by $S(u) = (u_2, \ldots, u_k) \in X^{k-1}$ is well defined.

The operator S, whose domain is D, is linear, closed and $(T, T^{(k-1)})$-intertwining. It is not scalar because $S(x_1) = (x_2, \ldots, x_k)$ and the k-tuple (x_j) is linearly independent. Thus the first assertion follows.

In the same way for every $1 \le j \le k$ we can obtain a densely defined closed operator associating to u_j the value $(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k) \in X^{k-1}$ when $(u_1, \ldots, u_k) \in G$. This means that in the representation $S(x) = (S_1(x), \ldots, S_{k-1}(x))$ the operators S_j, $1 \le j \le k$, are invertible and commute with the elements of $\mathcal{T}$. ■

Theorem 1.3 in its version for locally convex spaces follows immediately supplemented with the statement of injectivity of the coordinate operators S_j.

While proving Theorem 3.2 we have seen that in the case of a representation which is $(k-1)$-irreducible but not k-irreducible, there is in X^k a set G which is the graph of a family of densely defined intertwining operators. For linearly independent $(x_1, \ldots, x_k) \in G$ and $i, j \le k$ every mapping $S_{ij} \colon T_a x_i \to T_a x_j$ is well defined and injective. This implies in particular that all left ideals $\mathcal{A}_j = \{a \in \mathcal{A} : T_a x_j = 0\}$ coincide for $1 \le j \le k$.

We obtain another sufficient condition for an irreducible representation to be totally irreducible.

PROPOSITION 3.3. *Let (T, X) be an irreducible representation of $\mathcal{A}$ on a locally convex space X. If $\mathcal{A}_1 \ne \mathcal{A}_2$ for every linearly independent pair $x_1, x_2 \in X$ then the representation T is totally irreducible.*

4. On the Fell and Doran problem. In the case of the problem of Fell and Doran the topological structure of the algebra $\mathcal{A}$ is not relevant, so it is convenient to formulate the problem as a conjecture concerning properties of subalgebras of $L(X)$.

Problem. Let X be a locally convex space and let $\mathcal{T}$ be a subalgebra of $L(X)$ whose action on X is irreducible and $\mathcal{T}^{\mathrm{com}} = \mathbb{K}\,I$.

Is the action of $\mathcal{T}$ on X totally irreducible and consequently $\overline{\mathcal{T}} = L(X)$?

By Corollary 1.4 the total irreducibility holds under the assumption of algebraic irreducibility of the action of $\mathcal{T}$ on X if moreover the Closed Graph Theorem is valid in X.

The algebraic irreducibility assumption can be weakened to $\mathcal{T}$ having no non-trivial invariant injective operator ranges.

The concept of an injective operator range of order k in a Banach space was defined by Rosenthal and Troitsky [RT] where the corresponding theorem about the density of $\mathcal{T}$ in $L(X)$ was proved.

Definition. A subspace Y of a locally convex space X is called an *injective operator range of order* k if there exists a closed subspace $E \subset X^k$ and a continuous injective operator $B\colon X^k \to X$ such that $Y = B(E)$.

A subalgebra $\mathcal{T} \subset L(X)$ *has an invariant injective operator range of order* k if there is a proper subspace Y in X and a continuous injective $(\mathcal{T}^{(k)}, \mathcal{T})$-intertwining operator $B\colon X^k \to X$ such that $Y = B(E)$.

Obviously, if an algebra $\mathcal{T} \subset L(X)$ has no invariant operator ranges in X, its action in X is irreducible. On the other hand, if the action of $\mathcal{T}$ is algebraically irreducible, then $\mathcal{T}$ has no invariant operator ranges in X.

The lack of invariant operator ranges is a condition intermediate between algebraic and topological irreducibility.

Theorem 4.1. *Let X be a locally convex space in which the Closed Graph Theorem is valid and let $\mathcal{T} \subset L(X)$ be a subalgebra such that $\mathcal{T}^{\mathrm{com}} = \mathbb{K}\,I$. If $\mathcal{T} \subset L(X)$ has no non-trivial invariant injective operator ranges then the action of $\mathcal{T}$ on X is totally irreducible.*

Proof. If the action of $\mathcal{T}$ is not totally irreducible, there is $k \in \mathbb{N}$ such that this action is $(k-1)$-irreducible but not k-irreducible.

According to Theorem 3.3 there exists a non-trivial densely defined closed operator $S = S_1 \oplus \cdots \oplus S_{k-1}$ commuting with the action of $\mathcal{T}$. If G is the graph of S then the domain of S is the projection of G onto one of the coordinates. This projection P is injective on G by Theorem 3.1(iii). Since the image $P(G)$ is an injective invariant operator range, by the assumption that $P(G) = X$ and by the Closed Graph Theorem the operator S is continuous. By Theorem 1.3 the action of $\mathcal{T}$ is totally irreducible. This contradiction proves the statement. ■

In the next result the representation is supposed to be irreducible but the assumption about the commutant is strengthened.

Theorem 4.2. *Let X be a topological vector space and let (T, X) be an irreducible representation of an algebra $\mathcal{A}$. Then T is totally irreducible if and only if every densely defined $\overline{\mathcal{T}}$-intertwining operator is scalar.*

Proof. Assume that T is totally irreducible. Then $\overline{\mathcal{T}} = L(X)$. For every $\overline{\mathcal{T}}$-intertwining operator S we have $D(S) = X$. Since S commutes with all elements of $L(X)$, it is scalar.

Suppose now that T is not totally irreducible. By Theorem 1.3 there exists for some $k \in \mathbb{N}$ a densely defined closed operator $S\colon D(S) \to X^k$ which commutes with the action of $\mathcal{T}$. Let $T \in \overline{\mathcal{T}}$ and suppose that $T = \lim_\alpha T_{a_\alpha}$ in the strong operator topology with $T_{a_\alpha} \in \mathcal{T}$. For every $x \in D(S)$ we have $Tx = \lim_\alpha T_{a_\alpha} x$ and

$$\lim_\alpha ST_{a_\alpha} x = \lim_\alpha T_{a_\alpha} Sx = TSx.$$

The operator S is closed, hence $Tx \in D(S)$ and $STx = TSx$. This shows that the domain of S is invariant under $\overline{\mathcal{T}}$, and S commutes with the elements of $\overline{\mathcal{T}}$. We can represent $S = S_1 \oplus \cdots \oplus S_k$, where the operators S_j, $1 \le j \le k$, are $\overline{\mathcal{T}}$-intertwining. Since at least one of the operators S_j is not scalar, the conclusion follows. ■

Recall the well known example of an irreducible representation of an algebra which does not satisfy the condition $\mathcal{T}^{\mathrm{com}} = \mathbb{K}\,\mathrm{id}_X$ and hence is not totally irreducible. Let X be a Banach space such that there exists $A \in L(X)$ which has no closed invariant subspaces. Let $\mathcal{P}(A)$ be the algebra generated by A in $L(X)$. Then for every $0 \ne x \in X$ the orbit $\mathcal{P}(A)x$ is a non-zero A-invariant subspace, so its closure is equal to X. The action of $\mathcal{P}(A)$ on X is irreducible. Obviously, being commutative, the algebra $\mathcal{P}(A)$ is not dense in $L(X)$.

We are going to prove that in this case $L(X)$ is generated by two elements, namely by A and an arbitrary rank one operator. This is a consequence of the following more general fact.

Proposition 4.3. *Let X be a locally convex topological space and let $\mathcal{T}$ be a subalgebra of $L(X)$ acting irreducibly on X. If F is an arbitrary rank one operator, then $\mathcal{T}$ and F generate $L(X)$.*

Proof. Let $\mathcal{A}$ be the algebra generated by $\mathcal{T}$ and F. The operator F can be represented in the form $F(y) = f \otimes x(y) = \varphi(y)x$, where $\varphi \in X'$ and $0 \ne x \in X$. By Theorem 3.1(ii), the action of the algebra $\mathcal{T}^*$ on X' is irreducible, so

$$\forall f \in X',\ \varepsilon > 0,\ x_1, \ldots, x_k \in X\ \exists T \in \mathcal{T}, \quad |T^*\varphi(x_i) - f(x_i)| < \varepsilon,\ 1 \le i \le k.$$

On the other hand, by the irreducibility of the action of $\mathcal{T}$ on X, for every $x \in X$ and every neighbourhood U of zero in X there exists $A \in \mathcal{T}$ such that $Av - x \in U$. Then for every x_i,

$$\begin{aligned} f \otimes x(x_i) - AFT(x_i) &= f(x_i)x - \varphi(Tx_i)Av \\ &= (f(x_i) - \varphi(Tx_i))x + \varphi(Tx_i)(x - Av) \\ &= (f(x_i) - \varphi(Tx_i))x + (\varphi(Tx_i) - f(x_i))(x - Av) + f(x_i)(x - Av). \end{aligned}$$

The formula shows that for every neighbourhood O of zero in X we can find $A, T \in \mathcal{T}$ such that $f \otimes x(x_i) - AFT(x_i) \in O$ for all $1 \leq i \leq k$. It follows that every rank one operator $f \otimes x$ belongs to the strong closure $\overline{\mathcal{A}}$. The action of the set of rank one operators on X is algebraically irreducible. A linear operator S on X which commutes with all rank one operators is proportional to the identity.

By the JDT, the action of $\mathcal{A}$ on X is totally irreducible, hence $\mathcal{A}$ is strongly dense in $L(X)$. ■

COROLLARY 4.4. *If X is a locally convex space such that some $A \in L(X)$ has no non-trivial closed invariant subspace then $L(X)$ is generated by two elements.*

We close this section with some examples of irreducible subalgebras $\mathcal{A}$ of $L(X)$ such that the space of all densely defined operators commuting with the elements of the subalgebra is trivial. In this case Theorem 3.1 implies the strong density of $\mathcal{A}$ in $L(X)$.

EXAMPLE 4.5. Denote by $\mathcal{E}$ the space of entire functions on the complex plane provided with its natural topology of almost uniform convergence. The paper [Z4] is dedicated to generators of the space $L(\mathcal{E})$. The principal conclusion is that the translations and pointwise multiplications by elements of $\mathcal{E}$ strongly generate $L(\mathcal{E})$.

The proof uses the Jacobson Density Theorem; as part of the proof it is shown that every element of $L(\mathcal{E})$ which commutes with translations and multiplications by elements of $\mathcal{E}$ is scalar. It is easily seen that all densely defined intertwining operators are scalar.

First observe that the operator of derivation $f' = \frac{d}{dz}f$ generates translations in $\mathcal{E}$.

The Taylor series

$$f(z + w) = \sum_{n=0}^{\infty} f^{(n)}(z)\frac{w^n}{n!}$$

is convergent in $\mathcal{E}$, which implies that every translation can be strongly approximated by linear combinations of powers of $\frac{d}{dz}$.

Let S be a densely defined operator in $\mathcal{E}$ which commutes with $\frac{d}{dz}$ and with multiplication by any element of $\mathcal{E}$.

For f in the domain of S set $g = S(f)/f$, which is a function analytic outside the discrete set of zeros of f. Now,

$$g' = \frac{S(f)'f - S(f)f'}{f^2} = \frac{S(f'f) - S(ff')}{f^2} = 0$$

on the domain of g. It follows that g is constant and $S(f) = \alpha f$.

In [Z4] Żelazko asked whether $L(\mathcal{E})$ is generated by two operators.

The answer is "yes": multiplication by z and $\frac{d}{dz}$ generate $L(\mathcal{E})$.

With similar arguments one can prove that z and $\frac{d}{dz}$ generate the algebra $L(\mathcal{H}(O))$, where $\mathcal{H}(O)$ is the space of analytic functions on a connected, simply connected subset O of the complex plane.

Moreover, using the Corona Theorem the same fact can be proved for the Hardy space $H^\infty(\mathbb{D})$ on the disc.

Example 4.6. Let $(H, (\cdot|\cdot))$ be a Hilbert space. Berntzen and Sołtysiak [BS] construct two C^*-commutative subalgebras of $L(H)$ which generate $L(H)$ in the strong operator topology. The construction uses a simple but useful fact that an arbitrary set Z admits the structure of an abelian group. If $\{e_i\}_{i\in Z}$ is an orthonormal basis in H, and Z is provided with the structure of an abelian group, we can define a family of shift operators in H by the formula

$$S_i(e_j) = e_{i+j}.$$

Let $P_0(x) = (x|e_0)e_0$, where 0 means the neutral element of the group Z. The operator P_0 is the orthogonal projection on the line generated by e_0.

The main result of [BS] states that the subalgebra $\mathcal{A}$ generated by the operators S_i, $i \in I$, and P_0 is dense in $L(X)$ in the strong operator topology.

Let us verify how this result can be deduced from Theorem 1.3.

Note that for every $i \in Z$ we have $S_iP_0S_{-i} = P_i$, the orthogonal projection on the space generated by e_i. Thus the algebra $\mathcal{A}$ contains all projections on the basic vectors.

If $0 \neq V \subset H$ is a closed $\mathcal{A}$-invariant space, then V is invariant under all P_i, $i \in I$. There is i such that $P_iV \neq 0$, hence V contains at least one of the basic vectors. The invariance under all S_i implies that all basic vectors are in V, so $V = H$ and the action of $\mathcal{A}$ in H is irreducible.

Now, let A be a densely defined operator which commutes with $\mathcal{A}$ on its domain D_A. By the argument above, D_A contains all elements e_i, $i \in I$.

The formula $AP_i = P_iA$ implies $A(e_i) = AP_ie_i = P_iA(e_i) = \lambda_ie_i$ for some $\lambda_i \in \mathbb{C}$, so every element of the commutant $\mathcal{A}'$ is a diagonal operator.

Next, we have

$$\lambda_ie_{i+j} = \lambda_iS_je_i = S_jA(e_i) = AS_je_i = Ae_{i+j} = \lambda_{i+j}e_{i+j}$$

for all i, $j \in I$. It follows that $\lambda_i = \lambda_0$, $i \in I$, so the operator A is scalar.

By Theorem 1.3 the action of $\mathcal{A}$ in H is totally irreducible, hence $\mathcal{A}$ is dense in $L(H)$ in the strong operator topology.

EXAMPLE 4.7. Let X be a Q-algebra and let $\mathcal{T}$ be a subalgebra of $L(X)$. Suppose that left multiplication by each $a \in X$ belongs to $\mathcal{T}$. Then the algebraic commutant $\mathcal{T}_{\mathrm{a}}^{\mathrm{com}}$ coincides with $\mathcal{T}^{\mathrm{com}}$.

In fact, if $R \in \mathcal{T}_{\mathrm{a}}^{\mathrm{com}}$, then D_R has a non-trivial intersection with the (open) set of elements invertible in X. Let $a \in D_R$ be invertible. For every $x \in X$ we have $x = xa^{-1}a \in D_R$ and $R(x) = R(xa^{-1}a) = xa^{-1}R(a)$, which implies that R is continuous.

If moreover the action of $\mathcal{T}$ is irreducible and $\mathcal{T}^{\mathrm{com}} = \mathbb{K}\,I$, then $\mathcal{T}$ is dense in $L(X)$ by Theorem 1.3.

Doubtless, the state of investigations on the problem of Fell and Doran is not satisfactory. It seems that the problem is quite delicate, so the construction of a (possibly existing) counterexample is difficult. To this end it is sufficient to find for some locally convex space X a strongly closed subalgebra $\mathcal{A} \subset L(X)$ satisfying the following conditions:

(i) the action of $\mathcal{A}$ on X is irreducible,
(ii) $\mathcal{A}^{\mathrm{com}} = \mathbb{K}\,I$,
(iii) there is a non-trivial operator S defined on an orbit $\mathcal{O}(\mathcal{A}, x)$ commuting with all elements of $\mathcal{A}$.

So far, no example of this type is known.

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