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# Sets of bounded remainder for a continuous irrational rotation on $[0,1]^2$

by

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**1. Introduction.** In this paper we will be concerned with bounded remainder sets for a two-dimensional irrational rotation on the unit square $I^2 = [0,1)^2$.

Definition 1.1. Let $\mathbf{x} = (x_1, x_2) \in I^2$, and let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We say that the function $X : [0,\infty) \to I^2$ defined by

$$X(t) = (\{x_1 + t\}, \{x_2 + \alpha t\})$$

is the *two-dimensional continuous irrational rotation with slope* $\alpha$ *and starting point* $\mathbf{x}$.

Definition 1.2. Let $S \subset I^2$ be an arbitrary measurable subset of the unit square with Lebesgue measure $\lambda(S)$. We say that $S$ is a *bounded remainder set* for the continuous irrational rotation with slope $\alpha > 0$ and starting point $\mathbf{x} = (x_1, x_2) \in I^2$ if the distributional error

$$\Delta_T(S,\alpha,\mathbf{x}) = \int_0^T \chi_S(\{x_1 + t\}, \{x_2 + \alpha t\})\, dt - T\lambda(S) \tag{1.1}$$

is uniformly bounded for all $T > 0$. Here, $\chi_S$ denotes the characteristic function for the set $S$.

Bounded remainder sets have been extensively studied for the discrete analogue of continuous irrational rotation, that is, for Kronecker sequences $(\{n\alpha_1\}, \ldots, \{n\alpha_s\})_{n=1}^{\infty}$ in $[0,1)^s$, where $\alpha_1, \ldots, \alpha_s$ are given reals. In this context, a bounded remainder set $S \subseteq [0,1)^s$ is a measurable set for which

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the difference

$$\Big|\sum_{n=1}^{N} \chi_S(\{x_1+n\alpha_1\},\ldots,\{x_s+n\alpha_s\}) - N\lambda(S)\Big|$$

is uniformly bounded for all integers $N \geq 1$ and for almost every point $(x_1,\ldots,x_s) \in [0,1)^s$. In the simplest case when $s = 1$ and $S$ is just an interval, bounded remainder sets for the Kronecker sequences were explicitly characterized by Hecke [8], Ostrowski [13, 14] and Kesten [10]. In the general multi-dimensional case, a characterization of bounded remainder sets in terms of equidecomposability to certain parallelepipeds was recently given in [7].

Without going into further detail on the known results for the Kronecker sequences, let us simply emphasize that in the discrete case, a given set $S \subset [0,1)^s$ is a bounded remainder set for only "very few" choices of $(\alpha_1,\ldots,\alpha_s)$. Likewise, given a vector $(\alpha_1,\ldots,\alpha_s)$, the class of sets $S$ which are of bounded remainder with respect to this vector is, in some sense, small. Once we consider bounded remainder sets for *continuous* irrational rotations, the situation turns out to be quite different. In light of recent work by József Beck, this is not entirely unexpected. Beck studied distributional properties of continuous irrational rotations in [1, 2, 3], and showed in particular:

Theorem (Beck [3, Theorem 1]). *Let $S \subseteq I^2$ be an arbitrary Lebesgue measurable set in the unit square with positive measure. Then for every $\varepsilon > 0$, almost all $\alpha > 0$ and every starting point $\mathbf{x} = (x_1, x_2) \in I^2$, we have*

$$\int_0^T \chi_S(\{x_1+t\},\{x_2+\alpha t\})\,dt - T\lambda(S) = o((\log T)^{3+\varepsilon}).$$

As pointed out by Beck, the polylogarithmic error term is shockingly small compared to the linear term $T\lambda(S)$. Moreover, it is so for *all* measurable sets $S$. It is thus natural to ask if imposing certain regularity conditions on $S$ could give an even lower bound on the error term.

The aim of this paper is to show that the estimate of Beck can be significantly improved for a large collection of sets $S$. We show that:

Theorem 1.3. *For almost all $\alpha > 0$ and every $\mathbf{x} \in I^2$, every polygon $S \subset I^2$ with no edge of slope $\alpha$ is a bounded remainder set for the continuous irrational rotation with slope $\alpha$ and starting point $\mathbf{x}$.*

Theorem 1.4. *For almost all $\alpha > 0$ and every $\mathbf{x} \in I^2$, every convex set $S \subset I^2$ whose boundary $\partial S$ is a twice continuously differentiable curve with positive curvature at every point is a bounded remainder set for the continuous irrational rotation with slope $\alpha$ and starting point $\mathbf{x}$.*

We will see from the proofs that Theorems 1.3 and 1.4 hold for all $\alpha$ whose continued fraction expansion $\alpha = [a_0; a_1, a_2, \ldots]$ satisfies

$$\sum_{l=0}^{s} \frac{a_{l+1}}{q_l^{1/2}} \sum_{k=1}^{l+1} a_k < C,$$

where $C$ is a constant independent of $s$. Here, $(q_l)_{l\geq 0}$ is the sequence of best approximation denominators for $\alpha$.

The following are immediate consequences of Theorems 1.3 and 1.4.

Corollary 1.5. *Let $S$ be a polygon in $I^2$. Then $S$ is a bounded remainder set with respect to the continuous irrational rotation for almost every slope $\alpha > 0$ and every starting point $\mathbf{x} \in I^2$.*

Corollary 1.6. *Let $S$ be a convex set in $I^2$ whose boundary $\partial S$ is a twice continuously differentiable curve with positive curvature at every point. Then $S$ is a bounded remainder set with respect to the continuous irrational rotation for almost every slope $\alpha > 0$ and every starting point $\mathbf{x} \in I^2$.*

In light of Corollaries 1.5 and 1.6, it is tempting to raise the question of whether *every* convex set $S \subset I^2$ is a bounded remainder set with respect to the continuous irrational rotation for almost every slope $\alpha > 0$ and every starting point $\mathbf{x} \in I^2$. We leave this question open.

Theorems 1.3 and 1.4 above are, in a certain sense, optimal. First of all, the slope condition in Theorem 1.3 on the edges of the polygon $S$ cannot be omitted. To see this, fix some $\alpha > 0$, and let $S$ be the parallelogram shown in Figure 1 with $p \notin \mathbb{Z}\alpha \pmod 1$ and $\lambda(S) = p$. It is not difficult to show

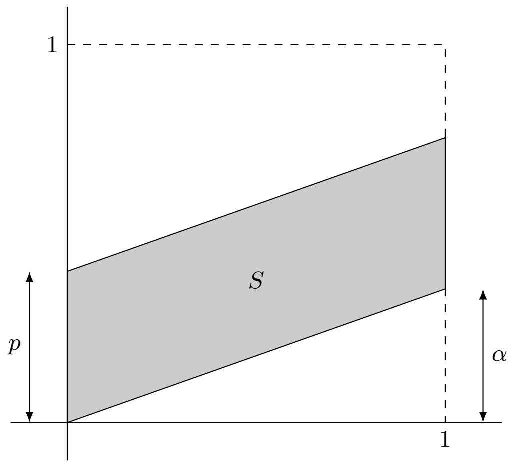


Fig. 1. The parallelogram $S$ with two edges of slope $\alpha$

that for such a set $S$, with two edges of slope $\alpha$, we have

$$\Big|\int_0^T \chi_S(\{t\},\{\alpha t\})\,dt - \sum_{n=1}^{\lfloor T\rfloor} \chi_{[0,p)}(\{n\alpha\})\Big| \le 1.$$

We recall from the discrete setting that if $p \notin \mathbb{Z}\alpha \pmod 1$, then the difference

$$\Big|\sum_{n=1}^{\lfloor T\rfloor} \chi_{[0,p)}(\{n\alpha\}) - p\lfloor T\rfloor\Big|$$

is unbounded as $T \to \infty$ [10], and accordingly so is

$$|\Delta_T(S,\alpha,0)| = \Big|\int_0^T \chi_S(\{t\},\{\alpha t\})\,dt - pT\Big|.$$

Thus, the set $S$ in Figure 1 is not of bounded remainder for the continuous irrational rotation with slope $\alpha$ starting at the origin. By an equivalent argument, all sets $S'$ similar to the examples shown in Figure 2 with $p \notin \mathbb{Z}\alpha \pmod 1$ are *not* bounded remainder sets.

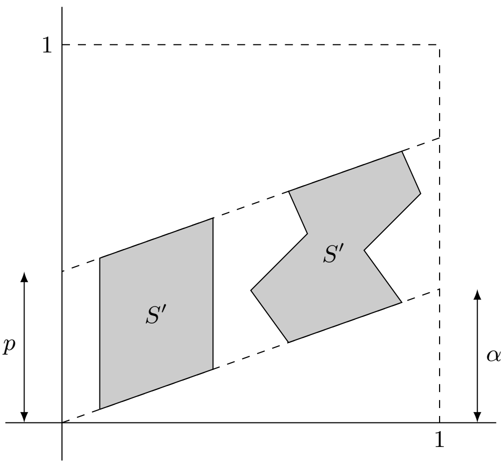


Fig. 2. Sets $S'$ which are not of bounded remainder for the continuous irrational rotation with slope $\alpha$ (given $p \notin \mathbb{Z}\alpha \pmod 1$)

Secondly, in neither Theorem 1.3 nor 1.4 can we replace "for almost all $\alpha$" by "for all irrational $\alpha$". This is clarified by the following:

THEOREM 1.7.

(i) *For uncountably many $\alpha > 0$ there exist triangles in $I^2$ with no edge of slope $\alpha$ which are not bounded remainder sets for the continuous irrational rotation with slope $\alpha$ and an arbitrary starting point.*

(ii) *For uncountably many $\alpha > 0$ there exist discs in $I^2$ which are not bounded remainder sets for the continuous irrational rotation with slope $\alpha$ and an arbitrary starting point.*

(iii) *The triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ is a bounded remainder set for every slope $\alpha > 0$ and every starting point $\mathbf{x} \in I^2$.*

Theorem 1.7(iii) illustrates that for very special polygons $S$, Theorem 1.3 does actually hold for all irrational $\alpha$. Other trivial examples of such special sets are rectangles of the form $[0,\gamma) \times [0,1)$ (or $[0,1) \times [0,\gamma)$), where $0 < \gamma \le 1$.

Finally, let us point out that Theorems 1.3 and 1.4, and their proofs, give information on the behavior of discrepancies of continuous irrational rotations on the unit square. Let $\mathcal{B}$ denote a certain class of measurable subsets of $I^2$. Then by the *discrepancy* $D_T^{(\mathcal{B})}$ of the continuous irrational rotation with slope $\alpha > 0$ and starting point $\mathbf{x} \in I^2$ with respect to $\mathcal{B}$ we mean

$$D_T^{(\mathcal{B})} := \sup_{S \in \mathcal{B}} \Delta_T(S, \alpha, \mathbf{x}),$$

with $\Delta_T(S, \alpha, \mathbf{x})$ defined in (1.1). The most extensively studied case in the classical theory of irregular distribution is when $\mathcal{B}$ is the class of axis-parallel rectangles. Theorem 1.3 tells us that in this case,

$$\Delta_T(S, \alpha, \mathbf{x}) = O(1)$$

for all $\mathbf{x}$, almost all $\alpha$ and all $S \in \mathcal{B}$. Moreover, by a careful consideration of the constants involved in the proof of Theorem 1.3, one can verify that the $O$-constant will depend only on $\alpha$, and not on the choice of the rectangle $S$. As a consequence, we obtain the following result, previously shown by Drmota [4] (see also [5]).

COROLLARY 1.8. *The discrepancy $D_T^{(\mathcal{B})}$ of the continuous irrational rotation with slope $\alpha$ and starting point $\mathbf{x}$ with respect to the class $\mathcal{B}$ of axis-parallel rectangles in $I^2$ is*

$$D_T^{(\mathcal{B})} = O(1) \quad \text{for all } \mathbf{x} \in I^2 \text{ and almost all } \alpha > 0.$$

As clarified by the example in Figure 1, an analogous result does not hold if $\mathcal{B}$ is the class of *all* rectangles. It follows that the *isotropic discrepancy*, i.e. the discrepancy with respect to the class of all convex sets, cannot be bounded. However, if we let $\mathcal{B}$ be the class $\mathcal{D}$ of all discs in $I^2$, then we can attain a result analogous to Corollary 1.8. Theorem 1.4 tells us that for all $S \in \mathcal{D}$, we have

$$\Delta_T(S, \alpha, \mathbf{x}) = O(1)$$

for all $\mathbf{x}$ and almost all $\alpha$, and from the proof of Theorem 1.4 it is not difficult to see that the $O$-constant can be made independent of the size and position of the disc $S$. We thus get:

Corollary 1.9. *The discrepancy* $D_T^{(\mathcal{D})}$ *of the continuous irrational rotation with slope* $\alpha$ *and starting point* $\mathbf{x}$ *with respect to the class* $\mathcal{D}$ *of discs in* $I^2$ *is*

$$D_T^{(\mathcal{D})} = O(1) \quad \text{for all } \mathbf{x} \in I^2 \text{ and almost all } \alpha > 0.$$

The rest of the paper is organized as follows. In Section 2 we present necessary preliminary material, and give the proofs of Theorems 1.3 and 1.4. Section 3 is devoted to the proof of Theorem 1.7.

## 2. Preliminaries and proofs of Theorems 1.3 and 1.4

**2.1. Continued fractions.** We begin by briefly reviewing some well-known facts about continued fractions. For an irrational $\alpha \in (0,1)$, let

$$[0; a_1, a_2, a_3, \ldots]$$

be its continued fraction expansion, and denote by $p_n/q_n$ its $n$th convergent. The numerators $p_n$ and denominators $q_n$ are given recursively by

$$\begin{aligned} q_0 &= 1, \quad q_1 = a_1, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \\ p_0 &= 0, \quad p_1 = 1, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}. \end{aligned}$$

It follows readily from these recurrences that

$$p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}. \tag{2.1}$$

The $n$th convergent $p_n/q_n$ is greater than $\alpha$ for every odd value of $n$, and smaller than $\alpha$ for every even value of $n$. It is easy to see that $\lim_{n\to\infty} p_n/q_n = \alpha$, and moreover we have the error bounds

$$\frac{1}{(a_{n+1}+2)q_n^2} \le \left|\alpha - \frac{p_n}{q_n}\right| \le \frac{1}{a_{n+1}q_n^2}. \tag{2.2}$$

Every non-negative integer $N$ has a unique expansion

$$N = \sum_{i=0}^{s} b_i q_i \quad \text{with } b_s > 0,\ 0 \le b_i \le a_{i+1},\ 0 \le i \le s.$$

We will refer to this as the *Ostrowski expansion* of $N$ to base $\alpha$.

Finally, we will need the following result, which follows from well-known facts in metric theory of continued fractions (see e.g. [11]).

Lemma 2.1. *For almost every irrational* $\alpha \in (0,1)$ *and every* $m > 0$,

$$\sum_{l=0}^{s} \frac{a_{l+1}}{q_l^{1/m}} \sum_{k=1}^{l+1} a_k \quad \textit{is uniformly bounded in } s.$$

**2.2. Functions of bounded remainder.** It is not difficult to show that the question of whether $S \subset I^2$ is a bounded remainder set for a continuous two-dimensional irrational rotation is essentially a one-dimensional problem.

By making an appropriate projection, the question can be restated as that of whether a certain associated *function* is of bounded remainder.

DEFINITION 2.2. Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic function which is integrable over $[0, 1]$. We say that $f$ is a *bounded remainder function* with respect to $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ if there is a constant $C = C(f, \alpha)$ such that

$$\Big|\sum_{k=0}^{N-1} f(k\alpha) - N\int_0^1 f(x)\,dx\Big| \le C \quad \text{for all integers } N > 0.$$

Bounded remainder functions have been studied by several authors (see e.g. [9], or [15] and the references therein).

We will consider two special classes of functions: hat functions and dome functions.

DEFINITION 2.3. We say that $T : \mathbb{R} \to [0, \infty)$ is a *hat function* if $T$ is supported on the interval $[0, b]$, with $b > 0$, and

$$T(x) = \begin{cases} \dfrac{H}{a}x, & 0 \le x \le a, \\ -\dfrac{H}{b-a}(x-b), & a < x \le b, \end{cases} \tag{2.3}$$

for some $0 < a < b$ and $H > 0$.

DEFINITION 2.4. We say that a continuous function $T : \mathbb{R} \to [0, \infty)$ supported on $[0, B]$, with $B > 0$, is a *dome function* if it satisfies the following conditions:

(i) $T$ is concave and twice differentiable on the open interval $(0, B)$.
(ii) There exist $\varepsilon, m, c > 0$ such that

$$|T(x)| \le c \cdot x^{1/m}, \quad |T(B-x)| \le c \cdot x^{1/m} \quad \text{for all } 0 \le x < \varepsilon. \tag{2.4}$$

We will establish and prove the following two results, which will be crucial for the proofs of Theorems 1.3 and 1.4 later on.

PROPOSITION 2.5. *Let $\tau(x) = \sum_{m\in\mathbb{Z}} T(x+m)$, where $T$ is a hat function. Then $\tau$ is a bounded remainder function for almost every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

PROPOSITION 2.6. *Let $\tau(x) = \sum_{m\in\mathbb{Z}} T(x+m)$, where $T$ is a dome function. Then $\tau$ is a bounded remainder function for almost every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.*

REMARK 2.7. For sufficiently regular functions, including periodizations of hat and dome functions, the bounded remainder property is not affected by shifting the function (see [15, pp. 128–129]). It thus follows from Propositions 2.5 and 2.6 that for almost every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have

$$\Big|\sum_{k=0}^{N-1} \tau(k\alpha + x_0) - N\int_0^1 \tau(x)\,dx\Big| \le C$$

for all $N > 0$ and every $x_0 \in \mathbb{R}$ whenever $\tau$ is the periodization of a hat or a dome function. The constant $C$ may depend on $\tau$ and $\alpha$, but not on $N$ or $x_0$.

Later on we explain how Theorems 1.3 and 1.4 follow from the results above.

For the proof of Proposition 2.5, we will need the following lemma.

LEMMA 2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function, $\alpha$ be irrational and $N$ be a non-negative integer with Ostrowski expansion*

$$N = b_s q_s + \cdots + b_0 q_0$$

*to base $\alpha$. We then have*

$$\sum_{k=0}^{N-1} f(k\alpha) = \sum_{l=0}^{s} \sum_{b=0}^{b_l-1} \sum_{k=0}^{q_l-1} f\left(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\right) \tag{2.5}$$

*for some $\rho_{k,l}$ satisfying $-1 < \rho_{k,l} < 2$.*

*Proof.* Let $n(0) = 0$ and $n(l) = b_{l-1}q_{l-1} + \cdots + b_0 q_0$ for $1 \le l \le s$. It is straightforward to show that

$$\sum_{k=0}^{N-1} f(k\alpha) = \sum_{l=0}^{s} \sum_{b=0}^{b_l-1} \sum_{k=0}^{q_l-1} f\big(k\alpha + (n(l) + bq_l)\alpha\big). \tag{2.6}$$

We define $\theta_l$ via the equation

$$\frac{\theta_l}{a_{l+1}q_l^2} = \alpha - \frac{p_l}{q_l},$$

and observe that by (2.2) we have $1/3 \le |\theta_l| \le 1$. Moreover, we find $x_l \in [0,1)$ and $m_l \in \{0,1,\ldots,q_l-1\}$, $m_l = m_l(b,x,\alpha)$, such that

$$\{(n(l) + bq_l)\alpha\} = \frac{m_l}{q_l} + \frac{x_l}{q_l}.$$

We can then rewrite the summand on the right hand side in (2.6) as

$$f\big(k\alpha + (n(l) + bq_l)\alpha\big) = f\left(\frac{kp_l + m_l}{q_l} + \frac{k\theta_l}{a_{l+1}q_l^2} + \frac{x_l}{q_l}\right). \tag{2.7}$$

Using the substitution $kp_l + m_l = t \pmod{q_l}$, which by (2.1) gives

$$k = (t - m_l)q_{l-1}(-1)^{l-1} \pmod{q_l},$$

we get

$$\left\{\frac{kp_l + m_l}{q_l} + \frac{k\theta_l}{a_{l+1}q_l^2} + \frac{x_l}{q_l}\right\} = \left\{\frac{t}{q_l} + \frac{\rho_{t,l}}{q_l}\right\}, \tag{2.8}$$

where

$$\rho_{t,l} := \left\{(t-m_l)(-1)^{l-1}\frac{q_{l-1}}{q_l}\right\}\frac{\theta_l}{a_{l+1}} + x_l. \tag{2.9}$$

With this definition we have

$$-\frac{1}{a_{l+1}} < \rho_{t,l} < \frac{1}{a_{l+1}} + 1,$$

and hence $-1 < \rho_{t,l} < 2$. Combining (2.6)–(2.8), we thus arrive at (2.5). ∎

*Proof of Proposition 2.5.* It suffices to handle the case when $b \le 1$ in Definition 2.3. To see this, observe that any general hat function $T$ can be written as a sum of shifted hat functions $T_i$ with support $[0,b]$, $b \le 1$. Since any finite sum of bounded remainder functions is again a bounded remainder function, the general case follows from the special case $\tau(x) = \sum_{m\in\mathbb{Z}} T_i(x+m)$.

Our goal is to show that for almost every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, we can find a constant $C = C(\alpha,\tau)$ such that

$$\Big|\sum_{k=0}^{N-1}\tau(k\alpha) - N\int_0^1 \tau(x)\,dx\Big| \le C \tag{2.10}$$

for every integer $N > 0$. It will be enough to verify this for $\alpha \in (0,1)$, as the sum in (2.10) depends only on the fractional part of $\alpha$. By Lemma 2.8 we may rewrite this sum as

$$\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1}\tau\Big(\frac{k}{q_l}+\frac{\rho_{k,l}}{q_l}\Big),$$

where $N = b_s q_s + \cdots + b_0 q_0$ is the Ostrowski expansion of $N$ to base $\alpha$ and $-1 < \rho_{k,l} < 2$. We verify (2.10) in two steps: First we show that

$$\Big|\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1}\tau\Big(\frac{k}{q_l}\Big) - N\int_0^1\tau(x)\,dx\Big| \le C, \qquad N = 1,2,\ldots, \tag{2.11}$$

for almost every irrational $\alpha \in (0,1)$. We then show that

$$\Big|\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1}\Big(\tau\Big(\frac{k}{q_l}+\frac{\rho_{k,l}}{q_l}\Big) - \tau\Big(\frac{k}{q_l}\Big)\Big)\Big| \le C, \qquad s = 1,2,\ldots, \tag{2.12}$$

for almost every irrational $\alpha \in (0,1)$. Combining (2.11) and (2.12), we immediately obtain (2.10).

Let us verify that (2.11) holds. On the interval $I$, the function $\tau$ is of the form (2.3) with $b \le 1$, so we can find $u_l, v_l \in \{0,1,\ldots,q_l-1\}$ and $\xi_l, \eta_l \in (0,1]$ such that

$$a = \frac{u_l+\xi_l}{q_l} \quad\text{and}\quad b = \frac{v_l+\eta_l}{q_l}. \tag{2.13}$$

For sufficiently large $l > l_0$ (where $l_0 = l_0(\tau)$ depends only on $\tau$), we have $u_l < v_l$, and a straightforward calculation gives

$$\sum_{k=0}^{q_l-1} \tau\left(\frac{k}{q_l}\right) = \frac{Hb}{2} q_l + \frac{Ha\eta_l(1-\eta_l) - Hb\xi_l(1-\xi_l)}{2a(b-a)q_l}.$$

Thus we have

$$\left|\sum_{k=0}^{q_l-1} \tau\left(\frac{k}{q_l}\right) - q_l \int_0^1 \tau(x)\,dx\right| \le C\frac{1}{q_l},$$

where $C = C(\tau)$ (this is trivially true also when $l \le l_0$), and so

$$\left|\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1} \tau\left(\frac{k}{q_l}\right) - N\int_0^1 \tau(x)\,dx\right| \le C\sum_{l=0}^{s}\frac{b_l}{q_l}.$$

Since $b_l < a_{l+1}$, it follows from Lemma 2.1 that the right hand side above is uniformly bounded in $s$ for almost every $\alpha \in (0,1)$. This confirms (2.11).

We go on to verify (2.12). We will assume below that $b < 1$ in (2.3); the proof when $b = 1$ is slightly simpler, but essentially the same. Let $Q_l = \{0, 1, \ldots, q_l - 1\}$, and define $u_l, v_l \in Q_l$ as in (2.13). Denote by $E$ a set of "exceptional" indices

$$E = \{0, u_l - 1, u_l, u_l + 1, v_l - 1, v_l, v_l + 1, q_l - 1\}$$

(for sufficiently large $l \ge l_0$, these are all distinct). We have

$$\sum_{k=0}^{q_l-1} \tau\left(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\right) = \sum_{k\in Q_l\setminus E} \tau\left(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\right) + \sum_{k\in E} \tau\left(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\right), \tag{2.14}$$

and since $\tau$ is everywhere linear (with bounded slope), it is clear that

$$\sum_{k\in E} \tau\left(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\right) = \sum_{k\in E} \tau\left(\frac{k}{q_l}\right) + O\left(\frac{1}{q_l}\right). \tag{2.15}$$

The second sum on the right hand side in (2.14) can be rewritten using the specific form (2.3) of $\tau$ on $I$. We get

$$\sum_{k\in Q_l\setminus E} \tau\left(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\right) = \sum_{k\in Q_l\setminus E} \tau\left(\frac{k}{q_l}\right) + \Sigma_1, \tag{2.16}$$

where

$$\Sigma_1 := \frac{1}{q_l}\left(\frac{H}{a}\sum_{k=1}^{u_l-2} \rho_{k,l} - \frac{H}{b-a}\sum_{k=u_l+2}^{v_l-2} \rho_{k,l}\right),$$

and $\rho_{k,l}$ is defined in (2.9). To verify (2.12), we will need to find an appropriate bound on $\Sigma_1$.

We now show that $\varSigma_1 = O(\sum_{i=1}^{l} a_i/q_l)$. By defining $\alpha_l$ and $\gamma_l$ as

$$\alpha_l := (-1)^{l-1}\frac{q_{l-1}}{q_l}, \qquad \gamma_l := -m_l(-1)^{l-1}\frac{q_{l-1}}{q_l}, \tag{2.17}$$

we can rewrite $\rho_{k,l}$ of (2.9) as

$$\rho_{k,l} = \omega_{k,l}\frac{\theta_l}{a_{l+1}} + x_l,$$

where $\omega_{k,l} := \{k\alpha_l + \gamma_l\}$. Using (2.13) and the fact that $x_l \in [0, 1)$, it is an easy task to show that

$$\frac{H}{a}\sum_{k=1}^{u_l-2} x_l - \frac{H}{b-a}\sum_{k=u_l+2}^{v_l-2} x_l = O(1).$$

We thus have

$$\begin{aligned}\varSigma_1 &= \frac{\theta_l}{q_l a_{l+1}}\left(\frac{H}{a}\sum_{k=1}^{u_l-2}\omega_{k,l} - \frac{H}{b-a}\sum_{k=u_l+2}^{v_l-2}\omega_{k,l}\right) + O\left(\frac{1}{q_l}\right) \\ &= \frac{H\theta_l}{q_l a_{l+1}}\left(\frac{1}{a}\sum_{k=0}^{u_l-1}\omega_{k,l} - \frac{1}{b-a}\sum_{k=u_l}^{v_l-1}\omega_{k,l}\right) + O\left(\frac{1}{q_l}\right),\end{aligned} \tag{2.18}$$

where the last equality follows from the boundedness of the terms $\omega_{k,l}$.

To further approximate $\varSigma_1$, we employ Koksma's inequality for the sequence $\{\omega_{k,l}\}_{k=0}^{q_l-1}$ and the linear function $f(x) = \{x\}$ (see [12, Theorem 5.1]). For $1 \le N \le q_l$, we have

$$\left|\sum_{k=0}^{N-1}\omega_{k,l} - N\int_0^1 x\,dx\right| = \left|\sum_{k=0}^{N-1}\omega_{k,l} - \frac{N}{2}\right| \le ND_N^*(\omega_{k,l})V_I(f),$$

where $V_I(f) = 1$ is the total variation of $f$ over $I$, and $D_N^*(\omega_{k,l})$ denotes the star-discrepancy of the point set $\{\omega_{k,l}\}_{k=0}^{N-1}$. The extreme discrepancy $D_N$ of $\{\omega_{k,l}\}_{k=0}^{N-1}$ equals that of $\{k\alpha_l\}_{k=0}^{N-1}$. Note that $|\alpha_l| = q_{l-1}/q_l$ has continued fraction expansion

$$|\alpha_l| = [0; a_l, a_{l-1}, \ldots, a_1].$$

It follows that

$$ND_N^*(\omega_{k,l}) \le ND_N(\omega_{k,l}) = ND_N(k\alpha_l) \le 1 + 2\sum_{i=1}^{l} a_i \tag{2.19}$$

for $1 \le N \le q_l$ (see [12, p. 126] for the last inequality). Hence, we have

$$\left|\sum_{k=0}^{N-1}\omega_{k,l} - \frac{N}{2}\right| \le 1 + 2\sum_{i=1}^{l} a_i,$$

and from this and (2.18) it follows that

$$\begin{aligned}\Sigma_1 &= \frac{H\theta_l}{q_l a_{l+1}}\bigg(\Big(\frac{1}{a}+\frac{1}{b-a}\Big)\sum_{k=0}^{u_l-1}\omega_{k,l} - \frac{1}{b-a}\sum_{k=0}^{v_l-1}\omega_{k,l}\bigg) + O\Big(\frac{1}{q_l}\Big)\\ &= \frac{H\theta_l}{q_l a_{l+1}}\Big(\frac{b}{a(b-a)}\cdot\frac{u_l}{2} - \frac{1}{b-a}\cdot\frac{v_l}{2}\Big) + O\Big(\frac{\sum_{i=1}^{l} a_i}{q_l}\Big)\\ &= \frac{H\theta_l}{2q_l a_{l+1}}\Big(\frac{b}{a(b-a)}(q_l a-\xi_l) - \frac{1}{b-a}(q_l b-\eta_l)\Big) + O\Big(\frac{\sum_{i=1}^{l} a_i}{q_l}\Big)\\ &= O\Big(\frac{\sum_{i=1}^{l} a_i}{q_l}\Big).\end{aligned}$$

Let us finally see that this bound on $\Sigma_1$ implies (2.12). Inserting (2.15) and (2.16) in (2.14), we get

$$\bigg|\sum_{k=0}^{q_l-1}\tau\Big(\frac{k}{q_l}+\frac{\rho_{k,l}}{q_l}\Big) - \sum_{k=0}^{q_l-1}\tau\Big(\frac{k}{q_l}\Big)\bigg| \le \frac{C}{q_l}\sum_{i=1}^{l} a_i$$

for $l \ge l_0 = l_0(\tau)$ and some constant $C$ which depends only on $\tau$ and $\alpha$ (this bound holds trivially also when $l < l_0$). We thus have

$$\begin{aligned}\bigg|\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1}\Big(\tau\Big(\frac{k}{q_l}+\frac{\rho_{k,l}}{q_l}\Big)-\tau\Big(\frac{k}{q_l}\Big)\Big)\bigg| &\le C'a_1 + C\sum_{l=1}^{s}\frac{b_l}{q_l}\sum_{i=1}^{l} a_i\\ &\le C\sum_{l=0}^{s}\frac{a_{l+1}}{q_l}\sum_{i=1}^{l+1} a_i.\end{aligned}$$

By Lemma 2.1, the sum on the right hand side above is bounded uniformly in $s$ for almost every irrational $\alpha \in (0,1)$. This verifies (2.12), and completes the proof of Proposition 2.5. ∎

Before we embark on the proof of Proposition 2.6, we establish the following preliminary result.

LEMMA 2.9. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a dome function as in Definition* 2.4, *and let $q > 2/B$. Denote by $f'_q$ the function*

$$f'_q(x) = \begin{cases} f'(x) & \text{if } 1/q \le x \le B-1/q,\\ 0 & \text{otherwise.}\end{cases} \tag{2.20}$$

*Then for $q > 1/\varepsilon$, with $\varepsilon$ as in* (2.4), *the total variation $V_I(f'_q)$ of $f'_q$ over $I$ satisfies*

$$V_I(f'_q) \le Cq^{1-1/m},$$

*where $C = C(c)$ with $c$ as in* (2.4).

*Proof.* The function $f$ is concave and twice differentiable on $(0, B)$, so $f'$ is non-increasing and

$$V_I(f'_q) = 2\left(f'\left(\frac{1}{q}\right) - f'\left(B - \frac{1}{q}\right)\right).$$

Moreover, we have

$$f'\left(\frac{1}{q}\right) \le \frac{f(1/q) - f(0)}{1/q} = qf\left(\frac{1}{q}\right),$$

and likewise

$$f'\left(B - \frac{1}{q}\right) \ge -qf\left(B - \frac{1}{q}\right).$$

By the conditions (2.4) on $f$ it thus follows that

$$V_I(f'_q) \le 4cq^{1-1/m} \quad \text{for all } q > 1/\varepsilon. \ \blacksquare$$

*Proof of Proposition 2.6.* It suffices to handle the case when $B \le 1$ in Definition 2.4. To see this, observe that any general dome function $T$ can be written as a sum of shifted hat functions, and shifted dome functions with support in $I$. For $1 < B \le 2$ this is illustrated in Figure 3; we may write

$$T = T_1 + T_2 + T_3,$$

where $T_1$ is the hat function in (2.3) with $a = 1$, $b = B$ and $H = T(1)$, and $T_2$ and $T_3$ are the dome functions $T_2 = \chi_{[0,1]} \cdot (T - T_1)$ and $T_3 = \chi_{[1,B]} \cdot (T - T_1)$. As the sum of finitely many bounded remainder functions is again a bounded remainder function, the general case follows from the special case $T = T_3$ and Proposition 2.5. In other words, it is sufficient to consider the case when, restricted to the unit interval, $\tau$ is simply a dome function with support $[0, B]$, $B \le 1$.

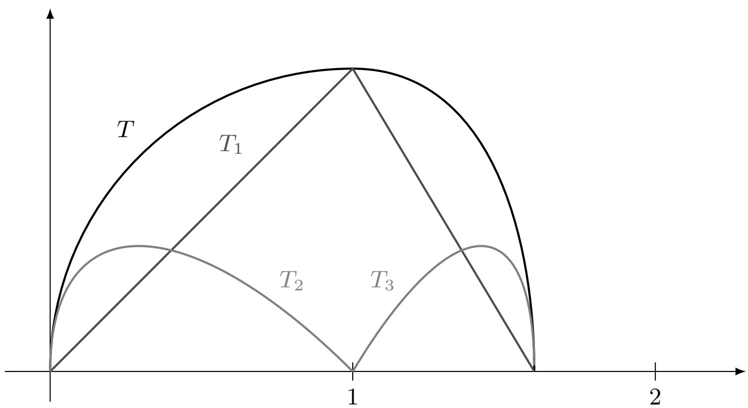


Fig. 3. The dome function $T$ decomposed as the sum of a hat function $T_1$, and two dome functions $T_2$ and $T_3$ supported on intervals of length at most one

Let $\tau$ be such a function. We want to show that for almost every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we can find a constant $C = C(c, m, \alpha)$ such that

$$\Big|\sum_{k=0}^{N-1} \tau(k\alpha) - N\int_0^1 \tau(x)\,dx\Big| \le C \tag{2.21}$$

for every integer $N > 0$. Again, it will be enough to verify this for $\alpha \in (0,1)$, as the sum in (2.21) depends only on the fractional part of $\alpha$. By Lemma 2.8, we may rewrite this sum as

$$\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1} \tau\Big(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\Big),$$

where $N = b_s q_s + \cdots + b_0 q_0$ is the Ostrowski expansion of $N$ to base $\alpha$ and $-1 < \rho_{k,l} < 2$. We verify (2.21) in two steps: First we prove that

$$\Big|\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1} \tau\Big(\frac{k}{q_l}\Big) - N\int_0^1 \tau(x)\,dx\Big| \le C, \qquad N = 1, 2, \ldots, \tag{2.22}$$

for almost every irrational $\alpha \in (0,1)$. Then we show that

$$\Big|\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1}\Big(\tau\Big(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\Big) - \tau\Big(\frac{k}{q_l}\Big)\Big)\Big| \le C, \qquad s = 1, 2, \ldots, \tag{2.23}$$

for almost every irrational $\alpha \in (0,1)$. Combining (2.22) and (2.23), we immediately obtain (2.21).

Let us see that (2.22) holds. On $I$, the function $\tau$ is supported on $[0, B]$ with $0 < B \le 1$, so we can find $u_l \in \{0, 1, \ldots, q_l - 1\}$ and $\xi_l \in (0,1]$ such that

$$B = \frac{u_l + \xi_l}{q_l}. \tag{2.24}$$

Consider the inner sum

$$\sum_{k=0}^{q_l-1} \tau\Big(\frac{k}{q_l}\Big) = \sum_{k=1}^{u_l-1} \tau\Big(\frac{k}{q_l}\Big) + \tau\Big(\frac{u_l}{q_l}\Big).$$

It is not difficult to show, for instance using integration by parts, that

$$\sum_{k=1}^{u_l-1} \tau\Big(\frac{k}{q_l}\Big) = q_l \int_{1/q_l}^{(u_l-1)/q_l} \tau(x)\,dx + \frac{1}{2}\Big(\tau\Big(\frac{1}{q_l}\Big) + \tau\Big(\frac{u_l-1}{q_l}\Big)\Big)$$
$$+ \int_{1/q_l}^{(u_l-1)/q_l} \Big(\{q_l x\} - \frac{1}{2}\Big)\tau'(x)\,dx,$$

and hence

$$\sum_{k=0}^{q_l-1}\tau\Big(\frac{k}{q_l}\Big)-q_l\int_0^1\tau(x)\,dx=\tau\Big(\frac{u_l}{q_l}\Big)+\frac{1}{2}\Big(\tau\Big(\frac{1}{q_l}\Big)+\tau\Big(\frac{u_l-1}{q_l}\Big)\Big)$$
$$-q_l\Big(\int_0^{1/q_l}\tau(x)\,dx+\int_{(u_l-1)/q_l}^{B}\tau(x)\,dx\Big)$$
$$+\int_{1/q_l}^{(u_l-1)/q_l}\Big(\{q_lx\}-\frac{1}{2}\Big)\tau'(x)\,dx.$$

Now let $l>l_0=l_0(\tau)$ be sufficiently large for $q_l>2/\varepsilon$. It is then clear from the conditions (2.4) on $\tau$ that all but the last term on the right hand side above are bounded by $Cq_l^{-1/m}$ in absolute value (where $C=C(c,m)$). In fact, the same bound also holds for the last term, as

$$\Big|\int_{1/q_l}^{(u_l-1)/q_l}\Big(\{q_lx\}-\frac{1}{2}\Big)\tau'(x)\,dx\Big|$$
$$\le\sum_{i=1}^{u_l-2}\Big|\int_{i/q_l}^{(i+1)/q_l}\Big(\{q_lx\}-\frac{1}{2}\Big)\tau'(x)\,dx\Big|$$
$$\le\sum_{i=1}^{u_l-2}\Big|\max_{x\in[i/q_l,(i+1)/q_l]}\tau'(x)-\min_{x\in[i/q_l,(i+1)/q_l]}\tau'(x)\Big|\int_{(2i+1)/2q_l}^{(i+1)/q_l}\Big(\{q_lx\}-\frac{1}{2}\Big)\,dx$$
$$\le\frac{1}{8q_l}V_I(\tau'_{q_l}),$$

with $\tau'_{q_l}$ defined as in (2.20). Since $q_l>2/\varepsilon$, it follows from Lemma 2.9 that

$$\frac{1}{8q_l}V_I(\tau'_{q_l})\le Cq_l^{-1/m},$$

where $C=C(c)$, and hence

$$\Big|\sum_{k=0}^{q_l-1}\tau\Big(\frac{k}{q_l}\Big)-q_l\int_0^1\tau(x)\,dx\Big|\le Cq_l^{-1/m}$$

for some constant $C(c,m)$ and $l>l_0$ (and this bound holds trivially also when $l\le l_0$). It follows that

$$\Big|\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1}\tau\Big(\frac{k}{q_l}\Big)-N\int_0^1\tau(x)\,dx\Big|\le C\sum_{l=0}^{s}\frac{b_l}{q_l^{1/m}},$$

and by Lemma 2.1 the last sum is uniformly bounded in $s$ for almost every irrational $\alpha\in(0,1)$. This confirms (2.22).

We now show that (2.23) holds. We assume below that $B < 1$; the proof when $B = 1$ is slightly simpler, but essentially the same. Again we begin by treating the inner sum

$$\sum_{k=0}^{q_l-1}\left(\tau\left(\frac{k}{q_l}+\frac{\rho_{k,l}}{q_l}\right)-\tau\left(\frac{k}{q_l}\right)\right), \tag{2.25}$$

which we will show is bounded in absolute value by

$$\sum_{i=1}^{l} a_i(C_1 q_l^{-1} + C_2 q_l^{-1/m}) \tag{2.26}$$

for constants $C_1 = C_1(m, c, \alpha)$ and $C_2 = C_2(m, c, \alpha)$.

Let $u_l$ be defined as in (2.24), and denote by $E$ a set of "exceptional" indices

$$E = \{0, 1, u_l - 2, u_l - 1, u_l, u_l + 1, q_l - 1\}$$

(for sufficiently large $l$, these are all distinct). We split the sum (2.25) into

$$\Sigma_1 := \sum_{k\in E}\left(\tau\left(\frac{k}{q_l}+\frac{\rho_{k,l}}{q_l}\right)-\tau\left(\frac{k}{q_l}\right)\right),$$

$$\Sigma_2 := \sum_{k=2}^{u_l-3}\left(\tau\left(\frac{k}{q_l}+\frac{\rho_{k,l}}{q_l}\right)-\tau\left(\frac{k}{q_l}\right)\right).$$

Now let $l > l_1$ be sufficiently large for $q_l > 4/\varepsilon$. Since $-1 < \rho_{k,l} < 2$, it follows from the conditions (2.4) on $\tau$ that

$$|\Sigma_1| \le C q_l^{-1/m}, \tag{2.27}$$

where $C = C(c)$. To find a bound on $\Sigma_2$, we first rewrite the sum using the mean value theorem. We have

$$\Sigma_2 = \sum_{k=2}^{u_l-3} \tau'(r_k)\frac{\rho_{k,l}}{q_l},$$

where $r_k \in (k/q_l, (k+\rho_{k,l})/q_l)$ if $\rho_{k,l} > 0$ and $r_k \in ((k+\rho_{k,l})/q_l, k/q_l)$ if $\rho_{k,l} < 0$. It follows that

$$\begin{aligned}\left|\Sigma_2 - \sum_{k=2}^{u_l-3}\tau'\left(\frac{k}{q_l}\right)\frac{\rho_{k,l}}{q_l}\right| &= \left|\sum_{k=2}^{u_l-3}\left(\tau'(r_k)-\tau'\left(\frac{k}{q_l}\right)\right)\frac{\rho_{k,l}}{q_l}\right| \\ &\le \frac{2}{q_l}\sum_{k=2}^{u_l-3}\max_{x,y\in[(k-1)/q_l,(k+2)/q_l]}|\tau'(x)-\tau'(y)| \\ &\le \frac{6}{q_l}V_I(\tau'_{q_l}) \le C q_l^{-1/m},\end{aligned} \tag{2.28}$$

where $C = C(c)$, and for the latter inequality we have used Lemma 2.9.

Finally, we need to find a bound on

$$\sum_{k=2}^{u_l-3} \tau'\left(\frac{k}{q_l}\right)\frac{\rho_{k,l}}{q_l}.$$

Recall from the proof of Proposition 2.5 that we may write $\rho_{k,l}$ as

$$\rho_{k,l} = \omega_{k,l}\frac{\theta_l}{a_{l+1}} + x_l,$$

where $\omega_{k,l} = \{k\alpha_l + \gamma_l\}$, and $\alpha_l$ and $\gamma_l$ are given in (2.17). Let us define the two-dimensional sequence $\omega := (\omega_1(k), \omega_2(k))_{k=0}^{q_l-1}$, where

$$\omega_1(k) = k/q_l, \qquad \omega_2(k) = \omega_{k,l}.$$

Moreover, let $G : I^2 \to \mathbb{R}$ be given by

$$G(x,y) := \chi_{[2/q_l,(u_l-3)/q_l]}(x)\tau'(x)\cdot h(y),$$

where $h : I \to \mathbb{R}$ is the linear function

$$h(y) := \frac{\theta_l}{a_{l+1}}y + x_l.$$

We then have

$$\sum_{k=2}^{u_l-3} \tau'\left(\frac{k}{q_l}\right)\frac{\rho_{k,l}}{q_l} = \frac{1}{q_l}\sum_{k=0}^{q_l-1} G(\omega_1(k),\omega_2(k)). \tag{2.29}$$

The two-dimensional Koksma–Hlawka inequality [12, pp. 151, 100] yields

$$\begin{aligned}\left|\frac{1}{q_l}\sum_{k=0}^{q_l-1} G(w_1(k),w_2(k)) - \int_0^1\!\!\int_0^1 G(x,y)\,dx\,dy\right| &\\ \le D^*_{q_l}(\omega_1)V_I(\chi_{[2/q_l,(u_l-3)/q_l]}\tau') &+ D^*_{q_l}(\omega_2)V_I(h) + D^*_{q_l}(\omega)V_{I^2}(G)\\ \le D^*_{q_l}(\omega)(V_I(\chi_{[2/q_l,(u_l-3)/q_l]}\tau') &+ V_I(h) + V_{I^2}(G)).\end{aligned} \tag{2.30}$$

We now use this inequality to find a bound on the sum (2.29). It is not difficult (see e.g. [12, p. 106]) to show that

$$q_l D^*_{q_l}(\omega) \le 2q_l D^*_{q_l}(\omega_2) \le 2\Big(1 + 2\sum_{i=1}^{l} a_i\Big), \tag{2.31}$$

where for the second inequality we have used (2.19). Moreover, we have

$$V_I(h) = \frac{|\theta_l|}{a_{l+1}} \le 1, \tag{2.32}$$

and using monotonicity of $\tau'$ and Lemma 2.9 we get

$$V_I(\chi_{[2/q_l,(u_l-3)/q_l]}\tau') \le V_I(\tau'_{q_l}) \le Cq_l^{1-1/m}, \tag{2.33}$$

where $C = C(c)$ with $c$ as in (2.4). It follows that

$$V_{I^2}(G) \le V_I(\chi_{[2/q_l,(u_l-3)/q_l]}\tau') \cdot V_I(h) \le C q_l^{1-1/m}. \tag{2.34}$$

Lastly, we have

$$\Big|\int_0^1\!\!\int_0^1 G(x,y)\,dx\,dy\Big| = \Big|\Big(\tau\Big(\frac{u_l-3}{q_l}\Big) - \tau\Big(\frac{2}{q_l}\Big)\Big)\Big(\frac{\theta_l}{2a_{l+1}} + x_l\Big)\Big|,$$

which by (2.4) is bounded by $Cq_l^{-1/m}$, $C = C(c,m)$, when $l > l_1$. Inserting (2.31)–(2.34) and this integral estimate in (2.30), we get

$$\begin{aligned}\Big|\frac{1}{q_l}\sum_{k=0}^{q_l-1} G(\omega_1(k),\omega_2(k))\Big| &\le Cq_l^{-1/m} + \frac{2}{q_l}\Big(1 + 2\sum_{i=1}^{l} a_i\Big)(1 + 2Cq_l^{1-1/m}) \\ &\le \sum_{i=1}^{l} a_i (C_1 q_l^{-1} + C_2 q_l^{-1/m}),\end{aligned}$$

where the constants $C_1$ and $C_2$ depend only on $c$ and $m$ in (2.4).

It thus follows from (2.29) and (2.28) that $|\Sigma_2|$ satisfies the bound (2.26) for $l > l_1$. The same is true for $|\Sigma_1|$ by (2.27), and hence $\Sigma_1 + \Sigma_2$ in (2.25) obeys the bound (2.26) as well. We get

$$\begin{aligned}&\Big|\sum_{l=0}^{s}\sum_{b=0}^{b_l-1}\sum_{k=0}^{q_l-1}\Big(\tau\Big(\frac{k}{q_l} + \frac{\rho_{k,l}}{q_l}\Big) - \tau\Big(\frac{k}{q_l}\Big)\Big)\Big| \\ &\qquad \le C'a_1 + C_1\sum_{l=1}^{s}\frac{b_l}{q_l}\sum_{i=1}^{l} a_i + C_2\sum_{l=1}^{s}\frac{b_l}{q_l^{1/m}}\sum_{i=1}^{l} a_i \\ &\qquad \le C_1\sum_{l=0}^{s}\frac{a_{l+1}}{q_l}\sum_{i=1}^{l+1} a_i + C_2\sum_{l=0}^{s}\frac{a_{l+1}}{q_l^{1/m}}\sum_{i=1}^{l+1} a_i,\end{aligned}$$

and Lemma 2.1 implies that the latter expression is bounded uniformly in $s$ for almost every irrational $\alpha \in (0,1)$. This verifies (2.23), and completes the proof of Proposition 2.6. ∎

**2.3. Proof of Theorems 1.3 and 1.4.** We will begin by proving a lemma showing that the question of whether $S \subset I^2$ is a bounded remainder set can be restated as a question of whether an associated function is of bounded remainder.

Let $S \subset I^2$ be either a polygon or a set satisfying the conditions in Theorem 1.4. We can then associate to $S$ a function $\tau_S : [0,1) \to [0,\infty)$ defined as

$$\tau_S(x) := \int_0^1 \chi_S(t, \{t\alpha + x\})\,dt. \tag{2.35}$$

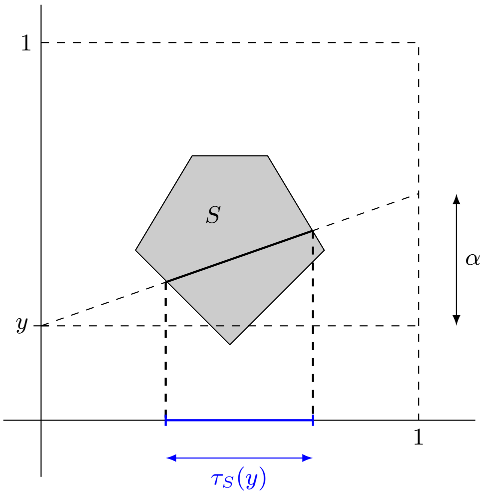


Fig. 4. Geometric interpretation of the function $\tau_S$ associated to the set $S$

A geometric interpretation of $\tau_S$ is illustrated in Figure 4. It is easy to show that

$$\int_0^1 \tau_S(x)\,dx = \lambda(S).$$

Moreover, we have the following:

LEMMA 2.10. *The set $S \subset I^2$ is a bounded remainder set for the irrational rotation with slope $\alpha > 0$ and starting point $\mathbf{x} = (x_1, x_2) \in I^2$ if and only if $\tau_S$ is a bounded remainder function with respect to $\alpha$.*

*Proof.* By Remark 2.7, it will be sufficient to show that $S \subset I^2$ is a bounded remainder set if and only if $\tau_S(x + x_0)$ is a bounded remainder function for some shift $x_0 \in I$. We will verify this for $x_0 = \{x_2 - x_1\alpha\}$.

Recall from Definition 1.2 that $S$ is a bounded remainder set if the difference

$$\Delta_T(S, \alpha, \mathbf{x}) = \int_0^T \chi_S(\{x_1 + t\}, \{x_2 + t\alpha\})\,dt - T\lambda(S)$$

is uniformly bounded in $T$. For a given $T > 0$ we let $N = \lfloor T \rfloor$ and

$$S_N(\alpha, x_0) = \sum_{k=0}^{N-1} \tau_S(\{k\alpha + x_0\}) - N\lambda(S).$$

Definition 2.2 says that the function $\tau_S(x + x_0)$ is of bounded remainder if $S_N(\alpha, x_0)$ is bounded uniformly in $N$. Thus, to prove Lemma 2.10 it is sufficient to show that

$$|S_N(\alpha, x_0) - \Delta_T(S, \alpha, \mathbf{x})| \le C, \tag{2.36}$$

where $C$ is a constant independent of $T$ (or equivalently, of $N$).

To verify (2.36), we observe that

$$\begin{aligned} S_N(\alpha, x_0) &= \sum_{k=0}^{N-1} \int_0^1 \chi_S(t, \{(t+k)\alpha + x_0\})\, dt - N\lambda(S) \\ &= \int_0^N \chi_S(\{t\}, \{t\alpha + x_0\})\, dt - N\lambda(S) \\ &= \int_{-x_1}^{\lfloor T \rfloor - x_1} \chi_S(\{x_1 + t\}, \{x_2 + t\alpha\})\, dt - \lfloor T \rfloor \lambda(S). \end{aligned}$$

It is now easy to see that the difference in (2.36) must be bounded by

$$\begin{aligned} &\Big| \int_{-x_1}^{0} \chi_S(\{x_1 + t\}, \{x_2 + t\alpha\})\, dt \Big| \\ &\qquad + \Big| \int_{\lfloor T \rfloor - x_1}^{T} \chi_S(\{x_1 + t\}, \{x_2 + t\alpha\})\, dt \Big| + \{T\}\lambda(S) \le 4, \end{aligned}$$

thus verifying (2.36) and completing the proof of Lemma 2.10. ∎

*Proof of Theorem 1.3.* It will be sufficient to consider the special case when $S$ is a triangle. This is easy to see when $S$ is a convex polygon; $S$ can then be partitioned into finitely many triangles which are disjoint (up to boundaries), and which all have the property that no edge has slope $\alpha$. Finally, since any union of finitely many disjoint bounded remainder sets is again a bounded remainder set for the irrational rotation with slope $\alpha$, the result follows. A similar, but slightly more involved argument can be given to show that also the case when $S$ is non-convex follows from the triangle case. We thus aim to prove that for almost all $\alpha > 0$ and every $\mathbf{x} \in I^2$, every triangle $S$ with no edge of slope $\alpha$ is a bounded remainder set for the continuous irrational rotation with slope $\alpha$ and starting point $\mathbf{x}$.

Fix some $\alpha$, and let $S$ be a triangle with no edge of slope $\alpha$. Denote by $l(y)$ the intersection of $S$ and the straight line with slope $\alpha$ through the point $(0, y)$, and let $T_S : \mathbb{R} \to [0, \infty)$ be the function

$$T_S(y) = \frac{|l(y)|}{\sqrt{1 + \alpha^2}}.$$

Then $T_S$ is a (possibly shifted) hat function as defined in (2.3), and $\tau_S$ in

(2.35) is given by

$$\tau_S(x) = \sum_{m\in\mathbb{Z}} T_S(x+m).$$

Let $\mathbf{x} \in I^2$ be any given starting point for the irrational rotation. By Lemma 2.10, the triangle $S$ is a bounded remainder set if and only if $\tau_S$ is a bounded remainder function with respect to $\alpha$. By Proposition 2.5, this is indeed the case for every irrational $\alpha > 0$ whose continued fraction expansion satisfies

$$\sum_{l=0}^{s} \frac{a_{l+1}}{q_l^{1/2}} \sum_{k=1}^{l+1} a_k \le C \tag{2.37}$$

for some constant $C$ independent of $s$, i.e. a set of full measure. This completes the proof of Theorem 1.3. ∎

We conclude this section with the proof of Theorem 1.4. Recall that this result says that for every $\mathbf{x} \in I^2$ and almost all $\alpha > 0$, every convex set $S$ whose boundary is a twice differentiable curve with positive curvature at every point is a bounded remainder set for the continuous irrational rotation with slope $\alpha$ and starting point $\mathbf{x}$.

*Proof of Theorem 1.4.* We have seen in Lemma 2.10 that the set $S$ is of bounded remainder for the irrational rotation with slope $\alpha$ and starting point $\mathbf{x} \in I^2$ if and only if the associated function $\tau_S$ in (2.35) is of bounded remainder with respect to $\alpha$. Suppose that $\tau_S$ is of the form

$$\tau_S(x) = \sum_{m\in\mathbb{Z}} T_S(x+m), \tag{2.38}$$

where $T_S$ is the shift of a dome function as given in Definition 2.4. Then this property would be an immediate consequence of Proposition 2.6 and Remark 2.7 for every $\mathbf{x} \in I^2$ and every irrational $\alpha > 0$ satisfying (2.37). Our proof is thus complete if we can show that $\tau_S$ is of the form (2.38) for some shifted dome function $T_S$.

As in the proof of Theorem 1.3, we let $l(y)$ be the intersection of the set $S$ and the straight line with slope $\alpha$ through $(0, y)$, and we let $T_S : \mathbb{R} \to [0,\infty)$ be the function

$$T_S(y) = \frac{|l(y)|}{\sqrt{1+\alpha^2}}.$$

Then $\tau_S$ is as in (2.38). It is clear that $T_S$ is a continuous function supported on some interval $[B_1, B_2]$, and that an appropriate shift of $T_S$ satisfies condition (i) in Definition 2.4. We will show that also condition (ii) is satisfied for this shift of $T_S$; that is, we can find $c, m, \varepsilon > 0$ such that

$$T_S(B_1 + x) \le cx^{1/m} \quad \text{and} \quad T_S(B_2 - x) \le cx^{1/m}$$

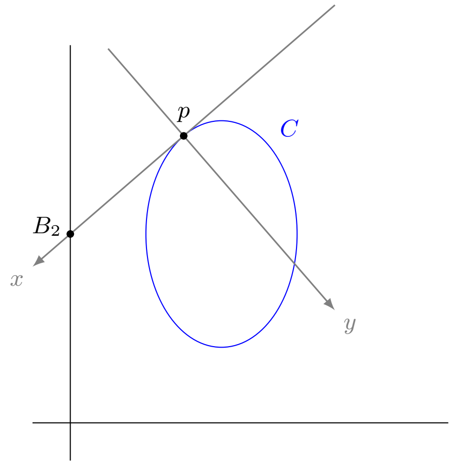


Fig. 5. The curve $C$ and the new coordinate axes $x$ and $y$

whenever $0 \leq x < \varepsilon$. We only verify the latter inequality (the argument for the former is analogous).

Let $C = (C_1(s), C_2(s))$ denote the boundary of $S$ parametrized by arc length, and denote by $L$ its total length. We then have $|C'(s)| = 1$ and $C'(s) \perp C''(s)$ for all $s \in [0, L]$. The curvature $\kappa(s)$ at the point $C(s)$ is given by $\kappa(s) = |C''(s)|$, and assumed positive for all $s \in [0, L]$. We let

$$k := \min_{s \in [0,L]} \kappa(s). \tag{2.39}$$

The line with slope $\alpha$ through the point $(0, B_2)$ in the plane will intersect the curve $C$ at a single point $p$. We let this line be the $x$-axis in a new coordinate system $(x, y)$ where $p$ is the origin (see Figure 5), and view $C$ as a curve in these coordinates with $C(0) = (C_1(0), C_2(0)) = (0, 0)$. We may then think of a section of $C$ around $p$ as the graph of the function $H : (-\delta, \delta) \to [0, \infty)$ given by

$$H(x) = C_2(C_1^{-1}(x)).$$

We have $C_2'(0) = 0$ and $C_1'(0) = 1$, and since $C_1$ and $C_2$ are both twice continuously differentiable it follows that

$$H'(x) = \frac{C_2'(C_1^{-1}(x))}{C_1'(C_1^{-1}(x))}$$

and

$$H''(x) = \frac{C_2''(s)C_1'(s) - C_2'(s)C_1''(s)}{(C_1'(s))^3}, \qquad s = C_1^{-1}(x),$$

are both well-defined and continuous on some interval $(-\delta, \delta)$. By choosing

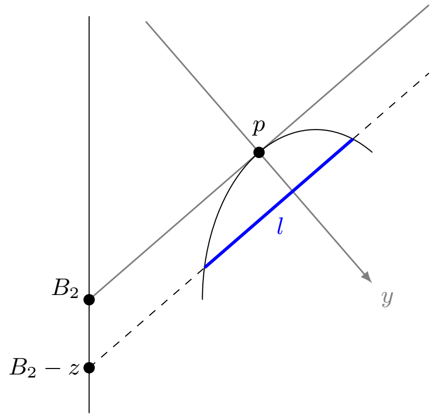


Fig. 6. The intersection $l$ of $S$ and the line of slope $\alpha$ through $(0, B_2 - z)$

$\delta$ sufficiently small we can ensure that

$$|C_1'(C_1^{-1}(x))| \geq 1/2, \qquad x \in (-\delta, \delta),$$

which (for $s = C_1^{-1}(x)$ and recalling that $C'(s) \perp C''(s)$) in turn implies

$$|H''(x)| = \frac{|C''(s)| \cdot |C'(s)|}{|C_1'(s)|^3} \geq \frac{k}{8}, \qquad x \in (-\delta, \delta), \tag{2.40}$$

with $k$ given in (2.39).

We now use this lower bound on $|H''(x)|$ to find an upper bound on $T_S(B_2 - z)$ for sufficiently small $z > 0$. We have

$$T_S(B_2 - z) = \frac{|l|}{\sqrt{1+\alpha^2}}, \tag{2.41}$$

where $l$ is the intersection of $S$ with the line of slope $\alpha$ through $(0, B_2 - z)$, illustrated in Figure 6. The line segment $l$ is at height $y = z/\sqrt{1+\alpha^2}$ above $p$ (see Figure 6). If $y < \min\{H(\delta), H(-\delta)\}$, then we denote by $x_1, x_2$ the two values of $x \in (-\delta, \delta)$ satisfying $H(x) = y$, and

$$|l| \leq 2\max\{|x_1|, |x_2|\}. \tag{2.42}$$

By Taylor's theorem we have

$$y = H(x_i) = H(0) + \frac{H'(0)}{1!}x_i + \frac{H''(r_i)}{2!}x_i^2 = \frac{H''(r_i)}{2!}x_i^2$$

for $i = 1, 2$ and some $r_i \in (-\delta, \delta)$, and from (2.40) it thus follows that

$$|x_i| = \left(\frac{2y}{H''(r_i)}\right)^{1/2} \leq \frac{4}{\sqrt{k}} \cdot y^{1/2}, \qquad i = 1, 2.$$

Hence, from (2.42) we get

$$|l| \leq \frac{8}{\sqrt{k}} \cdot y^{1/2} \leq \frac{8}{\sqrt{k}} \cdot z^{1/2}, \tag{2.43}$$

and by (2.41) and (2.43) we have

$$T_S(B_2 - z) \leq \frac{8}{\sqrt{k(1+\alpha^2)}} \cdot z^{1/2}.$$

This verifies that a shift of the function $T_S$ satisfies the growth condition (ii) in Definition 2.4 with $c = 8/\sqrt{k(1+\alpha^2)}$, $m = 2$ and some $\varepsilon > 0$ (for instance, $\varepsilon = \min\{H(\delta), H(-\delta)\}$ will suffice). The function $\tau_S$ is thus of the form (2.38), where $T_S$ is the shift of a dome function, and this completes the proof of Theorem 1.4. ∎

**3. Proof of Theorem 1.7.** For the proof of (i) we just give an outline, as this proof largely follows the proof given above for Proposition 2.5. Part (ii), on the other hand, is proven in full detail. Lastly, we present the proof of (iii).

*Proof of Theorem 1.7(i).* Fix an irrational $\alpha > 0$ with continued fraction expansion $\alpha = [0; a_1, a_2, \ldots]$ satisfying $a_1 = 1$ and $a_{l+1} \geq q_l^7$. One can show that there are uncountably many such irrationals.

Let $S$ be the triangle with vertices $(0,0)$, $(0,1)$ and $(K,1)$ for some $0 < K < 1$ to be determined. We will assume that $1 - K\alpha > 0$. Denote by $\tau_S$ the function in (2.35) associated to $S$; this is a hat function as defined in (2.3), with $a = 1 - K\alpha$ and $b = 1$. By Lemma 2.10, the triangle $S$ is a bounded remainder set for the continuous irrational rotation with slope $\alpha$ and some arbitrary starting point $\mathbf{x} \in I^2$ if and only if $\tau_S$ is a bounded remainder function with respect to $\alpha$. In what follows, we show that the latter is *not* the case, and so $S$ is not a bounded remainder set.

For $N = \sum_{l=0}^{s} b_l q_l$, one can show by calculations analogous to those in the proof of Proposition 2.5 that

$$\left| \sum_{k=0}^{N-1} \tau_S(\{k\alpha\}) - \frac{NK}{2} \right| = C \sum_{l=0}^{s} \xi_l(1-\xi_l)\frac{b_l}{q_l} + O(1), \tag{3.1}$$

where $C$ depends only on $K$ and $\alpha$, and $\xi_l = \{q_l a\} = \{q_l(1-K\alpha)\}$. For $x \in \mathbb{R}$, let $\|x\|$ denote the minimal distance from $x$ to an integer, and note that

$$\xi_l(1-\xi_l) \geq \tfrac{1}{2}\|q_l a\|.$$

It is a well-known fact (see e.g. [11, p. 69]) that for almost all $a \in (0,1)$ one can find a positive constant $c$ such that

$$\|n \cdot a\| \geq c/n^2$$

for all $n \geq 2$. Thus, one can indeed find $K \in (0,1)$ such that $a = 1 - K\alpha > 0$, and moreover

$$C\sum_{l=0}^{s}\xi_l(1-\xi_l)\frac{b_l}{q_l} \ge C\sum_{l=0}^{s}\|q_l a\|\frac{b_l}{q_l} > C\sum_{l=0}^{s}\frac{b_l}{q_l^3}.$$

Now let $b_l := q_l^4$. Then the sum on the right hand side of (3.1) is bounded from below by $C\sum_{l=0}^{s} q_l$, which tends to infinity as $s \to \infty$. For the sequence of integers $N_s = \sum_{l=0}^{s} q_l^5$, we thus have

$$\Big|\sum_{k=0}^{N_s-1}\tau_S(\{k\alpha\}) - N_s\lambda(S)\Big| \to \infty$$

as $s \to \infty$. This shows that $\tau_S$ is not a bounded remainder function with respect to $\alpha$, and completes the proof of Theorem 1.7(i). ∎

*Proof of Theorem 1.7(ii).* Fix an irrational $\alpha \in (1/4, 1/2)$ with continued fraction expansion $\alpha = [0; a_1, a_2, \ldots]$ satisfying $a_{l+1} > q_l^{100}$ and $p_l$ even for an infinite number of odd indices $l$, say for the sequence $l_1 < l_2 < \cdots$. One can show that there are uncountably many such irrationals.

Let $S$ be the disc with diameter $d := \alpha/\sqrt{1+\alpha^2}$ illustrated in Figure 7. By Lemma 2.10, the set $S$ is of bounded remainder for the continuous irrational rotation with slope $\alpha$ and arbitrary starting point $\mathbf{x} \in I^2$ if and only if the associated function $\tau_S$ in (2.35) is a bounded remainder function with respect to $\alpha$. In what follows, we will show that there exists an $x \in I$ and a sequence of integers $N_1 < N_2 < \cdots$ such that

$$\Big|\sum_{k=0}^{N_i-1}\tau_S(\{k\alpha + x\}) - N_i\lambda(S)\Big| \to \infty$$

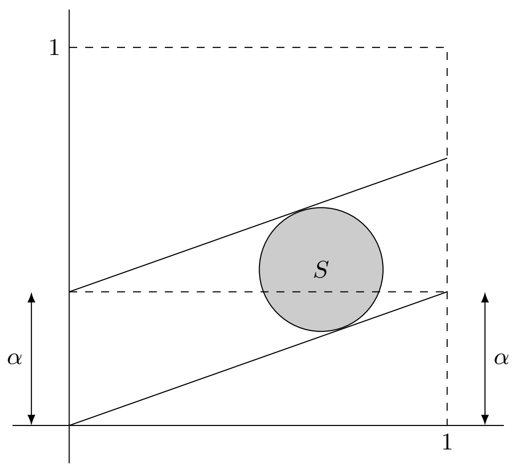


Fig. 7. The disc $S$ with diameter $d = \alpha/\sqrt{1+\alpha^2}$

as $i \to \infty$. By Remark 2.7, this proves that $\tau_S$ is not a bounded remainder function, and so $S$ is not a bounded remainder set.

The function $\tau_S$ associated to $S$ is given explicitly by

$$\tau_S(y) = \begin{cases} \dfrac{\alpha}{1+\alpha^2}\sqrt{1-(1-2y/\alpha)^2}, & 0 \le y \le \alpha, \\ 0, & \alpha < y \le 1, \end{cases}$$

and we note that

$$\lambda(S) = \int_0^1 \tau_S(y)\,dy = \frac{\pi}{4} \cdot \frac{\alpha^2}{1+\alpha^2}. \tag{3.2}$$

We set

$$S_N(x) := \sum_{k=0}^{N-1} \tau_S(\{k\alpha + x\}).$$

Let us now fix some $i$ (and thereby an odd index $l_i$), set

$$p := p_{l_i} = 2m \quad (m \in \mathbb{N}), \qquad q := q_{l_i},$$

and evaluate $S_N(x)$ for $N := q^{11}$ and some $x \in [0, 1/q]$. We then have

$$S_N(x) = \sum_{j=0}^{q^{10}-1} \sum_{k=0}^{q-1} \tau_S(\{(jq+k)\alpha + x\}). \tag{3.3}$$

Recall from (2.2) that

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^2 a_{l_i+1}} \le \frac{1}{q^{102}}.$$

Using this fact, we get

$$\|jq\alpha\| < j\|q\alpha\| \le \frac{j}{q^{101}} < \frac{1}{q^{91}}.$$

It follows that

$$\begin{aligned} \left\| \{(jq+k)\alpha + x\} - \left\{k \cdot \frac{p}{q} + x\right\} \right\| &\le \|jq\alpha\| + k\left\|\alpha - \frac{p}{q}\right\| \\ &< \frac{1}{q^{91}} + \frac{q}{q^{102}} < \frac{1}{q^{90}}, \end{aligned}$$

and hence

$$\left|\tau_S(\{(jq+k)\alpha + x\}) - \tau_S\left(\left\{k \cdot \frac{p}{q} + x\right\}\right)\right| \le \left|\tau_S\left(\frac{1}{q^{90}}\right)\right| < \frac{1}{q^{44}}.$$

Combining this bound with (3.3), we get

$$\left|S_N(x) - q^{10}\sum_{k=0}^{q-1} \tau_S\left(\left\{k \cdot \frac{p}{q} + x\right\}\right)\right| < q^{11} \cdot \frac{1}{q^{44}} = \frac{1}{q^{33}}. \tag{3.4}$$

In light of (3.4), we introduce the function

$$\sigma(y) := \begin{cases} \dfrac{\alpha}{1+\alpha^2}\sqrt{1-(1-2qy/p)^2}, & 0 \le y \le p/q, \\ 0, & p/q < y \le 1. \end{cases}$$

Since the index $l_i$ is odd, we have $\alpha < p/q$ and $\sigma(y) = \tau_S(\alpha qy/p)$ for all $y \in [0,1)$. From

$$\left|\frac{\alpha q}{p} - 1\right| = \frac{q}{p}\left|\alpha - \frac{p}{q}\right| < \frac{1}{\alpha}\cdot\frac{1}{q^{102}} < \frac{1}{q^{101}}$$

it thus follows that

$$|\sigma(y) - \tau_S(y)| = \left|\tau_S\left(\frac{\alpha p}{q}y\right) - \tau_S(y)\right| < \left|\tau_S\left(\frac{1}{q^{101}}\right)\right| < \frac{1}{q^{50}}.$$

Combining this bound with (3.4), we get

$$\left|S_N(x) - q^{10}\sum_{k=0}^{q-1}\sigma\left(\left\{k\cdot\frac{p}{q} + x\right\}\right)\right| < \frac{1}{q^{33}} + \frac{q^{11}}{q^{50}} < \frac{1}{q^{32}}. \tag{3.5}$$

Note that some of the above estimates hold only for $q$ greater than some lower threshold $q > q_0$.

Let us now have a closer look at the sum of $\sigma$ in (3.5). We have

$$\begin{aligned} \sum_{k=0}^{q-1}\sigma\left(\left\{k\cdot\frac{p}{q}+x\right\}\right) &= \sum_{k=0}^{p-1}\sigma\left(\frac{k}{q}+x\right) \\ &= \frac{\alpha}{1+\alpha^2}\sum_{k=0}^{p-1}\sqrt{1-\left(1-\frac{2k}{p}-\frac{2q}{p}x\right)^2} \\ &= \frac{\alpha}{1+\alpha^2}\sum_{k=0}^{2m-1}\sqrt{1-\left(1-\frac{k}{m}-\frac{q}{m}x\right)^2} \\ &= \frac{\alpha}{1+\alpha^2}\cdot 2mG_m\left(\frac{q}{m}x\right), \end{aligned} \tag{3.6}$$

where

$$G_m(x) := \frac{1}{2m}\sum_{k=0}^{2m-1}\sqrt{1-\left(1-\frac{k}{m}-x\right)^2}, \quad x \in [0, 1/m).$$

The function $G_m$ is illustrated in Figure 8. It is clear that $G_m(x) = G_m(1/m - x)$, and by elementary analysis one can show that $G_m$ increases on $[0, 1/(2m))$ in such a way that

$$G_m\left(\frac{1}{3m}\right) > G_m\left(\frac{1}{6m}\right) + \frac{2c}{m^{3/2}}$$

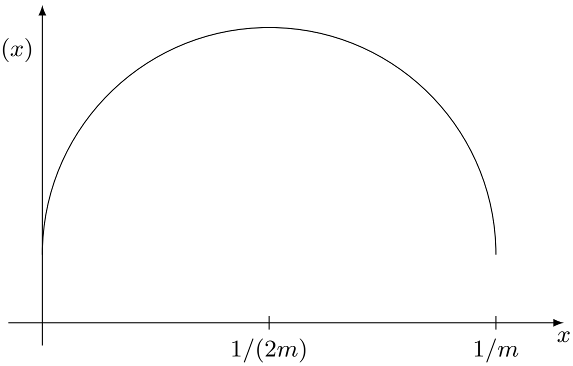


Fig. 8. The function $G_m(x)$

for some $c > 0$. From this one can deduce that there exists a subinterval $\Lambda \subset [0, 1/(2m)]$ of length at least $1/(6m)$ such that either

$$G_m(x) > \frac{1}{2}\int_0^2 \sqrt{1-(1-y)^2}\,dy + \frac{c}{m^{3/2}} = \frac{\pi}{4} + \frac{c}{m^{3/2}} \tag{3.7}$$

for all $x \in \Lambda$, or

$$G_m(x) < \frac{1}{2}\int_0^2 \sqrt{1-(1-y)^2}\,dy - \frac{c}{m^{3/2}} = \frac{\pi}{4} - \frac{c}{m^{3/2}} \tag{3.8}$$

for all $x \in \Lambda$. We assume in what follows that (3.7) holds for all $x \in \Lambda$ (the case when (3.8) holds is treated similarly). Then for $\tilde{x} \in \tilde{\Lambda}$, where

$$\tilde{\Lambda} := (m/q)\Lambda \subset [0, 1/(2q)),$$

we have $q\tilde{x}/m \in \Lambda$, and from (3.6) and (3.7) we deduce that

$$\sum_{k=0}^{q-1} \sigma\left(\left\{k \cdot \frac{p}{q} + \tilde{x}\right\}\right) > \frac{\alpha}{1+\alpha^2} \cdot 2m\left(\frac{\pi}{4} + \frac{c}{m^{3/2}}\right). \tag{3.9}$$

In the following we let $c_1, c_2, \dots$ denote positive absolute constants. From (3.9) and (3.5) we get

$$\begin{aligned} S_N(\tilde{x}) &> q^{10} \sum_{k=0}^{q-1} \sigma\left(\left\{k \cdot \frac{p}{q} + \tilde{x}\right\}\right) - \frac{1}{q^{32}} \\ &> q^{10} \cdot 2m \cdot \frac{\alpha}{1+\alpha^2}\left(\frac{\pi}{4} + \frac{c}{m^{3/2}}\right) - \frac{1}{q^{32}} > N \cdot \frac{p}{q} \cdot \frac{\pi\alpha}{4(1+\alpha^2)} + c_1 q^9 \\ &> N \cdot \frac{\pi\alpha^2}{4(1+\alpha^2)} + c_1 q^9 = N\lambda(S) + c_1 q^9, \end{aligned}$$

where we recall from (3.2) that $\lambda(S)$ is the integral of $\tau_S$ and the measure of the disc $S$ in Figure 7. Thus, we have shown that

$$S_N(\tilde{x}) - N\lambda(S) > c_1 q^9, \quad \tilde{x} \in \tilde{\Lambda}. \tag{3.10}$$

Finally, we define the set $\bar{\Lambda} \subset I$ by

$$\tilde{\Lambda}^{(j)} := \tilde{\Lambda} + j/q, \quad \bar{\Lambda} := \bigcup_{j=0}^{q-1} \tilde{\Lambda}^{(j)}.$$

Since $\lambda(\tilde{\Lambda}) \geq 1/(6q)$, we have $\lambda(\bar{\Lambda}) \geq 1/6$. Choose some $x \in \bar{\Lambda}$, and find $j \in \{0, 1, \ldots, q-1\}$ such that

$$x = \tilde{x} + j/q, \quad \tilde{x} \in \tilde{\Lambda}.$$

Furthermore, choose $k_j \in \{0, 1, \ldots, q-1\}$ such that $k_j p \equiv q - j \pmod q$, and note that

$$\left\| k_j\alpha + \frac{j}{q} \right\| = \left\| k_j\alpha - \frac{k_j p}{q} \right\| \leq k_j \left\| \alpha - \frac{p}{q} \right\| < \frac{1}{q^{101}}.$$

From this and the fact that $|\tau_S| \leq 1$, it follows that

$$\begin{aligned} S_N(x) &> \sum_{k=0}^{k_j} \tau_S(\{k\alpha + x\}) + \sum_{k=0}^{N-1} \tau_S(\{k\alpha + k_j\alpha + x\}) - q \\ &> \sum_{k=0}^{N-1} \tau_S\left(\left\{k\alpha + x - \frac{j}{q}\right\}\right) - c_2 q = \sum_{k=0}^{N-1} \tau_S(\{k\alpha + \tilde{x}\}) - c_2 q \\ &= S_N(\tilde{x}) - c_2 q, \end{aligned}$$

and from (3.10) we thus get

$$S_N(x) - N\lambda(S) > c_3 q^9 \tag{3.11}$$

for all $x \in \bar{\Lambda}$.

The above analysis can be carried out for each $l_i$ (given that $q_{l_i}$ is above the threshold, $q_{l_i} > q_0$). That is, for each $i$, we find $\bar{\Lambda}_i \subset I$ of measure $\lambda(\bar{\Lambda}_i) \geq 1/6$ such that (3.11) holds for all $x \in \bar{\Lambda}_i$ with $q = q_{l_i}$ and $N = q^{11}$. Now fix $x \in I$ such that $x \in \bar{\Lambda}_i$ for infinitely many $i$, and for each such $i$ let $q_i = q_{l_i}$ and $N_i = q_i^{11}$. Then for these $N_i$, we have

$$|S_{N_i}(x) - N_i\lambda(S)| = \left| \sum_{k=0}^{N_i-1} \tau_S(\{k\alpha + x\}) - N_i\lambda(S) \right| \to \infty$$

as $i \to \infty$. This verifies that $\tau_S$ is not a bounded remainder function with respect to $\alpha$, and completes the proof of Theorem 1.7(ii). ∎

*Proof of Theorem 1.7(iii).* Let $S$ be the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$. Fix some slope $\alpha > 0$ and starting point $\mathbf{x} \in I^2$. For simplicity we assume that $\alpha < 1$ (the proof is similar when $\alpha \geq 1$). By Lemma 2.10, the set $S$ is of bounded remainder for the continuous irrational rotation with slope $\alpha$ and starting point $\mathbf{x}$ if and only if the associated function $\tau_S$ in (2.35) is of bounded remainder with respect to $\alpha$. For the specific triangle $S$, we have

$$\tau_S(x) = \begin{cases} \dfrac{1-x}{1+\alpha}, & 0 \le x \le 1-\alpha, \\ \dfrac{1-x}{1+\alpha} + \dfrac{2-x}{1+\alpha} - \dfrac{1-x}{\alpha}, & 1-\alpha < x \le 1. \end{cases} \tag{3.12}$$

It is a well-known fact that a 1-periodic function $f$ which is integrable over the unit interval $I$ is a bounded remainder function with respect to $\alpha$ if and only if there exists a bounded and measurable 1-periodic function $g$ satisfying the equation

$$f(x) - \int_0^1 f(t)\,dt = g(x) - g(x+\alpha)$$

for almost every $x$. This is known as the *cohomological equation* for $f$. By a classical result of Gottschalk and Hedlund [6, Theorem 14.11], the function $g$ can be chosen to be continuous whenever $f$ is continuous. Thus, our proof is complete if we can find a continuous 1-periodic function $g$ such that

$$\tau_S(x) - \int_0^1 \tau_S(t)\,dt = g(x) - g(x+\alpha), \tag{3.13}$$

where $\tau_S$ is given in (3.12).

Let $g$ be the continuous 1-periodic function defined on $I$ by

$$g(x) = \frac{x(x-1)}{2\alpha(1+\alpha)}.$$

It is straightforward to check that this function satisfies (3.13). This confirms that $\tau_S$ is a bounded remainder function with respect to $\alpha$, and completes the proof of Theorem 1.7(iii). ∎

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