ANNALES
POLONICI MATHEMATICI
118.1 (2016)

On delta m-subharmonic functions

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Abstract. Let $p > 0$, and let $\mathcal{E}_{p,m}$ be the cone of negative m-subharmonic functions with finite m-pluricomplex p-energy. We will define a quasi-norm on the vector space $\delta\mathcal{E}_{p,m} = \mathcal{E}_{p,m} - \mathcal{E}_{p,m}$ and prove that this vector space with this quasi-norm is a quasi-Banach space. Furthermore, we characterize its topological dual.

Introduction. The δ-plurisubharmonic functions were studied by Cegrell [Ce1] and Kiselman [Ki]. Cegrell and Wiklund [CW] investigated the vector space $\delta\mathcal{F} = \mathcal{F} - \mathcal{F}$ equipped with a suitable norm. They proved that it is a nonseparable Banach space and provided the characterization of its dual space. Hai and Hiep [HH] introduced a metric which defines a locally convex topology on the space $\delta\mathcal{E}$ of δ-plurisubharmonic functions from the Cegrell class $\mathcal{E}$ (see [Ce3] for the definition of this class). They proved that with this topology, $\delta\mathcal{E}$ is a nonseparable and nonreflexive Fréchet space.

The cone $\mathcal{E}_p$ of negative plurisubharmonic functions with finite pluricomplex p-energy was introduced by Cegrell [Ce2] for $p \geq 1$, and for $0 < p < 1$ in [ACH] (see also [CKZ], [K2]). Åhag and Czyż [AC] proved that the vector space $\delta\mathcal{E}_p$ with the vector ordering induced by the cone $\mathcal{E}_p$ is σ-Dedekind complete, and with a suitable quasi-norm this space is a nonseparable quasi-Banach space. They also characterized its topological dual. Recently, Åhag, Cegrell and Czyż [ACC] generalized these results to cones $\mathcal{K}$ of negative plurisubharmonic functions with $\mathcal{E}_0 \subset \mathcal{K} \subset \mathcal{E}$.

The complex Hessian operator for m-subharmonic functions has been studied by Błocki, Dinew, Kołodziej, Nguyen, Lu, and others (see [Bl], [DK], [Ng], [Lu] for more details). In his Ph.D thesis, Lu extended the results from [Ce2], [Ce3], [ACH] to m-subharmonic functions.

2010 *Mathematics Subject Classification*: Primary 32U05; Secondary 06F30.
Key words and phrases: cone, delta m-subharmonic function, ordered vector space, quasi-Banach space, topological dual.
Received 9 March 2016; revised 6 September 2016.
Published online 27 October 2016.

DOI: 10.4064/ap3959-9-2916

In this article, we extend the results of [AC] to m-subharmonic functions. We give some background on m-subharmonic functions in Section 1. We consider the vector space $\delta\mathcal{E}_{p,m} = \mathcal{E}_{p,m} - \mathcal{E}_{p,m}$ generated by the cone $\mathcal{E}_{p,m}$. By straightforward calculations, $\delta\mathcal{E}_{p,m}$ is a vector space under pointwise addition and usual scalar multiplication, with the convention $-\infty - (-\infty) = -\infty$. We shall consider $\delta\mathcal{E}_{p,m}$ with two vector orders: the order induced by the positive cone $\succcurlyeq$, and the classical pointwise ordering $\geq$. The two order relations on $\delta\mathcal{E}_{p,m}$ are related as follows: if $u \succcurlyeq v$, then $u \leq v$, but there are functions u, v in $\delta\mathcal{E}_{p,m}$ with $u \geq v$ such that u and v are not comparable with respect to $\succcurlyeq$ (see Example 2.10).

In Section 3, for $u \in \delta\mathcal{E}_{p,m}$ we define

$$\|u\|_{p,m} = \inf_{\substack{u=u_1-u_2\\ u_1,u_2\in\mathcal{E}_{p,m}}} \left\{ \left(\int_{\Omega} [-(u_1+u_2)]^p H_m(u_1+u_2) \right)^{\frac{1}{m+p}} \right\}, \tag{0.1}$$

where $H_m(\cdot) = [dd^c(\cdot)]^m \wedge \beta^{n-m}$ is the m-complex Hessian operator. Our aim is to show that $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ is a quasi-Banach space, and for $p = 1$ a Banach space (see Theorem 3.8). We also prove that there exists a decomposition of each element in $\delta\mathcal{E}_{p,m}$ with control of the quasi-norm (see Theorem 3.9).

In Section 4, we study the dual space of $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$. The main results are Theorems 4.6 and 4.7.

In Section 5, we construct an inner product on $\delta\mathcal{E}_{1,1}$. We give two examples. The first shows that the norm defined by this inner product and the norm $\|\cdot\|_{1,1}$ defined by (0.1) are not equivalent (see Example 5.2). The second proves that on $\delta\mathcal{E}_{1,m}$, $m > 1$, the norm $\|\cdot\|_{1,m}$ defined by (0.1) cannot come from any inner product (see Example 5.3).

1. Preliminaries. Let Ω be an open set in $\mathbb{C}^n$ and let m be a natural number with $1 \leq m \leq n$. As usual let $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, and let $\beta = dd^c\|z\|^2$ be the canonical Kähler form in $\mathbb{C}^n$. We denote by $\mathbb{C}_{(1,1)}$ the space of $(1,1)$-forms with constant coefficients. One defines the positive cone

$$\Gamma_m = \{\eta \in \mathbb{C}_{(1,1)} : \eta \wedge \beta^{n-1} \geq 0, \ldots, \eta^m \wedge \beta^{n-m} \geq 0\}.$$

If $u \in C^2(\Omega)$ then u is an m-subharmonic function if

$$dd^c u \wedge \beta^{n-1} \geq 0, \ \ldots, \ (dd^c u)^m \wedge \beta^{n-m} \geq 0$$

at every point in Ω.

Definition 1.1. Let u be a subharmonic function in Ω. Then u is called *m-subharmonic* if

$$dd^c u \wedge \eta_1 \wedge \cdots \wedge \eta_{m-1} \wedge \beta^{n-m} \geq 0$$

in the sense of currents for all $\eta_1,\dots,\eta_{m-1}\in\Gamma_m$. Denote by $\mathrm{SH}_m(\Omega)$ the set of all m-subharmonic functions in Ω, and by $\mathrm{SH}_m^-(\Omega)$ the set of all nonpositive m-subharmonic functions in Ω.

Remark 1.2. By the definition, we have

$$\mathrm{PSH}(\Omega)=\mathrm{SH}_n(\Omega)\subset\mathrm{SH}_{n-1}(\Omega)\subset\cdots\subset\mathrm{SH}_1(\Omega)=\mathrm{SH}(\Omega).$$

In [Bl] (see also [DK]), Błocki used the method of Bedford and Taylor [BT1], [BT2] to define the complex Hessian operators. For $u_1,\dots,u_m\in\mathrm{SH}_m(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$, the operator

$$\begin{aligned}H_m(u_1,\dots,u_m)&:=dd^cu_1\wedge\cdots\wedge dd^cu_m\wedge\beta^{n-m}\\&=dd^c(u_1dd^cu_2\wedge\cdots\wedge dd^cu_m\wedge\beta^{n-m})\end{aligned}$$

is a nonnegative Radon measure. In particular, when $u=u_1=\cdots=u_m$, the measures

$$H_m(u):=(dd^cu)^m\wedge\beta^{n-m}$$

are well-defined for $u\in\mathrm{SH}_m(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$.

We list some elementary facts for m-subharmonic functions.

Proposition 1.3 ([Ng, Proposition 1.3]). *Let $\Omega\subset\mathbb{C}^n$ be a bounded domain.*

1. *If $u,v\in\mathrm{SH}_m(\Omega)$ then $\lambda u+\mu v\in\mathrm{SH}_m(\Omega)$ for all $\lambda,\mu\ge 0$.*
2. *If $u\in\mathrm{SH}_m(\Omega)$ then the standard regularization $u\star\chi_\epsilon$ is also m-subharmonic in $\Omega_\epsilon:=\{x\in\Omega:d(x,\partial\Omega)>\epsilon\}$.*
3. *If $u\in\mathrm{SH}_m(\Omega)$ and $\gamma:\mathbb{R}\to\mathbb{R}$ is a convex nondecreasing function then $\gamma\circ u\in\mathrm{SH}_m(\Omega)$.*
4. *If $u,v\in\mathrm{SH}_m(\Omega)$ then $\max\{u,v\}\in\mathrm{SH}_m(\Omega)$.*
5. *Let $\{u_\alpha\}\subset\mathrm{SH}_m(\Omega)$ be a sequence locally uniformly bounded from above, and let $u=\sup u_\alpha$. Then the upper semicontinuous regularization $u^\star$ is m-subharmonic and equal to u almost everywhere.*

Now we recall some definitions and basic properties related to m-subharmonic functions.

Definition 1.4. A bounded domain $\Omega\subset\mathbb{C}^n$ is said to be *m-hyperconvex* if there exists a continuous m-subharmonic function $\rho:\Omega\to\mathbb{R}^-$ such that $\{\rho<-c\}\Subset\Omega$ for all $c>0$.

Let

$$\mathcal{E}_{0,m}\ (=\mathcal{E}_{0,m}(\Omega))=\Big\{u\in\mathrm{SH}_m(\Omega)\cap L^\infty(\Omega):\lim_{z\to\partial\Omega}u(z)=0\ \text{ and }\int_\Omega H_m(u)<\infty\Big\}.$$

The following theorem essentially follows from [Ce3, Lemma 3.1] for $n=m$, and can be found in [Lu, Lemma 1.7.13].

Theorem 1.5.

$$C_0^\infty(\Omega) \subset \mathcal{E}_{0,m}(\Omega) \cap C(\Omega) - \mathcal{E}_{0,m}(\Omega) \cap C(\Omega).$$

Definition 1.6. For each $p > 0$, we define $\mathcal{E}_{p,m}$ to be the class of all functions $u \in \mathrm{SH}_m^-(\Omega)$ such that there exists a decreasing sequence $\{u_j\} \subset \mathcal{E}_{0,m}$ such that

(i) $\lim_{j\to\infty} u_j = u$,
(ii) $\sup_j \int_\Omega (-u_j)^p H_m(u_j) < \infty$.

From the following theorem we see that the Hessian operator is well-defined on the class $\mathcal{E}_{p,m}$.

Theorem 1.7. *Let $u_1, \ldots, u_m \in \mathcal{E}_{p,m}$ and $\{u_k^j\}_j \subset \mathcal{E}_{0,m}$ with $u_k^j \downarrow u_k$ be as in Definition* 1.6 *$k = 1, \ldots, m$. Then the sequence of measures*

$$dd^c u_1^j \wedge \cdots \wedge dd^c u_m^j \wedge \beta^{n-m}$$

weakly converges to a Radon measure and the limit measure does not depend on the choice of the sequence $\{u_k^j\}$. We denote this limit by

$$H_m(u_1, \ldots, u_m) := dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}.$$

Integration by parts is valid for $\mathcal{E}_{p,m}$ (see [Lu, Theorem 1.7.19]).

Theorem 1.8. *Let $u, v, \phi_j \in \mathcal{E}_{p,m}$ for $j = 1, \ldots, m-1$. Then*

$$\int_\Omega u dd^c v \wedge T = \int_\Omega v dd^c u \wedge T,$$

where $T = dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_{m-1} \wedge \beta^{n-m}$.

Definition 1.9. For $u \in \mathcal{E}_{p,m}$, we define the *m-pluricomplex p-energy* of u by

$$e_{p,m}(u) := \int_\Omega (-u)^p H_m(u).$$

The following theorem (see [Lu, Theorem 1.7.24, Proposition 1.8.9], see also [CKZ, Lemma 2.1]) states that $e_{p,m}(u)$ is finite for $u \in \mathcal{E}_{p,m}$.

Theorem 1.10. *If $u \in \mathcal{E}_{p,m}$ then $e_{p,m}(u) < \infty$, and there exists a sequence $\{u_j\} \subset \mathcal{E}_{0,m}$ with $u_j \downarrow u$ such that $e_{p,m}(u_j) \to e_{p,m}(u)$.*

Proposition 1.11.

(i) *If $u, v \in \mathcal{E}_{0,m}$ $[u, v \in \mathcal{E}_{p,m}]$, then $\lambda u + \mu v \in \mathcal{E}_{0,m}$ $[\lambda u + \mu v \in \mathcal{E}_{p,m}]$ for all $\lambda, \mu \geq 0$.*
(ii) *If $u \in \mathcal{E}_{0,m}$ $[u \in \mathcal{E}_{p,m}]$ and $v \in \mathrm{SH}_m^-(\Omega)$, then $\max(u, v) \in \mathcal{E}_{0,m}$ $[\max(u, v) \in \mathcal{E}_{p,m}]$.*
(iii) *If $u, v \in \mathcal{E}_{p,m}$, then*

$$e_{p,m}(u) + e_{p,m}(v) \leq e_{p,m}(u + v) < \infty.$$

Proof. See [Lu, Theorem 1.7.12] and [Ce2, Theorem 3.3, Lemma 3.4]. ■

The comparison principle is an important tool in pluripotential theory (see [BT2], [Ce2], [Ce3], etc). For our purposes, we record the following theorem (see [Lu, Theorem 1.7.27]).

THEOREM 1.12. *Let $u, v \in \mathcal{E}_{p,m}$ with $H_m(u) \le H_m(v)$. Then $u \ge v$ in Ω.*

The following theorem solves the Dirichlet problem in $\mathcal{E}_{p,m}$. For its proof we refer to [Lu, Theorem 0.0.1] (see also [Ce2, Theorem 6.2], [ACH, Theorem 3.6]).

THEOREM 1.13. *Let μ be a Radon measure in Ω. Then there exists a unique $u \in \mathcal{E}_{p,m}$ such that $H_m(u) = \mu$ if and only if there exists a constant $C > 0$ satisfying*

$$\int_\Omega (-v)^p \, d\mu \le C e_{p,m}(v)^{p/(m+p)}, \qquad \forall v \in \mathcal{E}_{0,m}.$$

2. Riesz spaces. Let us start by giving some background on ordered vector spaces. For further information and duality we refer the readers to [AT].

DEFINITION 2.1. A binary relation $\succcurlyeq$ on a set X is said to be an *order relation* if it has the following three properties:

(1) reflexivity: $x \succcurlyeq x$,
(2) antisymmetry: $x \succcurlyeq y$ and $y \succcurlyeq x$ imply $x = y$,
(3) transitivity: $x \succcurlyeq y$ and $y \succcurlyeq z$ imply $x \succcurlyeq z$.

DEFINITION 2.2. A nonempty subset $\mathcal{K}$ of a vector space X is a *cone* if:

(1) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$,
(2) $r\mathcal{K} \subseteq \mathcal{K}$ for all $r \ge 0$, and
(3) $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$.

DEFINITION 2.3. An order relation $\succcurlyeq_X$ on a vector space X is said to be a *vector ordering* if $\succcurlyeq_X$ is compatible with the algebraic structure of X:

(i) if $x \succcurlyeq_X y$, then $x + z \succcurlyeq_X y + z$ for all $z \in X$,
(ii) if $x \succcurlyeq_X y$, then $rx \succcurlyeq_X ry$ for all $r \ge 0$.

An *order vector space* $(X, \succcurlyeq_X)$ is a vector space X with a vector ordering $\succcurlyeq_X$.

We denote by $X^+ = \{x \in X : x \succcurlyeq_X 0\}$ the positive cone of X. Let $\mathcal{K}$ be any cone in X then it generates a vector ordering $\succcurlyeq_\mathcal{K}$ on X defined by letting $x \succcurlyeq_\mathcal{K} y$ whenever $x - y \in \mathcal{K}$. To simplify the notation we shall write $\succcurlyeq$ instead of $\succcurlyeq_\mathcal{K}$.

Definition 2.4. An ordered vector space $(X, \succcurlyeq)$ is a *Riesz space* (or a *vector lattice*) if every pair of vectors x, y of X have a supremum $x \vee_{\succcurlyeq} y$ and an infimum $x \wedge_{\succcurlyeq} y$ in X.

Remark 2.5. Since $x \wedge_{\succcurlyeq} y = -((-x) \vee_{\succcurlyeq} (-y))$, to show that an ordered vector space is a Riesz space it is enough to prove that any two vectors have a supremum.

Definition 2.6. An ordered vector space $(X, \succcurlyeq)$ is *Dedekind σ-complete* if every increasing sequence bounded from above has a supremum.

Let $\delta\mathcal{E}_{p,m} = \mathcal{E}_{p,m} - \mathcal{E}_{p,m}$. We make the convention that $-\infty - (-\infty) = -\infty$. Then $\delta\mathcal{E}_{p,m}$ is a vector space over $\mathbb{R}$ equipped with pointwise addition of functions and real scalar multiplication. We consider $\delta\mathcal{E}_{p,m}$ with the vector ordering induced by the positive cone, i.e. for $u, v \in \delta\mathcal{E}_{p,m}$, we write $u \succcurlyeq v$ if $u - v \in \mathcal{E}_{p,m}$. Note that $u \succcurlyeq 0$ for all $u \in \mathcal{E}_{p,m}$ although $u(x) \leq 0$ for all $x \in \Omega$. One of the major advantages of this construction is that $(\delta\mathcal{E}_{p,m})^+ = \mathcal{E}_{p,m}$.

The usual pointwise vector ordering $\geq$ is defined as $u \geq v$ if and only if $u(x) \geq v(x)$ for all $x \in \Omega$. The two vector orderings on $\delta\mathcal{E}_{p,m}$ are related as follows: if $u \succcurlyeq v$ then $v \geq u$, but not conversely. Example 2.10 below (see also [AC, Example 3.1]) shows there are functions u, v in $\delta\mathcal{E}_{p,m}$ with $u \geq v$, but u, v are not comparable with respect to $\succcurlyeq$. In particular, $\delta\mathcal{E}_{p,m}$ is not a totally ordered vector space.

Along with $\mathcal{E}_{p,m}$, we are interested in the set of measures

$$\mathcal{H}_{p,m} = \{\mu : \mu = H_m(u) \text{ for some } u \in \mathcal{E}_{p,m}\}.$$

By Theorem 1.13, $\mathcal{H}_{p,m}$ is a cone. The ordered vector space $(\delta\mathcal{H}_{p,m}, \succcurlyeq)$ is defined similarly, i.e. for $\mu, \nu \in \delta\mathcal{H}_{p,m}$, $\mu \succcurlyeq \nu$ if $\mu - \nu \in \mathcal{H}_{p,m}$.

Remark 2.7. Theorem 1.13 implies that $\mathcal{H}_{p,m}$ is a cone, and if $\mu \in \mathcal{H}_{p,m}$ and ν is any positive Radon measure such that $\mu \geq \nu$ then $\nu \in \mathcal{H}_{p,m}$.

The usual ordering $\geq$ on $\delta\mathcal{H}_{p,m}$ is defined as follows: if $\mu, \nu \in \delta\mathcal{H}_{p,m}$, then $\mu \geq \nu$ if $\mu(A) \geq \nu(A)$ for every measurable subset $A \subseteq \Omega$.

Theorem 2.8.

(a) *The classical order and the order induced by the cone $\mathcal{H}_{p,m}$ coincide.*
(b) *$(\delta\mathcal{E}_{p,m}, \geq)$ and $(\delta\mathcal{H}_{p,m}, \geq)$ are Riesz spaces.*
(c) *$(\delta\mathcal{E}_{p,m}, \succcurlyeq)$ is Dedekind σ-complete.*

Proof. We use an idea from [AC].

(a) Let $\mu, \nu \in \mathcal{H}_{p,m}$. If $\mu \succcurlyeq \nu$, then $\mu - \nu \in \mathcal{H}_{p,m}$, so $\mu \geq \nu$. Now suppose that $\mu \geq \nu$. As $\mu \geq \mu - \nu \geq 0$, Remark 2.7 implies $\mu - \nu \in \mathcal{H}_{p,m}$, so $\mu \succcurlyeq \nu$.

(b) Let $u, v \in (\delta\mathcal{E}_{p,m}, \geq)$. We have $u = u_1 - u_2$, $v = v_1 - v_2$ for some $u_j, v_j \in \mathcal{E}_{p,m}$, $j = 1, 2$. Then

$u \vee_{\geq} v = \max(u, v) = \max(u_1 - u_2, v_1 - v_2) = \max(u_1 + v_2, u_2 + v_1) - (u_2 + v_2)$. Since $\mathcal{E}_{p,m}$ is a cone, by Proposition 1.11 we get $u \vee_{\geq} v \in \delta\mathcal{E}_{p,m}$.

Similarly, let $\mu, \nu \in (\delta\mathcal{H}_{p,m}, \geq)$. Then there exist $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{H}_{p,m}$ such that $\mu = \mu_1 - \mu_2$ and $\nu = \nu_1 - \nu_2$. We have

$$\mu \vee_{\geq} \nu = \sup(\mu_1 - \mu_2, \nu_1 - \nu_2) = \sup(\mu_1 + \nu_2, \mu_2 + \nu_1) - (\mu_2 + \nu_2),$$

where $\sup(\alpha, \beta)(A) = \sup_{B \subset A}\{\alpha(B) + \beta(A \backslash B)\}$ for positive measures α, β. We can see that $\sup(\alpha, \beta)$ is the smallest measure majorant of α and β. Remark 2.7 implies that $\mu \vee_{\geq} \nu \in \delta\mathcal{H}_{p,m}$.

(c) Assume that $\{u_j\}$ is an increasing sequence in $(\delta\mathcal{E}_{p,m}, \succcurlyeq)$ which is bounded from above by ϕ, i.e. $\phi \succcurlyeq u_j$ for all $j \in \mathbb{N}$. By the definition, for each $j \in \mathbb{N}$, we have $u_{j+1} - u_j, \phi - u_j \in \mathcal{E}_{p,m}$. For $k \geq 2$,

$$\sum_{j=1}^{k-1}(u_{j+1} - u_j) \geq (\phi - u_k) + \sum_{j=1}^{k-1}(u_{j+1} - u_j) = \phi - u_1 \in \mathcal{E}_{p,m}.$$

Letting $k \to \infty$, we get $\sum_{j=1}^{\infty}(u_{j+1} - u_j) \geq \phi - u_1$. The function $\gamma = \sum_{j=1}^{\infty}(u_{j+1} - u_j)$ is the limit of a decreasing sequence of m-subharmonic functions, so it is a negative m-subharmonic function and $\gamma \geq \phi - u_1 \in \mathcal{E}_{p,m}$. By Proposition 1.11 we get $\gamma \in \mathcal{E}_{p,m}$. We set $u = u_1 + \gamma \in \delta\mathcal{E}_{p,m}$.

Now we prove that $u = \sup_j\{u_j\}$. First observe that by arguing much as above we get $\sum_{j=k}^{\infty}(u_{j+1} - u_j) \in \mathcal{E}_{p,m}$ for all $k \geq 2$, so

$$u - u_k = \gamma + u_1 - \sum_{j=1}^{k-1}(u_{j+1} - u_j) - u_1 = \sum_{j=k}^{\infty}(u_{j+1} - u_j) \in \mathcal{E}_{p,m}, \quad \forall k \geq 2.$$

Thus $u \succcurlyeq u_k$ for all k. Now suppose that $v \in \delta\mathcal{E}_{p,m}$ is any upper bound of $\{u_j\}$, so $v \succcurlyeq u_j$, or $v - u_j \in \mathcal{E}_{p,m}$, for all $j \in \mathbb{N}$. For all k we have $(v - u_{k+1}) - (v - u_k) = u_k - u_{k+1} \geq 0$, which means that $\{v - u_k\}$ is an increasing sequence of m-subharmonic functions with respect to the usual pointwise order $\geq$. Furthermore, the following limit exists:

$$\alpha = \lim_{k\to\infty}(v - u_k) = (v - u_1) - \sum_{j=1}^{\infty}(u_{j+1} - u_j) = (v - u_1) - \gamma.$$

Therefore $\alpha^* = (v - u_1) - \gamma \geq v - u_1$, where α^* denotes the upper semicontinuous regularization of α. Then Proposition 1.11 yields $\alpha^* \in \mathcal{E}_{p,m}$. Thus, $v - u = \alpha^*$, i.e. $v \succcurlyeq u$, which proves (c). ■

REMARK 2.9. Example 3.3 in [ACC] shows that $(\delta\mathcal{E}_{0,n}(\mathbb{B}), \succcurlyeq)$ is not a Riesz space.

EXAMPLE 2.10. Let $\rho \in \mathcal{E}_{0,m}$ be an m-subharmonic function defining Ω, and let $w_0 \in \Omega$. Select a, b such that $\inf_{\Omega} \rho < a < b < \rho(w_0) < 0$. Then the functions $u = \max(\rho, a)$ and $v = \max(\rho, b)$ are in $\mathcal{E}_{0,m}(\Omega)$, and $v \geq u$. But u and v are not comparable with respect to the order $\succcurlyeq$.

3. Normality. We want to show that the formula in (0.1) defines a quasi-norm on $\delta\mathcal{E}_{p,m}$ for $p \neq 1$, and a norm for $p = 1$. First, we prove a Hölder type inequality for functions in $\mathcal{E}_{p,m}$. For $m = n$ and $p \geq 1$, Theorem 3.1 below was proved in [Pe], and for $m = n$ and $0 < p < 1$ in [ACH]. The case $p \geq 1$ was handled in [Lu, Lemma 1.7.8]. By using the idea of [ACH, Lemma 2.1] we will prove it for $0 < p < 1$.

THEOREM 3.1. *Let $u_0, u_1, \ldots, u_m \in \mathcal{E}_{p,m}$. Then there exists a constant $D(p,m)$ depending only on p and m such that*

$$\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m} \leq D(p,m) e_{p,m}(u_0)^{\frac{p}{p+m}} e_{p,m}(u_1)^{\frac{1}{p+m}} \cdots e_{p,m}(u_m)^{\frac{1}{p+m}},$$

where

$$D(p,m) = \begin{cases} p^{-\frac{\alpha(p,m)}{1-p}} & \text{if } 0 < p < 1, \\ 1 & \text{if } p = 1, \\ p^{\frac{p\alpha(p,m)}{p-1}} & \text{if } p > 1, \end{cases}$$

and $\alpha(p,m) = (p+2)\big(\frac{p+1}{p}\big)^{m-1} - (p+1)$.

Proof. By standard approximation, without loss of generality we can assume that $u_0, u_1, \ldots, u_m \in \mathcal{E}_{0,m}$. If $0 < p < 1$, then $-(-u_0)^p \in \mathcal{E}_{0,m}$ (see [Ng, Proposition 1.3]). Now let $w = -(-u_1)^p \in \mathcal{E}_{0,m}$ and $T = dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}$. We have

$$\begin{aligned} \int_\Omega (-u_0)^p dd^c u_1 \wedge T &= -\int_\Omega (-u_0)^p dd^c (-w)^{1/p} \wedge T \\ &= -\frac{1}{p} \int_\Omega (-u_0)^p (-w)^{1/p-1} dd^c(-w) \wedge T \\ &\quad - \frac{1-p}{p^2} \int_\Omega (-u_0)^p (-w)^{1/p-2} d(-w) \wedge d^c(-w) \wedge T \\ &\leq \frac{1}{p} \int_\Omega (-u_0)^p (-w)^{1/p-1} dd^c w \wedge T = \frac{1}{p} \int_\Omega (-u_0)^p (-u_1)^{1-p} dd^c w \wedge T. \end{aligned} \tag{3.1}$$

Applying the Hölder inequality and integration by parts in $\mathcal{E}_{0,m}$ we obtain

$$\begin{aligned} \int_\Omega (-u_0)^p dd^c u_1 \wedge T &\leq \frac{1}{p} \Big[\int_\Omega (-u_0) dd^c w \wedge T \Big]^p \Big[\int_\Omega (-u_1) dd^c w \wedge T \Big]^{1-p} \\ &= \frac{1}{p} \Big[\int_\Omega (-w) dd^c u_0 \wedge T \Big]^p \Big[\int_\Omega (-w) dd^c u_1 \wedge T \Big]^{1-p} \\ &= \frac{1}{p} \Big[\int_\Omega (-u_1)^p dd^c u_0 \wedge T \Big]^p \Big[\int_\Omega (-u_1)^p dd^c u_1 \wedge T \Big]^{1-p}. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we get

$$\begin{aligned}\int_\Omega(-u_0)^p dd^c u_1\wedge T &\le \frac{1}{p}\Big[\int_\Omega(-u_1)^p dd^c u_0\wedge T\Big]^p\Big[\int_\Omega(-u_1)^p dd^c u_1\wedge T\Big]^{1-p}\\ &\le \frac{1}{p^{1+p}}\Big[\int_\Omega(-u_0)^p dd^c u_1\wedge T\Big]^{p^2}\Big[\int_\Omega(-u_0)^p dd^c u_0\wedge T\Big]^{p(1-p)}\\ &\quad\times\Big[\int_\Omega(-u_1)^p dd^c u_1\wedge T\Big]^{1-p}.\end{aligned}$$

This implies that

$$\begin{aligned}\int_\Omega(-u_0)^p dd^c u_1\wedge T \le p^{-\frac{1}{1-p}}&\Big(\int_\Omega(-u_0)^p dd^c u_0\wedge T\Big)^{\frac{p}{1+p}}\\ &\times\Big(\int_\Omega(-u_1)^p dd^c u_1\wedge T\Big)^{\frac{1}{1+p}}.\end{aligned}\tag{3.3}$$

The function $F:(\mathcal{E}_{0,m})^{m+1}\to\mathbb{R}^+$ defined by

$$F(u_0,u_1,\dots,u_m)=\int_\Omega(-u_0)^p dd^c u_1\wedge\dots\wedge dd^c u_m\wedge\beta^{n-m}$$

is symmetric in the last m variables. By (3.3),

$$F(u_0,u_1,\dots,u_m)\le p^{-\frac{1}{1-p}}F(u_0,u_0,u_2,\dots,u_m)^{\frac{p}{1+p}}F(u_1,u_1,u_2,\dots,u_m)^{\frac{1}{1+p}}.$$

The rest of the proof goes verbatim as the proof of [Pe, Theorem 4.1] (see also [ACH, Theorem 2.2]). ∎

Lemma 3.2. *For $u,v\in\mathcal{E}_{p,m}$, we have*

$$e_{p,m}(u+v)^{\frac{1}{p+m}}\le C(p,m)\big(e_{p,m}(u)^{\frac{1}{p+m}}+e_{p,m}(v)^{\frac{1}{p+m}}\big),\tag{3.4}$$

where $C(p,m)>1$ is a constant depending only on m and $p\ne 1$, and $C(1,m)=1$.

Proof. By Theorem 3.1 we have

$$\begin{aligned}e_{p,m}(u+v)&=\int_\Omega(-u-v)^p[dd^c(u+v)]^m\wedge\beta^{n-m}\\ &=\sum_{k=0}^m\binom{m}{k}\int_\Omega(-u-v)^p(dd^c u)^k\wedge(dd^c v)^{m-k}\wedge\beta^{n-m}\\ &\le D(p,m)\sum_{k=0}^m\binom{m}{k}e_{p,m}(u+v)^{\frac{p}{p+m}}e_{p,m}(u)^{\frac{k}{p+m}}e_{p,m}(v)^{\frac{m-k}{p+m}}\\ &=D(p,m)e_{p,m}(u+v)^{\frac{p}{p+m}}\big[e_{p,m}(u)^{\frac{1}{p+m}}+e_{p,m}(v)^{\frac{1}{p+m}}\big]^m.\end{aligned}$$

Hence

$$e_{p,m}(u+v) \le D(p,m)^{\frac{p+m}{m}} \left[e_{p,m}(u)^{\frac{1}{p+m}} + e_{p,m}(v)^{\frac{1}{p+m}}\right]^{m+p}.$$

Thus we get (3.4) with $C(p,m) = D(p,m)^{1/m}$. ■

REMARK 3.3. In general, if $u_1, \dots, u_k \in \mathcal{E}_{p,m}$, then

$$\begin{aligned} e_{p,m}&(u_1 + \dots + u_k)^{\frac{1}{p+m}} \\ &\le \sum_{j=1}^{k-2} C(p,m)^j e_{p,m}(u_j)^{\frac{1}{p+m}} + C(p,m)^{k-1}(e_{p,m}(u_{k-1}) + e_{p,m}(u_k))^{\frac{1}{p+m}} \\ &\le \sum_{j=1}^{k} C(p,m)^j e_{p,m}(u_j)^{\frac{1}{p+m}}. \end{aligned}$$

LEMMA 3.4. *Let $u, v \in \mathcal{E}_{p,m}$ with $v \le u$. Then*

$$e_{p,m}(u) \le D(p,m)^{\frac{p+m}{p}} e_{p,m}(v),$$

where $D(p,m)$ is the constant defined in Theorem 3.1. In addition if $p \le 1$, then $e_{p,m}(u) \le e_{p,m}(v)$.

Proof. By Theorem 3.1 we have

$$\begin{aligned} e_{p,m}(u) &= \int_\Omega (-u)^p (dd^c u)^m \wedge \beta^{n-m} \le \int_\Omega (-v)^p (dd^c u)^m \wedge \beta^{n-m} \\ &\le D(p,m) e_{p,m}(v)^{\frac{p}{p+m}} e_{p,m}(u)^{\frac{m}{p+m}}, \end{aligned}$$

which implies that

$$e_{p,m}(u) \le D(p,m)^{\frac{p+m}{p}} e_{p,m}(v).$$

If $p \le 1$, then by Theorem 1.10 there exist decreasing sequences $\{u_j\}, \{v_j\} \subset \mathcal{E}_{0,m}$ such that $u_j \ge v_j$ and

$$u_j \to u,\ v_j \to v,\ e_{p,m}(u_j) \to e_{p,m}(u) \text{ and } e_{p,m}(v_j) \to e_{p,m}(v) \quad \text{as } j \to \infty.$$

We have $-(-u_j)^p \in \mathcal{E}_{0,m}$ (see [Ng, Proposition 1.3]). Integrating by parts we obtain

$$e_{p,m}(u_j) = \int_\Omega (-u_j)^p (dd^c u_j)^m \wedge \beta^{n-m} \le \int_\Omega (-u_j)^p (dd^c v_j)^m \wedge \beta^{n-m} \le e_{p,m}(v_j).$$

By letting $j \to \infty$ we get $e_{p,m}(u) \le e_{p,m}(v)$. ■

For $u \in \delta\mathcal{E}_{p,m}$, the formula in (0.1) can be rewritten as follows:

$$\|u\|_{p,m} = \inf\{e_{p,m}(u_1 + u_2)^{\frac{1}{p+m}}\ u = u_1 - u_2,\ u_1, u_2 \in \mathcal{E}_{p,m}\}. \tag{3.5}$$

LEMMA 3.5. *If $u \in \mathcal{E}_{p,m}$ then $\|u\|_{p,m} = e_{p,m}(u)^{\frac{1}{p+m}}$.*

Proof. Since $u = u - 0$, then $\|u\|_{p,m} \le e_{p,m}(u)^{\frac{1}{p+m}}$. Let $u_1, u_2 \in \mathcal{E}_{p,m}$ be such that $u = u_1 - u_2$. Then $u \ge u_1 - u_2 + 2u_2$. We have

$$\begin{aligned}e_{p,m}(u) &= \int_{\Omega}(-u)^p(dd^c u)^m \wedge \beta^{n-m} \le \int_{\Omega}(-u)^p[dd^c(u+2u_2)]^m \wedge \beta^{n-m} \\ &\le \int_{\Omega}(-u_1-u_2)^p[dd^c(u_1+u_2)]^m \wedge \beta^{n-m} = e_{p,m}(u_1+u_2).\end{aligned}$$

Hence

$$e_{p,m}(u_1+u_2)^{\frac{1}{p+m}} \ge e_{p,m}(u)^{\frac{1}{p+m}}.$$

Taking the infimum over $u_1, u_2 \in \mathcal{E}_{p,m}$ with $u_1 - u_2 = u$, we get

$$\|u\|_{p,m} \ge e_{p,m}(u)^{\frac{1}{p+m}}. \ \blacksquare$$

Now we recall the definition of a quasi-Banach space.

Definition 3.6. A function $\|\cdot\| : X \to [0,\infty)$ is called a *quasi-norm* on a vector space X if it has the following properties:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\,\|x\|$ for all $x \in X$, $r \in \mathbb{R}$;
(iii) there exists a constant $C \ge 1$ such that

$$\|x+y\| \le C(\|x\| + \|y\|), \quad \forall x, y \in X.$$

Aoki [Ao] and Rolewicz [Ro] characterized quasi-norms as follows:

Theorem 3.7. *Let $\|\cdot\|$ be a quasi-norm on X. Then there exist $0 < q \le 1$ and an equivalent quasi-norm $|||\cdot|||$ on X such that, for all $x, y \in X$,*

$$|||x+y|||^q \le |||x|||^q + |||y|||^q.$$

Hence for a given quasi-norm $\|\cdot\|$ on X, we can define the metric $d(x,y) = |||x-y|||^q$ on X. The vector space X is called a *quasi-Banach space* if it is complete with respect to the metric induced by the quasi-norm $\|\cdot\|$.

Theorem 3.8. *$(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ is a quasi-Banach space for $p \ne 1$ and $(\delta\mathcal{E}_{1,m}, \|\cdot\|_{1,m})$ is a Banach space.*

Proof. (i) If $u = 0 \in \mathcal{E}_{p,m}$, then Lemma 3.5 implies $\|u\|_{p,m} = 0$. Assume that $u \in \delta\mathcal{E}_{p,m}$ with $\|u\|_{p,m} = 0$. Let $\epsilon > 0$. Then by the definition of $\|u\|_{p,m}$, there exist $u_1, u_2 \in \mathcal{E}_{p,m}$ such that $u = u_1 - u_2$ and $e_{p,m}(u_1+u_2) < \epsilon$. Since $u_1 + u_2 \in \mathcal{E}_{p,m}$, by Theorem 1.10 there exists a sequence $\{v_j\} \subset \mathcal{E}_{0,m}$ with $v_j \downarrow (u_1+u_2)$ and $\sup_j e_{p,m}(v_j) < \epsilon$. Let $\phi \in \mathcal{E}_{p,m}$ be such that $H_m(\phi) = d\lambda_n$ (see [Lu, Theorem 1.8.18]), where λ_n is the Lebesgue measure on $\mathbb{C}^n$. It follows from Theorem 3.1 that

$$\begin{aligned}\|v_j\|_{L^p}^p &= \int_{\Omega}(-v_j)^p\,d\lambda_n = \int_{\Omega}(-v_j)^p H_m(\phi) \\ &\le D(p,m)e_{p,m}(v_j)^{\frac{p}{p+m}} e_{p,m}(\phi)^{\frac{m}{p+m}} \le C\epsilon^{\frac{p}{p+m}},\end{aligned}$$

where C is a constant that does not depend on j. Hence

$$\|u\|_{L^p}^p \le \|u_1+u_2\|_{L^p}^p \le C\epsilon^{\frac{p}{p+m}}.$$

Letting $\epsilon \to 0^+$ yields $\|u\|_{L^p}=0$, thus $u=0$ almost everywhere. This means that $u_1=u_2$ almost everywhere in Ω. Moreover, u_1 and u_2 are subharmonic on Ω (see Remark 1.2), so $u_1=u_2$ in Ω, i.e. $u=0$ in Ω.

(ii) Let $u\in\delta\mathcal{E}_{p,m}$. For $t\in\mathbb{R}$, $t>0$, we have

$$\begin{aligned}\|tu\|_{p,m} &= \inf\{e_{p,m}(u_1+u_2)^{\frac{1}{p+m}} : tu=u_1-u_2,\ u_1,u_2\in\mathcal{E}_{p,m}\}\\ &= \inf\{e_{p,m}(tv_1+tv_2)^{\frac{1}{p+m}} : u=v_1-v_2,\ v_1,v_2\in\mathcal{E}_{p,m}\} = t\|u\|_{p,m}.\end{aligned}$$

The case $t<0$ is similar, and the case $t=0$ is clear.

(iii) Let $u,v\in\delta\mathcal{E}_{p,m}$ and $\epsilon>0$. Then there exist $u_1,u_2,v_1,v_2\in\mathcal{E}_{p,m}$ such that $u=u_1-u_2, v=v_1-v_2$ and

$$e_{p,m}(u_1+u_2)^{\frac{1}{p+m}} \le \|u\|_{p,m}+\epsilon, \qquad e_{p,m}(v_1+v_2)^{\frac{1}{p+m}} \le \|v\|_{p,m}+\epsilon.$$

By Lemma 3.2,

$$\begin{aligned}&\|u+v\|_{p,m} \le e_{p,m}(u_1+u_2+v_1+v_2)^{\frac{1}{p+m}}\\ &\le C\big(e_{p,m}(u_1+u_2)^{\frac{1}{p+m}} + e_{p,m}(v_1+v_2)^{\frac{1}{p+m}}\big) \le C(\|u\|_{p,m}+\|v\|_{p,m})+2C\epsilon,\end{aligned}$$

where $C=C(p,m)$ is given in Lemma 3.2. Letting $\epsilon\to 0^+$, we obtain

$$\|u+v\|_{p,m} \le C(\|u\|_{p,m}+\|v\|_{p,m}).$$

If $p=1$ then $C=C(1,m)=1$. This implies that $\|\cdot\|_{1,m}$ is a norm.

(iv) Now we shall prove that the space $(\delta\mathcal{E}_{p,m},\|\cdot\|_{p,m})$ is complete. Assume that $\{u_j\}$ is a Cauchy sequence in $(\delta\mathcal{E}_{p,m},\|\cdot\|_{p,m})$. For each integer i, there is an integer j_i such that

$$\|u_{j_{i+1}}-u_{j_i}\|_{p,m} \le (2C)^{-i}. \tag{3.6}$$

We can choose the j_i to form an increasing sequence. Moreover, for each i, there exist $v_i,w_i\in\mathcal{E}_{p,m}$ such that

$$u_{j_{i+1}}-u_{j_i}=v_i-w_i, \qquad e_{p,m}(v_i+w_i)^{\frac{1}{p+m}} \le \|u_{j_{i+1}}-u_{j_i}\|_{p,m}+(2C)^{-i}. \tag{3.7}$$

Note that

$$\begin{aligned}u_{j_{k+1}} &= u_{j_1}+\sum_{i=1}^k(u_{j_{i+1}}-u_{j_i}) = u_{j_1}+\sum_{i=1}^k(v_i-w_i)\\ &= u_{j_1}+\sum_{i=1}^k v_i-\sum_{i=1}^k w_i.\end{aligned} \tag{3.8}$$

By combining Proposition 1.11, Remark 3.3, (3.7) and (3.6) we get

$$\begin{aligned}\max\Big\{e_{p,m}\Big(\sum_{i=1}^{k} v_i\Big)^{\frac{1}{p+m}}, e_{p,m}\Big(\sum_{i=1}^{k} w_i\Big)^{\frac{1}{p+m}}\Big\} &\le e_{p,m}\Big(\sum_{i=1}^{k}(v_i+w_i)\Big)^{\frac{1}{p+m}}\\ &\le \sum_{i=1}^{k} C^i e_{p,m}(v_i+w_i)^{\frac{1}{p+m}} \le \sum_{i=1}^{k} C^i[(2C)^{-i} + \|u_{j_{i+1}} - u_{j_i}\|_{p,m}]\\ &\le \sum_{i=1}^{k} C^i[(2C)^{-i} + (2C)^{-i}] \le 2\sum_{i=1}^{\infty} 2^{-i} = 1.\end{aligned}$$

The sequences $\{\sum_{i=1}^{k} v_i\}_k$ and $\{\sum_{i=1}^{k} w_i\}_k$ are decreasing sequences in $\mathcal{E}_{p,m}$ with bounded m-pluricomplex p-energy. Thus there exist $\varphi, \psi \in \mathcal{E}_{p,m}$ such that $\sum_{i=1}^{k} v_i \to \varphi$, $\sum_{i=1}^{k} w_i \to \psi$ in $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$. By (3.8),

$$u_{j_k} \to u_{j_1} + \varphi - \psi := u \in \delta\mathcal{E}_{p,m}.$$

Since $\{u_j\}$ is Cauchy sequence, it follows that $u_j \to u$. ∎

The following theorem says that there exists a decomposition of each element in $\delta\mathcal{E}_{p,m}$ with explicit control of quasi-norms.

THEOREM 3.9. *For each $u \in \delta\mathcal{E}_{p,m}$, there exist unique $u^+, u^- \in \mathcal{E}_{p,m}$ such that $u = u^+ - u^-$ and*

$$\|u\|_{p,m} \le \|u^+ + u^-\|_{p,m} \le D(p,m)^{1/p}\|u\|_{p,m}.$$

Furthermore, if $p \le 1$, then $\|u\|_{p,m} = \|u^+ + u^-\|_{p,m}$.

Proof. Let $u = u_1 - u_2 \in \delta\mathcal{E}_{p,m}$, and define

$$u^+ = \sup\{\alpha \in \mathcal{E}_{p,m} : \text{there exists } \beta \in \mathcal{E}_{p,m} \text{ such that } u_2 + \alpha = u_1 + \beta\},$$
$$u^- = \sup\{\beta \in \mathcal{E}_{p,m} : \text{there exists } \alpha \in \mathcal{E}_{p,m} \text{ such that } u_2 + \alpha = u_1 + \beta\}.$$

Then $(u^+)^*, (u^-)^* \in \mathcal{E}_{p,m}$. By Choquet's lemma, there exist sequences $\{\alpha_j\}$, $\{\beta_j\} \subset \mathcal{E}_{0,m}$ such that $(\sup_j \alpha_j)^* = (u^+)^*$ and $(\sup_j \beta_j)^* = (u^-)^*$. Furthermore, we can assume $u_2 + \alpha_j = u_1 + \beta_j$. By passing to limits we obtain

$$u_2 + u^+ = u_1 + u^-.$$

Since $u^+ = (u^+)^*$ and $u^- = (u^-)^*$ almost everywhere, we obtain $u_2 + (u^+)^* = u_1 + (u^-)^*$. Hence

$$u^+ = (u^+)^* \quad \text{and} \quad u^- = (u^-)^*.$$

If $\alpha, \beta \in \mathcal{E}_{p,m}$ are such that $u = \alpha - \beta$, then $\alpha \le u^+$ and $\beta \le u^-$, so $\alpha + \beta \le u^+ + u^-$. By Lemmas 3.5 and 3.4,

$$\|u\|_{p,m} \le e_{p,m}(u^+ + u^-)^{\frac{1}{p+m}} = \|u^+ + u^-\|_{p,m} \le D(p,m)^{1/p} e_{p,m}(\alpha+\beta).$$

Taking the infimum over all decompositions $u = \alpha - \beta$, we get

$$\|u\|_{p,m} \le \|u^+ + u^-\|_{p,m} \le D(p,m)^{1/p}\|u\|_{p,m}.$$

If $p \le 1$, then by Lemma 3.4, $\|u\|_{p,m} = \|u^+ + u^-\|_{p,m}$. ∎

Remark 3.10. In general, let $u = u_1 - u_2$ be in $\delta\mathrm{SH}_m^-(\Omega)$, where Ω is a bounded domain in $\mathbb{C}^n$. Then

$$u^+ = \sup\{\alpha \in \mathrm{SH}_m^-(\Omega) : \text{there exists } \beta \in \mathrm{SH}_m^-(\Omega) \text{ with } u_2 + \alpha = u_1 + \beta\},$$
$$u^- = \sup\{\beta \in \mathrm{SH}_m^-(\Omega) : \text{there exists } \alpha \in \mathrm{SH}_m^-(\Omega) \text{ with } u_2 + \alpha = u_1 + \beta\}.$$

By reasoning as above, we can show that $u^+, u^- \in \mathrm{SH}_m^-(\Omega)$ and $u = u^+ - u^-$.

For $\mu \in \delta\mathcal{H}_{p,m}$, we define

$$|\mu|_{p,m} = \inf\{\|u_{\mu_1}\|_{p,m}^m + \|u_{\mu_2}\|_{p,m}^m : \mu = \mu_1 - \mu_2,\ \mu_1, \mu_2 \in \mathcal{H}_{p,m}\},$$

where $u_{\mu_j} \in \mathcal{E}_{p,m}$, $j = 1, 2$, are the unique solutions to $H_m(u_{\mu_j}) = \mu_j$, as in Theorem 1.13.

Lemma 3.11. *Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ, where*

$$\mu^+ = \tfrac{1}{2}(|\mu| + \mu) \quad \text{and} \quad \mu^- = \tfrac{1}{2}(|\mu| - \mu).$$

Then

$$|\mu|_{p,m} = \|u_{\mu^+}\|_{p,m}^m + \|u_{\mu^-}\|_{p,m}^m.$$

Proof. Suppose $\mu = \mu_1 - \mu_2$ is any representation of $\mu \in \delta\mathcal{H}_{p,m}$. Then $\mu^+ \le \mu_1$ and $\mu^- \le \mu_2$. This implies that $\mu^+, \mu^- \in \mathcal{H}_{p,m}$ by Theorem 1.13 and $H_m(u_{\mu^+}) \le H_{u_{\mu_1}}$. By Theorem 1.12, we have $u_{\mu^+} \ge u_{\mu_1}$. Now

$$\begin{aligned}\|u_{\mu^+}\|_{p,m}^m &= \Big(\int_\Omega (-u_{\mu^+})^p H_m(u_{\mu^+})\Big)^{\frac{m}{p+m}} \\ &\le \Big(\int_\Omega (-u_{\mu_1})^p H_m(u_{\mu_1})\Big)^{\frac{m}{p+m}} = \|u_{\mu_1}\|_{p,m}^m.\end{aligned}$$

Similarly, $\|u_{\mu^-}\|_{p,m}^m \le \|u_{\mu_2}\|_{p,m}^m$. Thus

$$|\mu|_{p,m} = \|u_{\mu^+}\|_{p,m}^m + \|u_{\mu^-}\|_{p,m}^m. \ \blacksquare$$

Theorem 3.12. *$(\delta\mathcal{H}_{p,m}, |\cdot|_{p,m})$ is a quasi-Banach space for $p \ne 1$, and it is a Banach space if $p = 1$.*

Proof. (i) Suppose that $\mu \in \delta\mathcal{H}_{p,m}$ and $|\mu|_{p,m} = 0$. From Lemma 3.11,

$$\|u_{\mu^+}\|_{p,m} = \|u_{\mu^-}\|_{p,m} = 0.$$

By Theorem 3.8(i) , we have $u_{\mu^+} = u_{\mu^-} = 0$. Thus $\mu^+ = \mu^- = 0$, so $\mu = 0$.

(ii) For $t \ge 0$, we have

$$(t\mu)^+ = t\mu^+, \quad (t\mu)^- = t\mu^-, \quad u_{t\mu^+} = t^{1/m} u_{\mu^+}, \quad u_{t\mu^-} = t^{1/m} u_{\mu^-}.$$

Hence

$$|t\mu|_{p,m} = \|u_{(t\mu)^+}\|_{p,m}^m + \|u_{(t\mu)^-}\|_{p,m}^m = \|t^{1/m}\|_{p,m}^m + \|t^{1/m} u_{\mu^-}\|_{p,m}^m = t|\mu|_{p,m}.$$

Similarly, if $t < 0$ then $|t\mu|_{p,m} = (-t)|\mu|_{p,m}$.

(iii) Let $\mu, \nu \in \delta\mathcal{H}_{p,m}$. We have

$$\mu + \nu = \mu^+ - \mu^- + \nu^+ - \nu^- = (\mu^+ + \nu^+) - (\mu^- + \nu^-).$$

Thus $(\mu+\nu)^+ \le \mu^+ + \nu^+$ and $(\mu+\nu)^- \le \mu^- + \nu^-$. By Theorem 1.13, there exist $u_{(\mu+\nu)^+}, u_{(\mu+\nu)^-} \in \mathcal{E}_{p,m}$ such that

$$H_m(u_{(\mu+\nu)^+}) = (\mu+\nu)^+ \quad \text{and} \quad H_m(u_{(\mu+\nu)^-}) = (\mu+\nu)^-.$$

Applying Theorem 3.1, we obtain

$$\begin{aligned} e_{p,m}(u_{(\mu+\nu)^+}) &= \int_\Omega (-u_{(\mu+\nu)^+})^p H_m(u_{(\mu+\nu)^+}) = \int_\Omega (-u_{(\mu+\nu)^+})^p (\mu+\nu)^+ \\ &\le \int_\Omega (-u_{(\mu+\nu)^+})^p (\mu^+ + \nu^+) = \int_\Omega (-u_{(\mu+\nu)^+})^p (H_m(u_{\mu^+}) + H_m(u_{\nu^+})) \\ &\le D(p,m) e_{p,m}(u_{(\mu+\nu)^+})^{\frac{p}{p+m}} \big(e_{p,m}(u_{\mu^+})^{\frac{m}{p+m}} + e_{p,m}(u_{\nu^+})^{\frac{m}{p+m}}\big). \end{aligned}$$

Thus

$$e_{p,m}(u_{(\mu+\nu)^+})^{\frac{m}{p+m}} \le D(p,m)\big(e_{p,m}(u_{\mu^+})^{\frac{m}{p+m}} + e_{p,m}(u_{\nu^+})^{\frac{m}{p+m}}\big).$$

Similarly,

$$e_{p,m}(u_{(\mu+\nu)^-})^{\frac{m}{p+m}} \le D(p,m)\big(e_{p,m}(u_{\mu^-})^{\frac{m}{p+m}} + e_{p,m}(u_{\nu^-})^{\frac{m}{p+m}}\big).$$

We have

$$\begin{aligned} |\mu+\nu|_{p,m} &= \|u_{(\mu+\nu)^+}\|_{p,m}^m + \|u_{(\mu+\nu)^-}\|_{p,m}^m \\ &= e_{p,m}(u_{(\mu+\nu)^+})^{\frac{m}{p+m}} + e_{p,m}(u_{(\mu+\nu)^-})^{\frac{m}{p+m}} \\ &\le D(p,m)\big(e_{p,m}(u_{\mu^+})^{\frac{m}{p+m}} + e_{p,m}(u_{\mu^-})^{\frac{m}{p+m}} + e_{p,m}(u_{\nu^+})^{\frac{m}{p+m}} + e_{p,m}(u_{\nu^-})^{\frac{m}{p+m}}\big) \\ &= D(p,m)(\|u_{\mu^+}\|_{p,m}^m + \|u_{\mu^-}\|_{p,m}^m + \|u_{\nu^+}\|_{p,m}^m + \|u_{\nu^-}\|_{p,m}^m) \\ &= D(p,m)(|\mu|_{p,m} + |\nu|_{p,m}), \end{aligned}$$

where $D(p,m)$ is the constant given in Theorem 3.1. Because $D(1,m) = 1$, $|\cdot|_{1,m}$ is a norm.

(iv) Now we prove that $(\delta\mathcal{H}_{p,m}, |\cdot|_{p,m})$ is complete. Assume that $\{\mu_j\}$ is a Cauchy sequence in $(\delta\mathcal{H}_{p,m}, |\cdot|_{p,m})$. For each integer i, there is an integer j_i such that

$$|\mu_{j_{i+1}} - \mu_{j_i}|_{p,m} = \|u_{(\mu_{j_{i+1}} - \mu_{j_i})^+}\|_{p,m}^m + \|u_{(\mu_{j_{i+1}} - \mu_{j_i})^-}\|_{p,m}^m \le (2C)^{-\frac{mi}{p+m}},$$

where $C = C(p,m)$ is the constant of Lemma 3.2. We can choose $\{j_i\}$ to be an increasing sequence. In particular,

$$\|u_{(\mu_{j_{i+1}} - \mu_{j_i})}\|_{p,m} \le (2C)^{-\frac{i}{p+m}}. \tag{3.9}$$

Define

$$\mu = \mu_{j_1} + \sum_{i=1}^{\infty} (\mu_{j_{i+1}} - \mu_{j_i}).$$

Then

$$\mu^+ \le \mu_{j_1}^+ + \sum_{i=1}^{\infty} (\mu_{j_{i+1}} - \mu_{j_i})^+. \tag{3.10}$$

Now, for any k we have

$$\begin{aligned} e_{p,m}\Big(\sum_{i=1}^{k} u_{(\mu_{j_{i+1}}-\mu_{j_i})^+}\Big) &\le \sum_{i=1}^{k} C^i e_{p,m}(u_{(\mu_{j_{i+1}}-\mu_{j_i})^+}) \quad \text{(by Remark 3.3)}\\ &= \sum_{i=1}^{k} C^i \|u_{(\mu_{j_{i+1}}-\mu_{j_i})^+}\|_{p,m}^{p+m} \quad \text{(by Lemma 3.5)}\\ &\le \sum_{i=1}^{k} C^i (2C)^{-i} \le 1 \quad \text{(by (3.9))}. \end{aligned}$$

Thus $\{\sum_{i=1}^{k} u_{(\mu_{j_{i+1}}-\mu_{j_i})^+}\}$ is a decreasing sequence in $\mathcal{E}_{p,m}$ with bounded m-pluricomplex p-energy. Then there is a function $u^+ \in \mathcal{E}_{p,m}$ such that $\sum_{i=1}^{k} u_{(\mu_{j_{i+1}}-\mu_{j_i})^+} \to u^+$. From this and (3.10) we obtain

$$\mu^+ \le H_m(u_{\mu_{j_1}} + u^+).$$

By Theorem 1.13, $\mu^+ \in \mathcal{H}_{p,m}$. In a similar way one can prove that $\mu^- \in \mathcal{H}_{p,m}$. Hence $\mu_{j_i} \to \mu = \mu^+ - \mu^-$ in $(\delta\mathcal{H}_{p,m}, |\cdot|_{p,m})$. ■

COROLLARY 3.13. *The cones $\mathcal{E}_{p,m}$ and $\mathcal{H}_{p,m}$ are closed in $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ and $(\delta\mathcal{H}_{p,m}, |\cdot|_{p,m})$ respectively.*

THEOREM 3.14. *Let $p > 0$. Then the interior of $\mathcal{E}_{p,m}$ in $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ is empty. The corresponding statement for $(\delta\mathcal{H}_{p,m}, |\cdot|_{p,m})$ is also valid.*

Proof. (i) First, 0 is not an interior point of $\mathcal{E}_{p,m}$. Assume that $0 \ne u \in \mathcal{E}_{p,m}$ is an interior point of $\mathcal{E}_{p,m}$. Then there exists $\epsilon > 0$ such that if $\|u - v\|_{p,m} < \epsilon$, then $v \in \mathcal{E}_{p,m}$. We can find a subset B in Ω such that $H_m(u)(B) > 0$ and $2^{1/m}(\int_B (-u)^p H_m(u))^{1/(p+m)} < \epsilon$. Let $w \in \mathcal{E}_{p,m}$ be such that $H_m(w) = 2\chi_B H_m(u)$. Then $H_m(w)(B) > H_m(u)(B)$, which implies that $v := u - w \notin \mathcal{E}_{p,m}$. Now we have

$$H_m(w) \le 2H_m(u) = H_m(2^{1/m}u).$$

Using Theorem 1.12 we obtain $2^{1/m}u \le w$. Hence

$$\begin{aligned} \|u - v\|_{p,m} = \|w\|_{p,m} = e_{p,m}(w)^{\frac{1}{p+m}} &= \Big(\int_\Omega (-w)^p H_m(w)\Big)^{\frac{1}{p+m}} \\ &= \Big(2\int_B (-w)^p H_m(u)\Big)^{\frac{1}{p+m}} \\ &\le \Big(2\int_B (-2^{1/m}u)^p H_m(u)\Big)^{\frac{1}{p+m}} < \epsilon. \end{aligned} \tag{3.11}$$

This contradicts our assumption that u is an interior point of $\mathcal{E}_{p,m}$.

(ii) We argue as above. The point $0 \in \mathcal{H}_{p,m}$ is not an interior point of $(\delta\mathcal{H}_{p,m}, |\cdot|_{p,m})$. Assume that $0 \neq \mu \in \mathcal{H}_{p,m}$ is an interior point of $\mathcal{H}_{p,m}$ in $(\mathcal{H}_{p,m}, |\cdot|_{p,m})$. Then there exists $\epsilon > 0$ such that if $|\mu - \nu|_{p,m} < \epsilon$, then $\nu \in \mathcal{H}_{p,m}$. Let $u_\mu \in \mathcal{E}_{p,m}$ be such that $H_m(u_\mu) = d\mu$. As before, we can find $B \subset \Omega$ such that $\mu(B) > 0$ and $2(\int_B (-u_\mu)^p d\mu)^{m/(p+m)} < \epsilon$. The measure $\nu = \chi_{\Omega\setminus B}\mu - \chi_B\mu$ is not an element of $\mathcal{H}_{p,m}$ since $\nu(B) < 0$. Theorem 1.12 implies that $u_\mu \leq u_{\chi_B\mu}$, where $u_{\chi_B\mu} \in \mathcal{E}_{p,m}$ is such that $H_m(u_{\chi_B\mu}) = \chi_B\mu$. Hence,

$$|\mu - \nu|_{p,m} = 2|\chi_B\mu|_{p,m} = 2\|u_{\chi_B\mu}\|_{p,m}^m = 2\Big(\int_\Omega (-u_{\chi_B\mu})^p H_m(u_{\chi_B\mu})\Big)^{\frac{m}{p+m}}$$
$$\leq 2\Big(\int_B (-u_\mu)^p d\mu\Big)^{\frac{m}{p+m}} < \epsilon. \blacksquare$$

4. Duality. Let us recall some notions related to duality (see [AT]). The algebraic dual of a vector space X is the vector space of all linear functions on X, and denoted by X^*. Let $(X, \succcurlyeq)$ be an ordered vector space. A linear functional $f : X \to \mathbb{R}$ is called:

- *positive* if $f(x) \geq 0$ for all $x \in X^+$;
- *regular* if f can be written as the difference of two positive operators;
- *ordered bounded* if $f([x, y])$ is bounded for all $x, y \in X$, where the order interval $[x, y]$ is defined by
$$[x, y] = \{z \in X : y \succcurlyeq z \succcurlyeq x\}.$$

Let X^r and X^b denote the sets of respectively all regular functionals and all bounded functionals on $(X, \succcurlyeq)$.

Remark 4.1. $X^r \subseteq X^b \subseteq X^*$.

The topological dual of a topological vector space (X, τ) is denoted by X' and it is the vector subspace of X^* consisting of all τ-continuous functionals. Let $\mathcal{K}$ be a cone in (X, τ). The dual cone $\mathcal{K}'$ of $\mathcal{K}$ is

$$\mathcal{K}' = \{f \in X^* : f(x) \geq 0, \ \forall x \in \mathcal{K}\}.$$

A cone $\mathcal{K}$ in a topological vector space (X, τ) is called τ-*normal* if τ has a base at zero consisting of $\mathcal{K}$ full sets.

Definition 4.2. Let X be a Banach space, and let $A \subset X'$. Then we say that the set A *separates the points of* X if for all $0 \neq x \in X$ there exists $f \in A$ such that $f(x) \neq 0$.

Remark 4.3. A set $A \subset X'$ separates the points of X if and only if the $\sigma(X', X)$-closure of the linear span of A is X', where $\sigma(X', X)$ is the usual weak*-topology of X' (see [Ru]).

In the context of normal cones we need the following result (see [AT, Theorem 2.23]).

LEMMA 4.4. *Let $\mathcal{K}$ be a cone in an ordered topological vector space $(X, \succcurlyeq, \tau)$. If for any two sequences $\{x_j\}$ and $\{y_j\}$ in $(X, \succcurlyeq, \tau)$ with $x_j \succcurlyeq y_j \succcurlyeq 0$ for each j, the condition $x_j \xrightarrow{\tau} 0$ implies that $y_j \xrightarrow{\tau} 0$, then K is a normal cone.*

By [AC, Lemma 5.2], Theorem 3.8, Theorem 3.12 and Corollary 3.13, we have

LEMMA 4.5.

$$\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})^b \subseteq (\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})',$$
$$(\delta\mathcal{H}_{p,m}, \succcurlyeq, |\cdot|_{p,m})^b \subseteq (\delta\mathcal{H}_{p,m}, \succcurlyeq, |\cdot|_{p,m})'.$$

For each nonpolar set $W \Subset \Omega$ we define $D_W : \mathcal{E}_{p,m} \to \mathbb{R}^+$ by $D_W(u) = \int_W \Delta u$. Then D_W is a positive linear functional on $\mathcal{E}_{p,m}$. Since $\mathcal{E}_{p,m} = (\delta\mathcal{E}_{p,m})^+$, D_W can be extended to a regular linear functional defined on $(\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})$. Let $\mathcal{D}$ denote the family of all functionals D_W together with the zero functional.

THEOREM 4.6.

(i) $\delta\mathcal{H}_{p,m} \subset (\delta\mathcal{E}_{p,m})'$ *and* $\mathcal{H}_{p,m}$ *separates the points of* $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ *if* $p \geq 1$.

(ii) $\delta\mathcal{E}_{p,m} \subset (\delta\mathcal{H}_{p,m})'$ *and* $\mathcal{E}_{p,m}$ *separates the points of* $(\delta\mathcal{H}_{p,m}, |\cdot|_{p,m})$ *if* $p \geq 1$.

(iii) *For* $p > 0$ *the family* $\mathcal{D}$ *separates the points of* $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$.

Proof. Fix $w \in \mathcal{E}_{0,m} \cap C^\infty(\Omega)$ such that $e_{p,m}(w) = D(p,m)^{\frac{p+m}{1-p}}$. If $p = 1$, we take $w = -1$.

(i) For each $\mu \in \delta\mathcal{H}_{p,m}$, let $T_\mu : \delta\mathcal{E}_{p,m} \to \mathbb{R}$ be defined by

$$T_\mu(u) = T_\mu(u_1 - u_2) = \int_\Omega (u_2 - u_1)(-w)^{p-1}\, d\mu.$$

We see that T_μ is well-defined and linear on $\delta\mathcal{E}_{p,m}$. Now we will show that T_μ is continuous. By Theorem 1.13, there exist unique $v^+, v^- \in \mathcal{E}_{p,m}$ such that $H_m(v^+) = \mu^+$ and $H_m(v^-) = \mu^-$. By combining the Hölder inequality and Theorem 3.1 we get

$$\begin{aligned}
|T_\mu(u)| &= \Big|\int_\Omega (u_2 - u_1)(-w)^{p-1}(d\mu^+ - d\mu^-)\Big| \\
&\leq \int_\Omega (-u_1 - u_2)(-w)^{p-1}(H_m(v^+) + H_m(v^-)) \\
&\leq \Big[\int_\Omega (-u_1 - u_2)^p (H_m(v^+) + H_m(v^-))\Big]^{1/p} \Big[\int_\Omega (-w)^p (H_m(v^+) + H_m(v^-))\Big]^{\frac{p-1}{p}}
\end{aligned}$$

$$\le D(p,m)e_{p,m}(u_1+u_2)^{\frac{1}{p+m}}e_{p,m}(w)^{\frac{p-1}{p+m}}\big(e_{p,m}(v^+)^{\frac{m}{p+m}}+e_{p,m}(v^-)^{\frac{m}{p+m}}\big)$$
$$= e_{p,m}(u_1+u_2)^{\frac{1}{p+m}}|\mu|_{p,m}.$$

Taking the infimum over all decompositions of u in $\mathcal{E}_{p,m}$, we get

$$|T_\mu(u)| \le |\mu|_{p,m}\|u\|_{p,m}.$$

This implies T_μ is continuous. We have constructed a continuous linear mapping $T : \delta\mathcal{H}_{p,m} \to (\delta\mathcal{E}_{p,m})'$ defined by $\mu \mapsto T_\mu$.

We now show that T is injective. Assume that $T_\mu = T_\nu$ for some $\mu,\nu \in \delta\mathcal{H}_{p,m}$. This means that for all $u \in \delta\mathcal{E}_{p,m}$,

$$\int_\Omega (-u)(-w)^{p-1}(d\mu^+ - d\mu^-) = \int_\Omega (-u)(-w)^{p-1}(d\nu^+ - d\nu^-).$$

For each $\varphi \in C_0^\infty(\Omega)$, we have $\varphi/(-w)^{p-1} \in C_0^\infty(\Omega)$. By Theorem 1.5, $C_0^\infty(\Omega) \subset \delta\mathcal{E}_{p,m}$, thus

$$\int_\Omega \varphi(d\mu^+ - d\mu^-) = \int_\Omega \varphi(d\nu^+ - d\nu^-).$$

So $\mu = \nu$.

Now we show that $\mathcal{H}_{p,m}$ separates the points of $\delta\mathcal{E}_{p,m}$. Take any $u = u_1 - u_2$ with distinct $u_1, u_2 \in \mathcal{E}_{p,m}$. Then at least one of the two sets

$$K \cap \{u_1 > u_2\} \quad \text{and} \quad K \cap \{u_1 < u_2\}$$

has positive Lebesgue measure for some $K \Subset \Omega$. Suppose $\lambda_n(K \cap \{u_1 > u_2\}) > 0$. By [Lu, Theorem 1.8.18], there exists $\phi \in \mathcal{E}_{p,m}$ such that $H_m(\phi) = \chi_{K\cap\{u_1>u_2\}}(-w)^{1-p}d\lambda_n$, where χ_A is the characteristic function of A. We have

$$|H_m(\phi)(u)| = \Big|\int_\Omega (u_2-u_1)(-w)^{p-1}H_m(\phi)\Big| = \int_{K\cap\{u_1>u_2\}} (u_1-u_2)\,d\lambda_n > 0.$$

(ii) We construct an injective, continuous linear map $L : \delta\mathcal{E}_{p,m} \to (\delta\mathcal{H}_{p,m})'$ by identifying $u \in \delta\mathcal{E}_{p,m}$ with L_u, where

$$L_u(\mu) = \int_\Omega (-u)(-w)^{p-1}\,d\mu.$$

As in (i), we have $|L_u(\mu)| \le \|u\|_{p,m}|\mu|_{p,m}$, thus $L_u \in (\delta\mathcal{H}_{p,m})'$. Since $\mathcal{H}_{p,m}$ separates the points of $\delta\mathcal{E}_{p,m}$, L is injective. And the fact that T is injective implies that $\mathcal{E}_{p,m}$ separates the points of $\delta\mathcal{H}_{p,m}$.

(iii) For $u \in \delta\mathcal{E}_{p,m}$, $u \neq 0$, there exist distinct $u_1, u_2 \in \mathcal{E}_{p,m}$ such that $u = u_1 - u_2$. The facts that $u_1, u_2 \in \mathrm{SH}(\Omega)$ and $u_1 = u_2 = 0$ on the boundary of Ω imply $\Delta u_1 \neq \Delta u_2$. Hence there exists a nonpolar set $W \Subset \Omega$ such that $D_W(u_1 - u_2) \neq 0$, i.e. $D_W(u) \neq 0$. ∎

THEOREM 4.7. *Let $p > 0$. Then:*

(1-i) $\mathcal{E}_{p,m}$ *is a normal cone in* $(\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})$.
(1-ii) $\mathcal{H}_{p,m}$ *is a normal cone in* $(\delta\mathcal{H}_{p,m}, \succcurlyeq, |\cdot|_{p,m})$.

(2-i) $(\delta\mathcal{H}_{p,m}, \succcurlyeq, |\cdot|_{p,m})^r = (\delta\mathcal{H}_{p,m}, \succcurlyeq, |\cdot|_{p,m})^b = (\delta\mathcal{H}_{p,m}, \succcurlyeq, |\cdot|_{p,m})'$.

(2-ii) $(\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})^r = (\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})^b = (\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})' = \mathcal{E}'_{p,m} - \mathcal{E}'_{p,m}$.

(3-i) *The space* $(\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})'$, $p \geq 1$, *is the closure of* $\delta\mathcal{H}_{p,m}$ *in* $\sigma((\delta\mathcal{E}_{p,m})', \delta\mathcal{E}_{p,m})$.

(3-ii) *The space* $(\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})'$ *is the* $\sigma((\delta\mathcal{E}_{p,m})', \delta\mathcal{E}_{p,m})$*-closure of the linear span of* $\mathcal{D}$.

(3-iii) *The space* $(\delta\mathcal{H}_{p,m}, \succcurlyeq, |\cdot|_{p,m})'$, $p \geq 1$, *is the closure of* $\delta\mathcal{E}_{p,m}$ *in* $\sigma((\delta\mathcal{H}_{p,m})', \delta\mathcal{H}_{p,m})$.

Proof. (1-i) Assume that $\{u_j\}$ and $\{v_j\}$ are sequences in $(\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})$ with

$$u_j \succcurlyeq v_j \succcurlyeq 0 \quad \text{and} \quad \|u_j\|_{p,m} \to 0.$$

From $u_j \succcurlyeq v_j \succcurlyeq 0$, we have $u_j, v_j \in \mathcal{E}_{p,m}$ and $u_j \leq v_j$. Hence by Lemmas 3.5 and 3.4,

$$\|u_j\|_{p,m} = e_{p,m}(u_j)^{\frac{1}{p+m}} \geq D(p,m)^{-1/p} e_{p,m}(v_j)^{\frac{1}{p+m}} = D(p,m)^{-1/p}\|v_j\|_{p,m}.$$

Thus $\|v_j\|_{p,m} \to 0$, and Lemma 4.4 implies that $\mathcal{E}_{p,m}$ is a normal cone.

(1-ii) We apply the same argument but use Lemma 3.11 instead of Lemma 3.5.

(2-i) By Theorem 2.8, $(\delta\mathcal{H}_{p,m}, \succcurlyeq)$ is a Riesz space, hence $(\delta\mathcal{H}_{p,m})^r = (\delta\mathcal{H}_{p,m})^b$. Thus, by Lemma 4.5 it is enough to prove that $(\delta\mathcal{H}_{p,m})' \subset (\delta\mathcal{H}_{p,m})^b$. For $\mu \in \delta\mathcal{H}_{p,m}$, we have

$$|T(\mu)| \leq \|T\|\,|\mu|_{p,m}, \quad \text{where} \quad \|T\| = \sup\{T(\nu) : \nu \in \mathcal{H}_{p,m} \text{ and } |\nu|_{p,m} \leq 1\}.$$

If $\nu \in [0,\mu]$ then $\nu \leq \mu$ and $\nu, \mu \in \mathcal{H}_{p,m}$. By Theorem 1.12, we have $u_\mu \leq u_\nu$, where $H_m(u_\mu) = \mu$ and $H_m(u_\nu) = \nu$. Hence by Lemma 3.11,

$$|T(\nu)| \leq \|T\|\,|\nu|_{p,m} = \|T\|\,\|u_\nu\|^m_{p,m} \leq \|T\|\,\|u_\mu\|^m_{p,m} = \|T\|\,|\mu|_{p,m}.$$

This means that $T([0,\mu])$ is bounded, or $T \in (\delta\mathcal{H}_{p,m})'$.

(2-ii) Let $(X, \|\cdot\|)$ be a quasi-Banach space such that X' separates the points of X. Then X' is a Banach space with the norm

$$\|x^*\| = \sup\{|x^*(x)| : \|x\| \leq 1\}.$$

We define an associated norm on X by

$$\|x\|_c = \sup\{\|x^*(x)\| : \|x^*\| \leq 1,\, x^* \in X'\}.$$

It can be shown that $\|\cdot\|_c$ is the largest norm on X dominated by the original quasi-norm. The completion X_c of X with this norm is called the *Banach envelope* of X. We know that X_c and X have the same topological dual space (see [KPR]). By Theorem 4.6(iii), we have $(\delta\mathcal{E}_{p,m})' = (\delta\mathcal{E}_{p,m})'_c$. For a functional $T \in (\delta\mathcal{E}_{p,m})'$, and fixed $u \in \mathcal{E}_{p,m}$, define $q : \mathcal{E}_{p,m} \to \mathbb{R}$ by

$$q(u) = \sup\{T(v) : v \in [0,u]\}.$$

Then $C = \{(t,u) \in \mathbb{R}\times \mathcal{E}_{p,m} : 0 \le t \le q(u)\}$ is a cone in $\mathbb{R}\times \delta\mathcal{E}_{p,m}$. We will show that $(1,0) \notin \overline{C}$, where $\overline{C}$ is the closure of C in $\mathbb{R}\times (\delta\mathcal{E}_{p,m})_c$.

Assume that $(1,0) \in \overline{C}$. Then there exists a sequence $\{(t_j,u_j)\} \subset C$ that converges to $(1,0)$ in the product topology. In particular,

$$\|u_j\|_c = \sup_{\substack{\|S\|\le 1\\ S\in(\delta\mathcal{E}_{p,m})'}} |S(u_j)| \to 0.$$

For each j we define

$$S_j(v) = \begin{cases} \|u_j\|_{p,m}^{1-p-m} \int_\Omega (-u_j)^p dd^c v \wedge H_{m-1}(u_j) & \text{if } 0<p<1,\\ \|u_j\|_{p,m}^{1-p-m} \int_\Omega (-v)(-u_j)^{p-1} H_m(u_j) & \text{if } p \ge 1. \end{cases}$$

Then $S_j \in (\delta\mathcal{E}_{p,m})'$. Theorem 3.1 implies that $\|S_j\| \le 1$. Thus $\|u_j\|_c \ge |S_j(u_j)| = \|u_j\|_{p,m}$. Hence $\|u_j\|_{p,m} \to 0$. Then for any $v \in [0,u_j]$ we see that $v \in \mathcal{E}_{p,m}$ and $v \ge u_j$. By Lemmas 3.4 and 3.5 we have

$$\|v\|_{p,m} = e_{p,m}(v)^{\frac{1}{p+m}} \le D(p,m)^{1/p} e_{p,m}(u_j)^{\frac{1}{p+m}} = D(p,m)^{1/p}\|u_j\|_{p,m} \to 0.$$

Thus $q(u_j)\to 0$, which implies $t_j \to 0$. This contradicts the assumption that $(1,0)\in\overline{C}$.

The Hahn–Banach theorem implies that there exists $H \in (\mathbb{R}\times(\delta\mathcal{E}_{p,m})_c)'$ such that $H \ge 0$ on C and $H(1,0) = -1$. Since $(\mathbb{R}\times(\delta\mathcal{E}_{p,m})_c)'$ is isomorphic to $\mathbb{R}'\oplus(\delta\mathcal{E}_{p,m})'_c = \mathbb{R}'\oplus(\delta\mathcal{E}_{p,m})'$ (see [SW, Theorem 4.3, p. 137]), we can write $H(t,u) = at + g(u)$, where $g \in (\delta\mathcal{E}_{p,m})'$. Now $H(1,0) = a = -1$, so $H(t,u) = -t + g(u)$. Since $(0,u) \in C$ for all $u \in \mathcal{E}_{p,m}$ we have $g(u) = H(0,u) \ge 0$ on $\mathcal{E}_{p,m}$. Moreover $(q(u),u) \in C$, hence $H(q(u),u) = -q(u) + g(u) \ge 0$, and we get $g(u) \ge q(u) \ge T(u)$. Thus $T = g - (g-T) \in \mathcal{E}'_{p,m} - \mathcal{E}'_{p,m} = (\delta\mathcal{E}_{p,m})^r$. Moreover, Lemma 4.5 implies $(\delta\mathcal{E}_{p,m})^b = (\delta\mathcal{E}_{p,m})'$, as desired.

(3-i) Theorem 4.6 shows that $\mathcal{H}_{p,m}$ separates the points of $(\delta\mathcal{E}_{p,m})'$, hence Remark 4.3 implies that the $\sigma((\delta\mathcal{E}_{p,m})', \delta\mathcal{E}_{p,m})$-closed linear span of $\mathcal{H}_{p,m}$ is $(\delta\mathcal{E}_{p,m})'$. Thus $(\delta\mathcal{E}_{p,m}, \succcurlyeq, \|\cdot\|_{p,m})'$, $p\ge 1$, is the closure of $\delta\mathcal{H}_{p,m}$ in $\sigma((\delta\mathcal{E}_{p,m})', \delta\mathcal{E}_{p,m})$.

(3-ii) As in (3-i), we use the fact that $\mathcal{D}$ separates the points of $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ for $p>0$.

(3-iii) As in (3-i), the result follows from Theorem 4.6(ii). ∎

EXAMPLE 4.8. We will show that $\mathcal{D}\cap\mathcal{H}_{p,m} = \{0\}$ for any $p\ge 1$. Suppose that there exists $0 \ne D_W \in \mathcal{D}\cap\mathcal{H}_{p,m}$, i.e. there exists a nonpolar set $W \Subset \Omega$, $w_0 \in \mathcal{E}_{0,m}$ (if $p>1$, while $w_0 = -1$ if $p = 1$), and $\mu \in \mathcal{H}_{p,m}$ such that

$$D_W(u) = \int_W \Delta u = \int_\Omega (-w_0)^{p-1}(-u)\,d\mu \quad \text{for any } u \in \mathcal{E}_{p,m}.$$

Take z_0 and $r>0$ such that $B(z_0,r) \Subset \Omega$. Fix $u \in \mathcal{E}_{p,m}$, and let $\epsilon > 0$ be such that $\sup\{u(z) : z \in W \cup B(z_0,r)\} + \epsilon < 0$. Define

$$v = \big(\sup\{w \in \mathcal{E}_{p,m} : w \le u+\epsilon \text{ on } W\cup B(z_0,r)\}\big)^*.$$

Then $v \in \mathcal{E}_{p,m}$, $v \geq u$ and $v = u + \epsilon$ on $W \cup B(z_0, r)$. Thus,

$$0 = D_W(u) - D_W(v) = \int_\Omega (-w_0)^{p-1}(v-u)\,d\mu.$$

Since $\mu\{v > u\} = 0$ we see that $\mu = 0$ on $W \cup B(z_0, r)$. The point z_0 was chosen arbitrarily, and so $\mu = 0$. Thus $D_W = 0$, a contradiction.

5. Inner product. In this section we define an inner product on $\delta\mathcal{E}_{1,1}$. We give an example to show that the norm defined by this inner product and the norm $\|\cdot\|_{1,1}$ defined by (3.5) are not equivalent.

On $\delta\mathcal{E}_{1,1}$ we define a bilinear map

$$\langle u, v\rangle = \int_\Omega (-u)dd^c v \wedge \beta^{n-1} = 4^{n-1}(n-1)! \int_\Omega (-u)\Delta v.$$

Theorem 5.1. *The form $\langle \cdot,\cdot\rangle$ defines an inner product on $\delta\mathcal{E}_{1,1}$.*

Proof. (i) The bilinearity of $\langle \cdot,\cdot\rangle$ is obvious.

(ii) By Theorem 1.8, we get the symmetry of $\langle \cdot,\cdot\rangle$.

(iii) For any $u = u_1 - u_2 \in \delta\mathcal{E}_{1,1}$, by Theorem 1.8,

$$\begin{aligned}(5.1)\qquad \langle u, u\rangle &= \int_\Omega (u_2 - u_1)dd^c(u_1 - u_2) \wedge \beta^{n-1}\\ &= \int_\Omega (-u_1)dd^c u_1 \wedge \beta^{n-1} + \int_\Omega (-u_2)dd^c u_2 \wedge \beta^{n-1} - 2\int_\Omega (-u_1)dd^c u_2 \wedge \beta^{n-1}\\ &= e_{1,1}(u_1) + e_{1,1}(u_2) - 2\int_\Omega (-u_1)dd^c u_2 \wedge \beta^{n-1}.\end{aligned}$$

By Theorem 3.1 and the Cauchy–Schwarz inequality

$$(5.2)\quad \int_\Omega (-u_1)dd^c u_2 \wedge \beta^{n-1} \leq e_{1,1}(u_1)^{1/2} e_{1,1}(u_2)^{1/2} \leq \tfrac{1}{2}(e_{1,1}(u_1) + e_{1,1}(u_2)).$$

(5.1) and (5.2) yield $\langle u, u\rangle \geq 0$. Now suppose that $u = u_1 - u_2 \in \delta\mathcal{E}_{1,1}$ with $\langle u, u\rangle = 0$. Since the smallest harmonic majorants of u_1 and u_2 are identically 0, by the Riesz decomposition theorem we have

$$u_i(z) = \frac{1}{\sigma_n \max\{1, 2n-2\}} \int_\Omega G_\Omega(z, y)\Delta u_i(y), \qquad i = 1, 2,$$

where $G_\Omega(z, y)$ is the Green function of Ω. Thus $\langle u, u\rangle$ is equal to

$$-\frac{1}{\sigma_n \max\{1, 2n-2\}} \int_\Omega\int_\Omega G_\Omega(z, y)(\Delta u_2(z) - \Delta u_1(z))(\Delta u_2(y) - \Delta u_1(y) = 0.$$

Applying [Do, Theorem XIII.7] with the signed measure $\mu = \Delta u_2 - \Delta u_1$ to the above identity we get $\mu = 0$, i.e. $\Delta u_1 = \Delta u_2$. This implies that $u_1 = u_2$ almost everywhere. By the subharmonicity of u_1, u_2 we get $u = 0$. ∎

We define the norm $|||u||| = \langle u, u\rangle^{1/2}$ on $\delta\mathcal{E}_{1,1}$. Then $|||u||| \leq \|u\|_{1,1}$, with equality when $u \in \mathcal{E}_{1,1}$. The following example shows that these two norms are not equivalent.

EXAMPLE 5.2. Let $E(z) = 1 - \|z\|^{2-2n}$ on the unit ball $\mathbb{B}$. Then $\Delta E = (2n-2)\sigma_n\delta_0$, where δ_0 is the Dirac measure at 0, and σ_n is the surface measure of $\mathbb{B}$ in $\mathbb{C}^n$. For $a < b < 0$ define the following functions on $\mathbb{B}$:

$$u_a(z) = \max(E(z), a), \quad u_b(z) = \max(E(z), b).$$

Then $u_a, u_b \in \mathcal{E}_{0,1}(\mathbb{B})$. If we take any $v, w \in \mathcal{E}_{0,1}(\mathbb{B})$ such that $u_a - u_b = v - w$ then

$$\Delta u_a + \Delta v = \Delta u_b + \Delta w$$

with

$$\operatorname{supp}(\Delta u_a) = \{E(z) = a\}, \qquad \operatorname{supp}(\Delta u_b) = \{E(z) = b\}.$$

Hence $\{E(z) = a\} \subseteq \operatorname{supp}(\Delta w)$. Therefore, $\Delta w \geq \Delta u_a$, so $u_a \geq w$. By Theorem 3.9, $(u_a - u_b)^+ = u_a$, $(u_a - u_b)^- = u_b$ and

$$\|u_a - u_b\|_{1,1} = \|u_a + u_b\|_{1,1} = e_{1,1}(u_a + u_b)^{1/2} \geq e_{1,1}(u_a)^{1/2}. \tag{5.3}$$

Choose any decreasing sequence $\{b_j\}$, $b_j < 0$, that converges to -1. Then $\{u_j\} = \{u_{-1} - u_{b_j}\} \subset \delta\mathcal{E}_{0,1}$, and by (5.3) we have

$$\|u_j\|_{1,1} \geq e_{1,1}(u_{-1})^{1/2} = [(2n-2)\sigma_n]^{1/2}, \quad \text{although} \quad \langle u_j, u_j\rangle \to 0.$$

The following example shows that the norm $\|\cdot\|_{1,m}$ defined on $\delta\mathcal{E}_{1,m}$ with $m > 1$ by (3.5) does not come from any inner product.

EXAMPLE 5.3. Let $m = n = 2$, and $\Omega = \mathbb{B}$ be the unit ball in $\mathbb{C}^2$. For $a < b < 0$ define the following functions on Ω:

$$u = \max(\log|z|, b), \quad v = \max(\log|z|, a) \in \mathcal{E}_{0,2}(\mathbb{B}).$$

We have

$$\begin{aligned}
&(dd^c u)^2 = d\sigma_{\{|z|=e^b\}}, \qquad (dd^c v)^2 = d\sigma_{\{|z|=e^a\}},\\
&[dd^c(u+v)]^2 = (dd^c u)^2 + 2dd^c u \wedge dd^c v + (dd^c v)^2\\
&\qquad\qquad\quad = 3(dd^c u)^2 + (dd^c v)^2 = 3d\sigma_{\{|z|=e^b\}} + d\sigma_{\{|z|=e^a\}},\\
&[dd^c(u-v)]^2 = d\sigma_{\{|z|=e^a\}} - d\sigma_{\{|z|=e^b\}},
\end{aligned}$$

where $d\sigma_A$ is the surface measure on A. It was proved in [AC] that $(u-v)^+ = u$ and $(u-v)^- = v$. Hence

$$e_{1,2}(u) = e_{1,2}((u+v)^+) = \int_{\mathbb{B}} (-u)(dd^c u)^2 = (-b)(2\pi)^2,$$

$$e_{1,2}(v) = e_{1,2}((u-v)^-) = \int_{\mathbb{B}} (-v)(dd^c v)^2 = (-a)(2\pi)^2.$$

Now we have

$$\begin{aligned}
\|u+v\|_{1,2}^2 &= e_{1,2}(u+v)^{2/3} = \Big(\int_{\mathbb{B}} (-u-v)[dd^c(u+v)]^2\Big)^{2/3}\\
&= [-(2\pi)^2(a+7b)]^{2/3},\\
\|u-v\|_{1,2}^2 &= \|(u-v)^+ + (u-v)^-\|_{1,2}^2 = \|u+v\|_{1,2}^2.
\end{aligned}$$

So

$$\|u+v\|_{1,2}^2 + \|u-v\|_{1,2}^2 = -2(2\pi)^{4/3}(a+7b)^{2/3},$$
$$2(\|u\|_{1,2}^2 + \|v\|_{1,2}^2) = 2(e_{1,2}(u)^{2/3} + e_{1,2}(v)^{2/3}) = -2(2\pi)^{4/3}(a^{2/3}+b^{2/3}).$$

This implies that $\|\cdot\|_{1,2}$ does not satisfy the parallelogram law, so it does not come from any inner product.

Acknowledgements. The author is supported by the Ph.D programme in the National Science Centre Poland grant DEC-2013/08/A/ST1/00312 “Hessian type equations in complex geometry”. I wish to thank Professor Sławomir Kołodziej for his help in accomplishing this work. I am indebted to my advisor, Dr. Rafał Czyż, for suggesting the problem and for many stimulating discussions. I am grateful to Sławomir Dinew and Ngoc Cuong Nguyen for many fruitful comments. I would like to thank the referee whose remarks helped to improve the exposition.

References

[ACC] P. Åhag, U. Cegrell and R. Czyż, *Vector spaces of delta-plurisubharmonic functions and extensions of the complex Monge–Ampère operator*, J. Math. Anal. Appl. 422 (2015), 960–980.

[AC] P. Åhag and R. Czyż, *Modulability and duality of certain cones in pluripotential theory*, J. Math. Anal. Appl. 361 (2010), 302–321.

[ACH] P. Åhag, R. Czyż and P. H. Hiep, *Concerning the energy class $\mathcal{E}_p$ for* $0<p<1$, Ann. Polon. Math. 91 (2007), 119–130.

[AT] C. D. Aliprantis and R. Tourky, *Cones and Duality*, Amer. Math. Soc., Providence, RI, 2007.

[Ao] T. Aoki, *Locally bounded linear topological spaces*, Proc. Imp. Acad. Tokyo 18 (1942), 588–594.

[BT1] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge–Ampère equation*, Invent. Math. 37 (1976), 1–44.

[BT2] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[Bl] Z. Błocki, *Weak solutions to the complex Hessian equation*, Ann. Inst. Fourier (Grenoble) 55 (2005), 1735–1756.

[Ce1] U. Cegrell, *Delta-plurisubharmonic functions*, Math. Scand. 43 (1978), 343–352.

[Ce2] U. Cegrell, *Pluricomplex energy*, Acta Math. 180 (1998), 187–217.

[Ce3] U. Cegrell, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.

[CKZ] U. Cegrell, S. Kołodziej and A. Zeriahi, *Subextension of plurisubharmonic functions with weak singularities*, Math. Z. 250 (2005), 7–22.

[CW] U. Cegrell and J. Wiklund, *A Monge–Ampère norm for delta-plurisubharmonic functions*, Math. Scand. 97 (2005), 201–216.

[DK] S. Dinew and S. Kołodziej, *A priori estimates for the complex Hessian equations*, Anal. PDE 1 (2014), 227–244.

[Do] J. L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart*, Classics Math., Springer, Berlin, 2001.

[HH] L. M. Hai and P. H. Hiep, *The topology on the space of δ-psh functions in the Cegrell classes*, Results Math. 49 (2006), 127–140.

[KPR] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, London Math. Soc. Lecture Note Ser. 89, Cambridge Univ. Press, 1984.

[Ki] C. O. Kiselman, *Fonctions delta-convexes, delta-sousharmoniques et delta-pluri-sousharmoniques*, in: Séminaire Pierre Lelong (Analyse), année 1975/76, Lecture Notes in Math. 578, Springer, Berlin, 1977, 93–107.

[K1] S. Kołodziej, *The range of the complex Monge–Ampère operator II*, Indiana Univ. Math. J. 44 (1995), 765–782.

[K2] S. Kołodziej, *The complex Monge–Ampère equation and pluripotential theory*, Mem. Amer. Math. Soc. 178 (2005), no. 840, 64 pp.

[Lu] H. C. Lu, *Complex Hessian equations*, Doctoral thesis, Univ. of Toulouse III Paul Sabatier, 2012.

[Ng] N. C. Nguyen, *Subsolution theorem for the complex Hessian equation*, Univ. Iagel. Acta Math. 50 (2013), 69–88.

[Pe] L. Persson, *A Dirichlet principle for the complex Monge–Ampère operator*, Ark. Mat. 37 (1999), 345–356.

[Ro] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci. Cl. III 5 (1957), 471–473.

[Ru] W. Rudin, *Functional Analysis*, Tata McGraw-Hill, New Delhi, 1990.

[SW] H. H. Schaefer and M. P. Wolff, *Topological Vector Spaces*, 2nd ed., Grad. Texts in Math. 3, Springer, 1999.

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