Zeros of the Derivatives of $L$-Functions Attached to Cusp Forms

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Summary. Let $f$ be a holomorphic cusp form of weight $k$ with respect to $\SL_2(\Z)$ which is a normalized Hecke eigenform, and $L_f(s)$ the $L$-function attached to $f$. We shall give a relation between the number of zeros of $L_f(s)$ and of the derivatives of $L_f(s)$ using Berndt’s method, and an estimate of zero-density of the derivatives of $L_f(s)$ based on Littlewood’s method.

1. Introduction. Let $f$ be a cusp form of weight $k$ for $\SL_2(\Z)$ which is a normalized Hecke eigenform. Let $a_f(n)$ be the $n$th Fourier coefficient of $f$ and set $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$. Rankin showed that $\sum_{n \leq x} |\lambda_f(n)|^2 = C_f x + O(x^{3/5})$ for $x \in \R_{>0}$, where $C_f$ is a positive constant depending on $f$ (see [8, (4.2.3), p. 364]). The $L$-function attached to $f$ is defined by

$$ L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p\text{ prime}} \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1} \quad (\Re s > 1), $$

where $\alpha_f(p)$ and $\beta_f(p)$ satisfy $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ and $\alpha_f(p) \beta_f(p) = 1$. By Hecke’s work [4], the function $L_f(s)$ is analytically continued to the whole $s$-plane by

$$ (2\pi)^{-s-(k-1)/2} \Gamma\left(s + \frac{k-1}{2}\right) L_f(s) = \int_0^{\infty} f(iy)y^{s+(k-1)/2-1} \, dy, $$

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and has a functional equation
\[ L_f(s) = \chi_f(s)L_f(1-s) \]
where \( \chi_f(s) \) is given by
\[
\chi_f(s) = (-1)^{-k/2}(2\pi)^{2s-1}\frac{\Gamma(1-s+k/2)}{\Gamma(s+k/2)} \]
\[
= 2(2\pi)^{-2(1-s)}\Gamma\left(s + \frac{k-1}{2}\right)\Gamma\left(s - \frac{k-1}{2}\right)\cos(\pi(1-s)).
\]
The second equality is deduced from the fact that \( \Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s) \) and \( \sin(\pi(s+(k-1)/2)) = (-1)^{k/2}\cos(\pi(1-s)) \). Similarly to the case of the Riemann zeta function \( \zeta(s) \), it is conjectured that all complex zeros of \( L_f(s) \) lie on the critical line \( \text{Re}\ s = 1/2 \), which is the Generalized Riemann Hypothesis (GRH). In order to support the truth of the GRH, the distribution and the density of complex zeros of \( L_f(s) \) are studied without assuming the GRH.

Lekkerkerker [6] proved an approximate formula for the number of complex zeros of \( L_f(s) \):
\[
N_f(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} + O(\log T),
\]
where \( T > 0 \) is sufficiently large, and \( N_f(T) \) denotes the number of complex zeros of \( L_f(s) \) in \( 0 < \text{Im}\ s \leq T \). The formula (1.4) is analogous to the formula for \( N(T) \), the number of complex zeros of \( \zeta(s) \) in \( 0 < \text{Im}\ s \leq T \), given by Riemann [9]:
\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).
\]
(Later von Mangoldt [14] proved (1.5) rigorously.) In the Riemann zeta function, the zeros of the derivative of \( \zeta(s) \) have a connection with the Riemann Hypothesis (RH). Speiser [10] showed that RH is equivalent to the non-existence of a complex zero of \( \zeta'(s) \) in \( \text{Re}\ s < 1/2 \). Levinson and Montgomery [7] proved that if RH is true, then \( \zeta^{(m)}(s) \) has at most finitely many complex zeros in \( 0 < \text{Re}\ s < 1/2 \) for any \( m \in \mathbb{Z}_{\geq 0} \).

There are many studies of the zeros of \( \zeta^{(m)}(s) \) without assuming RH. Spira [11, 12] showed that there exist \( \sigma_m \geq (7m+8)/4 \) and \( \alpha_m < 0 \) such that \( \zeta^{(m)}(s) \) has no zero with \( \text{Re}\ s \leq \sigma_m \) or \( \text{Re}\ s \leq \alpha_m \), and exactly one real zero in each open interval \((-1-2n, 1-2n)\) for \( 1-2n \leq \alpha_m \). Later, Yıldırım [16] showed that \( \zeta''(s) \) and \( \zeta'''(s) \) have no zeros in \( 0 \leq \text{Re}\ s < 1/2 \). Berndt [2] gave a relation between the numbers of complex zeros of \( \zeta(s) \) and of \( \zeta^{(m)}(s) \):
\[
N_m(T) = N(T) - \frac{T\log 2}{2\pi} + O(\log T),
\]
where \( m \in \mathbb{Z}_{\geq 1} \) is fixed and \( N_m(T) \) denotes the number of complex zeros of \( \zeta^{(m)}(s) \) in \( 0 < \text{Im} \, s \leq T \). Recently, Aoki and Minamide [1] studied the density of zeros of \( \zeta^{(m)}(s) \) in the right hand side of the critical line \( \text{Re} \, s = 1/2 \) by using Littlewood’s method. Let \( N_m(\sigma, T) \) be the number of zeros of \( \zeta^{(m)}(s) \) in \( \text{Re} \, s \geq \sigma \) and \( 0 < \text{Im} \, s \leq T \). They showed that

\[
N_m(\sigma, T) = O\left( \frac{T}{\sigma - 1/2} \log \frac{1}{\sigma - 1/2} \right),
\]

uniformly for \( \sigma > 1/2 \). From (1.6) and (1.7), we see that almost all complex zeros of \( \zeta^{(m)}(s) \) lie in the neighbourhood of the critical line.

The purpose of this paper is to study the counterparts of the results of Berndt, Aoki and Minamide for the derivatives of \( L_f(s) \), namely, the relation between the number of complex zeros of \( L_f(s) \) and that of \( L_f^{(m)}(s) \), and the density of zeros of \( L_f^{(m)}(s) \) in the right half-plane \( \text{Re} \, s > 1/2 \). Let \( n_f \) be the smallest integer greater than 1 such that \( \lambda_f(n_f) \neq 0 \). The \( m \)th derivative of \( L_f(s) \) is given by

\[
L_f^{(m)}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^s} = \sum_{n=n_f}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^s} \quad (\text{Re} \, s > 1),
\]

Differentiating both sides of (1.2), we find that \( L_f^{(m)}(s) \) is holomorphic in the whole \( s \)-plane and has the functional equation

\[
L_f^{(m)}(s) = \sum_{r=0}^{m} \binom{m}{r} (-1)^r \chi_f^{(m-r)}(s)L_f^{(r)}(1-s).
\]

First, we shall exhibit zero free regions for \( L_f^{(m)}(s) \) by following Berndt’s method (see [2]) and Spira’s method (see [11], [12]).

**Theorem 1.1.** The following assertions hold for any \( m \in \mathbb{Z}_{\geq 0} \).

(i) There exists \( \sigma_{f,m} \in \mathbb{R}_{>1} \) such that \( L_f^{(m)}(s) \) has no zero in \( \text{Re} \, s \geq \sigma_{f,m} \).

(ii) For any \( \varepsilon \in \mathbb{R}_{>0} \), there exists \( \delta_{f,m,\varepsilon} \in \mathbb{R}_{>(k-1)/2+1} \) such that \( L_f^{(m)}(s) \) has no zero with \( |s| \geq \delta_{f,m,\varepsilon} \) satisfying \( \text{Re} \, s \leq -\varepsilon \) and \( |\text{Im} \, s| \geq \varepsilon \).

(iii) There exists \( \alpha_{f,m} \in \mathbb{R}_{<(k-1)/2-1} \) such that \( L_f^{(m)}(s) \) has only real zeros in \( \text{Re} \, s \leq \alpha_{f,m} \), and one real zero in each interval \( (n-1, n) \) for \( n \in \mathbb{Z}_{\leq \alpha_{f,m}} \).

Next, based on Berndt’s proof, we can obtain the following formula for the number of complex zeros of \( L_f^{(m)}(s) \):

**Theorem 1.2.** For any fixed \( m \in \mathbb{Z}_{\geq 1} \), let \( N_{f,m}(T) \) be the number of complex zeros of \( L_f^{(m)}(s) \) in \( 0 < \text{Im} \, s \leq T \). Then for any large \( T > 0 \), we
have
\[ N_{f,m}(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} - \frac{T}{2\pi} \log n_f + O(\log T). \]

Moreover, the relation between \( N_f(T) \) and \( N_{f,m}(T) \) is given by
\[ N_{f,m}(T) = N_f(T) - \frac{T}{2\pi} \log n_f + O(\log T). \]

Finally, using the mean value formula for \( L_f^{(m)}(s) \) obtained in \[15\] and Littlewood’s method, we obtain the estimate of the density of zeros:

**Theorem 1.3.** For any \( m \in \mathbb{Z}_{\geq 0} \), let \( N_{f,m}(\sigma,T) \) be the number of complex zeros of \( L_f^{(m)}(s) \) with \( \Re s \geq \sigma \) and \( 0 < \Im s \leq T \). For any large \( T > 0 \), we have

\[ N_{f,m}(\sigma,T) = O\left( \frac{T}{\sigma - 1/2} \log \frac{1}{\sigma - 1/2} \right) \]

uniformly for \( 1/2 < \sigma \leq 1 \). More precisely,

\[ N_{f,m}(\sigma,T) \leq \frac{2m + 1}{2\pi} \frac{T}{\sigma - 1/2} \log \frac{1}{\sigma - 1/2} + \frac{1}{2\pi} \frac{T}{\sigma - 1/2} \log \frac{(2m)!n_f C_f}{|\lambda_f(n_f)|^2 (\log n_f)^{2m}} \]

\[ + O\left( \frac{\log T}{\sigma - 1/2} \right) \]

\[ + \frac{1}{2\pi} \frac{T}{\sigma - 1/2} \begin{cases} \log \left( 1 + O\left( \frac{(2\sigma - 1)^{2m+1}(\log T)^{2m}}{T^{2\sigma - 1}} \right) \right), & 1/2 < \sigma < 1, \\ \log \left( 1 + O\left( \frac{(2\sigma - 1)^{2m+1}(\log T)^{2m+2}}{T} \right) \right), & \sigma = 1, \\ \log \left( 1 + O\left( \frac{(2\sigma - 1)^{2m+1}}{T} \right) \right), & 1 < \sigma < \sigma_{f,m}, \end{cases} \]

where \( \sigma_{f,m} \) is given by Theorem [1.1][1].

**2. Proof of Theorem [1.1]** To show (1), we write

\[ L_f^{(m)}(s) = \lambda_f(n_f)(-\log n_f)^m F(s)n_f^{-s} \]

where

\[ F(s) = 1 + \sum_{n=n_f+1}^{\infty} \frac{\lambda_f(n)}{\lambda_f(n_f)} \left( \frac{\log n}{\log n_f} \right)^m \left( \frac{n_f}{n} \right)^s \quad (\Re s > 1). \]
Deligne’s result $|\lambda_f(n)| \leq d(n) \ll n^\epsilon$ implies that there exist $c_f \in \mathbb{R}_{>0}$ and $\sigma_{f,m} \in \mathbb{R}_{>1}$ depending on $f$ and $m$ such that

$$\sum_{n=n_f+1}^{\infty} \left| \frac{\lambda_f(n)}{\lambda_f(n_f)} \right| \left( \frac{\log n}{\log n_f} \right)^m \left( \frac{n_f}{n} \right)^\sigma \leq c_f \sum_{n=n_f+1}^{\infty} \left( \frac{\log n/\log n_f}{n/n_f} \right)^{\sigma-\epsilon} \leq \frac{1}{2}$$

for $\sigma \in \mathbb{R}_{\geq \sigma_{f,m}}$ and $t \in \mathbb{R}$, where $\epsilon$ is an arbitrary positive number. Hence $L_f^{(m)}(s)$ has no zeros with $\text{Re } s \geq \sigma_{f,m}$, that is, (i) is proved.

Next we shall show (ii) and (iii). Replacing $s$ with $1 - s$ in (1.3) and (1.8), we have

$$\sum_{r=0}^{m} \binom{m}{r} L_f^{(m-r)}(s) \left[ 2(2\pi)^{-2s} \cos(\pi s) \Gamma\left(s - \frac{k-1}{2}\right) \Gamma\left(s + \frac{k-1}{2}\right) \right]^{(r)}.$$

Since $(\cos(\pi s))^{(r)} = \pi^r \left( a_r \cos(\pi s) + b_r \sin(\pi s) \right)$ where $a_r, b_r \in \{0, \pm 1\}$, and $((2\pi)^{-2s})^{(r)} = (-2 \log 2\pi)^r (2\pi)^{-2s}$ for $r \in \mathbb{Z}_{\geq 0}$, the formula (2.3) can be written as

$$\sum_{r=0}^{m} \binom{m}{r} L_f^{(m-r)}(s) \left[ 2(2\pi)^{-2s} \cos(\pi s) \Gamma\left(s - \frac{k-1}{2}\right) \Gamma\left(s + \frac{k-1}{2}\right) \right]^{(r)}.$$

where

$$R_{m-r}(s) = \cos(\pi s) \sum_{j=0}^{m-r} a'_j L_f^{(j)}(s) + \sin(\pi s) \sum_{j=0}^{m-r} b'_j L_f^{(j)}(s)$$

and $a'_r, b'_r \in \mathbb{R}$. It is clear that $a'_0 = 1, b'_0 = 0$ and $R_0(s) = L_f(s) \cos(\pi s)$. Moreover we write (2.4) as

$$\frac{(-1)^m L_f^{(m)}(1-s)}{2(2\pi)^{-2s}} = f(s) + g(s)$$

where

$$f(s) = R_0(s) \left( \Gamma\left(s - \frac{k-1}{2}\right) \Gamma\left(s + \frac{k-1}{2}\right) \right)^{(m)},$$

$$g(s) = \sum_{r=0}^{m-1} R_{m-r}(s) \left( \Gamma\left(s - \frac{k-1}{2}\right) \Gamma\left(s + \frac{k-1}{2}\right) \right)^{(r)}.$$
The formula (2.5) implies that if \(|f(s)| > |g(s)|\) in some region then \(L_f^{(m)}(s)\) has no zero in that region.

In order to investigate the behavior of \(f(s)\) and \(g(s)\), we shall consider the approximate formula for \((\Gamma(s - \frac{k-1}{2}) \Gamma(s + \frac{k-1}{2}))^{(r)}/(\Gamma(s - \frac{k-1}{2}) \Gamma(s + \frac{k-1}{2}))\).

By Stirling’s formula, it is known that

\[
\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + \int_0^{\infty} \frac{\{u\} - 1/2}{(u + s)^2} \, du
\]

for \(s \in \mathbb{C}\) such that \(|\arg s| \leq \pi - \delta\) where \(\delta \in \mathbb{R}_{>0}\) is fixed (see [5, Theorem A.5 b])). Let 

\(D := \mathbb{C} \setminus \{s \in \mathbb{C} \mid \Re s < \varepsilon, |\Im s| < \varepsilon\}\), where \(\varepsilon\) is any fixed positive number. Setting \(G^{(j)}(s) := (dj^{-1}/ds^{j-1})G^{(1)}(s)\) for \(j \in \mathbb{Z}_{\geq 1}\) and \(s \in D\) where \(G^{(1)}(s)\) is the right-hand side of (2.6), we shall use the following lemma:

**Lemma 2.1 ([15, Lemma 2.3]).** Let \(F\) and \(G\) be holomorphic functions such that \(F(s) \neq 0\) and \(\log F(s) = G(s)\) for \(s \in D\), where \(D \subset \mathbb{C}\) is a domain. Then for any fixed \(r \in \mathbb{Z}_{\geq 1}\), there exist \(l_1, \ldots, l_r \in \mathbb{Z}_{\geq 0}\) and \(C_{l_1, \ldots, l_r} \in \mathbb{Z}_{\geq 0}\) such that

\[
\frac{F^{(r)}}{F}(s) = \sum_{1l_1+\cdots+rl_r=r} C_{l_1, \ldots, l_r} (G^{(1)}(s))^{l_1} \cdots (G^{(r)}(s))^{l_r}
\]

for \(s \in D\). In particular, \(C_{(r,0,\ldots,0)} = 1\).

The estimates

\[
|u + s|^2 = u^2 + |s|^2 + 2u|s| \cos \arg s
\]

give

\[
\int_0^\infty \frac{\{u\} - 1/2}{(u + s)^{j+1}} \, du \ll \int_0^{|s|} \frac{du}{|s|^{j+1}} + \int_0^\infty \frac{du}{s^{j+1}} \ll \frac{1}{|s|^j}
\]

for \(j \in \mathbb{Z}_{\geq 1}\) and \(s \in D\). Then

\[
G^{(1)}(s) = \log s + O\left(\frac{1}{|s|}\right),
\]

\[
G^{(j)}(s) = \frac{(-1)^{j-1}(j - 2)!}{s^{j-1}} + \frac{(-1)^j(j - 1)!}{2s^j} + (-1)^{j+1}j! \int_0^\infty \frac{\{u\} - 1/2}{(u + s)^{j+1}} \, du
\]

\[= O\left(\frac{1}{|s|^{j-1}}\right)\]
for \( j \in \mathbb{Z}_{\geq 2} \). Hence the approximate formula for \((\Gamma^{(r)}/\Gamma)(s)\) is

\[
(2.7) \quad \frac{\Gamma^{(r)}}{\Gamma}(s) = (G^{(1)}(s))^r + O\left( \sum_{1q_1+\cdots+r_q=r, q_1 \neq r, j=1}^r |G^{(j)}(s)|^{q_j} \right) \\
= \left( \log s + O\left( \frac{1}{|s|} \right) \right)^r + O\left( \frac{|\log s|^{r-1}}{|s|} \right) = (\log s)^r \sum_{j=0}^r \frac{M_j(s)}{(\log s)^j}
\]

for \( s \in D \) and \( r \in \mathbb{Z}_{\geq 0} \), where \( M_j(s) = O(1/|s|^j) \) for \( j \in \mathbb{Z}_{\geq 1} \) and \( M_0(s) = 1 \). Using (2.7) and the approximate formula

\[
(2.8) \quad \log \left( s \pm \frac{k-1}{2} \right) = \log s + \log \left( 1 \pm \frac{k-1}{2s} \right) = \log |s| + i \arg s + O(1/|s|)
\]

for \( |s| > (k-1)/2 \), we can write

\[
(2.9) \quad \frac{(\Gamma(s - k-1/2) \Gamma(s + k-1/2))^{(l)}}{\Gamma(s - k-1/2) \Gamma(s + k-1/2)} \\
= \sum_{j=0}^l \binom{l}{j} \frac{(\Gamma^{(j)}(s - k-1/2)^l \Gamma^{(l-j)}(s + k-1/2)^j}{\Gamma(s - k-1/2) \Gamma(s + k-1/2)} \\
= \sum_{j=0}^l \binom{l}{j} \left( \log \left( s - \frac{k-1}{2} \right) \right)^{j-1} \log \left( s + \frac{k-1}{2} \right) \\
\times \sum_{0 \leq j_1 + j_2 \leq l, 0 \leq j_1 \leq j, 0 \leq j_2 \leq l-j} \frac{M_{j_1}(s - k-1/2)^{j_1}}{(\log(s - k-1/2))^{j_1}} \frac{M_{j_2}(s + k-1/2)^{j_2}}{(\log(s + k-1/2))^{j_2}} \\
= S_l(s) + T_l(s)
\]

for \( l \in \mathbb{Z}_{\geq 0} \), \( s \in D' \) and \( |s| > (k-1)/2 \), where \( D' := \mathbb{C} \setminus \{ s \in \mathbb{C} \mid \Re s < (k-1)/2 + \varepsilon, |\Im s| < \varepsilon \} \) and

\[
S_l(s) = \left( \log \left( s - \frac{k-1}{2} \right) + \log \left( s + \frac{k-1}{2} \right) \right)^l,
\]

\[
T_l(s) = O\left( \frac{1}{|s| \log |s|} \sum_{j=0}^l (\log |s|)^j (\log |s|)^{l-j} \right) = O\left( \frac{(\log |s|)^{l-1}}{|s|} \right)
\]

for \( l \in \mathbb{Z}_{\geq 1} \), in particular \( S_0(s) = 1 \) and \( T_0(s) = 0 \).

Next using \( R_r(s) \), \( S_r(s) \) and \( T_r(s) \), we shall give a condition which implies \(|f(s)| > |g(s)|\) for some region. From (2.5) and (2.9), the inequality \(|f(s)| > |g(s)|\) is equivalent to

\[
|S_m(s) + T_m(s)| > \left| \sum_{r=0}^{m-1} \frac{R_r}{R_0} (s)(S_r(s) + T_r(s)) \right|.
\]
Dividing both sides by $S_{m-1}(s)$ and applying the triangle inequality, we see that if
\begin{equation}
|S_1(s)| > \left| \frac{T_m}{S_{m-1}}(s) + \sum_{r=0}^{m-1} \frac{R_r}{R_0}(s) \left( \frac{1}{S_{m-1-r}} + \frac{T_r}{S_{m-1}}(s) \right) \right|,
\end{equation}
then $|f(s)| > |g(s)|$ for $s \in D'$ and $|s| > (k-1)/2$. To prove (2.11), we shall obtain upper bounds of $(1/S_r)(s)$ and $(R_r/R_0)(s)$. The formula (2.8) gives
\begin{equation}
\left| \frac{1}{S_r}(s) \right| \leq \frac{C_1}{(\log |s|)^r}
\end{equation}
for the above $s$ and $r \in \mathbb{Z}_{\geq 0}$; here and later, $C_1, C_2, \ldots$ denote positive constants depending on $f, r$ and $\varepsilon$. Since $L_f^{(j)}(s)$ and $(1/L_f)(s)$ are absolutely convergent for $\text{Re } s > 1$, it follows that
\begin{equation}
\left| \frac{R_r}{R_0}(s) \right| = \left| \sum_{j=0}^{r} a_j L_f^{(j)}(s) + \tan(\pi s) \sum_{j=0}^{r} b_j L_f^{(j)}(s) \right| \leq C_2 + C_3 |\tan(\pi s)|
\end{equation}
for $\text{Re } s \geq 1 + \varepsilon$. Here $\tan(\pi s)$ is estimated as
\begin{equation}
|\tan(\pi s)| = \left| \frac{e^{-2t e^{2\pi i \sigma}} - 1}{e^{-2t e^{2\pi i \sigma}} + 1} \right| \leq \begin{cases} 2/(1- e^{-2\varepsilon}) & \text{if } |t| \geq \varepsilon, \\ 3 & \text{if } \sigma \in \mathbb{Z}. \end{cases}
\end{equation}
Combining (2.10) and (2.12)–(2.14), we see that the right-hand side of (2.11) is estimated as
\begin{equation}
\left| \frac{T_m}{S_{m-1}}(s) \right| + \left| \sum_{r=0}^{m-1} \frac{R_r}{R_0}(s) \left( \frac{1}{S_{m-1-r}} + \frac{T_r}{S_{m-1}}(s) \right) \right| \leq C_4 + C_5 |\tan(\pi s)| \sum_{r=0}^{m-1} \left( \frac{1}{(\log |s|)^{m-1-r}} + \frac{1}{|s|(\log |s|)^{m-r}} \right) \leq C_f, m, \varepsilon
\end{equation}
for $|s| > (k-1)/2$ and $\text{Re } s \geq 1 + \varepsilon$ provided $|\text{Im } s| \geq \varepsilon$ or $\text{Re } s \in \mathbb{Z}$, where $C_f, m, \varepsilon$ is a positive constant depending on $f, m$ and $\varepsilon$. Choose $r_{f, m, \varepsilon} \in \mathbb{R}_{>(k-1)/2}$ such that $C_{f, m, \varepsilon} < (\log r_{f, m, \varepsilon})/C_1$. The inequalities (2.12) and (2.15) imply that (2.11) is true, that is, $L_f^{(m)}(1-s)$ has no zero for $s \in \mathbb{C}$ such that $|s| \geq r_{f, m, \varepsilon}$, $\text{Re } s \geq 1 + \varepsilon$ and $|\text{Im } s| \geq \varepsilon$. Therefore, we conclude that for any $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta_{f, m, \varepsilon} \in \mathbb{R}_{>(k-1)/2+1}$ such that $L_f^{(m)}(s)$ has no zero with $|s| \geq \delta_{f, m, \varepsilon}$, $\text{Re } s \leq -\varepsilon$ and $|\text{Im } s| \geq \varepsilon$, that is, the proof of (iii) is complete.

Finally, we show (iii) by applying Rouché’s theorem to $f(s)$ and $g(s)$. For $n \in \mathbb{Z}_{\geq 1}$ let $D_n$ be the region where $n \leq \text{Re } s \leq n+1$ and $|\text{Im } s| \leq 1/2$. By (2.14) and (2.15), we see that there exists $\delta_{f, m, 1/2} \in \mathbb{R}_{>(k-1)/2}$ such that $|f(s)| > |g(s)|$ on the boundary of $D_n$ and in the region where $|s| > \delta_{f, m, 1/2}$ and $\text{Re } s \geq 1 + 1/2$. Then the number of zeros of $f(s)$ is equal to that of
\( f(s) + g(s) \) in the interior of \( D_n \). From (2.4), (2.5) and (2.9),
\[
(2.16) \quad f(s) = L_f(s) \cos(\pi s) \Gamma\left(s - \frac{k - 1}{2}\right) \Gamma\left(s + \frac{k - 1}{2}\right) (S_m(s) + T_m(s)).
\]
When \( R_{f,m} \) is chosen such that \( R_{f,m} \geq \delta_{f,m,1/2} \) and \( C_6/(R_{f,m} \log R_{f,m}) < 1 \), (2.12) gives
\[
(2.17) \quad \left| \frac{T_m}{S_m}(s) \right| \leq \frac{C_6}{|s| \log |s|} < 1,
\]
that is, \( S_m(s) + T_m(s) \) has no zero with \( |s| \geq R_{f,m} \). Hence \( f(s) \) has the only real zero \( s = n + 1/2 \) in \( D_n \). It is clear that \( \Re \bar{f}(s) = f(\bar{s}) \) and \( g(s) = g(\bar{s}) \) for \( s \in \mathbb{C} \), which implies that \( L_f^{(m)}(1 - s) \) can only have a real zero in the interior of \( D_n \). Replacing \( 1 - s \) with \( s \), we conclude that there exists \( \alpha_{f,m} \in \mathbb{R}_{<-(k-1)/2-1} \) such that \( L_f^{(m)}(s) \) has no complex zero in \( \Re s < \alpha_{f,m} \) and one real zero in each open interval \( (n - 1, n) \) for \( n \in \mathbb{Z}_{\leq \alpha_{f,m}} \). The proof of (iii) is complete.

3. Proof of Theorem 1.2. Using Theorem 1.1, we can choose \( \alpha_{f,m} \in \mathbb{R}_{<-(k-1)/2} \) and \( \sigma_{f,m} \in \mathbb{R}_{>1} \) such that \( L_f^{(m)}(s) \) has no zeros with \( \Re s \leq \alpha_{f,m} \) or \( \Re s \geq \sigma_{f,m} \). Moreover, choose \( \tau_{f,m} \in \mathbb{R}_{>2} \) and \( T \in \mathbb{R}_{>0} \) such that \( L_f^{(m)}(s) \) has no zeros with \( 0 < \Im s \leq \tau_{f,m} \) or \( \Im s = T \). Using the residue theorem in the region where \( \alpha_{f,m} \leq \Re s \leq \sigma_{f,m} \) and \( \tau_{f,m} \leq \Im s \leq T \), we get
\[
(3.1) \quad N_{f,m}(T) = \frac{1}{2\pi i} \left( \int_{\alpha_{f,m} + iT}^{\sigma_{f,m} + iT} \int_{\sigma_{f,m} - iT}^{\alpha_{f,m} + iT} \int_{\alpha_{f,m} - iT}^{\sigma_{f,m} - iT} \int_{\alpha_{f,m} - iT}^{\alpha_{f,m} + iT} (\log L_f^{(m)}(s))' ds \right)
\]
First, it is clear that
\[
(3.2) \quad I_1 = \frac{\log L_f^{(m)}(\sigma_{f,m} + iT) - \log L_f^{(m)}(\alpha_{f,m} + iT)}{2\pi i} = O(1).
\]
To approximate \( I_2 \), we write \( L_f^{(m)}(s) = \lambda_f(n_f)(-\log n_f)^m F(s)n_f^{-s} \) where \( F(s) \) is given by (2.1). Using (2.2) we find that \( 1/2 \leq |F(s)| \leq 3/2 \), \( \Re F(s) \geq 1/2 \) and \( |\arg F(s)| < \pi/2 \) for \( s = \sigma_{f,m} + it \) (\( t \in \mathbb{R} \)). Hence
\[
(3.3) \quad I_2 = \frac{1}{2\pi i} \left[ \log \frac{\lambda_f(n_f)(-\log n_f)^m}{n_f^s} + \log F(s) \right]_{\sigma_{f,m} - iT}^{\sigma_{f,m} + iT}
\]

\[= \frac{-(\sigma_{f,m} + iT) \log n_f}{2\pi i} + O(1) = \frac{T}{2\pi} \log n_f + O(1).\]
Next we shall estimate $I_3$. By (1.8), the approximate functional equation
for $L_f^{(m)}(s)$ (see [15, Theorem 1.2]) and Rankin’s result, there exists $A \in \mathbb{R}_{\geq 0}$
such that $L_f^{(m)}(\sigma + it) = O(|t|^A)$ uniformly for $\sigma \in [\alpha_{f,m}, \sigma_{f,m}]$. This implies that
\begin{equation}
I_3 = \frac{\log L_f^{(m)}(\alpha_{f,m} + iT) - \log L_f^{(m)}(\sigma_{f,m} + iT)}{2\pi i} \\
= \frac{\arg L_f^{(m)}(\alpha_{f,m} + iT) - \arg L_f^{(m)}(\sigma_{f,m} + iT)}{2\pi} + O(\log T).
\end{equation}
To estimate the first term of the right-hand side, we write $L_f^{(m)}(\sigma + iT) = (-1)^m e^{-iT \log n} \lambda_f(n) G(\sigma + iT)$ where
\[ G(\sigma + iT) = \frac{(\log n_f)^m}{n_f^\sigma} + \frac{1}{\lambda_f(n_f)} \sum_{n=n_f+1}^{\infty} \frac{\lambda_f(n)(\log n)^m}{n^\sigma} e^{iT \log n_f/n} \]
for $\sigma \in \mathbb{R}_{> 1}$. Let $Q$ be the number of zeros of $\Re G(s)$ on the line segment
$(\alpha_{f,m} + iT, \sigma_{f,m} + iT)$. Divide this line into $Q + 1$ subintervals by these
zeros. Then on each subinterval, the sign of $\Re G(s)$ is constant, and the
variation of $\arg G(s)$ is at most $\pi$. Hence, there exists a constant $C$ such that
$\arg G(s) = \arg L_f^{(m)}(s) + C$ on the divided line, and so
\begin{equation}
|\arg L_f^{(m)}(\alpha_{f,m} + iT) - \arg L_f^{(m)}(\sigma_{f,m} + iT)| \leq (Q + 1)\pi.
\end{equation}
In order to estimate $Q$, let $H(z) = (G(z + iT) + \overline{G(\overline{z} + iT)})/2$. Then we find that
\begin{equation}
H(\sigma) = \Re G(\sigma + iT) \\
= \frac{(\log n_f)^m}{n_f^\sigma} \left(1 + \sum_{n=n_f+1}^{\infty} \frac{\lambda_f(n)}{\lambda_f(n_f)} \left( \frac{\log n}{\log n_f} \right)^m \left( \frac{n_f}{n} \right)^\sigma \cos \left( T \log \frac{n_f}{n} \right) \right)
\end{equation}
for $\sigma \in \mathbb{R}_{> 1}$. Now (2.2) and (3.6) give
\begin{equation}
\frac{1}{2} \frac{(\log n_f)^m}{n_f^{\sigma_{f,m}}} \leq H(\sigma_{f,m}) \leq \frac{3}{2} \frac{(\log n_f)^m}{n_f^{\sigma_{f,m}}}.
\end{equation}
Take $T$ so large that $T - \tau_{f,m} > 2(\sigma_{f,m} - \alpha_{f,m})$. Since $\Im(z + iT) \geq T - (T - \tau_{f,m}) > 0$ for $z \in \mathbb{C}$ such that $|z - \sigma_{f,m}| < T - \tau_{f,m}$, it follows that
$H(z)$ is analytic in the disc $|z - \sigma_{f,m}| < T - \tau_{f,m}$. Note that there exists a
positive constant $B$ such that $H(z) = O(T^B)$ in that disc because $L_f(\sigma + it) = O(|t|^A)$. For $u \in \mathbb{R}_{\geq 0}$, let $P(u)$ be the number of zeros of $H(z)$ in
$|z - \sigma_{f,m}| \leq u$. Then using the trivial estimate
Jensen’s formula (see [13, Chapter 3.61]), the above remark and (3.7), we have

\[ P(\sigma_{f,m} - \alpha_{f,m}) \ll \int_{0}^{2\pi} \frac{P(u)}{u} \log \left| H(\sigma_{f,m} + 2(\sigma_{f,m} - \alpha_{f,m})e^{i\theta}) \right| d\theta - \log |H(\sigma_{f,m})| \]

\[ \ll \int_{0}^{2\pi} \log T^{B} d\theta + 1 \ll \log T, \]

Therefore

(3.8) \quad Q = \# \{ \sigma \in (\alpha_{f,m}, \sigma_{f,m}) \mid F(\sigma) = 0 \} \ll P(\sigma_{f,m} - \alpha_{f,m}) \ll \log T.

Combining (3.4), (3.5), (3.8), we obtain

(3.9) \quad I_{3} = O(\log T).

Finally, in order to approximate \( I_{4} \), we shall obtain an approximate formula for \( \log L^{(m)}_{f}(\alpha_{f,m} + iT) \) as \( T \to \infty \). By the proof of Theorem 1.1, there exists \( \delta_{f,m} \in \mathbb{R}_{>0} \) such that

(3.10) \quad \left| \frac{g}{f}(1 - s) \right| < 1, \quad \left| \frac{T_{m}}{S_{m}}(1 - s) \right| < 1

for \( s \in \mathbb{C} \) in the region where \( |s - (1 - (k - 1)/2)| > \delta_{f,m}, \text{ Re } s < 1 - (k - 1)/2 \) and \( |\text{Im } s| > 1/2 \). Choose \( \alpha_{f,m} \in \mathbb{R}_{<0} \) such that \( \alpha_{f,m} < 1 - (k - 1)/2 - \delta_{f,m} \). Then the path of \( I_{4} \) is contained in the above region. Replacing \( s \) with \( 1 - s \) and taking the logarithm of both sides of (2.5), we obtain

(3.11) \quad \log L^{(m)}_{f}(\alpha_{f,m} + iT)

\[ = -2(1 - \alpha_{f,m} - iT) \log 2\pi + \log f(1 - \alpha_{f,m} - iT) + \log \left( \frac{T_{m}}{S_{m}} \right) + O(1). \]

The first inequality of (3.10) gives \( |\arg((1 + (g/f)(1 - \alpha_{f,m} - iT)))| < \pi/2 \) and

(3.12) \quad \log \left( 1 + \frac{g}{f}(1 - \alpha_{f,m} - iT) \right)

\[ \ll \sqrt{\left| 1 + \frac{g}{f}(1 - \alpha_{f,m} - iT) \right|^{2} + \left( \arg \left( 1 + \frac{g}{f}(1 - \alpha_{f,m} - iT) \right) \right)^{2}} \ll 1. \]
By (2.16), the second term of the right-hand side of (3.11) can be written as

\[
\log f(1 - \alpha_{f,m} - iT) = \log \Gamma \left(1 - \alpha_{f,m} - \frac{k - 1}{2} - iT\right) + \log \Gamma \left(1 - \alpha_{f,m} + \frac{k - 1}{2} - iT\right) + \log S_m(1 - \alpha_{f,m} - iT) + \log \left(1 + \frac{T_m}{S_m}(1 - \alpha_{f,m} - iT)\right) + \log L_f(1 - \alpha_{f,m} - iT) + \log \cos \pi(1 - \alpha_{f,m} - iT).
\]

Now it is clear that

\[
\cos(\pi(1 - \alpha_{f,m} - iT)) = e^{\pi T}e^{i(1 - \alpha_{f,m}) - \log 2(1 + e^{-2\pi(1 - \alpha_{f,m})i/e^{\pi T}})}.
\]

\[
\log L_f(1 - \alpha_{f,m} - iT) = \sum_{n=1}^{\infty} \frac{b_f(n)}{n^{1 - \alpha_{f,m} - iT}}
\]

where \(b_f(n)\) is given by

\[
b_f(n) = \begin{cases} 
(\alpha_f(p)^r + \beta_f(p)^r)/r, & n = pr, \\
0, & \text{otherwise}, 
\end{cases}
\]

and \(\alpha_f(p)\), \(\beta_f(p)\) are given by (1.1). Hence for the last two terms of the right-hand side of (3.13) we have

\[
(3.14) \quad \log L_f(1 - \alpha_{f,m} - iT) + \log \cos(\pi(1 - \alpha_{f,m} - iT)) = \pi T + O(1).
\]

By a similar discussion to (3.12), the fourth term of the right-hand side of (3.13) is estimated as

\[
(3.15) \quad \log \left(1 + \frac{T_m}{S_m}(1 - \alpha_{f,m} - iT)\right) \ll 1.
\]

The trivial formula

\[
\log \left(1 - \alpha_{f,m} \pm \frac{k - 1}{2} - iT\right) = \log T - (\pi/2)i + O(1/T)
\]

shows that the third term of the right-hand side of (3.13) is approximated as

\[
(3.16) \quad \log S_m(1 - \alpha_{f,m} - iT) = m \log \log T + O(1).
\]

Using Stirling’s formula

\[
\log \Gamma(s) = (s - 1/2) \log s - s + \log \sqrt{2\pi} + O(1/|s|)
\]
and the approximate formula for $\log(1 - \alpha_{f,m} \pm (k - 1)/2 - iT)$, we approximate the sum of the first two terms of the right-hand side of (3.13) as

$$
(3.17) \quad \log \Gamma \left( 1 - \alpha_{f,m} + \frac{k - 1}{2} - iT \right) + \log \Gamma \left( 1 - \alpha_{f,m} - \frac{k - 1}{2} - iT \right) = (1 - 2\alpha_{f,m} - 2iT)(\log T - (\pi/2)i + O(1/T)) - 2(1 - \alpha_{f,m} - iT) + O(1) = -2iT \log(T/e) - \pi T + (1 - 2\alpha_{f,m}) \log T + O(1).
$$

Combining (3.11)–(3.17), we obtain the desired approximate formula

$$
\log L^{(m)}_{f}(\alpha_{f,m} + iT) = -2iT \log \frac{T}{2\pi e} + O(\log T),
$$

which implies that

$$
(3.18) \quad I_4 = \frac{T}{\pi} \log \frac{T}{2\pi e} + O(\log T).
$$

By (3.2), (3.3), (3.9) and (3.18), the proof of Theorem 1.2 is complete.

4. Proof of Theorem 1.3. Write $L^{(m)}_{f}(s) = \chi_{f}(n_f)(-\log n_f)^m F(s)/n_f$ where $F(s)$ is given by (2.1). By the proof of Theorem 1.1, we can choose $\sigma_{f,m} \in \mathbb{R}_{>1}$ such that $L_{f}(s)$ has no zero with Re $s > \sigma_{f,m}$, and

$$
\sum_{n=n_f+1}^{\infty} \left| \frac{\chi_{f}(n)}{\chi_{f}(n_f)} \right| \left( \log \frac{n}{\log n_f} \right)^m \left( \frac{n_f}{n} \right)^{\sigma_{f,m}/2} \leq \frac{1}{2}.
$$

Note that (2.2) and the above inequality give

$$
(4.1) \quad |F(s) - 1| \leq \sum_{n=n_f+1}^{\infty} \left| \frac{\chi_{f}(n)}{\chi_{f}(n_f)} \right| \left( \log \frac{n}{\log n_f} \right)^m \left( \frac{n_f}{n} \right)^{\sigma_{f,m}/2 + \sigma/2} \leq \frac{1}{2} \left( \frac{n_f}{n_f + 1} \right)^{\sigma/2}
$$

for Re $s \geq \sigma_{f,m}$. Applying Littlewood’s formula (see [13, Chapter 3.8]) to $F(s)$, we obtain

$$
(4.2) \quad 2\pi \sum_{\substack{F(\rho) = 0 \\ \sigma \leq \text{Re} \rho \leq \sigma_{f,m} \\ 1 \leq \text{Im} \rho \leq T}} (\text{Re} \rho - \sigma)
$$

\begin{align*}
&= \int_{1}^{T} \log |F(\sigma + it)| \, dt - \int_{1}^{T} \log |F(\sigma_{f,m} + it)| \, dt \\
&\quad + \int_{\sigma}^{\sigma_{f,m}} \log |F(u + iT)| \, dt - \int_{\sigma}^{\sigma_{f,m}} \log |F(u + i)| \, dt \\
&= : I_1 + I_2 + I_3 + I_4
\end{align*}
for $\sigma \in \mathbb{R}_{>1/2}$. We first estimate $I_2$. Cauchy’s theorem gives

$$I_2 = \int_1^T \log |F(v + it)| dt + \int_{\sigma_{f,m}}^v \log |F(u + i)| du - \int_{\sigma_{f,m}}^v \log |F(u + iT)| du$$

for all $v > \sigma_{f,m}$. The facts that $\log |X| \leq |X - 1|$ for $X \in \mathbb{C}$ and $-\log |Y| \leq 2|Y - 1|$ for $Y \in \mathbb{C}$ satisfying $|Y| \geq 1/2$, and (4.1), imply that

$$\int_1^T \log |F(v + it)| dt \leq \frac{T - 1}{2} \left( \frac{n_f}{n_f + 1} \right)^{v/2}$$

and

$$\int_{\sigma_{f,m}}^v \log |F(u + i)| du - \int_{\sigma_{f,m}}^v \log |F(u + iT)| du \ll \int_{\sigma_{f,m}}^v \left( \frac{n_f}{n_f + 1} \right)^{u/2} du \ll 1.$$  

Combining (4.3)–(4.5) we get

$$I_2 = O(1).$$

Following the estimation of $I_3$ in the proof of Theorem 1.2 we obtain

$$I_3 + I_4 = O(\log T).$$

To estimate $I_1$, we calculate

$$I_1 = \frac{T - 1}{2} \log \frac{n_f^{2\sigma}}{|\lambda_f(n_f)|^2(\log n_f)^2 m} + \frac{1}{2} \int_1^T \log |L_f^{(m)}(\sigma + it)|^2 dt.$$  

Jensen’s inequality gives

$$\int_1^T \log |L_f^{(m)}(\sigma + it)|^2 dt \leq (T - 1) \log \left( \frac{1}{T - 1} \int_1^T |L_f^{(m)}(\sigma + it)|^2 dt \right).$$

Combining (4.2) and (4.6)–(4.9), we obtain

$$\sum_{\substack{F(\rho) = 0 \\ \sigma \leq \text{Re} \rho \leq \sigma_{f,m} \\ 1 \leq \text{Im} \rho \leq T}} (\text{Re} \rho - \sigma) \leq \frac{T - 1}{4\pi} \log \left( \frac{1}{T - 1} \int_1^T |L_f^{(m)}(\sigma + it)|^2 dt \right) + \frac{T - 1}{4\pi} \log \frac{n_f^{2\sigma}}{|\lambda_f(n_f)|^2(\log n_f)^2 m} + O(\log T).$$

First, we consider the mean square of $L_f^{(m)}(s)$ for $\text{Re} s > 1$. We calculate as follows:
Next the mean square of $L_f^m(\sigma + it)$ for $1/2 < \Re s \leq 1$ is obtained as follows:

**Lemma 4.1** ([15, Theorem 1.3]). For any $m \in \mathbb{Z}_{\geq 0}$ and $T > 0$, we have

\begin{equation}
\int_1^T |L_f^m(\sigma + it)|^2 \, dt
\end{equation}

\begin{align*}
&= (T - 1) \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}} + O(T^{2(1 - \sigma)} (\log T)^{2m}), \quad 1/2 < \sigma < 1, \\
&= (T - 1) \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}} + O((\log T)^{2m+2}), \quad \sigma = 1.
\end{align*}

Using Rankin’s result mentioned in Introduction and the fact that

\[
\int_{n_f}^{\infty} \frac{(\log u)^{2m}}{u^{2\sigma}} \, du = \frac{(2m)!}{(2\sigma - 1)^{2m+1}} \sum_{j=0}^{2m} \frac{(\log n_f)^j (2\sigma - 1)^j}{j!},
\]
which is obtained by induction and integration by parts, we find that the series in the main term of (4.13) is approximated as

\[
\sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}}
\]

\[
= - \int_{n_f}^{\infty} \left( \frac{(\log u)^{2m}}{u^{2\sigma}} \right)' \sum_{n_f < n \leq u} |\lambda_f(n)|^2 \, du
\]

\[
= - \frac{C_f (\log n_f)^{2m}}{n_f^{2\sigma-1}} + C_f \int_{n_f}^{\infty} \frac{(\log u)^{2m}}{u^{2\sigma}} \, du + O\left( \int_{n_f}^{\infty} \frac{(\log u)^{2m}}{u^{2\sigma+2/5}} \, du \right)
\]

\[
= \frac{(2m)! n_f C_f}{n_f^{2\sigma}} \frac{1}{(2\sigma - 1)^{2m+1}} + O\left( \frac{1}{(2\sigma - 1)^{2m}} \right)
\]
as \( \sigma \to 1/2 + 0 \). From (4.10)–(4.14), the following approximate formula is obtained:

\[
\sum_{F(\rho)\neq 0, \sigma \leq \text{Re } \rho \leq \sigma_{f,m}, 1 \leq \text{Im } \rho \leq T} \frac{(\log u)^{2m}}{u^{2\sigma}} \sum_{n_f < n \leq u} |\lambda_f(n)|^2 \, du + O\left( \frac{(\log T)^{2m}}{T^{2\sigma-1}} \right), \quad 1/2 < \sigma < 1,
\]

\[
+ \frac{T - 1}{4\pi} \left\{ \begin{array}{ll}
\log \left( 1 + O\left( \frac{(2\sigma - 1)^{2m+1}(\log T)^{2m}}{T^{2\sigma-1}} \right) \right), & 1/2 < \sigma < 1, \\
\log \left( 1 + O\left( \frac{(2\sigma - 1)^{2m+1}(\log T)^{2m+2}}{T} \right) \right), & \sigma = 1, \\
\log \left( 1 + O\left( \frac{(2\sigma - 1)^{2m+1}}{T} \right) \right), & \sigma > 1.
\end{array} \right.
\]

Finally, we shall give an upper bound of \( N_{f,m}(\sigma, T) \). Since \( N_{f,m}(\sigma, T) \) is decreasing with respect to \( \sigma \), it follows that

\[
N_{f,m}(\sigma, T) = N_{f,m}(\sigma, T) - N_{f,m}(\sigma, 1) + C
\]

\[
\leq \frac{1}{\sigma - \sigma_1} \int_{\sigma_1}^{\sigma_{f,m}} (N_{f,m}(u, T) - N_{f,m}(u, 1)) \, du + C,
\]

where we set \( \sigma_1 = 1/2 + (\sigma - 1/2)/2 \). Note that \( \sigma - \sigma_1 = (\sigma - 1/2)/2 \), so \( 2\sigma_1 - 1 = \sigma - 1/2 \). Since the number of zeros of \( F_m(s) \) is equal to that of \( L_f^{(m)}(s) \), it follows that
(4.17) \[ \int_{\sigma_1}^{\sigma_{f,m}} (N_{f,m}(u,T) - N_{f,m}(u,1)) \, du \]

\[ = \int_{\sigma_1}^{\sigma_{f,m}} \sum_{F(\rho)=0} 1 \, du = \sum_{\sigma_1 \leq \text{Re} \rho \leq \sigma_{f,m}} \int_{1 \leq \text{Im} \rho \leq T} 1 \, du \]

\[ = \sum_{F(\rho)=0} (\text{Re} \rho - \sigma_1). \]

Combining (4.15)–(4.17) we obtain (1.10) and (1.9). Hence the proof of Theorem 1.3 is complete.

References


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