ACTA ARITHMETICA
176.4 (2016)

On a generalized identity connecting theta series associated with discriminants Δ and Δp^2

by

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1. Introduction. Let Δ be a discriminant of a positive definite binary quadratic form. When the discriminants Δ and Δp^2 have one form per genus, [8] gives an identity that connects the theta series associated to binary quadratic forms for each discriminant. This paper is mainly concerned with generalizing the central identity of [8] to discriminants which have multiple forms per genus. This generalized identity is stated in Theorem 5.1 where the discriminants Δ and Δp^2 are not required to have one form per genus. Theorem 5.1 gives an identity which connects a theta series associated to a binary quadratic form of discriminant Δ to a theta series associated to a subset of binary quadratic forms of discriminant Δp^2.

Section 2 sets the notation and discusses some preliminary results. Section 3 considers a map of Buell which connects the class groups $\mathrm{CL}(\Delta)$ and $\mathrm{CL}(\Delta p^2)$. Section 4 contains the lemmas and identities which are necessary for the proof of Theorem 5.1. Section 5 combines the results of the previous sections to prove Theorem 5.1. Section 6 employs Theorem 5.1 to prove a general result given by [8, Theorem 5.1]. Lastly, Section 7 gives an explicit example which employs Theorem 5.1 to derive a Lambert series decomposition and the corresponding product representation formula.

2. Preliminaries and notation. We use (a, b, c) to represent the class of binary quadratic forms which are equivalent to the binary quadratic form $ax^2+bxy+cy^2$. Equivalence of two binary quadratic forms means the transformation matrix which connects them is in $\mathrm{SL}(2,\mathbb{Z})$. The discriminant of (a, b, c) is defined as $\Delta := b^2 - 4ac$, and we only consider the case $\Delta < 0$ and $a > 0$. We say (a, b, c) is *primitive* when $\mathrm{GCD}(a, b, c) = 1$. The set of all

2010 *Mathematics Subject Classification*: 11E16, 11E25, 11F27, 11H55, 11R29.
Key words and phrases: binary quadratic forms, theta series, Lambert series.
Received 15 December 2015; revised 20 May 2016.
Published online 28 October 2016.

DOI: 10.4064/aa8383-6-2016

classes of primitive forms of discriminant Δ comprise what is known as the class group of discriminant Δ, denoted $\mathrm{CL}(\Delta)$. We will often use the term "form" to mean a class of binary quadratic forms.

We use $h(\Delta) := |\mathrm{CL}(\Delta)|$ to denote the class number of Δ. In his 1801 work [5], *Disquisitiones Arithmeticae*, Gauss develops much of the theory of binary quadratic forms, including the relation below between Δ and Δp^2:

$$h(\Delta p^2) = \frac{h(\Delta)\big(p - \big(\frac{\Delta}{p}\big)\big)}{w}, \tag{2.1}$$

where

$$w := \begin{cases} 3, & \Delta = -3, \\ 2, & \Delta = -4, \\ 1, & \Delta < -4. \end{cases} \tag{2.2}$$

The relation (2.1) as well as the number w in (2.2) appear in Section 3. Two binary quadratic forms of discriminant Δ are said to be *in the same genus* if they are equivalent over $\mathbb{Q}$ via a transformation matrix in $\mathrm{SL}(2,\mathbb{Q})$ whose entries have denominators coprime to 2Δ. An equivalent definition for the genera is given by introducing the concept of assigned characters. The *assigned characters* of a discriminant Δ are the functions $\big(\frac{r}{p}\big)$ for all odd primes $p \mid \Delta$, as well as possibly the functions $\big(\frac{-1}{r}\big)$, $\big(\frac{2}{r}\big)$, and $\big(\frac{-2}{r}\big)$. The details are discussed in Buell [2] and in Cox [3].

The genera are of equal size and partition the class group. We say a discriminant is *idoneal* when each genus contains only one form. The number of genera of discriminant Δp^2 is either equal to the number of genera of discriminant Δ, or double that number. Letting $v(\Delta)$ be the number of genera of discriminant Δ we have

$$\frac{v(\Delta p^2)}{v(\Delta)} = \begin{cases} 1, & p > 2,\ p \mid \Delta, \\ 2, & p > 2,\ p \nmid \Delta, \\ 1, & p = 2,\ p \nmid \Delta, \\ 1, & p = 2,\ \Delta \equiv 0, 12, 28 \pmod{32}, \\ 2, & p = 2,\ \Delta \equiv 4, 8, 16, 20, 24 \pmod{32}. \end{cases} \tag{2.3}$$

The theta series associated to (a, b, c) is

$$(a, b, c, q) := \sum_{x,y} q^{ax^2+bxy+cy^2} = \sum_{n\ge 0} (a, b, c; n) q^n,$$

where we use $(a, b, c; n)$ to denote the total number of representations of n by (a, b, c). We define the projection operator $P_{m,r}$ to be

$$P_{m,r} \sum_{n\ge 0} a(n) q^n = \sum_{n\ge 0} a(mn + r) q^{mn+r},$$

where we take $0 \le r < m$. Informally, the operator $P_{p,0}$ applied to (a,b,c,q) collects the terms of (a,b,c,q) which have the exponent of q congruent to 0 modulo p.

3. Connecting Δ to Δp^2. Let (a,b,c) be a primitive form of discriminant Δ. In [2, Chapter 7], Buell defines a map which sends (a,b,c) to a set of $p+1$ not necessarily distinct and not necessarily primitive forms of discriminant Δp^2. The image of (a,b,c) under this map is given by

$$\{(a, bp, cp^2)\} \cup \{(ap^2, pb + 2ahp, ah^2 + bh + c) : 0 \le h < p\}. \tag{3.1}$$

Buell devotes Section 7.1 of his book to determining the important properties of this map. He shows that if we cast out the nonprimitive forms of (3.1), then the remaining forms (all primitive) are repeated w times, where w is half the number of automorphs of Δ and is given in (2.2). We can map (a,b,c) to the set of distinct primitive forms of (3.1), and we call this set $\Psi_p(a,b,c)$.

Buell shows the images of Ψ_p are distinct, of the same size, and partition $\mathrm{CL}(\Delta p^2)$ [2, Section 7.1]. Moreover, there are exactly $1 + \left(\frac{\Delta}{p}\right)$ nonprimitive forms in (3.1). Thus there are $p + 1 - \left(1 + \left(\frac{\Delta}{p}\right)\right) = p - \left(\frac{\Delta}{p}\right)$ primitive forms in (3.1). In other words,

$$|\Psi_p(a,b,c)| = \frac{p - \left(\frac{\Delta}{p}\right)}{w}.$$

Combining these results, Buell derives the class number of Δp^2 to be

$$h(\Delta p^2) = \frac{h(\Delta)\left(p - \left(\frac{\Delta}{p}\right)\right)}{w}.$$

We emphasize that the only time there are repeated primitive forms in (3.1) is when $\Delta = -3, -4$. As an example we take $\Delta = -3$ and $p = 7$. The class group for $\Delta = -3$ consists of the single reduced form $(1,1,1)$. The class group for discriminant $\Delta p^2 = -3 \cdot 7^2 = 147$ consists of the two reduced forms $(1,1,37)$ and $(3,3,13)$. The forms in (3.1) (counting repetition) consist of $(1,1,37)$ union the forms listed in Table 1.

Table 1. $\Delta = -3$, $p = 7$

h	Corresponding form of (3.1)
0	$(1,1,37)$
1	$(3,3,13)$
2	$(7,7,7)$
3	$(3,3,13)$
4	$(7,7,7)$
5	$(3,3,13)$
6	$(1,1,37)$

As expected, the primitive forms are repeated $w = 3$ times and there are $1+\left(\frac{-3}{7}\right) = 2$ nonprimitive forms. We have $\Psi_7(1,1,1) = \{(1,1,37),(3,3,13)\}$. The preceding example illustrates the map Ψ_p when $w > 1$ and when (3.1) contains nonprimitive forms. If we apply Theorem 5.1 to this example, we obtain identities which are discussed in [8].

As another example we let $\Delta = -55$ and $p = 3$. The genus structure along with the assigned characters for the genera for the discriminants $\Delta = -55$ and $\Delta p^2 = -495$ are given below.

	$\mathrm{CL}(-55) \cong \mathbb{Z}_4$	$\left(\frac{r}{5}\right)$	$\left(\frac{r}{11}\right)$
g_1	$(1,1,14)$, $(4,3,4)$	$+1$	$+1$
g_2	$(2,1,7)$, $(2,-1,7)$	-1	-1

	$\mathrm{CL}(-495) \cong \mathbb{Z}_8 \times \mathbb{Z}_2$	$\left(\frac{r}{3}\right)$	$\left(\frac{r}{5}\right)$	$\left(\frac{r}{11}\right)$
G_1	$(1,1,124)$, $(9,9,16)$, $(4,1,31)$, $(4,-1,31)$,	$+1$	$+1$	$+1$
G_2	$(5,5,26)$, $(11,11,14)$, $(9,3,14)$, $(9,-3,14)$,	-1	$+1$	$+1$
G_3	$(2,1,62)$, $(2,-1,62)$, $(8,7,17)$, $(8,-7,17)$,	-1	-1	-1
G_4	$(7,3,18)$, $(7,-3,18)$, $(10,5,13)$, $(10,-5,13)$,	$+1$	-1	-1

We compute

$$\begin{aligned}
\Psi_3(1,1,14) &= \{(1,1,124),(9,9,16),(9,3,14),(9,-3,14)\},\\
\Psi_3(4,3,4) &= \{(5,5,26),(11,11,14),(4,1,31),(4,-1,31)\},\\
\Psi_3(2,1,7) &= \{(2,-1,62),(7,-3,18),(8,-7,17),(10,5,13)\},\\
\Psi_3(2,-1,7) &= \{(2,1,62),(7,3,18),(8,7,17),(10,-5,13)\}.
\end{aligned}$$

Also we see that

$$\Psi_3(g_1) = G_1 \cup G_2, \qquad \Psi_3(g_2) = G_3 \cup G_4.$$

As expected, the images are distinct, of equal size, and partition $\mathrm{CL}(\Delta p^2)$. Also we see in this example that $\Psi_p(f)$ is split evenly between two genera, and does not necessarily contain a form and its inverse. In general, the set $\Psi_p(f)$ will be either fully contained in one genus, or split equally between two genera. This behavior corresponds to whether $v(\Delta p^2)/v(\Delta) = 1, 2$, respectively. We refer the reader to (2.3) for the cases.

4. Lemmas and identities. This section contains several lemmas and identities which we use to prove Theorem 5.1. Lemma 4.1 shows exactly which forms in (3.1) are nonprimitive.

LEMMA 4.1. *Let (a,b,c) be a primitive form of discriminant Δ. There are exactly $1 + \left(\frac{\Delta}{p}\right)$ nonprimitive forms in the list*

$$\{(a, bp, cp^2)\} \cup \{(ap^2, pb + 2ahp, ah^2 + bh + c) : 0 \le h < p\}, \tag{4.1}$$

and they are given by

$$(4.2)\qquad \begin{cases} (a, bp, cp^2), & p \mid a,\ \left(\frac{\Delta}{p}\right) = 0, \\ f_1, & p \nmid a,\ \left(\frac{\Delta}{p}\right) = 0, \\ \emptyset, & \left(\frac{\Delta}{p}\right) = -1, \\ (a, pb, cp^2), f_2, & p \mid a,\ \left(\frac{\Delta}{p}\right) = 1, \\ f_3, f_4, & p \nmid a,\ \left(\frac{\Delta}{p}\right) = 1, \end{cases}$$

where $f_i := (ap^2, p(b + 2ah_i), ah_i^2 + bh_i + c)$. *For p odd we take*

$$\begin{aligned} h_1 &\equiv \tfrac{-b}{2a} \pmod p, \\ h_2 &\equiv \tfrac{-c}{b} \pmod p, \\ h_3 &\equiv \tfrac{-b+\sqrt{\Delta}}{2a} \pmod p, \\ h_4 &\equiv \tfrac{-b-\sqrt{\Delta}}{2a} \pmod p, \end{aligned}$$

and for $p = 2$ we take $h_4 \not\equiv h_3 \equiv h_2 \equiv h_1 \equiv c \pmod 2$. *We always take* $0 \le h_i < p$.

Proof. Let (a, b, c) be a primitive form of discriminant $\Delta < 0$. Since (a, b, c) is assumed primitive, the only way for a form in (4.1) to be nonprimitive is if p divides each of its entries. Hence (a, bp, cp^2) is nonprimitive if and only if $p \mid a$. This takes care of the form (a, bp, cp^2), and we are left with considering the forms $(ap^2, p(b + 2ah), ah^2 + bh + c)$ with $0 \le h < p$.

The form $(ap^2, p(b + 2ah), ah^2 + bh + c)$ is nonprimitive if and only if $p \mid (ah^2+bh+c)$, and so the remainder of the proof is devoted to determining exactly when $ah^2 + bh + c \equiv 0 \pmod p$.

Case $p \mid a$. If $p \mid a$ and $p \mid b$, then $ah^2+bh+c \equiv 0 \pmod p$ has no solutions since (a, b, c) is primitive. If $p \mid a$ and $p \nmid b$, then $ah^2 + bh + c \equiv 0 \pmod p$ has the unique solution $h \equiv -c/b \pmod p$, $0 \le h < p$. We note that when $p \mid a$ and $p \nmid b$, we have $\Delta \equiv b^2 \pmod p$, and so $\left(\frac{\Delta}{p}\right) = 1$. We have found the form f_2 in (4.2).

Case $p \nmid a$, $p \neq 2$. Since $p \nmid a$, we see that

$$ah^2 + bh + c \equiv 0 \pmod p$$

is equivalent to

$$(2ah + b)^2 \equiv \Delta \pmod p.$$

We find the following forms are nonprimitive:

$$\begin{cases} f_1, & p \nmid a,\ \left(\frac{\Delta}{p}\right) = 0, \\ \emptyset, & p \nmid a,\ \left(\frac{\Delta}{p}\right) = -1, \\ f_3, f_4, & p \nmid a,\ \left(\frac{\Delta}{p}\right) = 1, \end{cases}$$

where $f_i := (ap^2, p(b+2ah_i), ah_i^2 + bh_i + c)$ with

$$h_1 \equiv \tfrac{-b}{2a} \pmod p,$$
$$h_3 \equiv \tfrac{-b+\sqrt{\Delta}}{2a} \pmod p,$$
$$h_4 \equiv \tfrac{-b-\sqrt{\Delta}}{2a} \pmod p, \qquad 0 \le h_1, h_3, h_4 < p.$$

CASE $p \nmid a$, $p = 2$. Since $2 \nmid a$, we have

$$ah^2 + bh + c \equiv h + bh + c \equiv (b+1)h + c \pmod 2.$$

If $\left(\frac{\Delta}{2}\right) = 0$, then $2 \mid b$ and

$$(b+1)h + c \equiv 0 \pmod 2,$$

which implies $h \equiv c \pmod 2$. We have arrived at the nonprimitive form f_1 with $h_1 \equiv c \pmod 2$.

If $\left(\frac{\Delta}{2}\right) = -1$ then $2 \nmid b$ and we have

$$\Delta \equiv 1 - 4ac \equiv 5 \pmod 8,$$

and so c is odd in this subcase. Thus a, b, c are all odd and we see $(4a, 2b, c)$ and $(4a, 6b, a+b+c)$ are primitive. In other words, $(ap^2, pb+2ahp, ah^2+bh+c)$ with $h = 0, 1$ are both primitive forms. Hence we have only primitive forms in this subcase.

If $\left(\frac{\Delta}{p}\right) = 1$ then $2 \nmid b$ and

$$\Delta \equiv 1 - 4ac \equiv 1 \pmod 8,$$

so that c is even. Thus a, b are odd and c is even, which implies both $(4a, 2b, c)$ and $(4a, 6b, a+b+c)$ are nonprimitive. Hence $(ap^2, pb+2ahp, ah^2+bh+c)$ with $h = 0, 1$ are both nonprimitive forms.

We now list the nonprimitve forms found in this case:

$$\begin{cases} f_1, & 2 \nmid a,\ \left(\frac{\Delta}{2}\right) = 0, \\ \emptyset, & 2 \nmid a\ \left(\frac{\Delta}{2}\right) = -1, \\ f_3, f_4, & 2 \nmid a,\ \left(\frac{\Delta}{2}\right) = 1, \end{cases}$$

where $f_i := (ap^2, p(b+2ah_i), ah_i^2 + bh_i + c)$ with

$$h_4 \not\equiv h_3 \equiv h_1 \equiv c \pmod 2, \qquad 0 \le h_i < 2.$$

We have considered all possible cases and completed the proof of Lemma 4.1. ■

Lemma 4.1 is essential to finding which forms are in $\Psi_p(a, b, c)$, and we are a step closer to proving Theorem 5.1. Before proving it, we first consider $P_{p,0}(a, b, c, q)$ for an arbitrary primitive form (a, b, c) and prime p.

LEMMA 4.2. *Let (a,b,c) be a primitive form of discriminant Δ. Then*

$$P_{p,0}(a,b,c,q) = \begin{cases} (a,bp,cp^2,q), & p\,|\,a,\ \left(\frac{\Delta}{p}\right)=0,\\ f_1(q), & p\nmid a,\ \left(\frac{\Delta}{p}\right)=0,\\ (a,b,c,q^{p^2}), & \left(\frac{\Delta}{p}\right)=-1,\\ f_2(q)+(a,pb,cp^2,q)-(a,b,c,q^{p^2}), & p\,|\,a,\ \left(\frac{\Delta}{p}\right)=1,\\ f_3(q)+f_4(q)-(a,b,c,q^{p^2}), & p\nmid a,\ \left(\frac{\Delta}{p}\right)=1,\end{cases}$$

where $f_i(q) := (ap^2, p(b+2ah_i), ah_i^2+bh_i+c, q)$. For p odd we take

$$\begin{aligned} h_1 &\equiv \tfrac{-b}{2a} \pmod p,\\ h_2 &\equiv \tfrac{-c}{b} \pmod p,\\ h_3 &\equiv \tfrac{-b+\sqrt{\Delta}}{2a} \pmod p,\\ h_4 &\equiv \tfrac{-b-\sqrt{\Delta}}{2a} \pmod p,\end{aligned}$$

and for $p=2$ we take $h_4 \not\equiv h_3 \equiv h_2 \equiv h_1 \equiv c \pmod 2$. We always take $0 \le h_i < p$.

Proof. The proof is split into cases.

CASE $p\,|\,a$. If $p\,|\,a$ and $p\,|\,\Delta$, then $p\,|\,b$ and $p\nmid c$ since (a,b,c) is assumed primitive. The congruence

$$ax^2+bxy+cy^2 \equiv cy^2 \equiv 0 \pmod p$$

implies $y\equiv 0 \pmod p$, and we find $P_{p,0}(a,b,c,q) = (a,bp,cp^2,q)$.

If $p\,|\,a$ and $p\nmid\Delta$, then $\Delta \equiv b^2 \not\equiv 0 \pmod p$, and so we must have $\left(\frac{\Delta}{p}\right)=1$. Then

$$ax^2+bxy+cy^2 \equiv y(bx+cy) \equiv 0 \pmod p$$

if and only if either $y\equiv 0 \pmod p$ or $x \equiv \frac{-c}{b}y \pmod p$. We have

$$\begin{aligned} &P_{p,0}\sum_{x,y} q^{ax^2+bxy+cy^2}\\ &\quad= \sum_{\substack{x\\ y\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+bxy+cy^2} + \sum_{\substack{x\equiv\frac{-c}{b}y\,(\mathrm{mod}\,p)\\ y\not\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+bxy+cy^2}\\ &\quad= (a,pb,cp^2,q) + \sum_{x\equiv\frac{-c}{b}y\,(\mathrm{mod}\,p)} q^{ax^2+bxy+cy^2} - (a,b,c,q^{p^2})\\ &\quad= (a,pb,cp^2,q) + (ap^2, p(b+2ah_2), ah_2^2+bh_2+c, q) - (a,b,c,q^{p^2}),\end{aligned}$$

where $h_2 \equiv \frac{-c}{b} \pmod p$ and Identity 4.5 is employed in the last equality.

CASE $p\nmid a$, $p\neq 2$. In this case, the congruence

$$ax^2+bxy+cy^2 \equiv 0 \pmod p,$$

is equivalent to

$$(2ax+by)^2 \equiv \Delta y^2 \pmod{p}. \tag{4.3}$$

If $\left(\frac{\Delta}{p}\right)=0$ then (4.3) along with Identity 4.5 implies

$$\begin{aligned} P_{p,0}(a,b,c,q) &= \sum_{x\equiv h_1 y \,(\mathrm{mod}\, p)} q^{ax^2+bxy+cy^2} \\ &= (ap^2, p(b+2ah_1), ah_1^2+bh_1+c, q), \end{aligned}$$

where $h_1 \equiv \frac{-b}{2a} \pmod{p}$.

If $\left(\frac{\Delta}{p}\right)=1$ then (4.3) along with Identity 4.5 yields

$$P_{p,0}(a,b,c,q) = f_3(q)+f_4(q)-(a,b,c,q^{p^2}),$$

where

$$\begin{aligned} f_3(q) &= (ap^2, p(b+2ah_3), ah_3^2+bh_3+c, q), \\ f_4(q) &= (ap^2, p(b+2ah_4), ah_4^2+bh_4+c, q), \end{aligned}$$

and $h_3 \equiv \frac{-b+\sqrt{\Delta}}{2a} \pmod{p}$, $h_4 \equiv \frac{-b-\sqrt{\Delta}}{2a} \pmod{p}$.

Lastly, we note that if $\left(\frac{\Delta}{p}\right) = -1$, then the only solution to (4.3) is $x \equiv y \equiv 0 \pmod{p}$, and hence we have the conclusion in this case. We have now finished the proof for p odd.

Case $p \nmid a$, $p=2$. If $\left(\frac{\Delta}{2}\right)=0$ then $2 \mid b$ and we have

$$ax^2+bxy+cy^2 \equiv x+cy \equiv 0 \pmod{2},$$

which implies $x \equiv cy \pmod{2}$ is the only solution. Employing Identity 4.5 gives

$$\begin{aligned} &P_{p,0}(a,b,c,q) \\ &\quad = \sum_{x\equiv h_1 y \,(\mathrm{mod}\, 2)} q^{ax^2+bxy+cy^2} = (ap^2, p(b+2ah_1), ah_1^2+bh_1+c, q), \end{aligned}$$

where $h_1 \equiv c \pmod{2}$.

If $\left(\frac{\Delta}{2}\right)=-1$ then $2 \nmid b$ and $\Delta \equiv 1-4ac \equiv 5 \pmod{8}$. Thus c is odd and we have

$$ax^2+bxy+cy^2 \equiv x+xy+y \equiv 0 \pmod{2},$$

which implies $x \equiv y \equiv 0 \pmod{2}$ is the only solution, and we have finished this case.

If $\left(\frac{\Delta}{2}\right)=1$ then $2 \nmid b$ and $\Delta \equiv 1-4ac \equiv 1 \pmod{8}$. Thus c is even and we have

$$ax^2+bxy+cy^2 \equiv x+xy \equiv 0 \pmod{2},$$

which implies $x \equiv 0 \pmod 2$ is a solution or $y \equiv 1 \pmod 2$ is a solution. We find

$$\begin{aligned}
&P_{p,0}(a,b,c,q)\\
&\quad = (a,b,c,q^4) + \sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, 2),\\ y\not\equiv 0\,(\mathrm{mod}\, 2)}} q^{ax^2+bxy+cy^2} + \sum_{\substack{x\not\equiv 0 \,(\mathrm{mod}\, 2)\\ y\not\equiv 0\,(\mathrm{mod}\, 2)}} q^{ax^2+bxy+cy^2}\\
&\quad = (a,b,c,q^4) + \sum_{\substack{x\\ y\not\equiv 0\,(\mathrm{mod}\, 2)}} q^{ax^2+bxy+cy^2}.
\end{aligned}$$

Employing Identity 4.4 finishes the case, and hence the proof of Lemma 4.2. ∎

We now state and prove some identities which will be of use in our proof of Theorem 5.1.

IDENTITY 4.3. *Let (a,b,c) be a primitive form and $0 \le h < p$. Then*

$$\sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y\equiv j\,(\mathrm{mod}\, p)}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2} = \sum_{\substack{x\equiv hj \,(\mathrm{mod}\, p)\\ y\equiv j\,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2}. \tag{4.4}$$

Proof. Use the change of variables $(x,y) \mapsto (x-hy,y)$. ∎

IDENTITY 4.4. *Let (a,b,c) be a primitive form. Then*

$$\sum_{h=0}^{p-1} \sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y\not\equiv 0\,(\mathrm{mod}\, p)}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2} = \sum_{\substack{x\\ y\not\equiv 0\,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2}.$$

Proof. Sum (4.4) over $h = 0,1,\ldots,p-1$ and over $j = 1,\ldots,p-1$. Explicitly one gets

$$\begin{aligned}
&\sum_{h=0}^{p-1}\sum_{j=1}^{p-1} \sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y\equiv j\,(\mathrm{mod}\, p)}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2}\\
&\quad = \sum_{h=0}^{p-1}\sum_{j=1}^{p-1} \sum_{\substack{x\equiv hj \,(\mathrm{mod}\, p)\\ y\equiv j\,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} = \sum_{\substack{x\\ y\not\equiv 0\,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2}. \ \blacksquare
\end{aligned}$$

IDENTITY 4.5. *Let (a,b,c) be a primitive form and $0 \le h < p$. Then*

$$\sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2} = \sum_{x\equiv hy \,(\mathrm{mod}\, p)} q^{ax^2+bxy+cy^2}.$$

Proof. Sum (4.4) over $j = 0,1,\ldots,p-1$. Alternatively one may apply the change of variables $(x,y) \mapsto (x-hy,y)$ directly. ∎

Lemma 4.6. *Let* $(A,B,C) \in \mathrm{CL}(\Delta p^2)$. *Then*

$$P_{p,0}(A,B,C,q) = (a,b,c,q^{p^2}),$$

where $(a,b,c) \in \mathrm{CL}(\Delta)$ *and* $(A,B,C) \in \Psi_p(a,b,c)$.

Proof. By Section 3 we know $(A,B,C) \in \mathrm{CL}(\Delta p^2)$ implies there exists a unique $(a,b,c) \in \mathrm{CL}(\Delta)$ with $(A,B,C) \in \Psi_p(a,b,c)$. In other words, (A,B,C) is equivalent to either (a,bp,cp^2) or $(ap^2, p(b+2ah), ah^2+bh+c)$ for some $0 \le h < p$. Applying Lemma 4.2 along with Identity 4.5 completes the proof in both cases. ∎

5. Statement and proof of Theorem 5.1. We have arrived at the main theorem of our paper.

Theorem 5.1. *Let* (a,b,c) *be a primitive form of discriminant* $\Delta < 0$. *For any prime* p, *we have*

$$w \sum_{(A,B,C)\in\Psi_p(a,b,c)} (A,B,C,q) = \left[p - \left(\tfrac{\Delta}{p}\right)\right](a,b,c,q^{p^2}) + (a,b,c,q) - P_{p,0}(a,b,c,q). \tag{5.1}$$

We now prove Theorem 5.1. In all cases of the proof we start with the left hand side of (5.2)

$$w \sum_{(A,B,C)\in\Psi_p(a,b,c)} (A,B,C,q) - \left[p - \left(\tfrac{\Delta}{p}\right)\right](a,b,c,q^{p^2}) = (a,b,c,q) - P_{p,0}(a,b,c,q), \tag{5.2}$$

and using the results of the previous sections, we end with the right hand side of (5.2). The proof is split according to the sign of $\left(\frac{\Delta}{p}\right)$ and to whether $p \mid a$. Throughout the proof we will always take $0 \le h_i < p$.

Case $p \mid \Delta$, $p \mid a$. By Lemma 4.1 we have $|\Psi_p(a,b,c)| = p$ and (a,bp,cp^2) is the only nonprimitive form listed in (3.1). Employing Identity 4.3 (with $j=0$), Identity 4.4, and Lemma 4.2 we find that in this case we have

$$\begin{aligned} w \sum_{(A,B,C)\in\Psi_p(a,b,c)} (A,B,C,q) &- \left[p - \left(\tfrac{\Delta}{p}\right)\right](a,b,c,q^{p^2}) \\ &= \sum_{h=0}^{p-1} \sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y\not\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2} \\ &= \sum_{\substack{x\\ y\not\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} = (a,b,c,q) - P_{p,0}(a,b,c,q), \end{aligned}$$

as desired.

CASE $p \mid \Delta$, $p \nmid a$. By Lemma 4.1, the only nonprimitive form in (3.1) is $(ap^2, p(b+2ah_1), ah_1^2+bh_1+c, q)$, where for p odd we have $h_1 \equiv \frac{-b}{2a} \pmod{p}$ and for $p=2$ we have $h_1 \equiv c \pmod{2}$. By Identity 4.3 (with $j=0$), the left hand side of (5.2) is

(5.3)
$$\sum_{\substack{x\not\equiv 0 \,(\mathrm{mod}\, p)\\ y\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} + \sum_{\substack{h=0\\ h\not\equiv h_1 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y\not\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2}.$$

Employing Identity 4.4, we see (5.3) becomes

$$\sum_{\substack{x\not\equiv 0 \,(\mathrm{mod}\, p)\\ y\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} + \sum_{\substack{x\\ y\not\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} - \sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y\not\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+(b+2ah_1)xy+(ah_1^2+bh_1+c)y^2}. \tag{5.4}$$

Employing Identity 4.3 transforms (5.4) into

$$\sum_{\substack{x\\ y\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} + \sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} - (ap^2, p(b+2ah_1), ah_1^2+bh_1+c, q).$$

It is clear that this is $(a,b,c,q) - P_{p,0}(a,b,c,q)$, and we have finished this case.

CASE $\left(\frac{\Delta}{p}\right) = -1$. By Lemma 4.1 all forms of (3.1) are primitive in this case. By Identity 4.3 (with $j=0$) the left hand side of (5.2) is

$$\sum_{\substack{x\not\equiv 0 \,(\mathrm{mod}\, p)\\ y\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} + \sum_{h=0}^{p-1} \sum_{\substack{x\equiv 0 \,(\mathrm{mod}\, p)\\ y\not\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2}. \tag{5.5}$$

Employing Identity 4.4, we see (5.5) becomes

$$\sum_{\substack{x\not\equiv 0 \,(\mathrm{mod}\, p)\\ y\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2} + \sum_{\substack{x\\ y\equiv 0 \,(\mathrm{mod}\, p)}} q^{ax^2+bxy+cy^2}.$$

Adding and subtracting (a,b,c,q^{p^2}) and using Lemma 4.2 shows that this is $(a,b,c,q) - P_{p,0}(a,b,c,q)$, and we have finished this case.

CASE $\left(\frac{\Delta}{p}\right) = 1$, $p \mid a$. By Lemma 4.1 there are two nonprimitive forms in (3.1), and they are (a, pb, cp^2, q), and $(ap^2, p(b+2ah_2), ah_2^2+bh_2+c, q)$, where $h_2 \equiv \frac{-c}{b} \pmod{p}$ (note this h_2 holds for $p=2$ as well). Employing

Identity 4.3 (with $j=0$) we find the left hand side of (5.2) is

$$\sum_{h=0}^{p-1}\sum_{\substack{x\equiv 0\,(\mathrm{mod}\,p)\\ y\not\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+(b+2ah)xy+(ah^2+bh+c)y^2} - \sum_{\substack{x\equiv 0\,(\mathrm{mod}\,p)\\ y\not\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+(b+2ah_2)xy+(ah_2^2+bh_2+c)y^2}. \tag{5.6}$$

Employing Identity 4.4, we see (5.6) becomes

$$\sum_{\substack{x\\ y\not\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+bxy+cy^2} - \sum_{\substack{x\equiv 0\,(\mathrm{mod}\,p)\\ y\not\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+(b+2ah_2)xy+(ah_2^2+bh_2+c)y^2}. \tag{5.7}$$

Adding and subtracting (a,b,c,q^{p^2}) and employing Identity 4.3 (with $j=0$), we find (5.7) is

$$\sum_{\substack{x\\ y\not\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+bxy+cy^2} + (a,b,c,q^{p^2}) - \sum_{\substack{x\equiv 0\,(\mathrm{mod}\,p)\\ y}} q^{ax^2+(b+2ah_2)xy+(ah_2^2+bh_2+c)y^2}.$$

Lastly we add and subtract (a,bp,cp^2,q) to get

$$(a,b,c,q)-(a,pb,cp^2,q)-(ap^2,p(b+2ah_2),ah_2^2+bh_2+c,q)+(a,b,c,q^{p^2}),$$

and applying Lemma 4.2 finishes this case.

Case $\left(\frac{\Delta}{p}\right)=1$, $p\nmid a$. By Lemma 4.1 there are two nonprimitive forms in (3.1), and they are $(ap^2,p(b+2ah_3),ah_3^2+bh_3+c,q)$, and $(ap^2,p(b+2ah_4),ah_4^2+bh_4+c,q)$, where for p odd we take $h_3\equiv\frac{-b+\sqrt{\Delta}}{2a}$ (mod p), $h_4\equiv\frac{-b-\sqrt{\Delta}}{2a}$ (mod p), and for $p=2$ we can simply take $h_3\not\equiv h_4$ (mod 2). Employing Identity 4.3 (with $j=0$) along with Identity 4.4 shows the left hand side of (5.2) to be

$$\sum_{\substack{x\not\equiv 0\,(\mathrm{mod}\,p)\\ y\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+bxy+cy^2} + \sum_{\substack{x\\ y\not\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+bxy+cy^2} - \sum_{i=3}^{4}\sum_{\substack{x\equiv 0\,(\mathrm{mod}\,p)\\ y\not\equiv 0\,(\mathrm{mod}\,p)}} q^{ax^2+(b+2ah_i)xy+(ah_i^2+bh_i+c)y^2}. \tag{5.8}$$

By adding and subtracting $2(a,b,c,q^{p^2})$ and employing Identity 4.3 (with $j=0$), (5.8) becomes

$$(a,b,c,q)-(ap^2,p(b+2ah_3),ah_3^2+bh_3+c,q) - (ap^2,p(b+2ah_4),ah_4^2+bh_4+c,q)+(a,b,c,q^{p^2}).$$

Applying Lemma 4.2 finishes this case, and the theorem is proven.

6. Relating Theorem 5.1 to [8, Theorem 5.1]. In this section we use Theorem 5.1 to prove [8, Theorem 5.1]. First we give an example to illustrate the difference between the two theorems. In Section 3 we discuss the map Ψ_3 between the class groups $\mathrm{CL}(-55)$ and $\mathrm{CL}(-55\cdot 3^2)$. We continue this example by examining one of the identities of Theorem 5.1 with $\Delta=-55$ and $p=3$. Theorem 5.1 yields

$$(1,1,124,q)+(9,9,16,q)+2(9,3,14,q) = 4(1,1,14,q^9)+P_{3,1}(1,1,14,q)+P_{3,2}(1,1,14,q). \tag{6.1}$$

In general, Theorem 5.1 yields identities which are dissections modulo p of the theta series on the left hand side of (5.1). Equation (6.1) is a dissection modulo 3 of $(1,1,124,q)+(9,9,16,q)+2(9,3,14,q)$. Furthermore we see the forms $(1,1,124),(9,9,16)$ share a genus which is different than the genus containing $(9,3,14)$. See Section 3 for the genus structure of $\mathrm{CL}(-55\cdot 3^2)$. The forms $(1,1,124),(9,9,16)$ are in a different genus than $(9,3,14)$ because they have different assigned character values for the character $\left(\frac{\cdot}{3}\right)$. In other words, if $(1,1,124;n)+(9,9,16;n)>0$ and $3\nmid n$ then $n\equiv 1 \pmod 3$. Similarly if $(9,3,14;n)>0$ and $3\nmid n$ then $n\equiv 2 \pmod 3$. Employing Lemma 4.6 along with the above discussion allows us to separate (6.1) into the two identities

$$\begin{aligned}(1,1,124,q)+(9,9,16,q)&=2(1,1,14,q^9)+P_{3,1}(1,1,14,q),\\ 2(9,3,14,q)&=2(1,1,14,q^9)+P_{3,2}(1,1,14,q).\end{aligned} \tag{6.2}$$

Theorem 5.1 of [8] directly claims the identities of (6.2). That theorem is the present Theorem 5.1 with the addition that we consider the congruence conditions implied by the assigned characters of the genera. An example is when the left hand side of (5.1) contains theta series associated to forms of two genera; we then break (5.1) into two identities whose sum is (5.1).

We now state [8, Theorem 5.1].

THEOREM 6.1. *Let (a,b,c) be a primitive form of discriminant Δ, and G a genus of discriminant Δp^2 with $\Psi_{G,p}(a,b,c)$ nonempty. For p an odd prime, we have*

$$w\sum_{(A,B,C)\in\Psi_{G,p}(a,b,c)}(A,B,C,q)=w|\Psi_{G,p}(a,b,c)|(a,b,c,q^{p^2}) +\sum_{i=1}^{p-1}\frac{\left(\frac{ri}{p}\right)+1}{2}P_{p,i}(a,b,c,q), \tag{6.3}$$

and for $p=2$,

$$w\sum_{(A,B,C)\in\Psi_{G,2}(a,b,c)}(A,B,C,q) = w|\Psi_{G,2}(a,b,c)|(a,b,c,q^4)+P_{2^{t+1},r}(a,b,c,q) \tag{6.4}$$

where

$$w := \begin{cases} 3, & \Delta = -3, \\ 2, & \Delta = -4, \\ 1, & \Delta < -4, \end{cases}$$

r is coprime to Δp^2 and is represented by any form of $\Psi_{G,p}(a,b,c)$. When $\Delta \equiv 0 \pmod{16}$ we define $t = 2$, and for $\Delta \not\equiv 0 \pmod{16}$ we define $t = 0, 1$ according to whether Δ is odd or even.

Here $\Psi_{G,p}(a,b,c) := \Psi_p(a,b,c) \cap G$, and all other notation is consistent with our notation. We note that the coefficient $\frac{(\frac{ri}{p})+1}{2}$ of $P_{p,i}(a,b,c,q)$ is simply 0 or 1 depending on the congruence class of ri.

Proof of Theorem 6.1. Our proof naturally splits according to the parity of p and to whether $p \mid \Delta$. In general, both (6.3) and (6.4) are dissections modulo p of the theta series

$$\sum_{(A,B,C)\in\Psi_{G,p}(a,b,c)} (A,B,C,q).$$

Lemma 4.6 shows

$$P_{p,0} \sum_{(A,B,C)\in\Psi_{G,p}(a,b,c)} (A,B,C,q) = |\Psi_{G,p}(a,b,c)|(a,b,c,q^{p^2}),$$

which is the 0 modulo p dissection of (6.3) and (6.4). Our proof now breaks into cases.

Case $2 \nmid p$, $p \nmid \Delta$. Theorem 5.1 gives the identity

$$w \sum_{i=1}^{p-1} \sum_{(A,B,C)\in\Psi_p(a,b,c)} P_{p,i}(A,B,C,q) = \sum_{i=1}^{p-1} P_{p,i}(a,b,c,q). \tag{6.5}$$

When p is odd and $p \nmid \Delta$, we know from Section 3 that $\Psi_p(a,b,c)$ is split equally between two genera which have the same assigned characters except for the character $\left(\frac{\cdot}{p}\right)$. Let G_1 be the genus with assigned character $\left(\frac{\cdot}{p}\right) = 1$, G_2 the genus with assigned character $\left(\frac{\cdot}{p}\right) = -1$, and $\Psi_p(a,b,c)$ is contained in $G_1 \cup G_2$. The left hand side of (6.5) is

$$w \sum_{i=1}^{p-1} \sum_{(A,B,C)\in\Psi_{G_1,p}(a,b,c)} P_{p,i}(A,B,C,q) \\ + w \sum_{i=1}^{p-1} \sum_{(A,B,C)\in\Psi_{G_2,p}(a,b,c)} P_{p,i}(A,B,C,q).$$

The right hand side of (6.5) is

$$\sum_{\substack{i=1\\ \left(\frac{i}{p}\right)=1}}^{p-1} P_{p,i}(a,b,c,q) + \sum_{\substack{i=1\\ \left(\frac{i}{p}\right)=-1}}^{p-1} P_{p,i}(a,b,c,q).$$

If $(A,B,C) \in G_1$ and $(A,B,C;r) > 0$ for some r coprime to Δp^2, then $(A,B,C;n) = 0$ for any n with $\left(\frac{n}{p}\right) = -1$. Similarly if $(A,B,C) \in G_2$ and $(A,B,C;r) > 0$ for some r coprime to Δp^2, then $(A,B,C;n) = 0$ for any n with $\left(\frac{n}{p}\right) = 1$. We arrive at the identities

$$w\sum_{i=1}^{p-1} \sum_{(A,B,C)\in\Psi_{G_1,p}(a,b,c)} P_{p,i}(A,B,C,q) = \sum_{\substack{i=1\\ \left(\frac{i}{p}\right)=1}}^{p-1} P_{p,i}(a,b,c,q),$$

$$w\sum_{i=1}^{p-1} \sum_{(A,B,C)\in\Psi_{G_2,p}(a,b,c)} P_{p,i}(A,B,C,q) = \sum_{\substack{i=1\\ \left(\frac{i}{p}\right)=-1}}^{p-1} P_{p,i}(a,b,c,q),$$

which shows Theorem 6.1 when p is odd and $p \nmid \Delta$.

Case $2 \nmid p$, $p \mid \Delta$. In this case, $\Psi_p(a,b,c) \subseteq G$. Since p is odd and $p \mid \Delta$, the character $\left(\frac{\cdot}{p}\right)$ is one of the assigned characters for the discriminant Δ. If $(a,b,c) \in \mathrm{CL}(\Delta)$ is in a genus g which has $\left(\frac{\cdot}{p}\right) = 1$ then $P_{p,r}(a,b,c,q) = 0$ for any r with $\left(\frac{r}{p}\right) = -1$. In this case, showing (6.3) is equivalent to showing

$$\begin{aligned}(6.6)\qquad w \sum_{(A,B,C)\in\Psi_{G,p}(a,b,c)} &(A,B,C,q)\\ &= w|\Psi_p(a,b,c)|(a,b,c,q^{p^2}) + \sum_{i=1}^{p-1} P_{p,i}(a,b,c,q),\end{aligned}$$

where we have used $\Psi_{G,p}(a,b,c) = \Psi_p(a,b,c)$ and $P_{p,r}(a,b,c,q) = 0$ for any r with $\left(\frac{r}{p}\right) = -1$. Equation (6.6) is exactly (5.1). The case when $(a,b,c) \in \mathrm{CL}(\Delta)$ is in a genus g with assigned character $\left(\frac{\cdot}{p}\right) = -1$ follows similarly.

Case $p = 2$, $p \nmid \Delta$. Then (6.4) becomes

$$w \sum_{(A,B,C)\in\Psi_{G,2}(a,b,c)} (A,B,C,q) = w|\Psi_{G,2}(a,b,c)|(a,b,c,q^4) + P_{2,1}(a,b,c,q),$$

which is equivalent to (5.1).

Case $p = 2$, $p \mid \Delta$. Due to the nature of the assigned characters of a genus, there are several subcases to consider. This is apparent from (2.3) as well as examining whether the characters $\delta := \left(\frac{-1}{r}\right)$, $\epsilon := \left(\frac{2}{r}\right)$, and

$\delta\epsilon := \left(\frac{-2}{r}\right)$ are part of the assigned character list for Δ and 4Δ. Details regarding the congruence conditions when Δ contains the assigned characters δ, ϵ, and $\delta\epsilon$ are given in [2] and [3]. The assigned characters for even discriminants are given in Table 2, with $\chi_i := \left(\frac{\cdot}{p_i}\right)$ and p_i an odd prime dividing Δ where i runs up to the number of distinct odd primes dividing Δ.

Table 2

Δ	Assigned characters
$\Delta \equiv 4 \pmod{16}$	$\chi_1, \dots, \chi_r$
$\Delta \equiv 12 \pmod{16}$	$\chi_1, \dots, \chi_r, \delta$
$\Delta \equiv 24 \pmod{32}$	$\chi_1, \dots, \chi_r, \delta\epsilon$
$\Delta \equiv 8 \pmod{32}$	$\chi_1, \dots, \chi_r, \epsilon$
$\Delta \equiv 16 \pmod{32}$	$\chi_1, \dots, \chi_r, \delta$
$\Delta \equiv 0 \pmod{32}$	$\chi_1, \dots, \chi_r, \delta, \epsilon$

Our proof now splits according to whether $v(\Delta p^2)/v(\Delta) = 1, 2$ along with congruence conditions on Δ. In all of these cases we have $|\Psi_2(a,b,c)| = 2$ unless $\Delta = -4$. If $\Delta = -4$ then Theorem 6.1 directly reduces to Theorem 5.1, which reduces to the main theorem of [8] since both -4 and -16 are idoneal discriminants.

We first consider the case when $v(\Delta p^2)/v(\Delta) = 1$, which implies Ψ_2 maps into a single genus. Hence Theorem 6.1 reduces to Theorem 5.1 if can show

$$P_{2^{t+1},r}(a,b,c,q) = P_{2,1}(a,b,c,q), \tag{6.7}$$

where t is given in Theorem 6.1 and r is coprime to 2Δ and represented by (a,b,c). Equation (2.3) implies that we need to consider $\Delta \equiv 0 \pmod{32}$ or $\Delta \equiv 12 \pmod{16}$. When $\Delta \equiv 0 \pmod{32}$, (6.7) becomes

$$P_{8,r}(a,b,c,q) = P_{2,1}(a,b,c,q).$$

Proving this is equivalent to showing $(a,b,c;s) = 0$ for all odd s coprime to Δ and $s \not\equiv r \pmod{8}$. This congruence condition follows from the fact that when $\Delta \equiv 0 \pmod{32}$, both Δ and 4Δ have the same assigned characters, which are χ_p, δ, ϵ for all odd primes $p \mid \Delta$.

Similarly if $\Delta \equiv 12 \pmod{16}$ then (6.7) becomes

$$P_{4,r}(a,b,c,q) = P_{2,1}(a,b,c,q).$$

Proving this is equivalent to showing $(a,b,c;s) = 0$ for all odd s coprime to Δ and $s \not\equiv r \pmod{4}$. This congruence condition follows from the fact that when $\Delta \equiv 12 \pmod{16}$, both Δ and 4Δ have the same assigned characters, which are χ_p, δ for all odd primes $p \mid \Delta$.

We are now left with the cases which all have $v(\Delta p^2)/v(\Delta) = 2$, and so Ψ_2 consists of two forms in different genera. In other words, we are left with the cases in which Δp^2 has exactly one additional character besides those of Δ. Examining Table 2, we see these are the cases when $\Delta \equiv 4 \pmod{16}$ and $\Delta \equiv 8, 16, 24 \pmod{32}$. Let us call the additional character λ. For example, when $\Delta \equiv 4 \pmod{16}$, the assigned characters of Δ are $\chi_1, \ldots, \chi_m$ and those of 4Δ are $\chi_1, \ldots, \chi_m, \delta$. In this example, λ would be the character δ. By taking λ to be a general character we can prove the remaining cases together.

Fix $(a,b,c) \in \mathrm{CL}(\Delta)$. Let G_1 be the genus of 4Δ with assigned character $\lambda = 1$, G_2 the genus of 4Δ with assigned character $\lambda = -1$, and $\Psi_2(a,b,c) = \{(A,B,C),(D,E,F)\}$ so that $(A,B,C) \in G_1$ and $(D,E,F) \in G_2$. Theorem 5.1 gives

$$(A,B,C,q) + (D,E,F,q) = 2(a,b,c,q^4) + P_{2,1}(a,b,c,q).$$

We can write $P_{2,1}(a,b,c,q) = P_{2^k,r_1}(a,b,c,q) + P_{2^k,r_2}(a,b,c,q)$ where $\lambda(r_1) = 1$, $\lambda(r_2) = -1$, and k is 2 or 3 depending on the character λ. Employing Lemma 4.6 yields the identities

$$\begin{aligned}(A,B,C,q) &= (a,b,c,q^4) + P_{2^k,r_1}(a,b,c,q),\\ (D,E,F,q) &= (a,b,c,q^4) + P_{2^k,r_2}(a,b,c,q).\end{aligned}$$

These are the identities of Theorem 6.1, and we have finished the proof of the theorem. ∎

7. Lambert series and product representation formulas. One of the main applications of Theorem 6.1 is that we are often able to deduce a Lambert series decomposition of the left hand side of (6.3) and (6.4), and hence a product representation formula for the associated forms. Theorem 6.1 yields a Lambert series decomposition only when the theta series on the left hand side of (6.3) and (6.4) are associated to the entire genus. We illustrate this property with an example.

Let $\Delta = -23$ and $p = 3$. The class group and genus structure for the relevant discriminants is given by

$\mathrm{CL}(-23) \cong \mathbb{Z}_3$	$\left(\frac{r}{23}\right)$
$(1,1,6)$, $(2,1,3)$, $(2,-1,3)$	$+1$

$\mathrm{CL}(-207) \cong \mathbb{Z}_6$	$\left(\frac{r}{23}\right)$	$\left(\frac{r}{3}\right)$
$(1,1,52)$, $(4,1,13)$, $(4,-1,13)$	$+1$	$+1$
$(8,7,8)$, $(2,1,26)$, $(2,-1,26)$	$+1$	-1

We compute

$$\begin{aligned}\Psi_3(1,1,6) &= \{(1,1,52),(8,7,8)\},\\ \Psi_3(2,1,3) &= \{(2,-1,26),(4,1,13)\},\\ \Psi_3(2,-1,3) &= \{(2,1,26),(4,-1,13)\}.\end{aligned}$$

Employing Theorem 5.1 yields the identities

$$(7.1)\qquad \begin{aligned}(1,1,52,q)+(8,7,8,q) &= 2(1,1,6,q^9)+(P_{3,1}+P_{3,2})(1,1,6,q),\\ (2,1,26,q)+(4,1,13,q) &= 2(2,1,3,q^9)+(P_{3,1}+P_{3,2})(2,1,3,q).\end{aligned}$$

Either by Theorem 6.1 or by employing congruences directly to (7.1), we find

$$(7.2)\qquad \begin{aligned}(1,1,52,q) &= (1,1,6,q^9)+P_{3,1}(1,1,6,q),\\ (8,7,8,q) &= (1,1,6,q^9)+P_{3,2}(1,1,6,q),\\ (4,1,13,q) &= (2,1,3,q^9)+P_{3,1}(2,1,3,q),\\ (2,1,26,q) &= (2,1,3,q^9)+P_{3,2}(2,1,3,q).\end{aligned}$$

The identities of (7.1) and (7.2) do not directly yield Lambert series decompositions since the left hand sides are not associated with the entire genus. However, we can combine the identities of (7.2) to find

$$(7.3)\qquad \begin{aligned}(1,1,52,q)+2(4,1,13,q) &= f(q^9)+P_{3,1}f(q),\\ (8,7,8,q)+2(2,1,26,q) &= f(q^9)+P_{3,2}f(q),\end{aligned}$$

where $f(q) = (1,1,6,q) + 2(2,1,3,q)$ is the theta series associated with the principal genus of Δ. The identities of (7.3) yield Lambert series decompositions; we demonstrate how to derive these, together with product representation formulas.

Dirichlet's formula for quadratic forms gives $f(q)$ as a Lambert series

$$(7.4)\qquad f(q) := (1,1,6,q)+2(2,1,3,q) = 3+2\sum_{n=1}^{\infty}\left(\frac{-23}{n}\right)\frac{q^n}{1-q^n}.$$

Using (7.4) it is not hard to show

$$(7.5)\qquad (P_{3,1}-P_{3,2})f(q) = 2\sum_{n=1}^{\infty}\left(\frac{69}{n}\right)\frac{q^n(1-q^n)}{1-q^{3n}}.$$

For convenience we define the Lambert series

$$L_1(q) = \sum_{n=1}^{\infty}\left(\frac{-23}{n}\right)\frac{q^n}{1-q^n},\qquad L_2(q) = \sum_{n=1}^{\infty}\left(\frac{69}{n}\right)\frac{q^n(1-q^n)}{1-q^{3n}}.$$

It is easy to show

$$(7.6)\qquad P_{3,0}L_1(q) = 2L_1(q^3)-L_1(q^9).$$

Adding and subtracting the identities of (7.3) and employing (7.5), (7.6) gives the Lambert series decompositions for the theta series associated with the genera of discriminant -207:

$$(7.7)\qquad (1,1,52,q)+2(4,1,13,q) = 3+L_1(q)-2L_1(q^3)+3L_1(q^9)+L_2(q),$$

$$(7.8)\qquad (8,7,8,q)+2(2,1,26,q) = 3+L_1(q)-2L_1(q^3)+3L_1(q^9)-L_2(q).$$

Both (7.7) and (7.8) yield product representation formulas, as we now demonstrate. We use the notation

$$[q^k]\sum_{n\geq 0} a(n)q^n = a(k),$$

so that $[q^k]f(q)$ is simply the coefficient of q^k in the expansion of the series $f(q)$. The coefficient of q^n in $L_1(q)$ is given by

$$A(n) := [q^n]\sum_{n=1}^{\infty}\left(\frac{-23}{n}\right)\frac{q^n}{1-q^n} = \sum_{d\mid n}\left(\frac{-23}{d}\right).$$

We see that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{69}{n}\right)\frac{q^n(1-q^n)}{1-q^{3n}} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(\frac{69}{n}\right)(q^{n(3m+1)} - q^{n(3m+2)})\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{69}{n}\right)(q^{n(3m-1)} - q^{n(3m-2)})\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{69}{n}\right)\left(\frac{m}{3}\right)q^{nm}\\ &= \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{69}{d}\right)\left(\frac{n/d}{3}\right)\right)q^n,\end{aligned}$$

and so the coefficient of q^n in $L_2(q)$ is given by

$$B(n) := [q^n]\sum_{n=1}^{\infty}\left(\frac{69}{n}\right)\frac{q^n(1-q^n)}{1-q^{3n}} = \sum_{d\mid n}\left(\frac{69}{d}\right)\left(\frac{n/d}{3}\right).$$

It is easy to check that for a prime p,

$$A(p^\alpha) = \begin{cases} 1, & p = 23,\\ 1+\alpha, & \left(\frac{-23}{p}\right) = 1,\\ \frac{(-1)^\alpha+1}{2}, & \left(\frac{-23}{p}\right) = -1,\end{cases} \tag{7.9}$$

$$B(p^\alpha) = \begin{cases} 0, & p = 3,\ \alpha\neq 0,\\ (-1)^\alpha, & p = 23,\\ 1+\alpha, & \left(\frac{-23}{p}\right) = 1 \text{ and } \left(\frac{p}{3}\right) = 1,\\ (-1)^\alpha(1+\alpha), & \left(\frac{-23}{p}\right) = 1 \text{ and } \left(\frac{p}{3}\right) = -1,\\ \frac{(-1)^\alpha+1}{2}, & \left(\frac{-23}{p}\right) = -1.\end{cases} \tag{7.10}$$

Since $A(n)$ and $B(n)$ are multiplicative, we can use (7.9) and (7.10) along with (7.7) and (7.8) to give formulas for the number of representations of an integer by a given genus of discriminant -207.

Theorem 7.1. *Let the prime factorization of* n *be*

$$n = 3^a \cdot 23^b \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

where $p_i \neq 3$ *and* $\left(\frac{-23}{p_i}\right) = 1$ *and* $\left(\frac{-23}{q_j}\right) = -1$. *Let*

$$\Lambda(n) := \prod_{i=1}^{r}(1+v_i) \prod_{j=1}^{s} \frac{1+(-1)^{w_j}}{2}.$$

We have

$$(7.11)\qquad (1,1,52;n) + 2(4,1,13;n) = \begin{cases} (1+(-1)^{b+t})\Lambda(n), & a = 0,\\ 0, & a = 1,\\ 2\Lambda(n), & a \geq 2,\end{cases}$$

$$(7.12)\qquad (8,7,8;n) + 2(2,1,26;n) = \begin{cases} (1-(-1)^{b+t})\Lambda(n), & a = 0,\\ 0, & a = 1,\\ 2\Lambda(n), & a \geq 2,\end{cases}$$

where t *is the number of prime factors* p *of* n, *counting multiplicity, with* $\left(\frac{-23}{p}\right) = 1$ *and* $\left(\frac{p}{3}\right) = -1$.

Theorem 7.1 gives the total number of representations by all forms of a given genus of discriminant -207. To find $(a,b,c;n)$ for any particular form of discriminant -207, one can employ the techniques of [1].

Let $A = (2,1,26)$ and $A(q)$ the associated theta series. Recall $\mathrm{CL}(-207) \cong \mathbb{Z}_6$, and A is a generator of this group. Theorem 7.1 gives representation formulas for

$$(7.13)\qquad I(q) + 2A^2(q),$$

$$(7.14)\qquad A^3(q) + 2A(q),$$

where I is the principal form, and A^k corresponds to Gaussian composition k times. The techniques of [1] allow us to find representation formulas for

$$(7.15)\qquad I(q) - A^2(q),$$

$$(7.16)\qquad A(q) - A^3(q),$$

by using the fact that

$$(7.17)\qquad M(q) := \frac{I(q) - A^2(q) + [A(q) - A^3(q)]}{2}$$

is an eigenform for all Hecke operators and also employing congruences to separate $I(q) - A^2(q)$ and $A(q) - A^3(q)$. We note that one can use the formulas of Hecke [6, p. 794] to show $M(q)$ is an eigenform for all Hecke operators. A concise formula for the action of the Hecke operators on the theta series associated to a binary quadratic form is given by [1, (1.18)]. It is interesting to note that $L_2(q) = [I(q)+2A^2(q)-(A^3(q)+2A(q))]/2$ is an example of a Lambert series which is an eigenform for all Hecke operators.

[1] discusses the example $\mathrm{CL}(-135) \cong \mathbb{Z}_6 \cong \langle A\rangle$ which is very similar to our example except that for $\mathrm{CL}(-135)$ both $[I(q)-A^2(q)\pm(A(q)-A^3(q))]/2$ are eigenforms for all Hecke operators. In our example the combination $[I(q)-A^2(q)-(A(q)-A^3(q))]/2$ is not an eigenform for all Hecke operators and so congruences must be employed to derive (7.15). We do not give explicit representation formulas for (7.15) since the derivation process is similar to the example for $\Delta = -135$ in [1].

Another approach to proving Theorem 7.1 is to employ the general formula [7, Theorem 8.1, p. 289] proven by Huard, Kaplan, and Williams. This formula gives the total number of representations of an integer n by all the forms in a genus of discriminant $d < 0$. Section 8 of [9] discusses representations of n by an individual form. We conclude this paper by deriving Theorem 7.1 from [7, Theorem 8.1].

In the case that $9 \mid n > 0$, Theorem 8.1 of [7] gives

$$(1,1,52;n)+2(4,1,13;n) = (8,7,8;n)+2(2,1,26;n) = 2\sum_{\mu\mid\frac{n}{9}}\left(\frac{-23}{\mu}\right),$$

which is consistent with Theorem 7.1. When $3 \mid n$ and $9 \nmid n$ Theorem 8.1 of [7] gives

$$(1,1,52;n)+2(4,1,13;n) = (8,7,8;n)+2(2,1,26;n) = 0.$$

The last case to consider is when $3 \nmid n$. In the notation of [7], Theorem 8.1 of [7] gives the total number of representations by the forms of genus G to be

$$R_G(n,-207) = \frac{1}{2}\sum_{d_1\in\{1,-3,-23,69\}}\gamma_{d_1}(G)S\left(n,d_1,\frac{-207}{d_1}\right),$$

where

$$S\left(n,d_1,\frac{-207}{d_1}\right) = \sum_{\mu\nu=n}\left(\frac{d_1}{\mu}\right)\left(\frac{-207/d_1}{\nu}\right),$$

and $\gamma_{d_1}(G) = \left(\frac{d_1}{g}\right)$ with g a positive integer coprime to d_1 and represented by the genus G. Simplifying yields

$$(1,1,52;n)+2(4,1,13;n) = \sum_{\mu\mid n}\left(\frac{-23}{\mu}\right)+\sum_{\mu\nu=n}\left(\frac{-3}{\mu}\right)\left(\frac{69}{\nu}\right),$$

$$(8,7,8;n)+2(2,1,26;n) = \sum_{\mu\mid n}\left(\frac{-23}{\mu}\right)-\sum_{\mu\nu=n}\left(\frac{-3}{\mu}\right)\left(\frac{69}{\nu}\right),$$

which is consistent with Theorem 7.1 in this case.

Acknowledgments. The author would like to thank Alexander Berkovich for valuable discussions. Also the author is grateful to the anonymous referee for helpful comments and for bringing references [7] and [9] to his attention.

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