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A. W. Davis (Adelaide)

A NOTE ON INVARIANT POLYNOMIALS

Abstract. Some published results on integrals involving invariant polynomials of several matrix arguments are shown to be incorrect. This situation is related to the fact that a basic subspace in the theory of these polynomials is invariant and irreducible under a certain representation.

1. Introduction. Formulas for a class of integrals involving invariant polynomials have been proposed by J. A. Díaz-García (2011). Unfortunately, it is necessary to point out that these are in general incorrect. Comment to this effect has been made previously in the invited paper Hayakawa (2013, p. 237). See also a relevant remark in Davis (1980, p. 292).

Readers are referred to Davis (1980, 1981) and Chikuse (1980) for the theory and construction of invariant polynomials of two or more matrix arguments. The notation of Díaz-García (2011) will be followed in general, except that the arguments of the invariant polynomials will be written in symmetric form. This is because direct substitution of unsymmetric arguments in the formulas for invariant polynomials will in general give incorrect results.

Basically, the starting point in Díaz-García (2011) is the fundamental formula

$$(1)\qquad \int_{O(m)} \prod_{i=1}^{r} C_{\kappa_i}(A_i' H' X_i H A)\, dH = \sum_{\varphi\in\kappa_1\ldots\kappa_r} C_\varphi^{\kappa[r]}(A_1A_1',\ldots,A_rA_r')C_\varphi^{\kappa[r]}(X_1,\ldots,X_r)/C_\varphi(I_m)$$

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and its corollary

$$(2) \qquad \prod_{i=1}^{r} C_{\kappa_i}(X_i) = \sum_{\varphi \in \kappa_1 \dots \kappa_r} \theta_\varphi^{\kappa[r]} C_\varphi^{\kappa[r]}(X_1, \dots, X_r)$$

where $\theta_\varphi^{\kappa[r]} = C_\varphi^{\kappa[r]}(I_m, \dots, I_m)/C_\varphi(I_m)$, I_m being the $m \times m$ unit matrix.

The identity (2) follows from (1) by setting $A_{i=}I_m$ $(i = 1, \dots, m)$ and using the basic invariance property of the zonal polynomials $C_\kappa(H'XH) = C_\kappa(X)$ for all $H \in O(m)$ (Constantine (1963, equation (3))). Applying (2), the left-hand side of (1) takes the form

$$(3) \qquad \sum_{\varphi \in \kappa_1 \dots \kappa_r} \theta_\varphi^{\kappa[r]} \int_{O(m)} C_\varphi^{\kappa[r]}(A_1'H'X_1HA_1, \dots, A_r'H'X_rHA_r)\, dH.$$

The "φ-terms" in (1) and (3) are then equated to obtain

$$(4) \qquad \theta_\varphi^{\kappa[r]} \int_{O(m)} C_\varphi^{\kappa[r]}(A_1'H'X_1HA_1, \dots, A_r'H'X_rHA_r)\, dH$$
$$= C_\varphi^{\kappa[r]}(A_1'A_1, \dots, A_r'A_r) C_\varphi^{\kappa[r]}(X_1, \dots, X_r)/C_\varphi(I_m),$$

and division by $\theta_\varphi^{\kappa[r]}$ produces the claimed result of Díaz-García (2011, equation (7)).

In connection with the division by $\theta_\varphi^{\kappa[r]}$, however, it should be noted that available tabulations of invariant polynomials include polynomials for which the θ's are zero. For such polynomials division by $\theta_\varphi^{\kappa[r]}$ is obviously invalid, and (4) implies that the product of invariant polynomials on its right-hand side is identically zero.

Polynomials with zero θ's occur in cases where φ occurs with multiplicity greater than unity for a given $\kappa[r]$. See Davis (1980, p. 297). For $r = 2$ this first occurs when $\kappa[2] = ([2,1],[2,1])$, in which case $\varphi = [3,2,1]$ occurs three times (Hayakawa (2013, p. 245)).

The $r = 3$ tables in Davis (1981) contain a number of such cases. For $\kappa[3] = ([1],[1],[1])$ and $\varphi = [2,1]$ we have polynomials

$$3^{-1/2}[2(X_1X_2)(X_3) - (X_1X_3)(X_2) - (X_2X_3)(X_1)] \text{ and}$$
$$[(X_1X_3)(X_2) - (X_2X_3)(X_1)]$$

where $(X) = \operatorname{trace} X$. These polynomials clearly vanish when the X_i are set equal to the unit matrix, and hence have zero θ-values.

Furthermore, when zero θ-values occur in (3), equation (4) must also be invalid for the polynomials with nonzero θ's. In this case substitution of (4) in (3) yields an expansion which omits the polynomials with zero thetas, and hence cannot equal the right-hand side of (1). It follows that at least one of the integrals in (3) must be a linear combination of more than one invariant polynomial.

It is straightforward to show that in general, when the A_i are not constrained to be equal to or scalar multiples of a single matrix, integrals of the form given in (4) are linear combinations of more than one invariant polynomial in the X_i, contrary to (4). That they may be expanded in terms of invariant polynomials follows from the fact that they are clearly invariant under transformations $X_i \to K'X_iK$, $i = 1, \ldots, r$, where $K \in O(m)$.

If the integral in (4) is an invariant polynomial in the $V_\varphi^{\kappa[r]}[X_1, \ldots, X_r]$ invariant subspace for arbitrary $m \times m$ matrices A_i, then this holds in particular for arbitrary $A_i \in O(m)$, and hence also for the polynomial obtained by integrating the A_i over $O(m)$. In fact, however, we obtain by successive applications of Chikuse (1980, equation (3.15)),

$$\int_{O(m)} \cdots \int_{O(m)} C_\varphi^{\kappa[r]}(A_1'X_1A_1, \ldots, A_r'X_rA_r) \prod_{i=1}^{r} dA_i$$
$$= \prod_{i=1}^{r} \{C_{\kappa_i}(X_i)/C_{\kappa_i}(I_m)\} C_\varphi^{\kappa[r]}(I_m, \ldots, I_m),$$

which may be expanded by means of (2). This result implies that (4) is invalid for invariant polynomials with nonzero thetas, and that for such cases there exists an expansion

$$\text{(5)} \qquad \int_{O(m)} C_\varphi^{\kappa[r]}(A_1'H'X_1HA_1, \ldots, A_r'H'X_rHA_r)\, dH$$
$$= \sum_{\varphi \in \kappa_1 \ldots \kappa_r} \xi_\varphi^{\kappa[r]}(A_1, \ldots, A_r) C_\varphi^{\kappa[r]}(X_1, \ldots, X_r).$$

However, explicit expressions for the ξ coefficients are beyond the scope of existing methods.

2. The procedure of equating φ-terms has led to contradictions and is therefore not generally valid. To clarify the problem, let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two sets of vectors in Euclidean n-space. Let the reader consider conditions under which $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ implies $a_i = b_i$, $i = 1, \ldots, n$.

Essentially, the equating of φ-terms assumes that invariant polynomials $C_\varphi^{\kappa[r]}$ belong to the relevant invariant subspace $V_\varphi^{\kappa[r]}$ regardless of the structure of their arguments. To present a case in which the equation of φ-terms is valid, we consider a derivation of Chikuse (1980, equation (3.15)). Replacing X_1 by $L'K'X_1KL$ in equation (1) above, where $K \in O(m)$, we integrate with respect to K to obtain

$$\{C_{\kappa_1}(X_1)/C_{\kappa_1}(I_m)\}$$
$$\times \sum_{\varphi\in\kappa_1\ldots\kappa_r} C_\varphi^{\kappa[r]}(A_1'A_1,\ldots,A_r'A_r)C_\varphi^{\kappa[r]}(L'L,X_2,\ldots,X_r)$$
$$= \sum_{\varphi\in\kappa_1\ldots\kappa_r} C_\varphi^{\kappa[r]}(A_1'A_1,\ldots,A_r'A_r)\int_{O(m)} C_\varphi^{\kappa[r]}(L'K'X_1KL,X_2,\ldots,X_r)\,dK.$$

Each side of this equation is a sum of components in the invariant subspaces $V_\varphi^{\kappa[r]}[A_1'A_1,\ldots,A_r'A_r]$. We may therefore equate the φ-terms to obtain the required result.

3. The paper of Díaz-García is significant in having raised problems that have not been considered previously in the literature on invariant polynomials. The above situation regarding equation (4) leads to what may be regarded as a new basic result in the theory of these polynomials.

The subspaces $V_\varphi^{\kappa[r]}[X_1,\ldots,X_r]$ containing the invariant polynomials $C_\varphi^{\kappa[r]}(X_1,\ldots,X_r)$ respectively arise from the decomposition of the subspace

$$\bigotimes_{i=1}^{r} V_{\kappa_i}[X_i] \tag{6}$$

into irreducible invariant subspaces under the representation

$$\psi(X_1,\ldots,X_r)\to\psi(L'X_1L,\ldots,L'X_rL),\qquad L\in \mathrm{Gl}(m), \tag{7}$$

of the general linear group $\mathrm{Gl}(m)$. Here the $V_{\kappa_i}[X_i]$ are the irreducible invariant subspaces containing the zonal polynomials $C_{\kappa_i}(X_i)$ respectively, $i=1,\ldots,r$.

The invariance property (7) implies that

$$\int_{O(m)} C_\varphi^{\kappa[r]}(A'H'X_1HA,\ldots,A'H'X_rHA)\,dH = C_\varphi(A'A)C_\varphi^{\kappa[r]}(X_1\ldots X_r)/C_\varphi(I_m). \tag{8}$$

See Chikuse (1980, equation (3.14)). The integrand has the form in (7) with $L=HA$, and so is an element of $V_\varphi^{\kappa[r]}[X_1,\ldots,X_r]$ for all H and A. The integral is therefore an element of this subspace, and since it is invariant under transformations $X_i\to K'X_iK$, $i=1,\ldots,r$, for any $K\in O(m)$, it is a scalar multiple of the invariant polynomial $C_\varphi^{\kappa[r]}(X_1,\ldots,X_r)$.

Equation (3), however, has drawn attention to transformations

$$\psi(X_1,\ldots,X_r)\to\psi(L_1X_1L_1',\ldots,L_rX_rL_r'), \tag{9}$$

where $L_i\in \mathrm{Gl}(m)$ for all i, which basically constitute a representation of the n-fold product group $\mathrm{Gl}(m)\times\cdots\times \mathrm{Gl}(m)$.

Clearly the subspace (6) is invariant under (9). *We now show that it is also irreducible under this representation, i.e., it contains no proper subspace that is invariant under* (9). This applies, of course, to the subspaces $V_{\varphi}^{\kappa[r]}[X_1, \ldots, X_r]$ in particular. *The argument used to justify* (8) *in the case of equal A_i's is therefore not applicable to justify* (4) *for arbitrary A_i.*

Suppose on the contrary that Z is a proper subspace of $\bigotimes_{i=1}^{n} V_{\kappa_i}[X_i]$ that is invariant under the representation (9). If $\Psi(X_1, \ldots, X_r) \in Z$ then

$$\Psi^*(X_1, \ldots, X_r) = \int_{O(m)} \Psi(H'X_1H, \ldots, H'X_rH)\, dH \in Z.$$

Since Ψ^* is invariant under the transformations $(X_1, \ldots, X_r) \to (K'X_1K, \ldots, K'X_rK)$ for all $K \in O(m)$, it may be expressed as a linear combination of invariant polynomials

$$\Psi^*(X_1, \ldots, X_r) = \sum_{\varphi \in \kappa_1 \ldots \kappa_r} \alpha_{\varphi}^{\kappa[r]} C_{\varphi}^{\kappa[r]}(X_1, \ldots, X_r).$$

Again applying equation (3.15) of Chikuse (1980), it follows that

$$\int_{O(m)} \cdots \int_{O(m)} \Psi^*(H_1'X_1H_1, \ldots, H_r'X_rH_r) \prod_{i=1}^{r} dH_i$$
$$= \Big\{ \sum_{\varphi \in \kappa_1 \ldots \kappa_r} \alpha_{\varphi}^{\kappa[r]} \theta_{\varphi}^{\kappa[r]} C_{\varphi}(I_m) \Big\} \prod_{i=1}^{r} \{C_{\kappa_i}(X_i)/C_{\kappa_i}(I_m)\}$$

is an element of Z. That is, the assumed invariance of Z implies that

$$\prod_{i=1}^{r} C_{\kappa_i}(L_i'X_iL_i) \in Z \quad \text{for all } L_1, \ldots, L_r \in \mathrm{Gl}(m). \tag{10}$$

It is a property of zonal polynomials that, for each κ, $C_{\kappa}(A'XA)$ *spans* the subspace $V_{\kappa}[X]$ as A varies over $\mathrm{Gl}(m)$, i.e., $V_{\kappa}[X]$ is the minimal linear space containing these polynomials. It follows that the product (10) spans $\bigotimes_{i=1}^{r} V_{\kappa_i}[X_i]$. The latter is therefore irreducible under the representation (9).

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A. W. Davis
C.S.I.R.O. (retired)
E-mail: hawkes08@chariot.com.au