ACTA ARITHMETICA
176.3 (2016)

Cancellations amongst Kloosterman sums

by

Igor E. Shparlinski (Sydney) and Tianping Zhang (Xi'an)

1. Introduction. Let p be a sufficiently large prime. For integers m and n we define the Kloosterman sum

$$\mathcal{K}_p(m,n) = \sum_{x=1}^{p-1} \mathbf{e}_p(mx + n\overline{x}),$$

where $\overline{x}$ is the multiplicative inverse of x modulo p and

$$\mathbf{e}_p(z) = \exp(2\pi i z/p).$$

Furthermore, given two intervals

$$\mathcal{I} = [K+1, K+M],\ \mathcal{J} = [L+1, L+N] \subseteq [1, p-1],$$

and two sequences of weights $\mathcal{A} = \{\alpha_m\}_{m\in\mathcal{I}}$ and $\mathcal{B} = \{\beta_n\}_{n\in\mathcal{J}}$, we define the bilinear sums of Kloosterman sums

$$\mathcal{S}_p(\mathcal{A},\mathcal{B};\mathcal{I},\mathcal{J}) = \sum_{m\in\mathcal{I}}\sum_{n\in\mathcal{J}} \alpha_m\beta_n\mathcal{K}_p(mn,1).$$

We also consider the following special cases:

$$\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J}) = \mathcal{S}_p(\mathcal{A},\{1\}_{n=1}^{N};\mathcal{I},\mathcal{J}) = \sum_{m\in\mathcal{I}}\sum_{n\in\mathcal{J}} \alpha_m\mathcal{K}_p(mn,1),$$

$$\mathcal{S}_p(\mathcal{I},\mathcal{J}) = \mathcal{S}_p(\{1\}_{m=1}^{M},\{1\}_{n=1}^{N};\mathcal{I},\mathcal{J}) = \sum_{m\in\mathcal{I}}\sum_{n\in\mathcal{J}} \mathcal{K}_p(mn,1),$$

$$\mathcal{S}_p(\mathcal{I}) = \mathcal{S}_p(\{1\}_{m=1}^{M},\emptyset;\mathcal{I},\mathcal{J}) = \sum_{m\in\mathcal{I}} \mathcal{K}_p(m,1).$$

Making the change of variable $x \mapsto nx \pmod p$, one immediately observes that $\mathcal{K}_p(mn,1) = \mathcal{K}_p(m,n)$, thus we also have

$$\mathcal{S}_p(\mathcal{A},\mathcal{B};\mathcal{I},\mathcal{J}) = \sum_{m\in\mathcal{I}}\sum_{n\in\mathcal{J}} \alpha_m\beta_n\mathcal{K}_p(m,n). \tag{1.1}$$

2010 *Mathematics Subject Classification*: 11L05, 11L07.
Key words and phrases: Kloosterman sums, cancellation, bilinear form.
Received 27 November 2015; revised 19 May 2016.
Published online 28 October 2016.

DOI: 10.4064/aa8365-6-2016

We also define, for real $\sigma > 0$,

$$\|\mathcal{A}\|_\sigma = \Big(\sum_{m\in\mathcal{I}} |\alpha_m|^\sigma\Big)^{1/\sigma} \quad \text{and} \quad \|\mathcal{B}\|_\sigma = \Big(\sum_{n\in\mathcal{J}} |\beta_n|^\sigma\Big)^{1/\sigma}$$

with the usual convention

$$\|\mathcal{A}\|_\infty = \max_{m\in\mathcal{I}} |\alpha_m| \quad \text{and} \quad \|\mathcal{B}\|_\infty = \max_{n\in\mathcal{J}} |\beta_n|.$$

By the Weil bound we have

$$|\mathcal{K}_p(m,n)| \le 2p^{1/2}$$

(see [8, Theorem 11.11]). Hence

$$|\mathcal{S}_p(\mathcal{A},\mathcal{B};\mathcal{I},\mathcal{J})| \le 2\|\mathcal{A}\|_1\|\mathcal{B}\|_1 p^{1/2}. \tag{1.2}$$

We are interested in studying cancellations amongst Kloosterman sums and thus improvements of the trivial bound (1.2).

Throughout the paper, as usual $A \ll B$ is equivalent to the inequality $|A| \le cB$ with some constant $c > 0$ (all implied constants are absolute throughout the paper).

2. Previous results. First we note that by a very special case of a much more general result of Fouvry, Michel, Rivat and Sárközy [5, Lemma 2.3] we have

$$\mathcal{S}_p(\mathcal{I}) \ll p\log p,$$

which for $M \ge p^{1/2}\log p$ improves the trivial bound $|\mathcal{S}_p(\mathcal{I})| \le 2Mp^{1/2}$ following from (1.2). Recently, Fouvry, Kowalski, Michel, Raju, Rivat and Soundararajan [6, Corollary 1.6] have given the bound

$$\mathcal{S}_p(\mathcal{I}) \ll Mp^{1/2}(\log p)^{-\eta}$$

provided that $M \ge p^{1/2}(\log p)^{-\eta}$ with some absolute constant $\eta > 0$.

It is also easy to derive from [15, Theorem 7] that

$$\mathcal{S}_p(\mathcal{I},\mathcal{J}) \ll MNp^{1/4} + M^{1/2}N^{1/2}p^{1+o(1)},$$

which improves the trivial bound from (1.2) for $MN \ge p^{1+\varepsilon}$ for any fixed $\varepsilon > 0$.

The sums $\mathcal{S}_p(\mathcal{A},\mathcal{B};\mathcal{I},\mathcal{J})$ and $\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J})$ have been estimated by Fouvry, Kowalski and Michel [4, Theorem 1.17] as a part of a much more general result about sums of so-called *trace functions*. For example, by [4, Theorem 1.17(2)], for initial intervals $\mathcal{I} = [1,M]$ and $\mathcal{J} = [1,N]$, we have

$$|\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J})| \le \|\mathcal{A}\|_1 p^{1+o(1)}. \tag{2.1}$$

Furthermore, by a result of Blomer, Fouvry, Kowalski, Michel, and Milićević [1, Theorem 6.1], also for an initial interval $\mathcal{I}$ and an arbitrary interval $\mathcal{J}$ with

$$MN \le p^{3/2} \quad \text{and} \quad M \le N^2,$$

we have

$$|\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J})| \le (\|\mathcal{A}\|_1\|\mathcal{A}\|_2)^{1/2}M^{1/12}N^{7/12}p^{3/4+o(1)}. \tag{2.2}$$

One can also find in [1, 4, 11] a series of other bounds on the sums $\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J})$ and $\mathcal{S}_p(\mathcal{A},\mathcal{B};\mathcal{I},\mathcal{J})$ and also on more general sums.

Finally, Khan [10] has given a non-trivial estimate for the analogue of $\mathcal{S}_p(\mathcal{I})$ modulo a fixed prime power, which is non-trivial already for $M \ge p^{\varepsilon}$. In [13] this result has been extended to arbitrary prime powers.

3. New results. We start with the sums $\mathcal{S}_p(\mathcal{I},\mathcal{J})$ and present a bound which improves (1.2) already for $MN \ge p^{1/2+\varepsilon}$.

Theorem 3.1. *We have*

$$\mathcal{S}_p(\mathcal{I},\mathcal{J}) \ll (p+MN)p^{o(1)}.$$

We now estimate $\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J})$.

Theorem 3.2. *We have*

$$\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J}) \ll \|\mathcal{A}\|_2 N^{1/2}p.$$

We can rewrite the bounds (2.1) and (2.2) in terms of $\|\mathcal{A}\|_\infty$ as

$$\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J}) \ll \|\mathcal{A}\|_\infty M p^{1+o(1)} \tag{3.1}$$

and

$$\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J}) \ll \|\mathcal{A}\|_\infty M^{5/6}N^{7/12}p^{3/4+o(1)}, \tag{3.2}$$

respectively, and the bound of Theorem 3.2 as

$$\mathcal{S}_p(\mathcal{A};\mathcal{I},\mathcal{J}) \ll \|\mathcal{A}\|_\infty M^{1/2}N^{1/2}p. \tag{3.3}$$

We now see that Theorem 3.1 is non-trivial provided that $MN \ge p^{1/2+\varepsilon}$ for any fixed $\varepsilon > 0$, and thus extends the range (and strength) of all previously known bounds.

Furthermore, the bound (3.3) improves (3.1) and (3.2) for

$$N < Mp^{-\varepsilon} \quad \text{and} \quad M^4N \ge p^{3+\varepsilon}$$

respectively, and also applies to intervals $\mathcal{I}$ and $\mathcal{J}$ at arbitrary positions.

We note that Blomer, Fouvry, Kowalski, Michel and Milićević [2] have recently given several application of Theorem 3.1. Further applications of bounds of bilinear Kloosterman sums can be found in [1, 4, 11].

4. Preparations. We need the following simple result.

Lemma 4.1. *For any integers X and Y with $1 \le X, Y < p$, the congruence*

$$xy \equiv 1 \pmod p, \qquad 1 \le |x| \le X,\ 1 \le |y| \le Y,$$

has at most $(XY/p+1)p^{o(1)}$ solutions.

Proof. Writing $xy \equiv 1 \pmod p$ as $xy = 1 + kp$ for some integer k with $|k| \le XY/p$ and using the bound on the divisor function [8, (1.81)], we get the desired estimate. ■

We also need the following well-known result, which dates back to Vinogradov [17, Chapter 6, Problem 14.a].

LEMMA 4.2. *For arbitrary sets $\mathcal{U}, \mathcal{V} \subseteq \{0, \dots, p-1\}$ and complex numbers φ_u and ψ_v with*

$$\sum_{u\in\mathcal{U}} |\varphi_u|^2 \le \Phi \quad \text{and} \quad \sum_{v\in\mathcal{V}} |\psi_v|^2 \le \Psi,$$

we have

$$\Big|\sum_{u\in\mathcal{U}}\sum_{v\in\mathcal{V}} \varphi_u\psi_v\, \mathbf{e}_p(uv)\Big| \le \sqrt{\Phi\Psi p}.$$

5. Proof of Theorem 3.1. The proof rests on the specific properties of Kloosterman sums which lead to the identity (1.1), which allows us to sum over m and n independently.

For an integer u we define

$$\|u\|_p = \min_{k\in\mathbb{Z}} |u - kp|$$

as the distance to the closest integer which is a multiple of p.

Then, using (1.1) and changing the order of summation, we obtain

$$\mathcal{S}_p(\mathcal{I},\mathcal{J}) = \sum_{x=1}^{p-1}\sum_{m\in\mathcal{I}} \mathbf{e}_p(mx)\sum_{n\in\mathcal{J}} \mathbf{e}_p(n\overline{x}).$$

Hence,

$$\mathcal{S}_p(\mathcal{I},\mathcal{J}) \ll \sum_{x=1}^{p-1} \min\Big\{M, \frac{p}{\|x\|_p}\Big\} \min\Big\{N, \frac{p}{\|\overline{x}\|_p}\Big\}$$

(see [8, Bound (8.6)]). We now write

$$\mathcal{S}_p(\mathcal{I},\mathcal{J}) \ll MNS_1 + MpS_2 + NpS_3 + p^2S_4, \tag{5.1}$$

where

$$S_1 = \sum_{\substack{x=1\\ \|x\|_p\le p/M\\ \|\overline{x}\|_p\le p/N}}^{p-1} 1, \qquad S_2 = \sum_{\substack{x=1\\ \|x\|_p\le p/M\\ \|\overline{x}\|_p> p/N}}^{p-1} \frac{1}{\|\overline{x}\|_p},$$

$$S_3 = \sum_{\substack{x=1\\ \|x\|_p> p/M\\ \|\overline{x}\|_p\le p/N}}^{p-1} \frac{1}{\|x\|_p}, \quad S_4 = \sum_{\substack{x=1\\ \|x\|_p> p/M\\ \|\overline{x}\|_p> p/N}}^{p-1} \frac{1}{\|x\|_p\|\overline{x}\|_p}.$$

By Lemma 4.1 we immediately obtain

$$S_1 \le (p/MN + 1)p^{o(1)}. \tag{5.2}$$

To estimate S_2, we define $I = \lceil \log p \rceil$ and write

$$S_2 \le \sum_{i=0}^{I} S_{2,i},$$

where

$$S_{2,i} = \sum_{\substack{x=1 \\ \|x\|_p \le p/M \\ e^{i+1}p/N \ge \|\overline{x}\|_p > e^i p/N}}^{p-1} \frac{1}{\|\overline{x}\|_p}$$

$$\ll e^{-i}Np^{-1} \sum_{\substack{x=1 \\ \|x\|_p \le p/M \\ e^{i+1}p/N \ge \|\overline{x}\|_p > e^i p/N}}^{p-1} 1 \ll e^{-i}Np^{-1} \sum_{\substack{x=1 \\ \|x\|_p \le p/M \\ \|\overline{x}\|_p \le e^{i+1}p/N}}^{p-1} 1.$$

Now we use Lemma 4.1 again to derive

$$\begin{aligned} S_2 &\le \sum_{i=0}^{I} (M^{-1} + e^{-i}Np^{-1})p^{o(1)} \\ &\ll (I+1)M^{-1}p^{o(1)} + Np^{-1+o(1)} \sum_{i=0}^{I} e^{-i} \\ &\ll (I+1)M^{-1}p^{o(1)} + Np^{-1+o(1)} \ll M^{-1}p^{o(1)} + Np^{-1+o(1)}. \end{aligned} \tag{5.3}$$

Similarly we obtain

$$S_3 \le N^{-1}p^{o(1)} + Mp^{-1+o(1)}. \tag{5.4}$$

Finally, we write

$$S_4 \le \sum_{i,j=0}^{I} S_{4,i,j},$$

where

$$S_{4,i,j} = \sum_{\substack{x=1 \\ e^{i+1}p/M \ge \|x\|_p > e^i p/M \\ e^{j+1}p/N \ge \|\overline{x}\|_p > e^j p/N}}^{p-1} \frac{1}{\|x\|_p \|\overline{x}\|_p}$$

$$\ll e^{-i-j}MNp^{-2} \sum_{\substack{x=1 \\ e^{i+1}p/M \ge \|x\|_p > e^i p/M \\ e^{j+1}p/N \ge \|\overline{x}\|_p > e^j p/N}}^{p-1} 1 \ll e^{-i-j}MNp^{-2} \sum_{\substack{x=1 \\ \|x\|_p \le e^{i+1}p/M \\ \|\overline{x}\|_p \le e^{j+1}p/N}}^{p-1} 1.$$

Applying Lemma 4.1 one more time, we obtain

$$S_{4,i,j} \ll e^{-i-j} MNp^{-2}(e^{i+j}p/MN + 1)p^{o(1)} = (p^{-1} + e^{-i-j}MNp^{-2})p^{o(1)}.$$

Hence

$$(5.5) \qquad \begin{aligned} S_4 &\le \sum_{i,j=0}^{I} (p^{-1} + e^{-i-j}MNp^{-2})p^{o(1)} \\ &\le (I+1)^2 p^{-1+o(1)} + MNp^{-2+o(1)} \\ &\le p^{-1+o(1)} + MNp^{-2+o(1)}. \end{aligned}$$

Combining (5.2)–(5.5) we obtain the result.

6. Proof of Theorem 3.2. As in the proof of Theorem 3.2, using the identity (1.1) and changing the order of summation and then changing the variable $x \mapsto \overline{x}$, we obtain

$$\begin{aligned} \mathcal{S}_p(\mathcal{A}; \mathcal{I}, \mathcal{J}) &= \sum_{x=1}^{p-1} \sum_{m\in\mathcal{I}} \alpha_m \, \mathbf{e}_p(mx) \sum_{n\in\mathcal{J}} \mathbf{e}_p(n\overline{x}) \\ &= \sum_{x=1}^{p-1} \sum_{m\in\mathcal{I}} \alpha_m \, \mathbf{e}_p(m\overline{x}) \sum_{n\in\mathcal{J}} \mathbf{e}_p(nx). \end{aligned}$$

Hence

$$\mathcal{S}_p(\mathcal{A}; \mathcal{I}, \mathcal{J}) = \sum_{m\in\mathcal{I}} \sum_{x=1}^{p-1} \alpha_m \gamma_x \, \mathbf{e}_p(m\overline{x}),$$

where

$$|\gamma_x| \le \min\{N, p/\|x\|_p\}.$$

Thus, similarly to the proof of Theorem 3.1, we define $I = \lceil \log p \rceil$ and write

$$(6.1) \qquad \mathcal{S}_p(\mathcal{A}; \mathcal{I}, \mathcal{J}) \ll |S_0| + \sum_{i=1}^{I} |S_i|,$$

where

$$S_0 = \sum_{m\in\mathcal{I}} \sum_{\substack{x=1 \\ \|x\|_p \le p/N}}^{p-1} \alpha_m \gamma_x \, \mathbf{e}_p(m\overline{x}),$$

$$S_i = \sum_{m\in\mathcal{I}} \sum_{\substack{x=1 \\ e^{i+1}p/N \ge \|x\|_p > e^i p/N}}^{p-1} \alpha_m \gamma_x \, \mathbf{e}_p(m\overline{x}), \qquad i = 1, \ldots, I.$$

Now using Lemma 4.2, we have

$$(6.2) \qquad |S_0| \ll \|\mathcal{A}\|_2 N \sqrt{(p/N+1)p} \ll \|\mathcal{A}\|_2 N^{1/2} p.$$

Also, for $i = 1, \ldots, I$, using the fact that if $e^{i+1}p/N \ge \|x\|_p > e^i p/N$ then $\gamma_x \ll Ne^{-i}$, by Lemma 4.2 we obtain

$$S_i = \sum_{m\in\mathcal{I}} \sum_{\substack{x=1\\ e^{i+1}p/N\ge\|x\|_p>e^ip/N}}^{p-1} \alpha_m\gamma_x\,\mathbf{e}_p(m\overline{x})$$

$$\ll \|\mathcal{A}\|_2(N^2e^{-2i}e^ip/N)^{1/2}p^{1/2} = e^{-i/2}\|\mathcal{A}\|_2N^{1/2}p.$$

Therefore,

$$\sum_{i=1}^{I}|S_i| \ll \|\mathcal{A}\|_2N^{1/2}p\sum_{i=1}^{I}e^{-i/2} \ll \|\mathcal{A}\|_2N^{1/2}p. \tag{6.3}$$

Combining (6.2) and (6.3) gives the result.

7. Comments. It is easy to see that our estimates can be extended to the case of composite moduli at the cost of essentially typographical changes, while for the methods of [1, 4, 11] the primality of the modulus seems to be crucial.

It is also natural to consider cancellations between some other exponential and character sums. For example, in [16] one can find some bound on the sums

$$\mathcal{S}_p(f,\mathcal{A},\mathcal{B};\mathfrak{C}) = \sum_{(u,v)\in\mathfrak{C}} \alpha_u\beta_v\,\mathbf{e}_p(v/f(u)),$$

$$\mathcal{T}_p(f,\mathcal{A},\mathcal{B};\mathfrak{C}) = \sum_{(u,v)\in\mathfrak{C}} \alpha_u\beta_v\chi(v+f(u)),$$

(where χ is a multiplicative character modulo p), over a *convex* set $\mathfrak{C} \subseteq [1,U]\times[1,V]$, with some integers $1 \le U, V < p$.

Here we also note that one can also obtain a non-trivial cancellation for sums

$$\mathcal{H}_{k,p}(a;\mathcal{I}) = \sum_{m\in\mathcal{I}} \mathcal{G}_{k,p}(am)$$

of Gaussian sums

$$\mathcal{G}_{k,p}(a) = \sum_{x=0}^{p-1}\mathbf{e}_p(ax^k)$$

with a positive integer $k \mid p-1$. Indeed, we define

$$\tau_p(a;\chi) = \sum_{x=1}^{p-1}\chi(x)\,\mathbf{e}_p(ax),$$

where χ is a multiplicative character; we refer to [8, Chapter 3] for a background on multiplicative characters. Then by the orthogonality of characters, we have

$$\mathcal{G}_{k,p}(a) = \sum_{\substack{\chi^k=\chi_0\\ \chi\neq\chi_0}} \overline{\chi}(a)\tau_p(a;\chi),$$

where the summation is over all non-principal multiplicative characters χ modulo p such that χ^k is the principal character χ_0 (see also [12, Theorem 5.30]). Using $|\tau_p(a;\chi)| = p^{1/2}$ for any non-principal multiplicative characters χ and integer a with $\gcd(a,p) = 1$, we derive

$$|\mathcal{H}_{k,p}(a;\mathcal{I})| = p^{1/2} \sum_{\substack{\chi^k=\chi_0\\ \chi\neq\chi_0}} \overline{\chi}(a) \sum_{m\in\mathcal{I}} \overline{\chi}(m).$$

Thus applying the Burgess bound [8, (12.58)], we derive

$$\begin{aligned}\mathcal{H}_{k,p}(a;\mathcal{I}) &\ll M^{1-1/\nu}p^{1/2+(\nu+1)/(4\nu^2)}(\log p)^{1/\nu}\\ &= M^{1-1/\nu}p^{(2\nu^2+\nu+1)/(4\nu^2)}(\log p)^{1/\nu}\end{aligned} \tag{7.1}$$

for any fixed $k \mid p-1$ and $\nu = 1, 2, \ldots$.

Similarly, for general quadratic polynomials $f(X) = aX^2 + bX$ with $\gcd(a,p) = 1$, we can define the double sums

$$\mathcal{F}_p(a,b;\mathcal{I}) = \sum_{m\in\mathcal{I}} \sum_{x=0}^{p-1} \mathbf{e}_p(m(ax^2+bx)).$$

It is easy to see that

$$\begin{aligned}\sum_{x=0}^{p-1} \mathbf{e}_p(ax^2+bx) &= \mathbf{e}_p\left(-\frac{b^2}{4a}\right) \sum_{x=0}^{p-1} \mathbf{e}_p(a(x+b/(2a))^2)\\ &= \mathbf{e}_p\left(-\frac{b^2}{4a}\right) \sum_{x=0}^{p-1} \mathbf{e}_p(ax^2) = \left(\frac{a}{p}\right) \mathbf{e}_p\left(-\frac{b^2}{4a}\right) \mathcal{G}_{2,p}(1),\end{aligned}$$

where (a/p) is the Legendre symbol of a modulo p. Hence

$$\begin{aligned}\sum_{m\in\mathcal{I}} \sum_{x=0}^{p-1} \mathbf{e}_p(m(ax^2+bx)) &= \mathcal{G}_{2,p}(1) \sum_{m\in\mathcal{I}} \left(\frac{am}{p}\right) \mathbf{e}_p\left(-\frac{(bm)^2}{4am}\right)\\ &= \left(\frac{a}{p}\right)\mathcal{G}_{2,p}(1) \sum_{m\in\mathcal{I}} \left(\frac{m}{p}\right) \mathbf{e}_p\left(-\frac{b^2m}{4a}\right).\end{aligned}$$

Now, using the bound of Burgess [3] on short mixed sums (see [7, 9, 14] for various generalisations) we easily derive that for any fixed $\nu = 2, 3, \ldots$ we have

$$\mathcal{F}_p(a,b;\mathcal{I}) \ll M^{1-1/\nu}p^{1/2+1/(4(\nu-1))}(\log p)^2 = M^{1-1/\nu}p^{(2\nu-1)/(4(\nu-1))}(\log p)^2, \tag{7.2}$$

where the implied constant may depend on ν.

We note that the bounds (7.1) and (7.2) are non-trivial provided that $M \ge p^{1/4+\varepsilon}$ for any fixed $\varepsilon > 0$ and sufficiently large p.

Acknowledgments. The authors would like to thank the anonymous referee for the very careful reading of the manuscript and valuable comments and suggestions.

The second author gratefully acknowledge the support, the hospitality and the excellent conditions at the School of Mathematics and Statistics of UNSW during his visit.

This work was supported in part by ARC Grant DP140100118 (for I. E. Shparlinski), by NSFC Grant 11201275, the Natural Science Foundation of Shaanxi province of China Grant 2016JM1017, and the Fundamental Research Funds for the Central Universities Grant GK201503014 (for T. P. Zhang).

References

[1] V. Blomer, É. Fouvry, E. Kowalski, P. Michel and D. Milićević, *On moments of twisted L-functions*, Amer. J. Math., to appear.

[2] V. Blomer, É. Fouvry, E. Kowalski, P. Michel and D. Milićević, *Some applications of smooth bilinear forms with Kloosterman sums*, arXiv:1604.07664 (2016).

[3] D. A. Burgess, *Partial Gaussian sums*, Bull. London Math. Soc. 20 (1988), 589–592.

[4] É. Fouvry, E. Kowalski and P. Michel, *Algebraic trace functions over the primes*, Duke Math. J. 163 (2014), 1683–1736.

[5] É. Fouvry, P. Michel, J. Rivat and A. Sárközy, *On the pseudorandomness of the signs of Kloosterman sums*, J. Austral. Math. Soc. 77 (2004), 425–436.

[6] É. Fouvry, E. Kowalski, P. Michel, C. S. Raju, J. Rivat and K. Soundararajan, *On short sums of trace functions*, arXiv:1508.00512 (2015).

[7] D. R. Heath-Brown and L. Pierce, *Burgess bounds for short mixed character sums*, J. London Math. Soc. 91 (2015), 693–708.

[8] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc., Providence, RI, 2004.

[9] B. Kerr, *Some mixed character sums*, arXiv:1410.3587 (2014).

[10] R. Khan, *The divisor function in arithmetic progressions modulo prime powers*, Mathematika 62 (2016), 898–908.

[11] E. Kowalski, P. Michel and W. Sawin, *Bilinear forms with Kloosterman sums and applications*, arXiv:1511.01636 (2015).

[12] R. Lidl and H. Niederreiter, *Finite Fields*, Cambridge Univ. Press, Cambridge, 1997.

[13] K. Liu, I. E. Shparlinski and T. P. Zhang, *Divisor problem in arithmetic progressions modulo a prime power*, arXiv:1602.03583 (2016).

[14] L. B. Pierce, *Burgess bounds for multi-dimensional short mixed character sums*, J. Number Theory 163 (2016), 172–210.

[15] I. E. Shparlinski, *Distribution of inverses and multiples of small integers and the Sato–Tate conjecture on average*, Michigan Math. J. 56 (2008), 99–111.

[16] I. E. Shparlinski, *On bilinear exponential and character sums with reciprocals of polynomials*, Mathematika 62 (2016), 842–859.

[17] I. M. Vinogradov, *Elements of Number Theory*, Dover Publ., New York, 1954.

Igor E. Shparlinski
Department of Pure Mathematics
University of New South Wales
Sydney, NSW 2052, Australia
E-mail: igor.shparlinski@unsw.edu.au

Tianping Zhang (corresponding author)
School of Mathematics and Information Science
Shaanxi Normal University
Xi'an 710119 Shaanxi, P.R. China
E-mail: tpzhang@snnu.edu.cn