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Hardy spaces with variable exponents on RD-spaces and applications

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Contents

Abstract

In this article, the authors introduce Hardy spaces with variable exponents, $H^{*,p(\cdot)}(\mathcal{X})$, on RD-spaces with infinite measures via the grand maximal function. Then the authors characterize these spaces by means of the non-tangential maximal function or the dyadic maximal function. Characterizations in terms of atoms or Littlewood–Paley functions are also established. As applications, the authors prove an Olsen inequality for fractional integral operators and the boundedness of singular integral operators and quasi-Banach valued sublinear operators on these spaces. Finally, a duality theory of these spaces is developed.

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1. Introduction

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, with an exponent function

$$p(\cdot) : \mathbb{R}^n \to (0, \infty),$$

is a generalization of the classical Lebesgue space, which can be traced back to Birnbaum–Orlicz [4] and Orlicz [63] (see also Luxemburg [47] and Nakano [58, 59]). But the modern development was started with the articles [44] of Kováčik and Rákosník in 1991 and [19] of Fan and Zhao, in which the authors investigated Sobolev spaces based on Lebesgue spaces with variable exponents. The variable exponent function spaces have been widely used in harmonic analysis and partial differential equations; see, for example, [10, 14, 49].

Based on the boundedness of the Hardy–Littlewood maximal operator (see, for example, [9, 11, 13]) and other related operators (see, for example, [40, 51, 52, 64, 66]) on variable exponent Lebesgue spaces, the study of several function spaces with variable exponents developed rapidly (see, for example, [2, 12, 15, 56, 60–62, 79–81, 88, 89]). In particular, Nakai and Sawano [56] studied Hardy spaces with variable exponents, $H^{p(\cdot)}(\mathbb{R}^n)$, which are extensions of variable exponent Lebesgue spaces. Later, Sawano [68] gave more applications of these variable exponent Hardy spaces, and Zhuo et al. [90] established their equivalent characterizations in terms of intrinsic square functions. Independently, Cruz-Uribe and Wang [12] also investigated variable exponent Hardy spaces with some conditions weaker than those used in [56], which also extends variable exponent Lebesgue spaces. Recall that the classical Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ and their duals are well studied and play an important role in harmonic analysis and in partial differential equations (see, for example, [8, 20, 53, 74]).

On the other hand, variable exponent Lebesgue spaces on (quasi-)metric measure spaces seem to have appeared initially in [36], where Harjulehto et al. considered the boundedness of the Hardy–Littlewood maximal operator M on $L^{p(\cdot)}(\mathcal{X})$, under the assumption that $\mathcal{X}$ is a bounded doubling space and $p(\cdot)$ is locally log-Hölder continuous. Later, several papers appeared dealing with operators in variable exponent spaces on metric measure spaces (see, for example, [23, 27, 31, 41]); however, as was pointed out by Adamowicz et al. [1], all these papers had some restrictions that either the underlying space was bounded or an unnatural ball condition on $p(\cdot)$ was assumed. More precisely, in [27], the boundedness of fractional integral operators in weighted variable exponent spaces with non-doubling measures was investigated and, in [41], Kokilashvili and Samko considered the maximal operator in weighted variable exponent spaces on metric measure spaces. Hajibayov and Samko [31] studied generalized potential operators on bounded quasi-metric measure spaces with doubling measures satisfying the so-called upper Ahlfors N-regular condition. Moreover, very recently, Adamowicz et al. [1] studied the Hardy–Littlewood maximal operator M on $L^{p(\cdot)}(\mathcal{X})$ when $\mathcal{X}$ is an unbounded quasi-metric measure space with μ being a doubling measure (or an arbitrary, possibly non-doubling, Radon measure) and $p(\cdot)$ being log-Hölder continuous.

Recall that a metric measure space

$$(\mathcal{X}, d, \mu)$$

is called a *metric measure space of homogeneous type* if μ is a Borel regular measure and satisfies the doubling property. It is well known that spaces of homogeneous type in the sense of Coifman and Weiss [7] present a natural setting for the theory of Calderón–Zygmund operators. In [8], Coifman and Weiss introduced the atomic Hardy space $H^p_{\rm at}(\mathcal{X})$ with $p \in (0, 1]$ and, when $\mathcal{X}$ is an Ahlfors 1-regular metric measure space, they established a molecular characterization for $H^1_{\rm at}(\mathcal{X})$. Later, Macías and Segovia [48] obtained a grand maximal function characterization for $H^p_{\rm at}(\mathcal{X})$ with $p \in (1/2, 1]$ via distributions acting on certain spaces of Lipschitz functions; Han [32] established a Lusin-area function characterization for $H^p_{\rm at}(\mathcal{X})$; Duong and Yan [16] characterized these atomic Hardy spaces in terms of Lusin-area functions associated with certain Poisson semigroups; Li [46] also gave a characterization of $H^p_{\rm at}(\mathcal{X})$ by the grand maximal function defined via test functions introduced in [35].

A metric measure space of homogeneous type $\mathcal{X}$ is called an *RD-space* if it has a "dimension" n and satisfies some reverse doubling property (see Definition 2.1 below), which was originally introduced by Han, Müller and Yang [34]. The Littlewood–Paley theory of Hardy spaces on RD-spaces was established in [33], and the corresponding maximal function characterizations were obtained in [29]. Moreover, in [34] these Hardy spaces on RD-spaces were proved to coincide with Triebel–Lizorkin spaces on RD-spaces. To develop a real-variable theory of Hardy spaces or, more generally, Besov spaces and Triebel–Lizorkin spaces on RD-spaces, some basic tools, including spaces of test functions, approximations of the identity and various Calderón reproducing formulas on RD-spaces, were developed in [33, 34]. Now, it is well known that these basic tools play important roles in harmonic analysis on RD-spaces (see, for example, [28, 30, 33, 34, 42, 43, 85, 87]).

In this article, we introduce Hardy spaces with variable exponents on RD-spaces, denoted by $H^{*,p(\cdot)}(\mathcal{X})$, via the grand maximal function. We then prove that $H^{*,p(\cdot)}(\mathcal{X})$ coincides with Hardy spaces with variable exponents defined via the non-tangential maximal function or via the dyadic maximal function. This generalizes both Hardy spaces on RD-spaces with constant exponents $H^p(\mathcal{X})$ (see [29, 33]) and Hardy spaces on Euclidean spaces with variable exponents $H^{p(\cdot)}(\mathbb{R}^n)$ (see [56]). Characterizations of $H^{*,p(\cdot)}(\mathcal{X})$ in terms of atoms or Littlewood–Paley functions are also obtained in this article. As applications, we give an Olsen inequality for fractional integral operators, and consider the boundedness of singular integral operators and quasi-Banach valued sublinear operators on these Hardy spaces. Finally, we prove that the dual space of $H^{*,p(\cdot)}(\mathcal{X})$ is a special case of the space ${\rm BMO}_\phi(\mathcal{X})$ which is defined in Definition 7.1. Recall that Xu [82, Problem 2.1] pointed out that a real-variable theory of Hardy spaces with variable exponents on metric measure spaces was still unknown, and hence the results in this article make a step in this direction.

This article is organized as follows.

In Section 2, we first recall the notions of RD-spaces, including the space of test functions and approximations of identity, and variable exponent Lebesgue spaces on metric measure spaces. We then establish a Fefferman–Stein vector-valued inequality for the Hardy–Littlewood maximal operator on $L^{p(\cdot)}(\mathcal{X})$ (see Theorem 2.7) and another interesting inequality (see Proposition 2.11), which play important roles in this article.

In Section 3, we introduce the Hardy space $H^{*,p(\cdot)}(\mathcal{X})$ with variable exponent on RD-spaces via the grand maximal function. Then we establish the coincidence of $H^{*,p(\cdot)}(\mathcal{X})$ with $H^{p(\cdot)}_{\alpha}(\mathcal{X})$, the Hardy space with variable exponent defined via the non-tangential maximal function (see Theorem 3.11), as well as with $H^{p(\cdot)}_{\mathrm{d}}(\mathcal{X})$, the Hardy space with variable exponent defined via the dyadic maximal function (see Theorem 3.15).

Section 4 is devoted to atomic characterizations including infinite and finite atomic characterizations (see Theorems 4.3 and 4.24, respectively). To this end, we first prove that the subset $L^q(\mathcal{X}) \cap H^{*,p(\cdot)}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$ by the Calderón–Zygmund decomposition. We then show Theorem 4.3 by using the Calderón–Zygmund decomposition and some argument similar to that used in the proof of [21, Theorem 3.28]. As consequences of the above infinite atomic characterization, we show that the spaces

$$H^{*,p(\cdot)}(\mathcal{X}), \quad H^{p(\cdot)}_{\alpha}(\mathcal{X}) \quad \text{and} \quad H^{p(\cdot)}_{\mathrm{d}}(\mathcal{X})$$

are independent of the corresponding parameters (see Theorem 4.17). Moreover, we prove that, when $1 < p_- \le p_+ < \infty$, the space $H^{*,p(\cdot)}(\mathcal{X})$ coincides with $L^{p(\cdot)}(\mathcal{X})$, where p_- and p_+ are as in (2.3) below. By using the constructive proof of Theorem 4.3, we further prove Theorem 4.24. We point out that finite atomic characterizations of Hardy spaces were first considered by Meda et al. [50], who established a finite atomic characterization of $H^1(\mathbb{R}^n)$. We also point out that the approach used in the proof of Theorems 4.3 is different from that of [29, Theorem 4.16], in which the authors established an atomic characterization of the Hardy space $H^p(\mathcal{X})$ with p being a constant exponent. Another proof of Theorem 4.3(ii), similar to that of [29, Theorem 4.16], is given at the end of Section 4; however, the atoms constructed in this way do not seem to be well suited for establishing the finite atomic characterization of the Hardy space $H^{*,p(\cdot)}(\mathcal{X})$ with variable exponent in the setting of this article.

In Section 5, we mainly establish characterizations of $H^{*,p(\cdot)}(\mathcal{X})$ via Littlewood–Paley functions, including the Lusin area function, the g^*_λ-function and the g-function. In fact, we first introduce a Hardy space $H^{p(\cdot)}(\mathcal{X})$ with variable exponent via the Lusin area function, and then give its atomic characterization via $(p(\cdot),\infty)$-atoms (see Theorem 5.3) by using the Calderón reproducing formula from [33] (see also Lemma 5.10); this space is further proved to coincide with $H^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})$, where $q \in [1,\infty] \cap (p_+,\infty]$, and hence with $H^{*,p(\cdot)}(\mathcal{X})$ in Theorem 5.4. We point out that the method used in the proof of Theorem 5.3 is similar to the one used in the proof of the constant exponent case (see [33, Theorem 2.21]), with some subtle modifications in the construction of atoms and coefficients. Moreover, as a benefit of such subtle modifications, we obtain a characterization of $H^{p(\cdot)}(\mathcal{X})$ via $(p(\cdot),\infty)$-atoms, and not via $(p(\cdot),2)$-atoms, in Theorem 5.3, which, when $p(\cdot) \equiv p$ is a constant exponent, improves the result in [33, Theorem 2.21] where $H^p(\mathcal{X})$ was characterized via $(p,2)$-atoms.

As applications, in Section 6, we obtain an Olsen inequality via the atomic characterization established in Theorem 4.3, investigate the boundedness of singular integral operators by using the characterization via the Lusin area function obtained in Theorems 5.3 and 5.4, and consider the boundedness of quasi-Banach valued sublinear operators on $H^{*,p(\cdot)}(\mathcal{X})$ via the finite atomic characterization presented in Theorem 4.24.

In Section 7, using the atomic characterization of these spaces, we consider their duals when $p(x) \le 1$ for μ-almost every $x \in \mathcal{X}$.

Finally, we point out that it would be interesting to see whether or not the results of this article still hold true for spaces of homogeneous type or even for spaces of non-homogeneous type in the sense of Hytönen [37] (see [22, 84] for developments of the theory of Hardy spaces in this setting).

2. Preliminaries

We first give some notation which will be used in this article. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Throughout the article, we denote by C a positive constant which is independent of the main parameters, but may vary from line to line. The notation $A \lesssim B$ means $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For all $a, b \in \mathbb{R}$, let

$$a \vee b := \max\{a, b\} \quad \text{and} \quad a \wedge b := \min\{a, b\}.$$

If E is a subset of $\mathcal{X}$, we denote by χ_E its characteristic function. For $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the largest integer m such that $m \le a$.

In this section, we first recall the notions of RD-spaces and variable exponent Lebesgue spaces, respectively, in Subsections 2.1 and 2.2. Then, in Subsection 2.3, we consider the boundedness of the Hardy–Littlewood maximal function on variable exponent Lebesgue spaces on metric measure spaces of homogeneous type and, as a consequence, we obtain Proposition 2.11, which plays an important role in this article and is also of independent interest.

2.1. RD-spaces. In this subsection, we recall the notions of metric measure spaces of homogeneous type in the sense of Coifman and Weiss [7, 8], and RD-spaces in the sense of [34] (see also [57, 87]).

Definition 2.1. Let $(\mathcal{X}, d, \mu)$ be a metric space with a Borel regular measure μ such that all the balls defined by μ have finite and positive measures. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, denote by $B(x, r)$ the *ball centered at x with radius r*,

$$B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}.$$

(i) The triple $(\mathcal{X}, d, \mu)$ is called a *metric measure space of homogeneous type* if there exists a constant $C_1 \in [1, \infty)$ such that, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \le C_1 \mu(B(x, r)) \quad (\textit{doubling property}). \tag{2.1}$$

(ii) Let $0 < \kappa \le n < \infty$. The triple $(\mathcal{X}, d, \mu)$ is called a (κ, n)-*space* if there exist constants $C_2 \in (0, 1]$ and $C_3 \in [1, \infty)$ such that, for all $0 < r < \operatorname{diam}(\mathcal{X})/2$,

$1 \le \lambda < \mathrm{diam}(\mathcal{X})/(2r)$ and $x \in \mathcal{X}$,

$$C_2\lambda^\kappa \mu(B(x,r)) \le \mu(B(x,\lambda r)) \le C_3\lambda^n \mu(B(x,r)), \tag{2.2}$$

where $\mathrm{diam}(E) := \sup_{x,y\in E} d(x,y)$ for a subset $E \subset \mathcal{X}$.

A metric measure space of homogeneous type is called an *RD-space* if it is a (κ, n)-space for some $0 < \kappa \le n < \infty$, that is, if some "reverse" doubling condition holds true.

Recall that in [7, 8] a triple $(\mathcal{X}, d, \mu)$ is called a *space of homogeneous type* if it satisfies Definition 2.1(i) with d being a quasi-metric.

In this article, unless otherwise stated, we *always assume* that $\mathcal{X}$ is an RD-space and $\mu(\mathcal{X}) = \infty$. Moreover, for all balls and $a \in (0,\infty)$, we use aB to denote the ball with the same center as B but a times its radius. For any x, $y \in \mathcal{X}$ and $\delta \in (0,\infty)$, let

$$V_\delta(x) := \mu(B(x,\delta)) \quad \text{and} \quad V(x,y) := \mu(B(x,d(x,y))).$$

By (2.1), we see that $V(x,y) \sim V(y,x)$ for all $x, y \in \mathcal{X}$ with implicit positive constants independent of x and y.

Remark 2.2. (i) In some sense, κ and n measure the "dimension" of $\mathcal{X}$. Obviously, a (κ, n)-space is a metric measure space of homogeneous type with $C_1 := C_3 2^n$. Conversely, a metric measure space of homogeneous type satisfies the second inequality of (2.2) with $C_3 := C_1$ and $n := \log_2 C_1$.

(ii) If μ is doubling, then μ satisfies (2.2) if and only if there exist constants $a_0, \widetilde{C}_0 \in (1,\infty)$ such that, for all $x \in \mathcal{X}$ and $0 < r < \mathrm{diam}(\mathcal{X})/a_0$,

$$\mu(B(x,a_0 r)) \ge \widetilde{C}_0 \mu(B(x,r)) \quad (\textit{reverse doubling property}),$$

or, equivalently, for all $0 < r < \mathrm{diam}(\mathcal{X})/a_0$ and $x \in \mathcal{X}$, $B(x,a_0 r) \setminus B(x,r) \neq \emptyset$; see [34, 57, 87] for some other equivalent characterizations of RD-spaces.

2.2. Variable exponent Lebesgue spaces. In what follows, a measurable function $p(\cdot) : \mathcal{X} \to (0,\infty)$ is called a *variable exponent*. For any variable exponent $p(\cdot)$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\mathcal{X}} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathcal{X}} p(x). \tag{2.3}$$

Moreover, let $\underline{p} := \min\{1, p_-\}$. Denote by $\mathcal{P}(\mathcal{X})$ the set of all variable exponents on $\mathcal{X}$ with $0 < p_- \le p_+ < \infty$. For a measurable function $f\colon \mathcal{X} \to \mathbb{R}$, define the *modular* of f by setting

$$\varrho_{p(\cdot)}(f) := \int_{\mathcal{X}} |f(x)|^{p(x)}\, d\mu(x),$$

and define the *Luxemburg quasi-norm* to be

$$\|f\|_{L^{p(\cdot)}(\mathcal{X})} := \inf\{\lambda \in (0,\infty) : \varrho_{p(\cdot)}(f/\lambda) \le 1\}.$$

Then the *variable exponent Lebesgue space* on $(\mathcal{X}, d, \mu)$, denoted by $L^{p(\cdot)}(\mathcal{X})$, is defined to be the set of all measurable functions f such that $\varrho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}(\mathcal{X})}$. For more properties of variable exponent Lebesgue spaces, we refer the reader to [10, 14]. We point out that $L^{p(\cdot)}(\mathcal{X})$ is a special case of Musielak–Orlicz spaces (see [54]).

For any $q \in (0, \infty]$, let $L^q_{\rm loc}(\mathcal{X})$ be the space of locally q-integrable functions on $\mathcal{X}$, and $L^q(\mathcal{X})$ the space of all q-integrable functions on $\mathcal{X}$.

REMARK 2.3. (i) Variable exponent Lebesgue spaces on (quasi-)metric measure spaces have already been studied in several papers (see, for example, [1, 31, 36]).

(ii) Let $p(\cdot) \in \mathcal{P}(\mathcal{X})$. Then it is easy to see that $\|\cdot\|_{L^{p(\cdot)}(\mathcal{X})}$ is a quasi-norm.

(iii) (The *Hölder inequality*) Let $p_- \in (1, \infty)$. Then, for all $f \in L^{p(\cdot)}(\mathcal{X})$ and $g \in L^{p^*(\cdot)}(\mathcal{X})$,

$$\int_{\mathcal{X}} |f(x)g(x)|\, d\mu(x) \le 2\|f\|_{L^{p(\cdot)}(\mathcal{X})}\|g\|_{L^{p^*(\cdot)}(\mathcal{X})};$$

here and hereafter $p^*(\cdot)$ denotes the dual variable exponent of $p(\cdot)$ defined by $1/p(x) + 1/p^*(x) = 1$ for all $x \in \mathcal{X}$ (see [36]).

(iv) Let $p(\cdot) \in \mathcal{P}(\mathcal{X})$. Then, by an argument similar to that used in the proof of [10, Proposition 2.21], we conclude that, for all non-trivial functions $f \in L^{p(\cdot)}(\mathcal{X})$,

$$\varrho_{p(\cdot)}(f/\|f\|_{L^{p(\cdot)}(\mathcal{X})}) = 1.$$

Recall that the variable exponent $p(\cdot)$ is said to be *locally log-Hölder continuous* in $\mathcal{X}$ if there exists a positive constant $c_{\log}$ such that, for all $x, y \in \mathcal{X}$,

$$|p(x) - p(y)| \le \frac{c_{\log}}{\log(e + 1/d(x, y))};$$

and that $p(\cdot)$ is said to satisfy the *log-Hölder decay condition* with a basepoint $x_p \in \mathcal{X}$ if there exist $p_\infty \in \mathbb{R}$ and a positive constant c_∞ such that, for all $x \in \mathcal{X}$,

$$|p(x) - p_\infty| \le \frac{c_\infty}{\log(e + d(x, x_p))}.$$

Moreover, the variable exponent $p(\cdot)$ is said to be *log-Hölder continuous* if $p(\cdot)$ satisfies both the locally log-Hölder continuous condition and the log-Hölder decay condition.

In what follows, we always fix the basepoint x_p, which plays the same role as the origin of $\mathbb{R}^n$. For $0 \le a < b \le \infty$, denote by $C^{\log}_{(a,b)}(\mathcal{X})$ (resp., $C^{\log}_{(a,b]}(\mathcal{X})$) the set of all log-Hölder continuous variable exponents $p(\cdot)$ such that $p(\mathcal{X})$ is contained in a compact interval in (a, b) (resp., $(a, b]$).

REMARK 2.4. Let $p(\cdot) \in \mathcal{P}(\mathcal{X})$. Then it is easy to see that $p(\cdot) \in C^{\log}_{(0,\infty)}(\mathcal{X})$ if and only if $1/p(\cdot) \in C^{\log}_{(0,\infty)}(\mathcal{X})$. Moreover, if $p(\cdot) \in C^{\log}_{(0,\infty)}(\mathcal{X})$, then $p^*(\cdot) \in C^{\log}_{(0,\infty)}(\mathcal{X})$.

2.3. Boundedness of the Hardy–Littlewood maximal operator. In this subsection, we mainly consider the boundedness of the maximal operator on metric measure spaces. Recall that, for any $f \in L^1_{\rm loc}(\mathcal{X})$, the *Hardy–Littlewood maximal function* $M(f)$ of f is defined by setting, for all $x \in \mathcal{X}$,

$$M(f)(x) := \sup_{r \in (0,\infty)} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)|\, d\mu(y) =: \sup_{r \in (0,\infty)} m_{B(x,r)}(|f|);$$

here and hereafter, for any measurable set $E \subset \mathcal{X}$ and any measurable function g, we write

$$m_E(g) := \frac{1}{\mu(E)} \int_E g(x)\, d\mu(x). \tag{2.4}$$

Obviously, by the Hölder inequality for variable exponent Lebesgue spaces, we find that $L^{p(\cdot)}(\mathcal{X}) \subset L^1_{\mathrm{loc}}(\mathcal{X})$ as sets when $p_- \in [1,\infty)$. Therefore, for all $f \in L^{p(\cdot)}(\mathcal{X})$ with $p_- \in [1,\infty)$, $M(f)$ is finite almost everywhere.

The following lemma is just [1, Corollary 1.8].

Lemma 2.5. *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type and let $p(\cdot) \in C^{\log}_{(1,\infty)}(\mathcal{X})$. Then, for all $f \in L^{p(\cdot)}(\mathcal{X})$,*

$$\|M(f)\|_{L^{p(\cdot)}(\mathcal{X})} \le C\|f\|_{L^{p(\cdot)}(\mathcal{X})},$$

where C is a positive constant independent of f, but which may depend on the base-point x_p.

Remark 2.6. Let $p(\cdot) \in C^{\log}_{(1,\infty)}(\mathcal{X})$ and B be a ball of $\mathcal{X}$. Then it is easy to see that, for all $\lambda \in (1,\infty)$ and $r \in (0,\infty)$,

$$\chi_{\lambda B} \le (C\lambda^n)^{1/r}[M(\chi_B)]^{1/r} \quad \text{and} \quad \|\chi_{\lambda B}\|_{L^{p(\cdot)}(\mathcal{X})} \le C\lambda^{n/\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathcal{X})},$$

where C is a positive constant independent of B and λ.

By Lemma 2.5 and some duality argument, we obtain the following Fefferman–Stein vector-valued inequality for the Hardy–Littlewood maximal operator.

Theorem 2.7. *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type (here $\mu(\mathcal{X}) \in (0,\infty]$) and $p(\cdot) \in C^{\log}_{(1,\infty)}(\mathcal{X})$. Then there exists a positive constant C such that, for all $u \in (1,\infty]$ and measurable functions $\{f_j\}_{j\in\mathbb{N}} \subset L^{p(\cdot)}(\mathcal{X})$,*

$$\left\|\left\{\sum_{j\in\mathbb{N}}[M(f_j)]^u\right\}^{1/u}\right\|_{L^{p(\cdot)}(\mathcal{X})} \le C\left\|\left\{\sum_{j\in\mathbb{N}}|f_j|^u\right\}^{1/u}\right\|_{L^{p(\cdot)}(\mathcal{X})}, \tag{2.5}$$

where, when $u = \infty$, it is understood that (2.5) means

$$\left\|\sup_{j\in\mathbb{N}} M(f_j)\right\|_{L^{p(\cdot)}(\mathcal{X})} \le C\left\|\sup_{j\in\mathbb{N}}|f_j|\right\|_{L^{p(\cdot)}(\mathcal{X})}. \tag{2.6}$$

Remark 2.8. If $p(\cdot) \equiv p \in (1,\infty)$ is a constant exponent, the conclusion of Theorem 2.7 was proved in [30, Theorem 2.1].

To prove Theorem 2.7, we need the following technical lemma, whose proof is similar to that of [10, Theorem 2.34] (see also [39, Theorem 9.2]), in which the corresponding result on the Euclidean space is considered; the details are omitted.

Lemma 2.9. *Let $(\mathcal{X}, d, \mu)$ be a metric space with a Borel regular measure μ (here $\mu(\mathcal{X}) \in (0,\infty]$) and $p(\cdot) \in C^{\log}_{(1,\infty)}(\mathcal{X})$. If $f \in L^{p(\cdot)}(\mathcal{X})$, then there exists a positive constant C such that*

$$C^{-1}\|f\|_{L^{p(\cdot)}(\mathcal{X})} \le \widetilde{\|f\|}_{L^{p(\cdot)}(\mathcal{X})} \le C\|f\|_{L^{p(\cdot)}(\mathcal{X})},$$

where

$$\widetilde{\|f\|}_{L^{p(\cdot)}(\mathcal{X})} := \sup\left\{\left|\int_{\mathcal{X}} f(x)g(x)\,d\mu(x)\right| : g \in L^{p^*(\cdot)}(\mathcal{X}) \text{ and } \|g\|_{L^{p^*(\cdot)}(\mathcal{X})} \le 1\right\}.$$

Proof of Theorem 2.7. It suffices to show (2.5), since (2.6) is obviously true. To this end, let $1 < v < p_-$. Then, by Lemma 2.9,

$$\Big\|\Big\{\sum_{j\in\mathbb{N}}[M(f_j)]^u\Big\}^{1/u}\Big\|_{L^{p(\cdot)}(\mathcal{X})}=\Big[\Big\|\Big(\sum_{j\in\mathbb{N}}[M(f_j)]^u\Big)^{v/u}\Big\|_{L^{p(\cdot)/v}(\mathcal{X})}\Big]^v$$
$$\sim\Big\{\int_{\mathcal{X}}\Big(\sum_{j\in\mathbb{N}}[M(f_j)(x)]^u\Big)^{v/u}g(x)\,d\mu(x)\Big\}^{1/v}$$

for some non-negative measurable function g such that $\|g\|_{L^{(p(\cdot)/v)^*}(\mathcal{X})}\le 1$. By Remark 2.4 and Lemma 2.5, we know that, for all $h\in L^{(p(\cdot)/v)^*}(\mathcal{X})$,

$$\|M(h)\|_{L^{(p(\cdot)/v)^*}(\mathcal{X})}\le N\|h\|_{L^{(p(\cdot)/v)^*}(\mathcal{X})}$$

for some $N\in(1,\infty)$ independent of h. Define

$$G:=\sum_{k\in\mathbb{N}}\frac{1}{2^kN^k}M^k(g),$$

where M^k denotes the k-fold iteration of the Hardy–Littlewood maximal operator M. Then

$$M(G)\le 2NG \tag{2.7}$$

and

$$\Big\|\Big\{\sum_{j\in\mathbb{N}}[M(f_j)]^u\Big\}^{1/u}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\Big\{\int_{\mathcal{X}}\Big(\sum_{j\in\mathbb{N}}[M(f_j)(x)]^u\Big)^{v/u}G(x)\,d\mu(x)\Big\}^{1/v}.$$

Notice that, for all $j\in\mathbb{N}$ and $x\in\mathcal{X}$,

$$M(f_j)(x)\lesssim\sup_{r\in(0,\infty)}\frac{1}{\int_{B(x,22r)}G(y)\,d\mu(y)}\int_{B(x,r)}|f_j(y)|G(y)\,d\mu(y)=:\mathcal{M}_G(f_j)(x)$$

from (2.2) and (2.7). Therefore, from [67, Theorem 1.3] with μ replaced by $Gd\mu$, we deduce that

$$\Big\|\Big\{\sum_{j\in\mathbb{N}}[M(f_j)]^u\Big\}^{1/u}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\Big\{\int_{\mathcal{X}}\Big(\sum_{j\in\mathbb{N}}[\mathcal{M}_G(f_j)(x)]^u\Big)^{v/u}G(x)\,d\mu(x)\Big\}^{1/v}$$
$$\lesssim\Big\{\int_{\mathcal{X}}\Big[\sum_{j\in\mathbb{N}}|f_j(x)|^u\Big]^{v/u}G(x)\,d\mu(x)\Big\}^{1/v},$$

which, combined with the Hölder inequality and the fact that $\|g\|_{L^{(p(\cdot)/v)^*}(\mathcal{X})}\le 1$, implies that

$$\Big\|\Big\{\sum_{j\in\mathbb{N}}[M(f_j)]^u\Big\}^{1/u}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\Big\|\Big\{\sum_{j\in\mathbb{N}}|f_j|^u\Big\}^{1/u}\Big\|_{L^{p(\cdot)}(\mathcal{X})}.$$

This finishes the proof of Theorem 2.7. ∎

We transform Theorem 2.7 to the form we need in this article.

Corollary 2.10. *Let $0<\beta<1$, $u\in((n+\beta)/n,\infty)$, $(\mathcal{X},d,\mu)$ be a metric measure space of homogeneous type (here $\mu(\mathcal{X})\in(0,\infty]$) and $r(\cdot)\in C^{\log}_{(n/(n+\beta),\infty)}(\mathcal{X})$. Then there exists a positive constant C such that, for any sequence $\{f_j\}_{j\in\mathbb{N}}$ of μ-measurable functions,*

$$\Big\|\sum_{j\in\mathbb{N}}[M(f_j)]^u\Big\|_{L^{r(\cdot)}(\mathcal{X})}\le C\Big\|\sum_{j\in\mathbb{N}}|f_j|^u\Big\|_{L^{r(\cdot)}(\mathcal{X})}. \tag{2.8}$$

Proof. Notice that (2.8) is equivalent to

$$\Big\|\Big(\sum_{j\in\mathbb{N}}[M(f_j)]^u\Big)^{1/u}\Big\|_{L^{r(\cdot)u}(\mathcal{X})}\lesssim\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^u\Big)^{1/u}\Big\|_{L^{r(\cdot)u}(\mathcal{X})}. \tag{2.9}$$

Since $(r(\cdot)u)_->1$ by assumption, we are in a position to apply Theorem 2.7 with $p(\cdot)$ replaced by $r(\cdot)u$ to (2.9), and hence (2.8) holds true. This finishes the proof of Corollary 2.10. ■

Thanks to Lemma 2.5, we obtain the following conclusion, which plays an important role in this article and is also of independent interest.

PROPOSITION 2.11. *Let $(\mathcal{X},d,\mu)$ be a metric measure space of homogeneous type (here $\mu(\mathcal{X})\in(0,\infty]$), let $r(\cdot)\in\mathcal{P}(\mathcal{X})$ be a log-Hölder continuous variable exponent and let $q\in[1,\infty]\cap(r_+,\infty]$. Suppose that $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$, $\{B_j\}_{j\in\mathbb{N}}$ and $\{a_j\}_{j\in\mathbb{N}}$ are given collections of balls and $L^q(\mathcal{X})$-functions, respectively, such that, for all $j\in\mathbb{N}$, $\operatorname{supp}a_j\subset B_j:=B(x_j,r_j)$ for some $x_j\in\mathcal{X}$ and $r_j\in(0,\infty)$, and*

$$\|a_j\|_{L^q(\mathcal{X})}\le\frac{[\mu(B_j)]^{1/q}}{\|\chi_{B_j}\|_{L^{r(\cdot)}(\mathcal{X})}},$$

and that

$$\widetilde{\mathcal{A}}_{r(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}}):=\Big\|\Big\{\sum_{j\in\mathbb{N}}\Big[\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{r(\cdot)}(\mathcal{X})}}\chi_{B_j}\Big]^{\underline{r}}\Big\}^{1/\underline{r}}\Big\|_{L^{r(\cdot)}(\mathcal{X})}<\infty.$$

Then

$$\Big\|\Big\{\sum_{j\in\mathbb{N}}|\lambda_ja_j|^{\underline{r}}\Big\}^{1/\underline{r}}\Big\|_{L^{r(\cdot)}(\mathcal{X})}\le C\widetilde{\mathcal{A}}_{r(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}}),$$

where C is a positive constant independent of λ_j, B_j and a_j.

Proof. By Lemma 2.9, we find $g\in L^{(r(\cdot)/\underline{r})^*}(\mathcal{X})$ with norm not greater than 1 such that

$$\begin{aligned}\Big\|\Big\{\sum_{j\in\mathbb{N}}|\lambda_ja_j|^{\underline{r}}\Big\}^{1/\underline{r}}\Big\|_{L^{r(\cdot)}(\mathcal{X})}^{\underline{r}}&=\Big\|\sum_{j\in\mathbb{N}}|\lambda_ja_j|^{\underline{r}}\Big\|_{L^{r(\cdot)/\underline{r}}(\mathcal{X})}\\&\lesssim\int_{\mathcal{X}}\sum_{j\in\mathbb{N}}|\lambda_ja_j(x)|^{\underline{r}}\,|g(x)|\,d\mu(x).\end{aligned}$$

From the Hölder inequality, we deduce that

$$\begin{aligned}\int_{\mathcal{X}}\sum_{j\in\mathbb{N}}|\lambda_ja_j(x)|^{\underline{r}}|g(x)|\,d\mu(x)&\le\sum_{j\in\mathbb{N}}\frac{|\lambda_j|^{\underline{r}}[\mu(B_j)]^{\underline{r}/q}}{\|\chi_{B_j}\|^{\underline{r}}_{L^{r(\cdot)}(\mathcal{X})}}\|g\|_{L^{(q/\underline{r})^*}(B_j)}\\&\lesssim\sum_{j\in\mathbb{N}}\frac{|\lambda_j|^{\underline{r}}\mu(B_j)}{\|\chi_{B_j}\|^{\underline{r}}_{L^{r(\cdot)}(\mathcal{X})}}\inf_{z\in B_j}[M(|g|^{(q/\underline{r})^*})]^{1/(q/\underline{r})^*}\\&\lesssim\int_{\mathcal{X}}\sum_{j\in\mathbb{N}}\frac{|\lambda_j|^{\underline{r}}\chi_{B_j}(x)}{\|\chi_{B_j}\|^{\underline{r}}_{L^{r(\cdot)}(\mathcal{X})}}[M(|g|^{(q/\underline{r})^*})(x)]^{1/(q/\underline{r})^*}\,d\mu(x),\end{aligned}$$

which, together with Lemma 2.5, the Hölder inequality in Remark 2.3(iii) and the fact that $q \in (r_+, \infty]$, implies that

$$\begin{aligned}
&\int_{\mathcal{X}} \sum_{j\in\mathbb{N}} |\lambda_j a_j(x)|^{\underline{r}} |g(x)|\, d\mu(x)\\
&\qquad\lesssim \left\| \sum_{j\in\mathbb{N}} \frac{|\lambda_j|^{\underline{r}} \chi_{B_j}}{\|\chi_{B_j}\|_{L^{r(\cdot)}(\mathcal{X})}^{\underline{r}}} \right\|_{L^{r(\cdot)/\underline{r}}(\mathcal{X})} \|[M(|g|^{(q/\underline{r})^*})]^{1/(q/\underline{r})^*}\|_{L^{(r(\cdot)/\underline{r})^*}(\mathcal{X})}\\
&\qquad\lesssim \left\| \left\{ \sum_{j\in\mathbb{N}} \frac{|\lambda_j|^{\underline{r}} \chi_{B_j}}{\|\chi_{B_j}\|_{L^{r(\cdot)}(\mathcal{X})}^{\underline{r}}} \right\}^{1/\underline{r}} \right\|_{L^{r(\cdot)}}^{\underline{r}} \|g\|_{L^{(r(\cdot)/\underline{r})^*}(\mathcal{X})}.
\end{aligned}$$

This finishes the proof of Proposition 2.11. ∎

3. Hardy spaces with variable exponents

Based on the viewpoints of [29, 33], in Subsection 3.1, we introduce the Hardy space $H^{*,p(\cdot)}(\mathcal{X})$ via the grand maximal function. Then a non-tangential maximal function characterization for $H^{*,p(\cdot)}(\mathcal{X})$ is presented in Subsection 3.2 and, in Subsection 3.3, we consider another characterization of $H^{*,p(\cdot)}(\mathcal{X})$ in terms of the dyadic maximal function. As an application of these characterizations, in Subsection 3.4, we investigate relations between the spaces of test functions with different parameters.

3.1. Hardy spaces with variable exponents via the grand maximal function. Let us first recall the notion of test functions which suits RD-spaces and was introduced in [29]. Observe that this kind of test functions is a slight variant of the test functions originally introduced in [33] (see also [34]).

Definition 3.1. Let $z \in \mathcal{X}$, $r, \gamma \in (0, \infty)$ and $\beta \in (0, 1]$. A function φ on $\mathcal{X}$ is called a *test function of type* (z, r, β, γ) if, for all $x \in \mathcal{X}$,

$$|\varphi(x)| \le C \frac{1}{\mu(B(x, r + d(x, z)))} \left[\frac{r}{r + d(z, x)}\right]^\gamma \tag{3.1}$$

and, for all $x, y \in \mathcal{X}$ satisfying $d(x, y) \le [r + d(z, x)]/2$,

$$|\varphi(x) - \varphi(y)| \le C \left[\frac{d(x, y)}{r + d(z, x)}\right]^\beta \left[\frac{r}{r + d(z, x)}\right]^\gamma \frac{1}{\mu(B(x, r + d(x, z)))}, \tag{3.2}$$

where C is a positive constant independent of x, y and z.

Denote by $\mathcal{G}(z, r, \beta, \gamma)$ the set of all test functions of type (z, r, β, γ). For any φ in $\mathcal{G}(z, r, \beta, \gamma)$, define its norm by

$$\|\varphi\|_{\mathcal{G}(z,r,\beta,\gamma)} := \inf\{C : (3.1) \text{ and } (3.2) \text{ hold true}\}.$$

The space $\mathcal{G}(z, r, \beta, \gamma)$ is called the *space of test functions*.

Remark 3.2. In [33, Definition 2.2] or [34, Definition 2.8], another space of test functions was introduced in the same way as in Definition 3.1 but with $\mu(x, r + d(x, z))$ replaced by $V_r(z) + V(z, x)$. Observe that, by (2.2),

$$\mu(x, r + d(x, z)) \sim V_r(z) + V(z, x).$$

It follows that the spaces of test functions in Definition 3.1 and in [33, Definition 2.2] or [34, Definition 2.8] coincide with equivalent norms.

Let x_1 be a fixed point of $\mathcal{X}$. Then define

$$\mathcal{G}(\beta,\gamma) := \mathcal{G}(x_1,1,\beta,\gamma).$$

Let $\epsilon \in (\beta\wedge\gamma,1]$. Observe that $\mathcal{G}(\epsilon,\epsilon)\subset\mathcal{G}(\beta,\gamma)$. Define $\mathcal{G}_0^\epsilon(\beta,\gamma)$ to be the completion of $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$. The topological dual of $\mathcal{G}_0^\epsilon(\beta,\gamma)$ is denoted by $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. We fix the point x_1 throughout the present article. It turns out that x_1 plays the same role as the origin of $\mathbb{R}^n$.

Keeping the above definition of test functions, let us recall the notion of the grand maximal function.

Definition 3.3. Let $(\mathcal{X},d,\mu)$ be an RD-space, $\epsilon\in(0,1]$ and $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with some $\beta,\gamma\in(0,\epsilon)$. For any $x\in\mathcal{X}$, the *grand maximal function* of f is defined by

$$f^*(x) := \sup\{|\langle f,\varphi\rangle| : \varphi\in\mathcal{G}_0^\epsilon(\beta,\gamma),\ \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)}\le 1 \text{ for some } r\in(0,\infty)\}.$$

Remark 3.4. It was established in [29, (3.4)] that, for all $x\in\mathcal{X}$, $f^*(x)\le M(f)(x)$.

In the present setting, we fix $\beta,\gamma\in(n(1/p_- -1),\epsilon)$. Now we introduce the Hardy space $H^{*,p(\cdot)}(\mathcal{X})$ by using the grand maximal function.

Definition 3.5. Let $(\mathcal{X},d,\mu)$ be an RD-space, $p(\cdot)\in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and $\epsilon\in(0,1]$ satisfying $\epsilon>n(1/p_- -1)$, and $\beta,\gamma\in(0,\infty)$ be such that $\beta,\gamma\in(n(1/p_- -1),\epsilon)$. Then the *Hardy space* $H^{*,p(\cdot)}(\mathcal{X})$ *with variable exponent* is defined as the set of all $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ for which the quasi-norm $\|f\|_{H^{*,p(\cdot)}(\mathcal{X})} := \|f^*\|_{L^{p(\cdot)}(\mathcal{X})}$ is finite.

Obviously, when $p(\cdot)$ is a constant $p\in(0,\infty)$, we have $H^{*,p(\cdot)}(\mathcal{X}) = H^{*,p}(\mathcal{X})$, the space studied in [29, 33]. Similar to $H^{*,p}(\mathcal{X})$, we need to show that $H^{*,p(\cdot)}(\mathcal{X})$ is independent of the choice of ϵ and $\beta,\gamma\in(n(1/p_- -1),\epsilon)$. This will be proved in Theorem 4.17. Here let us content ourselves with checking the following fundamental inclusion.

Lemma 3.6. *Let* $p(\cdot)$, ϵ, β *and* γ *be as in Definition* 3.5. *Then, in the sense of continuous embedding,* $H^{*,p(\cdot)}(\mathcal{X})\hookrightarrow(\mathcal{G}_0^\epsilon(\beta,\gamma))'$, *namely,*

$$|\langle f,\varphi\rangle| \le C\|\varphi\|_{\mathcal{G}_0^\epsilon(x_1,1,\beta,\gamma)}\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}$$

for all $f\in H^{*,p(\cdot)}(\mathcal{X})$ *and* $\varphi\in\mathcal{G}_0^\epsilon(\beta,\gamma)$, *where* C *is a positive constant independent of* f *and* φ.

Proof. Let $\varphi\in\mathcal{G}_0^\epsilon(\beta,\gamma)$ be a test function. Then it is easy to see that, for all $x\in B(x_1,1)$,

$$\|\varphi\|_{\mathcal{G}_0^\epsilon(x,1,\beta,\gamma)} \lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(x_1,1,\beta,\gamma)},$$

where $x_1\in\mathcal{X}$ is the fixed point described above. Thus, for all $x\in B(x_1,1)$,

$$|\langle f,\varphi\rangle| \lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(x_1,1,\beta,\gamma)} f^*(x),$$

which, combined with the Hölder inequality, implies that

$$|\langle f,\varphi\rangle| \lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(x_1,1,\beta,\gamma)}\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

This finishes the proof of Lemma 3.6. ∎

3.2. Hardy spaces with variable exponents via the non-tangential maximal function. In this subsection, we first introduce the Hardy space with variable exponent via the non-tangential maximal function with aperture α, denoted by $H_\alpha^{p(\cdot)}(\mathcal{X})$, and then prove the coincidence of $H^{*,p(\cdot)}(\mathcal{X})$ and $H_\alpha^{p(\cdot)}(\mathcal{X})$.

The following notion of approximations of the identity on RD-spaces was first introduced in [34]; see also [33].

Definition 3.7. Let $\epsilon_1 \in (0,1]$, $\epsilon_2, \epsilon_3 \in (0,\infty)$ and $\{S_k\}_{k\in\mathbb{Z}}$ be a sequence of bounded linear integral operators on $L^2(\mathcal{X})$. Assume that $S_k(\cdot,\cdot) : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is the integral kernel of S_k for each $k \in \mathbb{Z}$. Then $\{S_k\}_{k\in\mathbb{Z}}$ is called an *approximation of the identity of order* $(\epsilon_1, \epsilon_2, \epsilon_3)$ (for short, $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI) if there exists a positive constant C such that, for all $k \in \mathbb{Z}$ and all $x, x', y, y' \in \mathcal{X}$,

(i) $$|S_k(x,y)| \le C \frac{2^{-k\epsilon_2}}{[2^{-k}+d(x,y)]^{\epsilon_2}} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)};$$

(ii) when $d(x,x') \le [2^{-k}+d(x,y)]/2$,
$$|S_k(x,y)-S_k(x',y)| \le C \frac{2^{-k\epsilon_2}}{[2^{-k}+d(x,y)]^{\epsilon_2}} \frac{[d(x,x')]^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)};$$

(iii) property (ii) holds true with x and y interchanged;

(iv) when $d(x,x') \le [2^{-k}+d(x,y)]/3$ and $d(y,y') \le [2^{-k}+d(x,y)]/3$,
$$\begin{aligned}&|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\\ &\quad\le C \frac{2^{-k\epsilon_3}}{[2^{-k}+d(x,y)]^{\epsilon_3}} \frac{[d(x,x')]^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}}\\ &\qquad\times \frac{[d(y,y')]^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)};\end{aligned}$$

(v) $\int_{\mathcal{X}} S_k(x,y)\,d\mu(y) = 1 = \int_{\mathcal{X}} S_k(x,y)\,d\mu(x)$.

Before we go further, a helpful remark may be in order.

Remark 3.8. (i) Let ϵ_1, ϵ_2 and $\{S_k\}_{k\in\mathbb{Z}}$ be as in Definition 3.7, and $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$. Then it was pointed out in [29, p. 2258] that, for any fixed $x \in \mathcal{X}$, we have $S_k(x,\cdot) \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ with $\beta, \gamma \in (0,\epsilon)$.

(ii) According to Definition 3.7(i), $S_k(x,\cdot) \in L^1(\mathcal{X})$ and $\|S_k(x,\cdot)\|_{L^1(\mathcal{X})} \lesssim 1$ with the implicit positive constant independent of x. Indeed, by using (i), we have

$$\begin{aligned}&\int_{\mathcal{X}} |S_k(x,y)|\,d\mu(y)\\ &\quad= \int_{B(x,2^{-k})} |S_k(x,y)|\,d\mu(y) + \sum_{l=1}^\infty \int_{B(x,2^{l-k})\setminus B(x,2^{l-k-1})} |S_k(x,y)|\,d\mu(y)\\ &\quad\lesssim \frac{\mu(B(x,2^{-k}))}{V_{2^{-k}}(x)} + \sum_{l=1}^\infty \frac{2^{-\epsilon_2 l}}{V_{2^{l-k-1}}(x)} \mu\big(B(x,2^{l-k}) \setminus B(x,2^{l-k-1})\big) \lesssim 1,\end{aligned}$$

which implies that the above claim holds true.

(iii) It was proved in [34, Theorem 2.6] that there exists an $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI with bounded support on $\mathcal{X}$, which means that there exists a positive constant C such that, for all $k\in\mathbb{Z}$ and $x,y\in\mathcal{X}$ with $d(x,y)>2^{-k}$, we have $S_k(x,y)=0$. Here let us recall the construction from the proof of [34, Theorem 2.6].

Let $h\in C^1(\mathbb{R})$ be such that $\chi_{[-3/2,3/2]}\le h\le \chi_{[-2,2]}$. For all $k\in\mathbb{Z}$, $f\in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x\in\mathcal{X}$, let

$$T_kf(x):=\int_{\mathcal{X}} h(2^kd(x,y))f(y)\,d\mu(y)$$

and, for all $x,y\in\mathcal{X}$, let

$$S_k(x,y):=\frac{1}{T_k1(x)T_k1(y)}\int_{\mathcal{X}} h(2^kd(x,z))h(2^k(d(y,z))\frac{1}{T_k[(T_k1)^{-1}](z)}\,d\mu(z).$$

In addition to properties (i) through (v) of Definition 3.7, S_k also satisfies

$$\frac{1}{CV_{2^{-k}}(x)}\le S_k(x,y)\le \frac{C}{V_{2^{-k}}(x)} \tag{3.3}$$

for all $x,y\in\mathcal{X}$ with $d(x,y)\le 2^{-k}$, where C is a positive constant independent of x, y and k.

Definition 3.9. Let $\epsilon_1\in(0,1]$, $\epsilon_2,\epsilon_3\in(0,\infty)$, $\epsilon\in(0,\epsilon_1\wedge\epsilon_2)$, $\beta,\gamma\in(0,\epsilon)$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI.

(i) For any $k\in\mathbb{Z}$, $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $x\in\mathcal{X}$, define

$$S_k(f)(x):=\langle f,S_k(x,\cdot)\rangle.$$

(ii) Let $\alpha\in(0,\infty)$ and $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Then the *non-tangential maximal function* of f with aperture α is defined by setting, for all $x\in\mathcal{X}$,

$$\mathcal{M}_\alpha(f)(x):=\sup_{k\in\mathbb{Z}}\Big[\sup_{y\in B(x,\alpha2^{-k})}|S_k(f)(y)|\Big].$$

(iii) Let $p(\cdot)\in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ satisfy $p_-\in(n/(n+\epsilon),\infty)$. In particular, let $p(\cdot)\in C^{\log}_{(n/(n+\epsilon),\infty)}(\mathcal{X})$. Then the *Hardy space* $H_\alpha^{p(\cdot)}(\mathcal{X})$ *with variable exponent via the non-tangential maximal function* is defined as the set of all $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ for which the quasi-norm $\|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})}:=\|\mathcal{M}_\alpha(f)\|_{L^{p(\cdot)}(\mathcal{X})}$ is finite.

Remark 3.10. Let $\{S_k\}_{k\in\mathbb{N}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI as above. Then, for any $f\in L^q(\mathcal{X})$ with $q\in(1,\infty)$, we have $\|S_k(f)\|_{L^q(\mathcal{X})}\to0$ as $k\to-\infty$ (here we need $\mu(\mathcal{X})=\infty$) and $\|S_k(f)-f\|_{L^q(\mathcal{X})}\to0$ as $k\to\infty$; see, for example, [29, Lemma 3.1].

Now let us show that the above two notions of Hardy spaces with variable exponents are equivalent.

Theorem 3.11. *Let $\alpha\in(0,\infty)$ and $p(\cdot)\in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$. Then*

$$H^{*,p(\cdot)}(\mathcal{X})=H_\alpha^{p(\cdot)}(\mathcal{X})$$

with equivalent quasi-norms.

Proof. Let $\epsilon\in(0,1]$ and $\beta,\gamma\in(0,\infty)$ satisfy $\epsilon\in(n[1/p_--1],1]$ and

$$\beta,\gamma\in(n[1/p_--1],\epsilon).$$

Observe that, by [29, Remark 2.9(ii)], there exists a positive constant $C_{(\alpha)}$, depending on α, such that, for all $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $x \in \mathcal{X}$,

$$\mathcal{M}_\alpha(f)(x) \le C_{(\alpha)} f^*(x).$$

Then we have

$$H^{*,p(\cdot)}(\mathcal{X}) \subset H_\alpha^{p(\cdot)}(\mathcal{X}) \quad \text{and} \quad \|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})} \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

Conversely, by [29, (3.13)], we see that, for all $\theta \in (n/(n+\beta\wedge\gamma),1]$, $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $x \in \mathcal{X}$,

$$f^*(x) \le C_{(\theta)} \{M([\mathcal{M}_\alpha(f)]^\theta)(x)\}^{1/\theta}, \tag{3.4}$$

where $C_{(\theta)}$ is a positive constant depending on θ, but independent of f and x. From (3.4) and Lemma 2.5, we further deduce that $H_\alpha^{p(\cdot)}(\mathcal{X}) \subset H^{*,p(\cdot)}(\mathcal{X})$ and

$$\|f\|_{H^{*,p(\cdot)}(\mathcal{X})} \lesssim \|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})}.$$

This finishes the proof of Theorem 3.11. ∎

We point out that there is no restriction on α in Theorem 3.11. Moreover, from the proof of Theorem 3.11, we easily deduce the following conclusion.

Corollary 3.12. *Let $\alpha_1, \alpha_2 \in (0,\infty)$ and $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$. Then $H_{\alpha_1}^{p(\cdot)}(\mathcal{X})$ and $H_{\alpha_2}^{p(\cdot)}(\mathcal{X})$ coincide with equivalent quasi-norms.*

3.3. Hardy spaces with variable exponents via the dyadic maximal function. As further applications of the $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI, we consider Hardy spaces with variable exponents in terms of the dyadic maximal function. The definition is based on the following dyadic cubes introduced in [6]. Here we are still fixing ϵ, β, γ as in Definition 3.9.

Lemma 3.13. *Let $\mathcal{X}$ be a metric measure space of homogeneous type. Then there exist a collection $\{Q_\tau^k \subset \mathcal{X} : k \in \mathbb{Z},\ \tau \in I_k\}$ of open subsets, where I_k denotes some index set, and constants $C, D \in (0,\infty)$ such that*

(i) $\mu(\mathcal{X} \setminus \bigcup_{\tau\in I_k} Q_\tau^k) = 0$ *for each fixed k and, if $\tau,\eta \in I_k$ and $\tau \ne \eta$, then $Q_\tau^k \cap Q_\eta^k = \emptyset$;*
(ii) *for any $\ell, k \in \mathbb{Z}$ with $\ell \ge k$, $\tau \in I_k$ and $\eta \in I_\ell$, either $Q_\eta^\ell \subset Q_\tau^k$ or $Q_\eta^\ell \cap Q_\tau^k = \emptyset$;*
(iii) *for all $\ell, k \in \mathbb{Z}$ with $\ell < k$ and $\tau \in I_k$, there uniquely exists $\eta \in I_\ell$ such that $Q_\tau^k \subset Q_\eta^\ell$;*
(iv) *for each $k \in \mathbb{Z}$,*

$$\mathrm{diam}(Q_\tau^k) \le D2^{-k}; \tag{3.5}$$

(v) *each Q_τ^k contains some ball $B(z_\tau^k, C2^{-k})$ with $z_\tau^k \in \mathcal{X}$.*

Indeed, for each $k \in \mathbb{Z}$ and $\tau \in I_k$, we can roughly regard Q_τ^k as a dyadic cube with diameter roughly 2^{-k} centered at z_τ^k as if we were placing ourselves in $\mathbb{R}^n$. In what follows, let j_0 be a positive integer large enough such that

$$2^{-j_0} D < 1/3. \tag{3.6}$$

For all $k \in \mathbb{Z}$ and $\tau \in I_k$, we denote by $Q_\tau^{k,\nu}$, $\nu \in \{1,\ldots,N(k,\tau)\}$, the set of all dyadic cubes $Q_{\tau'}^{k+j_0} \subset Q_\tau^k$. For all $k \in \mathbb{Z}$, define $D_k := S_k - S_{k-1}$.

Now we introduce Hardy spaces with variable exponents via the dyadic maximal function as follows.

DEFINITION 3.14. Let $\beta,\gamma\in(0,1)$, $\epsilon_1\in(0,1]$ and $\epsilon_2,\epsilon_3\in(0,\infty)$ satisfy

$$\epsilon_1\wedge\epsilon_2\in(\beta\wedge\gamma,\infty).$$

Assume that $\epsilon\in(\beta\wedge\gamma,\epsilon_1\wedge\epsilon_2)$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI.

(i) Let $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Then the *dyadic maximal function* of f is defined by setting

$$\mathcal{M}_{\mathrm{d}}(f)(x):=\sup_{k\in\mathbb{Z},\,\tau\in I_k} m_{Q_\tau^k}(|S_k(f)|)\chi_{Q_\tau^k}(x),\quad x\in\mathcal{X},$$

where $\{Q_\tau^k\}_{k\in\mathbb{Z},\,\tau\in I_k}$ is as in Lemma 3.13 and $m_{Q_\tau^k}$ is defined as in (2.4).

(ii) Let $p(\cdot)\in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and $\beta,\gamma\in(n(1/p_--1),\epsilon)$. Then the *Hardy space* $H_{\mathrm{d}}^{p(\cdot)}(\mathcal{X})$ *with variable exponent via the dyadic maximal function* collects all f in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ for which the quasi-norm $\|f\|_{H_{\mathrm{d}}^{p(\cdot)}(\mathcal{X})}:=\|\mathcal{M}_{\mathrm{d}}(f)\|_{L^{p(\cdot)}(\mathcal{X})}$ is finite.

The notion of Hardy spaces with variable exponents via the dyadic maximal function coincides with that via the non-tangential maximal function, as indicated by the following theorem.

THEOREM 3.15. *Let $\alpha\in(0,\infty)$ and $p(\cdot)\in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$. Then $H_{\mathrm{d}}^{p(\cdot)}(\mathcal{X})$ and $H_\alpha^{p(\cdot)}(\mathcal{X})$ coincide with equivalent quasi-norms.*

Proof. Let $f\in H_\alpha^{p(\cdot)}(\mathcal{X})$. Then, by [29, p. 2267], we know that, for all $x\in\mathcal{X}$,

$$\mathcal{M}_{\mathrm{d}}(f)(x)\lesssim\mathcal{M}_{\alpha_0}(f)(x)$$

for some $\alpha_0\in(0,\infty)$. This, combined with Corollary 3.12, implies that $H_\alpha^{p(\cdot)}(\mathcal{X})\subset H_{\mathrm{d}}^{p(\cdot)}(\mathcal{X})$ and

$$\|f\|_{H_{\mathrm{d}}^{p(\cdot)}(\mathcal{X})}\lesssim\|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})}.$$

Conversely, let $\beta,\gamma\in(0,1)$ and $f\in H_{\mathrm{d}}^{p(\cdot)}(\mathcal{X})$. We claim that, for all $\theta\in(\frac{n}{n+(\beta\wedge\gamma)},1]$ and $x\in\mathcal{X}$,

$$f^*(x)\le C_{(\theta)}\{M([\mathcal{M}_{\mathrm{d}}(f)]^\theta)(x)\}^{1/\theta},\tag{3.7}$$

where $C_{(\theta)}$ is a positive constant depending on θ, but independent of f and x. By combining (3.7) and Theorem 3.11, we obtain the desired result, namely,

$$H_{\mathrm{d}}^{p(\cdot)}(\mathcal{X})\subset H_\alpha^{p(\cdot)}(\mathcal{X})\quad\text{and}\quad\|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})}\lesssim\|f\|_{H_{\mathrm{d}}^{p(\cdot)}(\mathcal{X})}.$$

To complete the proof of Theorem 3.15, it thus remains to show (3.7). To this end, suppose that ϵ, ϵ_1' satisfy

$$\frac{n}{n+\beta\wedge\gamma}<\theta<\epsilon\quad\text{and}\quad n(\theta^{-1}-1)<\epsilon_1'<\beta\wedge\gamma.\tag{3.8}$$

Fix $x\in\mathcal{X}$ and a test function $\varphi\in\mathcal{G}_0^\epsilon(\beta,\gamma)$ satisfying $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)}\le 1$ with some $r\in(0,\infty)$. Let $\ell_0:=\lfloor-\log_2 r\rfloor$. Then there exists a positive constant C, independent of x and ℓ_0, such that

$$\|\varphi\|_{\mathcal{G}(x,2^{-\ell_0},\beta,\gamma)}\le C.$$

According to the reproducing formula in [29, Theorem 3.3] (see also [34, Theorem 4.16]), if j_0 as in (3.6) is large enough, then we know that, for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $k \in \mathbb{N}$, $\tau \in I_k$ and $\nu \in \{1, \ldots, N(k,\tau)\}$, and for all $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with $\beta, \gamma \in (0,\epsilon)$,

$$\begin{aligned} f = &\sum_{\tau \in I_{\ell_0}} \sum_{\nu=1}^{N(\ell_0,\tau)} \left[\frac{1}{\mu(Q_\tau^{\ell_0,\nu})} \int_{Q_\tau^{\ell_0,\nu}} \widetilde{D}_{\ell_0}(\cdot, y)\, d\mu(y)\right] \int_{Q_\tau^{\ell_0,\nu}} S_{\ell_0}(f)(w)\, d\mu(w) \\ &+ \sum_{k=\ell_0+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,v}) \widetilde{D}_k(\cdot, y_\tau^{k,v}) D_k(f)(y_\tau^{k,\nu}) \end{aligned}$$

converges in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$, where $\{\widetilde{D}_k(\cdot,\cdot)\}_{k=\ell_0}^\infty$ is a family of functions on $\mathcal{X} \times \mathcal{X}$ satisfying, for all $x, x', y \in \mathcal{X}$,

$$|\widetilde{D}_k(x,y)| \lesssim \frac{1}{V_1(x) + V_1(y) + V(x,y)} \frac{1}{[1+d(x,y)]^{\epsilon'}},$$

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \lesssim \left[\frac{d(x,x')}{1+d(x,y)}\right]^{\epsilon'} \frac{1}{V_1(x) + V_1(y) + V(x,y)} \frac{1}{[1+d(x,y)]^{\epsilon'}}$$

when $2d(x,x') \le 1 + d(x,y)$, and

$$\int_{\mathcal{X}} \widetilde{D}_k(z,y)\, d\mu(z) = \chi_{\{\ell_0\}}(k) = \int_{\mathcal{X}} \widetilde{D}_k(x,z)\, d\mu(z).$$

From this, we further deduce that, for any $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$,

$$\begin{aligned} |\langle f, \varphi\rangle| \le &\left|\sum_{\tau \in I_{\ell_0}} \sum_{\nu=1}^{N(\ell_0,\tau)} \left[\int_{Q_\tau^{\ell_0,\nu}} \widetilde{D}_{\ell_0}^*(\varphi)(y)\, d\mu(y)\right] \frac{1}{\mu(Q_\tau^{\ell_0,\nu})} \int_{Q_\tau^{\ell_0,\nu}} S_{\ell_0}(f)(w)\, d\mu(w)\right| \\ &+ \left|\sum_{k=\ell_0+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k^*(\varphi)(y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu})\right|, \end{aligned}$$

where $\widetilde{D}_k^*$ denotes the integral operator with kernel $\widetilde{D}_k^*(x,y) := \widetilde{D}_k(y,x)$ for all $x, y \in \mathcal{X}$.

By [29, (3.17)], we find that, for all $k \in \mathbb{Z}$ with $k \in [\ell_0, \infty)$,

$$\epsilon_1' \in (n(1/\theta - 1), \beta \wedge \gamma)$$

and for all $y \in \mathcal{X}$,

$$|\widetilde{D}_k^*(\varphi)(y)| \lesssim 2^{-(k-\ell_0)\epsilon_1'} \frac{1}{\mu(B(y, 2^{-\ell_0} + d(x,y)))} \frac{2^{-\ell_0\gamma}}{[2^{-\ell_0} + d(x,y)]^\gamma}.$$

Obviously, by Definition 3.14, we have, for all $y_\tau^{\ell_0,v} \in Q_\tau^{\ell_0,v}$,

$$\left|\frac{1}{\mu(Q_\tau^{\ell_0,\nu})} \int_{Q_\tau^{\ell_0,\nu}} S_{\ell_0}(f)(w)\, d\mu(w)\right| \le \mathcal{M}_{\mathrm{d}}(f)(y_\tau^{\ell_0,\nu})$$

and, when $k > \ell_0$, for any $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$,

$$\begin{aligned} |D_k(f)(y_\tau^{k,\nu})| &\le |S_k(f)(y_\tau^{k,\nu})| + |S_{k-1}(f)(y_\tau^{k,\nu})| \\ &\lesssim \mathcal{M}_{\mathrm{d}}(f)(y_\tau^{k,\nu}). \end{aligned}$$

Altogether, we then see that

$$|\langle f,\varphi\rangle|\lesssim\sum_{k=\ell_0+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\left[\frac{2^{-(k-\ell_0)\epsilon_1'}\mu(Q_\tau^{k,\nu})}{\mu(B(y_\tau^{k,\nu},2^{-\ell_0}+d(x,y_\tau^{k,\nu})))}\right.$$
$$\left.\times\frac{2^{-\ell_0\gamma}}{[2^{-\ell_0}+d(x,y_\tau^{k,\nu})]^\gamma}\mathcal{M}_{\rm d}(f)(y_\tau^{k,\nu})\right].$$

Notice that

$$M\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}[\mathcal{M}_{\rm d}(f)]^\theta\chi_{Q_\tau^{k,\nu}}\Big)(x)$$
$$\gtrsim\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{1}{\mu(B(x,2^{-\ell_0}+d(x,y_\tau^{k,\nu})))}\int_{Q_\tau^{k,\nu}}[\mathcal{M}_{\rm d}(f)(z)]^\theta\,d\mu(z)$$
$$\gtrsim\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\mu(Q_\tau^{k,\nu})}{\mu(B(x,2^{-\ell_0}+d(x,y_\tau^{k,\nu})))}\inf_{z\in Q_\tau^{k,\nu}}[\mathcal{M}_{\rm d}(f)(z)]^\theta.$$

Since, due to (2.2),

$$\mu\big(B(x,2^{-\ell_0}+d(x,y_\tau^{k,\nu}))\big)\lesssim[2^{k-\ell_0}+2^kd(x,y_\tau^{k,\nu})]^n\mu(Q_\tau^{k,\nu}),$$

it follows that

$$M\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}[\mathcal{M}_{\rm d}(f)]^\theta\chi_{Q_\tau^{k,\nu}}\Big)(x)$$
$$\gtrsim\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{[\mu(Q_\tau^{k,\nu})]^\theta}{[2^{k-\ell_0}+2^kd(x,y_\tau^{k,\nu})]^{n(1-\theta)}}\frac{\inf_{z\in Q_\tau^{k,\nu}}[\mathcal{M}_{\rm d}(f)(z)]^\theta}{[\mu(B(x,2^{-\ell_0}+d(x,y_\tau^{k,\nu})))]^\theta}$$
$$\sim 2^{-n(1-\theta)(k-\ell_0)}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\inf_{z\in Q_\tau^{k,\nu}}[\mathcal{M}_{\rm d}(f)(z)]^\theta}{[1+2^{-\ell_0}d(x,y_\tau^{k,\nu})]^{n(1-\theta)}}\frac{[\mu(Q_\tau^{k,\nu})]^\theta}{[\mu(B(x,2^{-\ell_0}+d(x,y_\tau^{k,\nu})))]^\theta}$$
$$\gtrsim 2^{-n(1-\theta)(k-\ell_0)}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\inf_{z\in Q_\tau^{k,\nu}}[\mathcal{M}_{\rm d}(f)(z)]^\theta}{[2^{\ell_0}+d(x,y_\tau^{k,\nu})]^{\theta\gamma}}\frac{2^{-\ell_0\theta\gamma}[\mu(Q_\tau^{k,\nu})]^\theta}{[\mu(B(x,2^{-\ell_0}+d(x,y_\tau^{k,\nu})))]^\theta}$$
$$\gtrsim 2^{-n(1-\theta)(k-\ell_0)}\left\{\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\inf_{z\in Q_\tau^{k,\nu}}[\mathcal{M}_{\rm d}(f)(z)]}{[2^{\ell_0}+d(x,y_\tau^{k,\nu})]^{\gamma}}\frac{2^{-\ell_0\gamma}\mu(Q_\tau^{k,\nu})}{\mu(B(x,2^{-\ell_0}+d(x,y_\tau^{k,\nu})))}\right\}^\theta,$$

where we have used (3.8) in the penultimate inequality and the fact that, for all $\{\xi_j\}_j\subset\mathbb{C}$ and $\delta\in(0,1]$,

$$\Big(\sum_j|\xi_j|\Big)^\delta\le\sum_j|\xi_j|^\delta \tag{3.9}$$

in the last inequality. From the arbitrariness of $y_\tau^{k,\nu}$ in $Q_\tau^{k,\nu}$ and (3.8), we deduce that, for any $\theta \in (n/(n+(\beta \wedge \gamma)), 1]$,

$$
\begin{aligned}
|\langle f, \varphi\rangle| &\lesssim \sum_{k=\ell_0}^{\infty} 2^{-(k-\ell_0)[\epsilon_1'+n(1-1/\theta)]} \Big\{ M\Big(\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} [\mathcal{M}_{\mathrm{d}}(f)]^\theta \chi_{Q_\tau^{k,\nu}} \Big)(x) \Big\}^{1/\theta} \\
&\lesssim M([\mathcal{M}_{\mathrm{d}}(f)]^\theta)(x),
\end{aligned}
$$

which further implies that (3.7) holds true, and hence completes the proof of Theorem 3.15. ■

We point out that the proof of (3.7) is similar to that of (3.4) (see [29, (3.13)]). As an immediate consequence of Theorems 3.11 and 3.15, we obtain the following conclusion.

Corollary 3.16. *Let $\alpha \in (0,\infty)$ and $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$. Then*

$$
H^{*,p(\cdot)}(\mathcal{X}) = H^{p(\cdot)}_{\alpha}(\mathcal{X}) = H^{p(\cdot)}_{\mathrm{d}}(\mathcal{X})
$$

with equivalent quasi-norms.

3.4. Relations between $(\mathcal{G}_0^\epsilon(\beta_1,\gamma_1))'$ and $(\mathcal{G}_0^\epsilon(\beta_2,\gamma_2))'$. In this subsection, we clarify the relations of the test function classes for different parameters by using the characterizations of $H^{*,p(\cdot)}(\mathcal{X})$ obtained in Subsection 3.2.

Proposition 3.17. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and $\epsilon \in (0,1]$ satisfy $p_- \in (n/[n+\epsilon], 1)$. Assume that $f \in (\mathcal{G}_0^\epsilon(\beta_1,\gamma_1))'$ with $\beta_1, \gamma_1 \in (n(1/p_- - 1), \epsilon)$ and $\|f\|_{H^{*,p(\cdot)}(\mathcal{X})} < \infty$. Then $f \in (\mathcal{G}_0^\epsilon(\beta_2,\gamma_2))'$ for every $\beta_2, \gamma_2 \in (n(1/p_- - 1), \epsilon)$.*

Proof. For all $\varphi \in \mathcal{G}_0^\epsilon(\beta_2,\gamma_2)$, let

$$
\begin{aligned}
\langle f, \varphi\rangle &:= \sum_{\tau \in I_{\ell_0}} \sum_{\nu=1}^{N(\ell_0,\tau)} \left[\int_{\mathcal{X} \times Q_\tau^{\ell_0,\nu}} \varphi(x) \widetilde{D}_{\ell_0}(x,y)\, d\mu(x)\, d\mu(y) \right] D_{\tau,1}^{\ell_0,\nu}(f) \\
&\quad + \sum_{k=\ell_0+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left[\int_{\mathcal{X}} \varphi(x) \widetilde{D}_k(x, y_\tau^{k,\nu})\, d\mu(x) \right] D_k(f)(y_\tau^{k,\nu}) \\
&=: \mathrm{I} + \mathrm{II},
\end{aligned}
$$

where ℓ_0, $\widetilde{D}_k$ are as in the proof of (3.7) and

$$
D_{\tau,1}^{\ell_0,\nu}(f) := \frac{1}{\mu(Q_\tau^{\ell_0,\nu})} \int_{Q_\tau^{\ell_0,\nu}} S_{\ell_0}(f)(w)\, d\mu(w).
$$

Next, we show that, for all $\varphi \in \mathcal{G}_0^\epsilon(\beta_2,\gamma_2)$,

$$
|\langle f, \varphi\rangle| \lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta_2,\gamma_2)} \|f\|_{H^{p(\cdot)}_{\alpha}(\mathcal{X})},
$$

where $\alpha \in (0,\infty)$. To this end, let $\gamma_2' \in (0,\gamma_2)$. Then we have, for all $k \in \mathbb{Z}_+$,

$$
\left| \int_{\mathcal{X}} \varphi(x) \widetilde{D}_k(x,y)\, d\mu(x) \right| \lesssim 2^{-k\beta_2} \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta_2,\gamma_2)} \frac{1}{[1+d(x_1,y)]^{\gamma_2}} \, \frac{1}{V_1(x_1)+V(x_1,y)} \tag{3.10}
$$

and, for all $k \in \mathbb{Z} \setminus \mathbb{Z}_+$,

$$
\left| \int_{\mathcal{X}} \varphi(x) \widetilde{D}_k(x,y)\, d\mu(x) \right| \lesssim 2^{k\gamma_2'} \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta_2,\gamma_2)} \frac{2^{-k\gamma_2}}{[1+d(x_1,y)]^{\gamma_2}} \, \frac{1}{V_{2^{-k}}(x_1)+V(x_1,y)};
$$

see [34, (5.24) and (5.25)].

To estimate II, by similarity, we only need to prove that

$$\sum_{k=(\ell_0+1)\vee 0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\left|\int_{\mathcal{X}}\varphi(x)\widetilde{D}_k(x,y_\tau^{k,\nu})\,d\mu(x)\right||D_k(f)(y_\tau^{k,\nu})|$$
$$\lesssim\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta_2,\gamma_2)}\|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})}. \tag{3.11}$$

Since we have (3.10), it suffices to show that

$$\sum_{k=(\ell_0+1)\vee 0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{\mu(Q_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|}{2^{k\beta_2}[V_1(x_1)+V(x_1,y_\tau^{k,\nu})][1+d(x_1,y_\tau^{k,\nu})]^{\gamma_2}}\lesssim\|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})}. \tag{3.12}$$

To this end, let us first consider the case that $p_-\in(1,\infty)$. By the Hölder inequality and [33, Lemma 2.1(ii)], we easily find that, for each $k\in(0,\infty)$,

$$\begin{aligned}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}&\frac{\mu(Q_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|}{2^{k\beta_2}[V_1(x_1)+V(x_1,y_\tau^{k,\nu})][1+d(x_1,y_\tau^{k,\nu})]^{\gamma_2}}\\
&\lesssim\int_{\mathcal{X}}\frac{1}{2^{k\beta_2}[V_1(x_1)+V(x_1,x)][1+d(x_1,x)]^{\gamma_2}}\\
&\qquad\times\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\left[\inf_{z\in Q_\tau^{k,\nu}}\mathcal{M}_\alpha(f)(z)\right]\chi_{Q_\tau^{k,\nu}}(x)d\mu(x)\\
&\lesssim\frac{1}{2^{k\beta_2}}\left\|\frac{1}{[V_1(x_1)+V(x_1,\cdot)][1+d(x_1,\cdot)]^{\gamma_2}}\right\|_{L^{p^*(\cdot)}(\mathcal{X})}\|f^*\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim\frac{1}{2^{k\beta_2}}\|\mathcal{M}_\alpha(f)\|_{L^{p(\cdot)}(\mathcal{X})}.\end{aligned}$$

Therefore, (3.12) and hence (3.11) hold true when $p_-\in(1,\infty)$.

Suppose instead that $p_-\le 1$. Let u be a positive constant slightly less than p_-. Then we shall prove that, for each k,

$$\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\left\{\frac{\mu(Q_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|}{[V_1(x_1)+V(x_1,y_\tau^{k,\nu})][1+d(x_1,y_\tau^{k,\nu})]^{\gamma_2}}\right\}^u\lesssim 2^{kn(1-u)}\|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})}^u, \tag{3.13}$$

which is stronger than (3.12) due to (3.9). By the definition of the grand maximal function, the arbitrariness of $y_\tau^{k,v}\in Q_\tau^{k,v}$ and Remark 3.8(i), we have

$$\begin{aligned}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}&\left\{\frac{\mu(Q_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|}{[V_1(x_1)+V(x_1,y_\tau^{k,\nu})][1+d(x_1,y_\tau^{k,\nu})]^{\gamma_2}}\right\}^u\\
&\lesssim\int_{\mathcal{X}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^{u-1}\chi_{Q_\tau^{k,\nu}}(x)\left\{\frac{|D_k(f)(x)|}{[V_1(x_1)+V(x_1,x)][1+d(x_1,x)]^{\gamma_2}}\right\}^u d\mu(x)\\
&\lesssim\int_{\mathcal{X}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{[\mu(Q_\tau^{k,\nu})]^{u-1}\chi_{Q_\tau^{k,\nu}}(x)}{\{[V_1(x_1)+V(x_1,x)][1+d(x_1,x)]^{\gamma_2}\}^u}[\mathcal{M}_\alpha(f)(x)]^u\,d\mu(x).\end{aligned}$$

Now, for any $x \in \mathcal{X}$, let $r(x) := p(x)/[p(x) - u]$ and

$$W(x) := \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \frac{[\mu(Q_\tau^{k,\nu})]^{u-1} \chi_{Q_\tau^{k,\nu}}(x)}{\{[V_1(x_1) + V(x_1, x)][1 + d(x_1, x)]^{\gamma_2}\}^u}.$$

Then the proof of (3.13) will be complete once we show that $W \in L^{r(\cdot)}(\mathcal{X})$. Indeed, since $\operatorname{diam}(Q_\tau^{k,\nu}) \sim 2^{-k}$, we have, for all $x \in Q_\tau^{k,v}$,

$$B(x_1, \max\{1, d(x_1, x)\}) \subset \widetilde{C} 2^k \max\{1, d(x_1, x)\} Q_\tau^{k,\nu}$$

for some positive constant $\widetilde{C}$, which, together with (2.2), implies that

$$\mu(B(x_1, \max\{1, d(x_1, x)\})) \lesssim [2^k \max\{1, d(x_1, x)\}]^n \mu(Q_\tau^{k,\nu}).$$

Inserting this estimate into the definition of W, we obtain

$$\begin{aligned} W(x) &\lesssim 2^{kn(1-u)} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \frac{\chi_{Q_\tau^{k,\nu}}(x)}{[V_1(x_1) + V(x_1, x)][1 + d(x_1, x)]^{\gamma_2 u - n(1-u)}} \\ &\lesssim 2^{kn(1-u)} \frac{1}{[V_1(x_1) + V(x_1, x)][1 + d(x_1, x)]^{\gamma_2 u - n(1-u)}}. \end{aligned}$$

From the assumption that $\gamma_2 > n(1/p_- - 1)$ and u is slightly less than p_-, we further deduce that $\gamma_2 u - n(1-u) > 0$. Due to this observation and [33, Lemma 2.1(ii)], we see that $W \in L^{r(\cdot)}(\mathcal{X})$.

It remains to handle term I. Indeed, we can prove

$$\mathrm{I} \lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta_2, \gamma_2)} \|f\|_{H_\alpha^{p(\cdot)}(\mathcal{X})}.$$

The proof is analogous, but simpler than that of (3.11), because there is no need to sum over $k \geq \ell_0 + 1$; the details are omitted. This finishes the proof of Proposition 3.17. ■

4. Atomic characterizations

In this section, we aim to establish an atomic characterization of the spaces $H^{*,p(\cdot)}(\mathcal{X})$ (see Theorem 4.3). To this end, in Subsection 4.1, we introduce the atomic Hardy spaces with variable exponents on RD-spaces. In Subsection 4.2, we present some auxiliary estimates which are needed in the proof of Theorem 4.3, and in Subsection 4.3, we conclude the proof of Theorem 4.3 by some arguments similar to those used in the proof of [21, Theorem 3.28]. As consequences of the atomic characterization, in Subsection 4.4, we prove that the space $H^{*,p(\cdot)}(\mathcal{X})$ is independent of the choice of the parameters β, γ, ϵ appearing in the space of test functions, $\mathcal{G}_0^\epsilon(\beta, \gamma)$, and then show that, when $p_- \in (1, \infty)$, $H^{*,p(\cdot)}(\mathcal{X})$ and $L^{p(\cdot)}(\mathcal{X})$ coincide with equivalent norms. Finally, in Subsection 4.5, we establish a finite atomic characterization of $H^{*,p(\cdot)}(\mathcal{X})$ (see Theorem 4.24). At the end of this section, we give another proof of Theorem 4.3(ii) by borrowing some ideas from [29, Lemma 4.15 and Theorem 4.16].

4.1. Atomic Hardy spaces with variable exponents. Again the parameters ϵ, β, γ are fixed till we prove Theorem 4.17. Let us start with the notion of atoms.

Definition 4.1. Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and let $q \in [1,\infty] \cap (p_+,\infty]$. A function $a \in L^q(\mathcal{X})$ is called a $(p(\cdot),q)$-*atom* if

(A1) $\operatorname{supp} a \subset B(x_0,r)$ for some $x_0 \in \mathcal{X}$ and $r \in (0,\infty)$;

(A2)
$$\|a\|_{L^q(\mathcal{X})} \le \frac{[\mu(B(x_0,r))]^{1/q}}{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathcal{X})}};$$

(A3)
$$\int_{\mathcal{X}} a(x)\,d\mu(x) = 0.$$

When it is necessary to specify the ball $B(x_0,r)$, then a is called a $(p(\cdot),q)$-*atom supported on* $B(x_0,r)$.

Via atoms, we introduce the atomic Hardy spaces with variable exponents.

Definition 4.2. Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and $q \in [1,\infty] \cap (p_+,\infty]$. Let $\epsilon \in (0,1]$ and $\beta,\gamma \in (0,\epsilon)$. Then the *atomic Hardy space with variable exponent*, $H^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})$, is defined to be the set of all distributions $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$ such that there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $(p(\cdot),q)$-atoms $\{a_j\}_{j\in\mathbb{N}}$ such that $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$, where, for any $j \in \mathbb{N}$, a_j is supported on $B_j := B(x_j,r_j)$ for some $x_j \in \mathcal{X}$ and $r_j \in (0,\infty)$, and

$$\begin{aligned}\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j\in\mathbb{N}}) &:= \widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}})\\ &:= \left\|\left(\sum_{j\in\mathbb{N}}\left[\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}}\right]^{\underline{p}}\right)^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})} < \infty.\end{aligned}$$

Moreover, let

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})} := \inf\bigl\{\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}})\bigr\},$$

where the infimum is taken over all the decompositions of f as above.

The Hardy spaces $H^{*,p(\cdot)}(\mathcal{X})$ with variable exponents have the following atomic characterizations.

Theorem 4.3. *Let $\epsilon \in (0,1]$ and $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ satisfy $\epsilon > n(1/p_- - 1)$. Assume that the parameters q, β, γ satisfy $q \in [1,\infty] \cap (p_+,\infty]$ and $\beta,\gamma \in (n(1/p_- - 1),\epsilon)$. Then*

(i) $H^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X}) \hookrightarrow H^{*,p(\cdot)}(\mathcal{X})$. *More precisely, suppose that $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $(p(\cdot),q)$-atoms $\{a_j\}_{j\in\mathbb{N}}$ satisfy*
$$\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j\in\mathbb{N}}) < \infty.$$
Then $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$ and f belongs to $H^{*,p(\cdot)}(\mathcal{X})$. Furthermore, there exists a positive constant C, independent of f, such that
$$\|f\|_{H^{*,p(\cdot)}(\mathcal{X})} \le C\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j\in\mathbb{N}}).$$

(ii) $H^{*,p(\cdot)}(\mathcal{X}) \hookrightarrow H^{p(\cdot),\infty}_{\mathrm{at}}(\mathcal{X})$. *More precisely, if $f \in H^{*,p(\cdot)}(\mathcal{X})$, then there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $(p(\cdot),\infty)$-atoms $\{a_j\}_{j\in\mathbb{N}}$ such that*
$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in } (\mathcal{G}^\epsilon_0(\beta,\gamma))' \tag{4.1}$$
and
$$\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j\in\mathbb{N}}) \le \widetilde{C}\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}$$
with $\widetilde{C}$ a positive constant independent of f.

4.2. Auxiliary estimates for the proof of Theorem 4.3. To prove Theorem 4.3, we need some auxiliary estimates. We begin with the following estimate.

LEMMA 4.4. *Let ϵ, $p(\cdot)$, q, β and γ be as in the assumptions of Theorem 4.3 and a be a $(p(\cdot), q)$-atom supported on $B(x_0, r)$ for some $x_0 \in \mathcal{X}$ and $r \in (0, \infty)$. Then the grand maximal function of a satisfies, for all $x \in \mathcal{X}$,*

$$a^*(x) \le C\chi_{B(x_0,3r)}(x)M(a)(x) + \frac{C}{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathcal{X})}}[M(\chi_{B(x_0,r)})(x)]^{\beta/n+1},$$

where C is a positive constant independent of x and a.

Proof. Let ϵ, β and γ be as in the assumptions of Theorem 4.3. We distinguish two cases: $x \in B(x_0, 3r)$ and $x \in \mathcal{X} \setminus B(x_0, 3r)$.

Suppose first that $x \in B(x_0, 3r)$. Then

$$a^*(x) = \sup\{|\langle a, \varphi\rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma),\ \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1 \text{ for some } r \in (0, \infty)\}$$

by Definition 3.3. Let $\varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$ satisfy $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1$ for some $r \in (0, \infty)$. Then, by (2.2) and (3.1), we obtain

$$\begin{aligned}
|\langle a, \varphi\rangle| &\lesssim \int_{B(x_0,r)} \frac{1}{\mu(B(y, r + d(y, x)))}|a(y)|\,d\mu(y) \\
&\lesssim \frac{1}{\mu(B(x_0, 3r))}\int_{B(x_0,r)} |a(y)|\,d\mu(y) \\
&\sim \frac{1}{\mu(B(x_0, 3r))}\int_{B(x_0,3r)} |a(y)|\,d\mu(y) \lesssim M(a)(x).
\end{aligned}$$

So, the estimate for $x \in B(x_0, 3r)$ is complete.

Now suppose instead that $x \in \mathcal{X} \setminus B(x_0, 3r)$. Observe that, when $y \in B(x_0, r)$, we have $d(y, x_0) \le [r + d(x, y)]/2$, and it follows from (3.2) that

$$|\varphi(y) - \varphi(x_0)| \lesssim \left[\frac{d(y, x_0)}{r + d(x, y)}\right]^\beta \frac{1}{\mu(B(y, r + d(y, x)))}\left[\frac{r}{r + d(x, y)}\right]^\gamma.$$

Then, by the vanishing moment condition on a and (2.2), we conclude that

$$\begin{aligned}
|\langle a, \varphi\rangle| &= \left|\int_{B(x_0,r)} a(y)[\varphi(y) - \varphi(x_0)]\,d\mu(y)\right| \\
&\lesssim \int_{B(x_0,r)} \left[\frac{d(y, x_0)}{r + d(x, y)}\right]^\beta \frac{1}{\mu(B(y, r + d(y, x)))}\left[\frac{r}{r + d(x, y)}\right]^\gamma |a(y)|\,d\mu(y) \\
&\lesssim \int_{B(x_0,r)} \left[\frac{r}{r + d(x, x_0)}\right]^\beta \frac{1}{\mu(B(x_0, r + d(x_0, x)))}|a(y)|\,d\mu(y) \\
&\lesssim \left[\frac{r}{r + d(x, x_0)}\right]^\beta \frac{\mu(B(x_0, r))}{\mu(B(x_0, r + d(x_0, x)))}\,\frac{1}{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathcal{X})}} \\
&\lesssim \frac{1}{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathcal{X})}}[M(\chi_{B(x_0,r)})(x)]^{\beta/n+1},
\end{aligned}$$

which implies the desired estimate in the case when $x \in \mathcal{X} \setminus B(x_0, 3r)$. This finishes the proof of Lemma 4.4. ∎

The following two lemmas are just [29, Lemma 4.6] and [29, Lemma 4.8], respectively.

LEMMA 4.5. *Let Ω be an open proper subset of $\mathcal{X}$ and, for all $x \in \mathcal{X}$, let*

$$d(x, \Omega) := \inf\{d(x, y) : y \notin \Omega\}.$$

For any $A \in [1, \infty)$ and $x \in \mathcal{X}$, let

$$r(x, \Omega) := d(x, \Omega)/(2A).$$

Then there exist a positive number L, independent of Ω, and a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{X}$ such that

(i) *$\{B(x_k, r_k/4)\}_{k\in\mathbb{N}}$ are pairwise disjoint, where $r_k := r(x_k, \Omega)$;*
(ii) *$\bigcup_{k\in\mathbb{N}} B(x_k, r_k) = \Omega$;*
(iii) *for any given $k \in \mathbb{N}$, $B(x_k, Ar_k) \subset \Omega$;*
(iv) *$Ar_k < d(x, \Omega) < 3Ar_k$ whenever $k \in \mathbb{N}$ and $x \in B(x_k, Ar_k)$;*
(v) *for any given $k \in \mathbb{N}$, there exists a $y_k \notin \Omega$ such that $d(x_k, y_k) < 3Ar_k$;*
(vi) *for any given $k \in \mathbb{N}$, the number of balls $B(x_i, Ar_i)$ which have non-empty intersections with the ball $B(x_k, r_k)$ is at most L_0.*

LEMMA 4.6. *Let Ω be an open subset of $\mathcal{X}$ with finite measure. Suppose that the sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{r_k\}_{k\in\mathbb{N}}$ are as in Lemma 4.5 with $A = 15$. Then there exist non-negative functions $\{\phi_k\}_{k\in\mathbb{N}}$ such that*

(i) *for any given $k \in \mathbb{N}$, we have $0 \le \phi_k \le 1$, $\operatorname{supp}\phi_k \subset B(x_k, 2r_k)$ and $\sum_{k\in\mathbb{N}} \phi_k = \chi_\Omega$;*
(ii) *for any given $k \in \mathbb{N}$ and $x \in B(x_k, r_k)$, we have $\phi_k(x) \ge 1/L_0$, where L_0 is as in Lemma 4.5;*
(iii) *there exists a positive constant $\widetilde{C}$ independent of Ω such that, for all $k \in \mathbb{N}$ and $\epsilon \in (0, 1]$,*

$$\|\phi_k\|_{\mathcal{G}(x_k, r_k, \epsilon, \epsilon)} \le \widetilde{C} V_{r_k}(x_k).$$

Let $\epsilon \in (0, 1]$, $p(\cdot) \in C^{\log}_{(n/(n+\epsilon),\infty)}(\mathcal{X})$ and $\beta, \gamma \in (0, \infty)$ satisfy $\epsilon > n(1/p_- - 1)$ and $\beta, \gamma \in (n(1/p_- - 1), \epsilon)$. For $f \in H^{*,p}(\mathcal{X})$ and $t \in (0, \infty)$, let

$$\Omega_t := \{x \in \mathcal{X} : f^*(x) > t\}.$$

Then $\mu(\Omega_t) < \infty$ and Ω_t is open (see [29, Remark 2.9(iii)]). Denote by $\{\phi_k^t\}_{k\in\mathbb{N}}$ the partition of unity associated to Ω_t as in Lemma 4.6. Let $\{\Phi_k^t\}_{k\in\mathbb{N}}$ be the corresponding linear operators defined by setting, for all $t \in (0, \infty)$, $k \in \mathbb{N}$, $\varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$ and $x \in \mathcal{X}$,

$$\Phi_k^t(\varphi)(x) := \phi_k^t(x)\left[\int_{\mathcal{X}} \phi_k^t(z)\, d\mu(z)\right]^{-1} \int_{\mathcal{X}} [\varphi(x) - \varphi(z)]\phi_k^t(z)\, d\mu(z).$$

Then Φ_k^t is bounded on $\mathcal{G}_0^\epsilon(\beta, \gamma)$ with the operator norm depending on k (see [29, Lemma 4.9]). For any $\varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$, define the distribution b_k^t by setting

$$\langle b_k^t, \varphi\rangle := \langle f, \Phi_k^t(\varphi)\rangle.$$

The following Calderón–Zygmund type decomposition is just [29, Proposition 4.11].

Proposition 4.7. *With the notation as above, there exists a positive constant C such that, for all $k \in \mathbb{N}$, $t \in (0,\infty)$ and $x \in \mathcal{X}$,*

$$(b_k^t)^*(x) \le C \frac{tV_{r_k}(x_k)}{\mu(B(x_k, r_k + d(x,x_k)))}\left[\frac{r_k}{r_k + d(x_k,x)}\right]^\beta \chi_{[B(x_k,10r_k)]^\complement}(x) \\ + Cf^*(x)\chi_{B(x_k,10r_k)}(x)$$

and the series $\sum_{k\in\mathbb{N}} b_k^t$ converges in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ to a distribution b^t satisfying, for all $t \in (0,\infty)$ and $x \in \mathcal{X}$,

$$(b^t)^*(x) \le Ct\sum_{k\in\mathbb{N}} \frac{V_{r_k}(x_k)}{\mu(B(x_k, r_k + d(x,x_k)))}\left[\frac{r_k}{r_k + d(x_k,x)}\right]^\beta + Cf^*(x)\chi_{\Omega_t}(x); \tag{4.2}$$

moreover, the distribution $g^t := f - b^t$ satisfies $g^t \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and, for all $t \in (0,\infty)$ and $x \in \mathcal{X}$,

$$(g^t)^*(x) \le Ct\sum_{k\in\mathbb{N}} \frac{V_{r_k}(x_k)}{\mu(B(x_k, r_k + d(x,x_k)))}\left[\frac{r_k}{r_k + d(x_k,x)}\right]^\beta + Cf^*(x)\chi_{(\Omega_t)^\complement}(x). \tag{4.3}$$

Lemma 4.8. *Let $q \in (p_+,\infty) \cap [1,\infty)$. With the notation as in Proposition 4.7,*

$$g^t \in H^{*,p(\cdot)}(\mathcal{X}) \cap L^q(\mathcal{X})$$

and g^t tends to f in $H^{*,p(\cdot)}(\mathcal{X})$ as $t \to \infty$. In particular, $H^{*,p(\cdot)}(\mathcal{X}) \cap L^q(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$.

Proof. Thanks to (4.3) and (2.2), we have, for all $t \in (0,\infty)$ and $x \in \mathcal{X}$,

$$(g^t)^*(x) \lesssim t\sum_{k\in\mathbb{N}}[M(\chi_{B(x_k,r_k)})(x)]^{\beta/n+1} + f^*(x)\chi_{(\Omega_t)^\complement}(x).$$

Then, by Theorem 2.7 and Lemma 4.5, we obtain

$$\begin{aligned}\|(g^t)^*\|_{L^q(\mathcal{X})} &\lesssim t\left\|\sum_{k\in\mathbb{N}}[M(\chi_{B(x_k,r_k)})]^{\beta/n+1}\right\|_{L^q(\mathcal{X})} + \|f^*\chi_{(\Omega_t)^\complement}\|_{L^q(\mathcal{X})} \\ &\lesssim t\left\|\sum_{k\in\mathbb{N}}\chi_{B(x_k,r_k)}\right\|_{L^q(\mathcal{X})} + \|f^*\chi_{(\Omega_t)^\complement}\|_{L^q(\mathcal{X})} \\ &\lesssim \|\min\{t,f^*\}\|_{L^q(\mathcal{X})}.\end{aligned}$$

Since

$$[\min\{t,f^*(x)\}]^q \le t^{q-p(x)}[f^*(x)]^{p(x)} \le (t^{q-p_-} + t^{q-p_+})[f^*(x)]^{p(x)}$$

for all $t \in (0,\infty)$ and $x \in \mathcal{X}$, it follows that $(g^t)^* \in L^q(\mathcal{X})$. Together with the fact that $H^{*,q}(\mathcal{X}) = L^q(\mathcal{X})$ (see [29, Corollary 3.11]), this implies $g^t \in L^q(\mathcal{X})$. Likewise, we can prove $g^t \in H^{*,p(\cdot)}(\mathcal{X})$. To see that g^t tends to f in $H^{*,p(\cdot)}(\mathcal{X})$, from Proposition 4.7 and Theorem 2.7, we deduce that

$$\begin{aligned}\|f-g^t\|_{H^{*,p(\cdot)}(\mathcal{X})} &= \|b^t\|_{H^{*,p(\cdot)}(\mathcal{X})} = \|(b^t)^*\|_{L^{p(\cdot)}(\mathcal{X})}\\ &\lesssim \Big\|t\sum_{k\in\mathbb{N}}[M(\chi_{B(x_k,r_k)})]^{\beta/n+1} + f^*\chi_{\Omega_t}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\ &\lesssim \Big\|t\sum_{k\in\mathbb{N}}\chi_{B(x_k,r_k)}\Big\|_{L^{p(\cdot)}(\mathcal{X})} + \|f^*\chi_{\Omega_t}\|_{L^{p(\cdot)}(\mathcal{X})}\\ &\lesssim \|f^*\chi_{\Omega_t}\|_{L^{p(\cdot)}(\mathcal{X})}.\end{aligned}$$

By the dominated convergence theorem, we conclude that

$$\lim_{t\to\infty} g^t = f \quad \text{ in } H^{*,p(\cdot)}(\mathcal{X}).$$

This finishes the proof of Lemma 4.8. ∎

By an argument similar to that used in the proof of [48, Lemma (3.36)], we deduce the following conclusion, which is a variant of [29, Proposition 4.13]; the details are omitted.

PROPOSITION 4.9. *Let $\epsilon\in(0,1)$, $p_-\in(n/(n+\epsilon),\infty)$, $\beta,\gamma\in(n(1/p_- -1),\epsilon)$, $q\in(1,\infty)$ and $f\in L^q(\mathcal{X})\cap H^{*,p(\cdot)}(\mathcal{X})$. Assume that there exists a positive constant $\widetilde{C}$ such that, for all $x\in\mathcal{X}$,*

$$|f(x)|\le \widetilde{C}f^*(x).$$

With the same notation as above, there exists a positive constant C, independent of f, k and t, such that

(i) *if*

$$\eta_k^t := \left[\int_{\mathcal{X}}\phi_k^t(\xi)\,d\mu(\xi)\right]^{-1}\int_{\mathcal{X}} f(\xi)\phi_k^t(\xi)\,d\mu(\xi)\in\mathbb{C},$$

then $|\eta_k^t|\le Ct$ for all k and t;

(ii) *if $b_k^t := (f-\eta_k^t)\phi_k^t$, then $\operatorname{supp} b_k^t\subset B(x_k,2r_k)$ and the distribution on $\mathcal{G}_0^\epsilon(\beta,\gamma)$ induced by b_k^t coincides with b_k^t in Proposition 4.7;*

(iii) *the series $\sum_k b_k^t$ converges in $L^q(\mathcal{X})$; it induces a distribution on $\mathcal{G}_0^\epsilon(\beta,\gamma)$ which coincides with b^t in Proposition 4.7 and is still denoted by b^t; moreover, $\operatorname{supp} b^t\subset\Omega_t$;*

(iv) *if $g^t := f-b^t$, then*

$$g^t = f\chi_{(\Omega_t)^\complement}+\sum_k \eta_k^t\phi_k^t$$

and, for all $x\in\mathcal{X}$,

$$|g^t(x)|\le \tilde{C}t; \tag{4.4}$$

moreover, g^t induces a distribution on $\mathcal{G}_0^\epsilon(\beta,\gamma)$ which agrees with g^t appearing in Proposition 4.7.

Going through an argument similar to that used in the proof of Lemma 4.8, we have the following density result.

COROLLARY 4.10. *Suppose that $\epsilon\in(0,1]$, $p(\cdot)\in C^{\log}_{((n/(n+\epsilon),\infty)}(\mathcal{X})$ and*

$$\beta,\gamma\in(n(1/p_- -1),\epsilon).$$

Then $L^\infty(\mathcal{X})\cap H^{*,p(\cdot)}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$.

Proof. Since $H^{*,p(\cdot)}(\mathcal{X}) \cap L^{1+p_+}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$ according to Lemma 4.8, we have only to show that any element $f \in H^{*,p(\cdot)}(\mathcal{X}) \cap L^{1+p_+}(\mathcal{X})$ lies in the closure of $H^{*,p(\cdot)}(\mathcal{X}) \cap L^{\infty}(\mathcal{X})$. Since $f \in L^{1+p_+}(\mathcal{X})$, we are in a position to apply Proposition 4.9 in order to obtain a function g^t satisfying (i) through (iv) of Proposition 4.9. As in the proof of Lemma 4.8, we can show that $g^t \to f$ in $H^{*,p(\cdot)}(\mathcal{X})$ as $t \to \infty$. Since $g^t \in L^{\infty}(\mathcal{X})$ according to Proposition 4.9(iv), it follows that f lies in the closure of $H^{*,p(\cdot)}(\mathcal{X}) \cap L^{\infty}(\mathcal{X})$, which completes the proof of Corollary 4.10. ■

4.3. Proof of Theorem 4.3. We now turn to the proof of Theorem 4.3. Assume that $f \in L^q(\mathcal{X}) \cap H^{*,p(\cdot)}(\mathcal{X})$. For each $k \in \mathbb{Z}$, let

$$\Omega^k := \{x \in \mathcal{X} : f^*(x) > 2^k\}.$$

Then, by Lemmas 4.5 and 4.6, we immediately obtain the following Lemmas 4.11, 4.12 and 4.14.

LEMMA 4.11. *Let $k \in \mathbb{Z}$ and Ω^k be as above. Then there exist a positive number L and sequences $\{x_j^k\}_{j\in\mathbb{N}} \subset \mathcal{X}$ and $\{r_j^k\}_{j\in\mathbb{N}} \subset (0,\infty)$ such that*

(i)
$$\Omega^k = \bigcup_{j\in\mathbb{N}} B(x_j^k, r_j^k) =: \bigcup_{j\in\mathbb{N}} B_j^k$$

and $\{B(x_j^k, r_j^k/4)\}_{j\in\mathbb{N}}$ are mutually disjoint balls;

(ii) *for any $j \in \mathbb{N}$, $B(x_j^k, 15r_j^k) \cap (\Omega^k)^{\complement} = \emptyset$ and $B(x_j^k, 45r_j^k) \cap (\Omega^k)^{\complement} \neq \emptyset$;*

(iii) *for any $j \in \mathbb{N}$, the number of balls $B(x_i^k, 15r_i^k)$ satisfying*
$$B(x_i^k, 15r_i^k) \cap B(x_j^k, 15r_j^k) \neq \emptyset$$
is at most L.

LEMMA 4.12. *Let $k \in \mathbb{Z}$. Then there exist non-negative functions $\{\phi_j^k\}_{j\in\mathbb{N}}$ satisfying, for any $j \in \mathbb{N}$,*

(i) $0 \le \phi_j^k \le 1$, $\operatorname{supp}\phi_i^k \subset B(x_j^k, 2r_j^k)$ *and* $\sum_{j\in\mathbb{N}} \phi_j^k = \chi_{\Omega^k}$*;*

(ii) *for any $x \in B_j^k := B(x_j^k, r_j^k)$, $\phi_j^k(x) \ge 1/L$;*

(iii) *for any $\epsilon \in (0,1)$, there exists a positive constant C, independent of j, k, such that, for all $j \in \mathbb{N}$,*
$$\|\phi_j^k\|_{\mathcal{G}(x_j^k, r_j^k, \epsilon, \epsilon)} \le C\mu(B(x_j^k, r_j^k)).$$

Moreover, we have the following conclusion.

REMARK 4.13. (i) If $(2B_j^{k+1}) \cap (2B_i^k) \neq \emptyset$, then $r_j^{k+1} < 4r_i^k$ and $2B_j^{k+1} \subset B(x_i^k, 15r_i^k)$.

Indeed, obviously, we have $d(x_j^{k+1}, x_i^k) < 2(r_j^{k+1} + r_i^k)$. By Lemma 4.11 and the fact that $\Omega^{k+1} \subset \Omega^k$, we see that

$$d(x_j^{k+1}, (\Omega^k)^{\complement}) \ge d(x_j^{k+1}, (\Omega^{k+1})^{\complement}) > 15r_j^{k+1}.$$

Thus,

$$15r_j^{k+1} < d(x_j^{k+1}, x_i^k) + d(x_i^k, (\Omega^k)^{\complement}) < 2(r_j^{k+1} + r_i^k) + 45r_i^k,$$

which implies that $r_j^{k+1} < \frac{47}{13} r_i^k < 4r_i^k$. On the other hand, for every $y \in 2B(x_j^{k+1}, r_j^{k+1})$, since

$$d(y, x_i^k) < d(x_j^{k+1}, x_i^k) + 2r_j^{k+1} < 15r_i^k,$$

it follows that $(2B_j^{k+1}) \subset B(x_i^k, 15r_i^k)$.

(ii) From (2.2) and Lemmas 4.12 and 4.11(ii), we deduce that there exists a positive constant $\widetilde{C}$ such that, for all $w \in B(x_j^k, 15r_j^k) \cap (\Omega^k)^\complement$,

$$\|\phi_j^k\|_{\mathcal{G}(w, r_j^k, \epsilon, \epsilon)} \le \widetilde{C} \mu(B(x_j^k, r_j^k)).$$

For any given $k \in \mathbb{Z}$, as in Proposition 4.9, we let, for each $j, i \in \mathbb{N}$,

$$\eta_j^k := \frac{1}{\|\phi_j^k\|_{L^1(\mathcal{X})}} \int_{\mathcal{X}} f(\xi) \phi_j^k(\xi)\, d\mu(\xi), \qquad b_j^k := (f - \eta_j^k)\phi_j^k$$

and

$$\ell_{i,j}^{k+1} := \frac{1}{\|\phi_j^{k+1}\|_{L^1(\mathcal{X})}} \int_{\mathcal{X}} [f(\xi) - \eta_j^{k+1}] \phi_i^k(\xi) \phi_j^{k+1}(\xi)\, d\mu(\xi).$$

Moreover, by Proposition 4.9, we have the following conclusion.

LEMMA 4.14. *With the same notation as above, there exists a positive constant C, independent of f, k and j, such that*

- (i) $|\eta_j^k| \le C2^k$;
- (ii) $\operatorname{supp} b_j^k \subset B(x_j^k, 2r_j^k)$;
- (iii) *the series* $\sum_{j\in\mathbb{N}} b_j^k$ *converges in* $L^q(\mathcal{X})$ *and induces a distribution* b^k *on* $\mathcal{G}_0^\epsilon(\beta, \gamma)$; *moreover,* $\operatorname{supp} b^k \subset \Omega^k$;
- (iv) *if* $g^k := f - b^k$, *then, for all* $x \in \mathcal{X}$, $|g^k(x)| \le C2^k$.

As an immediate consequence of Lemmas 4.11, 4.12, 4.14 and Remark 4.13, we obtain the following conclusion.

LEMMA 4.15.

(i) *There exists a positive constant C, independent of f, i, j and k, such that*

$$\sup_{x\in\mathcal{X}} |\ell_{i,j}^{k+1} \phi_j^{k+1}(x)| \le C2^{k+1}. \tag{4.5}$$

(ii) *For every* $k \in \mathbb{Z}$,

$$\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \ell_{i,j}^{k+1} \phi_j^{k+1} = 0, \tag{4.6}$$

where the series converges pointwise and also in $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$.

Proof. We first show (i). Obviously, by Lemma 4.14, we see that

$$\frac{1}{\|\phi_j^{k+1}\|_{L^1(\mathcal{X})}} \left| \int_{\mathcal{X}} \eta_j^{k+1} \phi_i^k(\xi) \phi_j^{k+1}(\xi)\, d\mu(\xi) \right| \lesssim 2^k. \tag{4.7}$$

On the other hand, by Lemma 4.12(ii), we find that

$$\|\phi_j^{k+1}\|_{L^1(\mathcal{X})} \ge \int_{B(x_j^{k+1}, r_j^{k+1})} \phi_j^{k+1}(\xi)\, d\mu(\xi) \ge \frac{1}{L} \mu(B(x_j^{k+1}, r_j^{k+1})). \tag{4.8}$$

Let

$$\varphi := \frac{\phi_i^k \phi_j^{k+1}}{\|\phi_j^{k+1}\|_{L^1(\mathcal{X})}}.$$

Then, by (4.8), Remark 4.13(ii) and the fact that $0 \le \phi_i^k \le 1$, we see that, for any $\epsilon \in (0,1)$ and any $w \in B(x_j^{k+1}, 15r_j^{k+1}) \cap (\Omega^{k+1})^{\complement}$,

$$\|\varphi\|_{\mathcal{G}(w, r_j^{k+1}, \epsilon, \epsilon)} \lesssim 1,$$

which further implies that

$$\frac{1}{\|\phi_j^{k+1}\|_{L^1(\mathcal{X})}} \left| \int_{\mathcal{X}} f(\xi) \phi_i^k(\xi) \phi_j^{k+1}(\xi)\, d\mu(\xi) \right| = |\langle f, \varphi \rangle| \lesssim f^*(w) \lesssim 2^{k+1}. \tag{4.9}$$

From (4.9), (4.7) and Lemma 4.12(i), we deduce that (4.5) holds true.

Next, we show (ii). Since $\operatorname{supp} \phi_j^{k+1} \subset B(x_j^{k+1}, 2r_j^{k+1})$, it follows from Lemma 4.11(iii) that, for any given $x \in \mathcal{X}$, the number j satisfying $\phi_j^{k+1}(x) \neq 0$ is at most L. Observe that, for such fixed j, in order to have $\ell_{i,j}^{k+1} \neq 0$, i must satisfy

$$B(x_i^k, 2r_i^k) \cap B(x_j^{k+1}, 2r_j^{k+1}) \neq \emptyset \tag{4.10}$$

by the definition of $\ell_{i,j}^{k+1}$. Moreover, by Lemma 4.11(iii) again, we see that the number i satisfying (4.10) is at most L. Thus, for any fixed $x \in \mathcal{X}$, the sum in (4.6) is actually finite and hence, by (i), we conclude that

$$\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |\ell_{i,j}^{k+1} \phi_j^{k+1}(x)| \lesssim L^2 2^{k+1}, \tag{4.11}$$

namely, the series in (4.6) is absolutely convergent. Therefore,

$$\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \ell_{i,j}^{k+1} \phi_j^{k+1}(x) = \sum_{j \in \mathbb{N}} \Big(\sum_{i \in \mathbb{N}} \ell_{i,j}^{k+1} \Big) \phi_j^{k+1}(x). \tag{4.12}$$

Since, for each $j \in \mathbb{N}$, the sum $\sum_{i \in \mathbb{N}} \ell_{i,j}^{k+1}$ is actually finite, by Lemma 4.12(i) and the facts that $\Omega^{k+1} \subset \Omega^k$ and

$$\operatorname{supp} \phi_j^{k+1} \subset B(x_j^{k+1}, 2r_j^{k+1}) \subset \Omega^{k+1} \tag{4.13}$$

we find that

$$\begin{aligned}
\sum_{i \in \mathbb{N}} \ell_{i,j}^{k+1} &= \int_{\mathcal{X}} [f(\xi) - \eta_j^{k+1}] \Big\{ \sum_{i \in \mathbb{N}} \phi_i^k(\xi) \Big\} \phi_j^{k+1}(\xi)\, d\mu(\xi) \\
&= \int_{\mathcal{X}} [f(\xi) - \eta_j^{k+1}] \chi_{\Omega^k}(\xi) \phi_j^{k+1}(\xi)\, d\mu(\xi) \\
&= \int_{\mathcal{X}} [f(\xi) - \eta_j^{k+1}] \phi_j^{k+1}(\xi)\, d\mu(\xi) = \int_{\mathcal{X}} b_j^{k+1}(\xi)\, d\mu(\xi) = 0,
\end{aligned}$$

which, combined with (4.12), implies that (4.6) converges pointwise.

On the other hand, by (4.11) and (4.13), we have

$$\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \int_{\mathcal{X}} |\ell_{i,j}^{k+1} \phi_j^{k+1}(\xi)|\, d\mu(\xi) \lesssim 2^{k+1} \mu(\Omega^{k+1}). \tag{4.14}$$

From (4.14) and the Lebesgue dominated convergence theorem, we deduce that (4.6) holds true in $L^1(\mathcal{X})$ and hence in $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$. This finishes the proof of Lemma 4.15. ■

Finally, we give the proof of Theorem 4.3 by using some ideas from [21, Theorem 3.28].

Proof of Theorem 4.3. To prove (i), let $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of $(p(\cdot), q)$-atoms, with $\operatorname{supp} a_j \subset B_j := B(x_j, r_j)$ for some $x_j \in \mathcal{X}$ and $r_j \in (0,\infty)$ and each $j \in \mathbb{N}$, satisfying

$$\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}) < \infty.$$

We first assume that $\lambda_j = 0$ when $j \geq N_0 + 1$ for some $N_0 \in \mathbb{N}$ and $g := \sum_{j=1}^{N_0} \lambda_j a_j$. By Lemma 4.4 and (3.9), we find that, for all $x \in \mathcal{X}$,

$$\begin{aligned} g^*(x) &\lesssim \sum_{j=1}^{N_0} |\lambda_j| M(a_j)(x)\chi_{B(x_j,3r_j)}(x) + \sum_{j=1}^{N_0} \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}} [M(\chi_{B_j})(x)]^{\beta/n+1} \\ &\lesssim \left\{\sum_{j=1}^{N_0} [|\lambda_j| M(a_j)(x)\chi_{B(x_j,3r_j)}(x)]^{\underline{p}}\right\}^{1/\underline{p}} \\ &\quad + \left\{\sum_{j=1}^{N_0}\left[\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}[M(\chi_{B_j})(x)]^{\beta/n+1}\right]^{\underline{p}}\right\}^{1/\underline{p}} \\ &=: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

For the first term I_1, since $[M(a_j)]^{\underline{p}} \in L^{q/\underline{p}}(\mathcal{X})$ for all $j \in \mathbb{N}$, we are in a position to use Proposition 2.11 with $r(\cdot) = p(\cdot)/\underline{p}$ to obtain

$$\mathrm{I}_1 \lesssim \widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j=1}^{N_0}, \{B_j\}_{j=1}^{N_0}).$$

Since $\beta \in (n/(n+1), \varepsilon)$, it follows that we are in a position to use Theorem 2.7 for the second term to obtain the same estimation as for I_1. Therefore,

$$\|g\|_{H^{*,p(\cdot)}(\mathcal{X})} = \|g^*\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j=1}^{N_0}, \{B_j\}_{j=1}^{N_0}),$$

which implies that $g \in H^{*,p(\cdot)}(\mathcal{X})$.

To consider the general case, for all $N \in \mathbb{N}$, let

$$f_N := \sum_{j=1}^{N} \lambda_j a_j.$$

Then, from what we have proved above, we deduce that, for all $N_1, N_2 \in \mathbb{N}$ with $N_1 < N_2$,

$$\|f_{N_1} - f_{N_2}\|_{H^{*,p(\cdot)}(\mathcal{X})} \lesssim \left\|\left(\sum_{j=N_1+1}^{N_2}\left[\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\right]^{\underline{p}}\right)^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}.$$

This implies that $\{f_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $H^{*,p(\cdot)}(\mathcal{X})$. Therefore, by Lemma 3.6, we find that $\sum_{j\in\mathbb{N}} \lambda_j a_j$ converges in $(\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$, with β and γ as in Theorem 4.3, and denote its limit by f. Finally, we go through the same argument as in the case where the sum is finite to obtain (i) for the general case.

Next, we show (ii). We first assume that $f \in L^q(\mathcal{X}) \cap H^{*,p(\cdot)}(\mathcal{X})$. In the remainder of the proof, we shall use the same notation as in Lemmas 4.11, 4.12, 4.14 and 4.15. Then

$$f = g^k + \sum_{j\in\mathbb{N}} b_j^k \quad \text{in } (\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$$

with β and γ as in Theorem 4.3. Observe that $g^k \to f$ in $H^{*,p(\cdot)}(\mathcal{X})$ as $k \to \infty$ by Lemma 4.8, and $g^k \to 0$ uniformly as $k \to -\infty$ by Lemma 4.14(iv). Thus, we have

$$f = \sum_{k=-\infty}^{\infty} (g^{k+1} - g^k) \quad \text{in } (\mathcal{G}_0^\epsilon(\beta,\gamma))'.$$

By Lemma 4.15(ii) and the fact that

$$\sum_{i\in\mathbb{N}} b_j^{k+1}\phi_i^k = \chi_{\Omega^k} b_j^{k+1} = b_j^{k+1},$$

we know that

$$\begin{aligned} g^{k+1} - g^k = b^k - b^{k+1} &= \sum_{i\in\mathbb{N}} b_i^k - \sum_{j\in\mathbb{N}} b_j^{k+1} + \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \ell_{i,j}^{k+1}\phi_j^{k+1} \\ &= \sum_{i\in\mathbb{N}} \Big[b_i^k - \sum_{j\in\mathbb{N}} (b_j^{k+1}\phi_i^k - \ell_{i,j}^{k+1}\phi_j^{k+1}) \Big] =: \sum_{i\in\mathbb{N}} h_i^k, \end{aligned}$$

where the series converge in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Moreover,

$$\begin{aligned} h_i^k &= (f-\eta_i^k)\phi_i^k - \sum_{j\in\mathbb{N}} [(f-\eta_j^{k+1})\phi_i^k - \ell_{i,j}^{k+1}]\phi_j^{k+1} \\ &= f\phi_i^k \chi_{(\Omega^{k+1})^\complement} - \eta_i^k\phi_i^k + \phi_i^k \sum_{j\in\mathbb{N}} \eta_j^{k+1}\phi_j^{k+1} + \sum_{j\in\mathbb{N}} \ell_{i,j}^{k+1}\phi_j^{k+1}. \end{aligned} \tag{4.15}$$

Now, let

$$\lambda_i^k := 2^k \|\chi_{B(x_i^k, 15r_i^k)}\|_{L^{p(\cdot)}(\mathcal{X})} \quad \text{and} \quad a_i^k := [\lambda_i^k]^{-1} h_i^k.$$

Then we have the following decomposition of f:

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \quad \text{in } (\mathcal{G}_0^\epsilon(\beta,\gamma))'.$$

We claim that a_i^k is a $(p(\cdot),\infty)$-atom up to a constant multiple. Indeed, from the first equality of (4.15), we deduce that $\int_{\mathcal{X}} h_i^k(\xi)\,d\mu(\xi) = 0$. Let $\{S_k\}_{k\in\mathbb{Z}}$ be as in Definition 3.7. By Remarks 3.8(i) and 3.10, the Riesz lemma and the definition of Ω^{k+1}, we find that there exists $\{k_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $k_l \to \infty$ as $l \to \infty$ and, for almost every $x \in (\Omega^{k+1})^\complement$,

$$|f(x)| = \lim_{l\to\infty} |S_{k_l}(f)(x)| \lesssim f^*(x) \lesssim 2^{k+1},$$

which, together with Lemma 4.14(i), the fact that $\sum_{j\in\mathbb{N}} \phi_j^{k+1} \le L$, Lemma 4.15(i) and the second equality of (4.15), further implies that

$$\|h_i^k\|_{L^\infty(\mathcal{X})} \lesssim 2^{k+1} + 2^k + L2^{k+1} + L2^{k+1} \lesssim 2^k.$$

Finally, since $\ell_{i,j}^{k+1} = 0$ unless $(2B_j^{k+1}) \cap (2B_i^k) \neq \emptyset$, it follows from Remark 4.13 that

$$\operatorname{supp}\Big(\sum_{j\in\mathbb{N}} \ell_{i,j}^{k+1}\phi_j^{k+1}\Big) \subset B(x_i^k, 15r_i^k).$$

From this and the second equality of (4.15), we deduce that

$$\operatorname{supp} h_i^k \subset B(x_i^k, 15r_i^k). \tag{4.16}$$

Therefore, for each $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, a_i^k is a $(p(\cdot),\infty)$-atom up to a constant multiple, and the above claim holds true.

Moreover, by Theorem 2.7 and Lemma 4.11(ii), we conclude that

$$\begin{aligned}
\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_i^k a_i^k\}_{k\in\mathbb{Z},\, i\in\mathbb{N}}) &\sim \Big\|\Big\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}[2^k\chi_{B(x_i^k,15r_i^k)}]^{\underline{p}}\Big\}^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim \Big\|\Big\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}[2^k\chi_{B(x_i^k,r_i^k/4)}]^{\underline{p}}\Big\}^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim \Big\|\Big\{\sum_{k\in\mathbb{Z}}[2^k\chi_{\Omega^k}]^{\underline{p}}\Big\}^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\sim \Big\|\Big\{\sum_{k\in\mathbb{Z}}[2^k\chi_{\Omega^k\setminus\Omega^{k+1}}]^{\underline{p}}\Big\}^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim \Big\|f^*\Big\{\sum_{k\in\mathbb{Z}}[\chi_{\Omega^k\setminus\Omega^{k+1}}]^{\underline{p}}\Big\}^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim \|f^*\|_{L^{p(\cdot)}(\mathcal{X})} \sim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.
\end{aligned}$$

Thus,

$$L^q(\mathcal{X})\cap H^{*,p(\cdot)}(\mathcal{X}) \subset H_{\mathrm{at}}^{p(\cdot),\infty}(\mathcal{X})$$

and

$$\|f\|_{H_{\mathrm{at}}^{p(\cdot),\infty}(\mathcal{X})} \lesssim \widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_i^k a_i^k\}_{k\in\mathbb{Z},\, i\in\mathbb{N}}) \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

Now, we let $f\in H^{*,p(\cdot)}(\mathcal{X})$. Then, by virtue of Lemma 4.8, there exists a sequence $\{f_l\}_{l\in\mathbb{N}}\subset L^q(\mathcal{X})\cap H^{*,p(\cdot)}(\mathcal{X})$ such that $f=\sum_{l\in\mathbb{N}} f_l$ in $H^{*,p(\cdot)}(\mathcal{X})$ and

$$\|f_l\|_{H^{*,p(\cdot)}(\mathcal{X})} \le 2^{2-l}\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

For each $l\in\mathbb{N}$, by the conclusion above, we find that f_l has an atomic decomposition

$$f_l=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^{l,k}a_i^{l,k} \quad \text{in } (\mathcal{G}_0^\epsilon(\beta,\gamma))',$$

where $\{\lambda_i^{l,k}a_i^{l,k}\}_{k\in\mathbb{Z},\, i\in\mathbb{N}}$ are constructed as above and hence $\{a_i^{l,k}\}_{k\in\mathbb{Z},\, l,i\in\mathbb{N}}$ are $(p(\cdot),\infty)$-atoms. Thus, we have

$$f=\sum_{l\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^{l,k}a_i^{l,k} \quad \text{in } (\mathcal{G}_0^\epsilon(\beta,\gamma))'$$

and

$$\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_i^{l,k}a_i^{l,k}\}_{k\in\mathbb{Z},\, l,i\in\mathbb{N}}) \le \Big\{\sum_{l\in\mathbb{N}}\|f_l\|_{H^{*,p(\cdot)}(\mathcal{X})}^{\underline{p}}\Big\}^{1/\underline{p}} \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})},$$

which further imply that $f\in H_{\mathrm{at}}^{p(\cdot),\infty}(\mathcal{X})$ and

$$\|f\|_{H_{\mathrm{at}}^{p(\cdot),\infty}(\mathcal{X})} \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

This finishes the proof of (ii) and hence of Theorem 4.3. ■

Before we further investigate atomic Hardy spaces, one remark, which is useful for later considerations, may be in order.

Remark 4.16. (i) Thanks to Theorem 4.3(i), the convergence in (4.1) takes place in $H^{*,p(\cdot)}(\mathcal{X})$ as well.

(ii) The construction in the proof of Theorem 4.3(ii) does not depend on $p(\cdot)$. This means that if $f \in H^{*,p_1(\cdot)}(\mathcal{X}) \cap H^{*,p_2(\cdot)}(\mathcal{X})$ for some $p_1(\cdot)$ and $p_2(\cdot)$ satisfying the same assumptions as $p(\cdot)$, then the convergence in (4.1) takes place in $H^{*,p_1(\cdot)}(\mathcal{X})$ and also in $H^{*,p_2(\cdot)}(\mathcal{X})$. In particular, since $L^q(\mathcal{X}) \sim H^q(\mathcal{X})$ with $q \in (1,\infty)$ (see [29, Corollary 3.11]), it follows that if $f \in L^q(\mathcal{X}) \cap H^{*,p(\cdot)}(\mathcal{X})$ with $q \in (1,\infty)$, then the summation in (4.1) converges in $L^q(\mathcal{X})$ and also in $H^{*,p(\cdot)}(\mathcal{X})$ according to the construction above.

4.4. Some consequences of the atomic characterization. Now we harvest some conclusions of the atomic decomposition theorem. Here we consider a problem left open: do the spaces depend on ϵ, α, β and γ? For this problem, we have the following answer.

THEOREM 4.17. *The spaces $H^{*,p(\cdot)}(\mathcal{X})$, $H^{p(\cdot)}_\alpha(\mathcal{X})$ and $H^{p(\cdot)}_{\mathrm{d}}(\mathcal{X})$ are independent of the parameters ϵ, α, β, γ satisfying the assumptions of Theorem* 3.11.

To prove Theorem 4.17, we need the following several lemmas.

LEMMA 4.18. *Let $R \in (1,\infty)$ be fixed and, for all $x \in \mathcal{X}$,*

$$A_R(x) := \min\{1, \max\{R^{-1}d(x_1,x) - 1, 0\}\}. \tag{4.17}$$

Then

$$|A_R(x) - A_R(y)| \le \frac{6d(x,y)}{R + d(x_1,x)} \tag{4.18}$$

for all $x, y \in \mathcal{X}$ satisfying $d(x,y) \le [1 + d(x_1,x)]/2$.

Proof. Observe that, by the triangle inequality, we have, for all $x, y \in \mathcal{X}$,

$$|A_R(x) - A_R(y)| \le R^{-1}d(x,y).$$

Hence, to prove this lemma, we may assume that $d(x_1,x) \ge 5R$. Then it follows from $d(x,y) \le [1 + d(x_1,x)]/2$ that

$$d(x_1,y) \ge d(x_1,x) - d(x,y) \ge \frac{d(x_1,x) - 1}{2} \ge \frac{5R - 1}{2} \ge 2R.$$

In this case, we have $A_R(x) = 1 = A_R(y)$, and hence (4.18) holds true. This finishes the proof of Lemma 4.18. ∎

In what follows, let $\mathcal{C}_{\mathrm{b}}(\mathcal{X})$ be the set of all continuous functions with bounded support.

LEMMA 4.19. *Let $\epsilon \in (0,1]$ and $\beta, \gamma \in (0,\epsilon)$. Then $\mathcal{C}_{\mathrm{b}}(\mathcal{X})$ is dense in $\mathcal{G}^\epsilon_0(\beta,\gamma)$.*

Proof. Since $\mathcal{G}^\epsilon_0(\beta,\gamma)$ is the completion of the space $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$, to prove this lemma it suffices to approximate any $\varphi \in \mathcal{G}(\epsilon,\epsilon)$ by $\mathcal{C}_{\mathrm{b}}(\mathcal{X})$ functions.

We claim that

$$\|A_R\varphi\|_{\mathcal{G}(\beta,\gamma)} = O(R^{\gamma-\epsilon}), \quad R \to \infty.$$

By this claim, we find that

$$\lim_{R\to\infty} \|(1 - A_R)\varphi - \varphi\|_{\mathcal{G}(\beta,\gamma)} = \lim_{R\to\infty} \|A_R\varphi\|_{\mathcal{G}(\beta,\gamma)} = 0.$$

Since $(1 - A_R)\varphi \in \mathcal{C}_{\mathrm{b}}(\mathcal{X})$ thanks to Lemma 4.18, it follows that $\mathcal{C}_{\mathrm{b}}(\mathcal{X})$ is dense in $\mathcal{G}(\epsilon,\epsilon)$.

It remains to prove the above claim. Obviously, by (3.1) and (4.17), we see that, for all $R\in(0,\infty)$ and $x\in\mathcal{X}$,

$$|\varphi(x)A_R(x)|\le R^{\gamma-\epsilon}\frac{1}{\mu(B(x,1+d(x,x_1)))}\left[\frac{1}{1+d(x_1,x)}\right]^{\gamma}. \tag{4.19}$$

Meanwhile, for all $R\in(0,\infty)$ and $x,y\in\mathcal{X}$ satisfying $d(x,y)\le[1+d(x_1,x)]/2$, we have

$$\begin{aligned}&|\varphi(x)[A_R(x)-A_R(y)]|\\&\qquad\lesssim R^{\gamma-\epsilon}\frac{d(x,y)}{1+d(x_1,x)}\,\frac{1}{\mu(B(x,1+d(x,x_1)))}\left[\frac{1}{1+d(x_1,x)}\right]^{\gamma}.\end{aligned} \tag{4.20}$$

To show (4.20), we may assume that either x or y lies outside $B(x_1,R)$; otherwise the left-hand side becomes 0. If one lies outside $B(x_1,R)$, then the other lies outside $B(x_1,R/4)$. So, we may assume that x and y lie outside $B(x_1,R/4)$. Then, by using (3.1) and (4.18), we obtain (4.20). Likewise, by (3.2), we conclude that, for all $R\in(0,\infty)$ and $x,y\in\mathcal{X}$ with $d(x,y)\le[1+d(x_1,x)]/2$,

$$|A_R(y)[\varphi(x)-\varphi(y)]|\lesssim R^{\gamma-\epsilon}\left[\frac{d(x,y)}{1+d(x_1,x)}\right]^{\beta}\frac{1}{\mu(B(x,1+d(x,x_1)))}\left[\frac{1}{1+d(x_1,x)}\right]^{\gamma},$$

which, combined with (4.20) and the fact that $\beta<\epsilon\le1$, implies that

$$\begin{aligned}&|\varphi(x)A_R(x)-\varphi(y)A_R(y)|\\&\qquad\le|\varphi(x)[A_R(x)-A_R(y)]|+|A_R(y)[\varphi(x)-\varphi(y)]|\\&\qquad\lesssim R^{\gamma-\epsilon}\left[\frac{d(x,y)}{1+d(x_1,x)}\right]^{\epsilon}\frac{1}{\mu(B(x,1+d(x,x_1)))}\left[\frac{1}{1+d(x_1,x)}\right]^{\gamma}.\end{aligned}$$

From this and (4.19), we further deduce that the above claim holds true, which completes the proof of Lemma 4.19. ■

Proof of Theorem 4.17. Observe that the spaces $H^{p(\cdot)}_{\alpha}(\mathcal{X})$ are independent of α; see Corollary 3.12. So, let us now concentrate on the independence from ϵ, β, γ. Let ϵ_1, β_1, γ_1 and ϵ_2, β_2, γ_2 satisfy the same assumptions as in Theorem 3.11. Let $k\in\{1,2\}$. Denote by H_k the Hardy space $H^{*,p(\cdot)}(\mathcal{X})$ defined via the grand maximal function generated by $(\mathcal{G}^{\epsilon_k}_0(\beta_k,\gamma_k))'$. We need to prove that H_1 and H_2 coincide with equivalent quasi-norms.

Let $f\in H_1$. Then $f\in(\mathcal{G}^{\epsilon_1}_0(\beta_1,\gamma_1))'$. By Lemma 4.19, we find that $f|_{\mathcal{C}_{\rm b}(\mathcal{X})}$ can be extended to an element in $(\mathcal{G}^{\epsilon_2}_0(\beta_2,\gamma_2))'$. According to Theorem 4.3(ii), f has an expression $f=\sum_{j\in\mathbb{N}}\lambda_ja_j$ in $(\mathcal{G}^{\epsilon_1}_0(\beta_1,\gamma_1))'$, where each λ_j is a non-negative number and a_j is a $(p(\cdot),\infty)$-atom supported on a ball B_j satisfying

$$\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}})\lesssim\|f\|_{H_1}.$$

However, by Theorem 4.3(i), we further know that

$$g=\sum_{j\in\mathbb{N}}\lambda_ja_j$$

converges in $(\mathcal{G}^{\epsilon_2}_0(\beta_2,\gamma_2))'$, $g\in H_2$ and

$$\|g\|_{H_2}\lesssim\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}})\lesssim\|f\|_{H_1}.$$

Since g and f coincide on a dense space $\mathcal{C}_{\mathrm{b}}(\mathcal{X})$, it follows that the mapping

$$f \in H_1 \mapsto g \in H_2$$

is a continuous injection. Likewise, we can show that $g \in H_2 \mapsto f \in H_1$ is a continuous injection. Therefore, H_1 and H_2 are isomorphic and have equivalent quasi-norms. This finishes the proof of Theorem 4.17. ■

COROLLARY 4.20. *Let* $p(\cdot) \in C^{\log}_{(0,\infty)}(\mathcal{X})$.

(i) *If* $1 \le p_- \le p_+ < \infty$, *then*

$$H^{*,p(\cdot)}(\mathcal{X}) \hookrightarrow L^{p(\cdot)}(\mathcal{X}). \tag{4.21}$$

(ii) *If* $1 < p_- \le p_+ < \infty$, *then* $H^{*,p(\cdot)}(\mathcal{X}) = L^{p(\cdot)}(\mathcal{X})$ *with equivalent norms.*

Proof. To prove (i), let $\epsilon \in (0,1]$, $\beta, \gamma \in (0,\epsilon)$ and $f \in H^{*,p(\cdot)}(\mathcal{X})$. Then, by Theorem 4.3(ii), we find that $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$, where $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $\{a_j\}_{j\in\mathbb{N}}$ are $(p(\cdot),\infty)$-atoms such that each a_j is supported on a ball B_j and

$$\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}) \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

By (3.9) and Proposition 2.11, we find that, for any $L \in \mathbb{N}$,

$$\begin{aligned}\Big\|\sum_{j=1}^{L} |\lambda_j a_j|\Big\|_{L^{p(\cdot)}(\mathcal{X})} &\le \Big\|\Big(\sum_{j=1}^{L} |\lambda_j a_j|^{\underline{p}}\Big)^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})} \\ &\lesssim \widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j=1}^{L}, \{B_j\}_{j=1}^{L}) \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})} < \infty. \end{aligned} \tag{4.22}$$

This implies $\sum_{j\in\mathbb{N}} |\lambda_j a_j(x)| < \infty$ for μ-a.e. $x \in \mathcal{X}$. Going back to (4.22) and using the absolute continuity of $L^{p(\cdot)}(\mathcal{X})$, by letting $L \to \infty$, we see that $g := \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $L^{p(\cdot)}(\mathcal{X})$ ($\hookrightarrow (\mathcal{G}^\epsilon_0(\beta,\gamma))'$). Since $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$, it follows that

$$f = g \in L^{p(\cdot)}(\mathcal{X}) \quad \text{and} \quad \|f\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

Conversely, we need to prove $L^{p(\cdot)}(\mathcal{X}) \subset H^{*,p(\cdot)}(\mathcal{X})$ in view of (4.21). Let $f \in L^{p(\cdot)}(\mathcal{X})$. Then, by the fact that $f^* \lesssim M(f)$ (see [29, (3.4)]) and the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathcal{X})$ (see Lemma 2.5), we see that $f \in H^{*,p(\cdot)}(\mathcal{X})$ and

$$\|f\|_{H^{*,p(\cdot)}(\mathcal{X})} \lesssim \|f\|_{L^{p(\cdot)}(\mathcal{X})}.$$

This finishes the proof of Corollary 4.20. ■

By combining Corollary 4.20 and Theorem 4.3, we obtain the following atomic characterization of $L^{p(\cdot)}(\mathcal{X})$.

REMARK 4.21. Let $p(\cdot) \in C^{\log}_{(1,\infty)}(\mathcal{X})$ and the parameters q, ϵ, β, γ be as in Theorem 4.3. Then:

(i) $H^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X}) \hookrightarrow L^{p(\cdot)}(\mathcal{X})$, namely, if $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $\{a_j\}_{j\in\mathbb{N}}$ are $(p(\cdot),q)$-atoms satisfying $\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j\in\mathbb{N}}) < \infty$, then $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$ and $f \in L^{p(\cdot)}(\mathcal{X})$. Furthermore,

$$\|f\|_{L^{p(\cdot)}(\mathcal{X})} \le C\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j\in\mathbb{N}}),$$

where C is a positive constant independent of $\{\lambda_j\}_{j\in\mathbb{N}}$ and $\{a_j\}_{j\in\mathbb{N}}$.

(ii) $L^{p(\cdot)}(\mathcal{X}) \hookrightarrow H^{p(\cdot),\infty}_{\mathrm{at}}(\mathcal{X})$. More precisely, if $f \in L^{p(\cdot)}(\mathcal{X})$, then there exist $(p(\cdot),\infty)$-atoms $\{a_j\}_{j\in\mathbb{N}}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^{\epsilon}_0(\beta,\gamma))'$ and that

$$\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j\in\mathbb{N}}) \le \widetilde{C}\|f\|_{L^{p(\cdot)}(\mathcal{X})},$$

where $\widetilde{C}$ is a positive constant independent of f.

4.5. Finite atomic characterizations. In this subsection, we consider a finite atomic characterization of $H^{*,p(\cdot)}(\mathcal{X})$.

DEFINITION 4.22. Let $p(\cdot) \in C^{\log}_{(0,\infty)}(\mathcal{X})$ with $p_- \in (\frac{n}{n+1},\infty)$ and $q \in [1,\infty] \cap (p_+,\infty]$. Then the *finite atomic Hardy space with variable exponent*, $H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$, is the set of all finite linear combinations of $(p(\cdot),q)$-atoms, and for all $f \in H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$, its quasi-norm is defined as

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})} := \inf\Big\{\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j=1}^N) : f = \sum_{j=1}^N \lambda_j a_j,\ N \in \mathbb{N}\Big\},$$

where the infimum is taken over all decompositions of f such that, for some $N \in \mathbb{N}$, $f = \sum_{j=1}^N \lambda_j a_j$, $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ and $\{a_j\}_{j=1}^N$ are $(p(\cdot),q)$-atoms.

Obviously, $H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$ is a dense subspace of $H^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})$, and hence of $H^{*,p(\cdot)}(\mathcal{X})$ by Theorem 4.3.

For all $q \in [1,\infty]$, let $L^{q,0}_{\mathrm{b}}(\mathcal{X})$ be the set of all functions $f \in L^q(\mathcal{X})$ with bounded support and zero average. We have the following lemma.

LEMMA 4.23. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and $q \in [1,\infty] \cap (p_+,\infty]$. Then*

$$L^{q,0}_{\mathrm{b}}(\mathcal{X}) = H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$$

as sets. Moreover, $L^{q,0}_{\mathrm{b}}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$.

Proof. Let $f \in L^{q,0}_{\mathrm{b}}(\mathcal{X})$ with $\operatorname{supp} f \subset B$. Then $\frac{[\mu(B)]^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}(\mathcal{X})}\|f\|_{L^q(\mathcal{X})}} f$ is a $(p(\cdot),q)$-atom supported on B. Thus, $f \in H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$ and

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})} \le [\mu(B)]^{-1/q}\|\chi_B\|_{L^{p(\cdot)}(\mathcal{X})}\|f\|_{L^q(\mathcal{X})},$$

which implies that $L^{q,0}_{\mathrm{b}}(\mathcal{X}) \subset H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$ as sets.

Conversely, since each $(p(\cdot),q)$-atom belongs to $L^{q,0}_{\mathrm{b}}(\mathcal{X})$, it follows that $H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X}) \subset L^{q,0}_{\mathrm{b}}(\mathcal{X})$ as sets. Therefore, $L^{q,0}_{\mathrm{b}}(\mathcal{X}) = H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$ as sets, and hence $L^{q,0}_{\mathrm{b}}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$. This finishes the proof of Lemma 4.23. ■

THEOREM 4.24. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$.*

(i) *If $q \in (1,\infty)\cap(p_+,\infty)$, then $\|\cdot\|_{H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})}$ and $\|\cdot\|_{H^{*,p(\cdot)}(\mathcal{X})}$ are equivalent quasi-norms on $H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$.*

(ii) *$\|\cdot\|_{H^{p(\cdot),\infty}_{\mathrm{fin}}(\mathcal{X})}$ and $\|\cdot\|_{H^{*,p(\cdot)}(\mathcal{X})}$ are equivalent quasi-norms on $H^{p(\cdot),\infty}_{\mathrm{fin}}(\mathcal{X}) \cap \mathcal{C}(\mathcal{X})$, where $\mathcal{C}(\mathcal{X})$ denotes the set of all continuous functions on $\mathcal{X}$.*

To prove Theorem 4.24, we need some auxiliary estimates.

Lemma 4.25. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$, $\epsilon \in (0,1]$ and $\beta, \gamma \in (n(1/p_- - 1), \epsilon)$. Let $r \in (0,\infty)$ and $f \in H^{*,p(\cdot)}(\mathcal{X})$ supported on $B(x_1, R)$ for some $R \in (0,\infty)$. Then there exists a positive constant C_0 such that, for all $x \in \mathcal{X}$ and $\varphi \in \mathcal{G}^\epsilon_0(\beta,\gamma)$ with $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1$,*

$$|\langle f, \varphi\rangle| \le C_0 \inf_{y\in B(x,d(x,x_1))} f^*(y); \tag{4.23}$$

moreover, for all $x \in [B(x_1, 16R)]^\complement$,

$$f^*(x) \le C_0 \|f\|_{H^{*,p(\cdot)}(\mathcal{X})} \|\chi_{B(x_1,R)}\|^{-1}_{L^{p(\cdot)}(\mathcal{X})}. \tag{4.24}$$

Proof. See [46, Lemma 2.2] for the proof of (4.23) with $r \ge 4d(x_1, x)/3$. If $r < 4d(x_1, x)/3$, then we invoke [29, (5.10)] to prove (4.23). Indeed, let $\zeta \in C^\infty(\mathbb{R})$ be chosen so that $\chi_{[-1,1]} \le \zeta \le \chi_{[-2,2]}$. Define

$$\widetilde{\varphi}(z) := \varphi(z)\zeta\left(\frac{16d(z,x_1)}{d(x,x_1)}\right)$$

for $z \in \mathcal{X}$. Notice that $\widetilde{\varphi}$ and φ agree on $B(x,R)$ because $d(x,x_1) \ge 16R$. Since f is supported on $B(x,R)$, we have $\langle f, \varphi\rangle = \langle f, \widetilde{\varphi}\rangle$. According to [29, (5.10)], we find that, for all $y \in B(x_1, d(x_1, x))$,

$$\|\widetilde{\varphi}\|_{\mathcal{G}(y,r,\beta,\gamma)} \lesssim \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)},$$

and hence $|\langle f, \varphi\rangle| = |\langle f, \widetilde{\varphi}\rangle| \lesssim f^*(y)$, which implies that (4.23) holds true also when $r < 4d(x_1, x)/3$.

By (4.23), Remark 2.6 and the fact that, when $x \in [B(x_1, 16R)]^\complement$,

$$B(x_1, R) \subset B(x, 2d(x_1, x)),$$

we further conclude that (4.24) holds true. This finishes the proof of Lemma 4.25. ∎

With these estimates in hand, let us prove Theorem 4.24.

Proof of Theorem 4.24. To show (i), let $f \in H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})$. Obviously, by Theorem 4.3(i), we see that

$$H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X}) \subset H^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X}) \subset H^{*,p(\cdot)}(\mathcal{X}).$$

Then

$$\|f\|_{H^{*,p(\cdot)}(\mathcal{X})} \lesssim \|f\|_{H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})}.$$

Therefore, to complete the proof of (i) it suffices to prove that

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})} \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

Without loss of generality, we may assume that $\operatorname{supp} f \subset B(x_1, R)$ for some R in $(0,\infty)$. According to the proof of Theorem 4.3(ii), we have an atomic decomposition

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} h^k_i \quad \text{in } (\mathcal{G}^\epsilon_0(\beta,\gamma))'$$

with β, γ as in Theorem 4.3. Moreover, from the construction of h^k_i, we further deduce that

$$\|h^k_i\|_{L^\infty(\mathcal{X})} \lesssim 2^k, \quad \operatorname{supp} h^k_i \subset B(x^k_i, 15r^k_i) \quad \text{and} \quad \sum_{i\in\mathbb{N}} \chi_{B(x^k_i,15r^k_i)} \lesssim 1. \tag{4.25}$$

Denote by k' the largest integer k satisfying

$$2^k < \frac{C_0\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}}{\|\chi_{B(x_1,R)}\|_{L^{p(\cdot)}(\mathcal{X})}},$$

where C_0 is the constant in (4.23). Then, by Lemma 4.25, we know that $\Omega^k \subset B(x_1, 16R)$ for all $k > k'$. Let

$$g := \sum_{k\le k'}\sum_{i\in\mathbb{N}} h_i^k \quad \text{and} \quad \zeta := \sum_{k>k'}\sum_{i\in\mathbb{N}} h_i^k,$$

where the series converge in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Obviously, $f = g + \zeta$. By (4.16) and Lemma 4.11(ii), we easily see that

$$\operatorname{supp}\zeta \subset \bigcup_{k>k'} \Omega^k \subset B(x_1, 16R), \tag{4.26}$$

and hence $\operatorname{supp} g \subset B(x_1, 16R)$. Since $f \in L_{\mathrm{b}}^{q,0}(\mathcal{X})$ due to Lemma 4.23, it follows that f is a multiple of a classical $H^1(\mathcal{X})$-atom (see [29, Definition 4.1]), and hence $f^* \in L^1(\mathcal{X})$. Recall that a measurable function a is called an $H^1(\mathcal{X})$-*atom* if $\operatorname{supp} a \subset B(x_0, r_0)$ with some $x_0 \in \mathcal{X}$ and $r_0 \in (0,\infty)$, $\|a\|_{L^q(\mathcal{X})} \le [\mu(B(x_0,r_0))]^{1/q-1}$ with $q \in (1,\infty)$ and $\int_{\mathcal{X}} a(x)\,d\mu(x) = 0$. Therefore, by (4.25) and Lemma 4.11(ii), we see that

$$\begin{aligned}\int_{\mathcal{X}}\sum_{k>k'}\sum_{i\in\mathbb{N}}|h_i^k(\xi)|\,d\mu(\xi) &\lesssim \int_{\mathcal{X}}\sum_{k>k'}2^k\sum_{i\in\mathbb{N}}\chi_{B(x_i^k,15r_i^k)}(\xi)\,d\mu(\xi)\\ &\lesssim \int_{\mathcal{X}}\sum_{k>k'}2^k\chi_{\Omega^k}(\xi)\,d\mu(\xi) \lesssim \|f^*\|_{L^1(\mathcal{X})} < \infty. \end{aligned}\tag{4.27}$$

From (4.27) and the vanishing moment condition on h_i^k, we further deduce that ζ satisfies the vanishing moment condition, and hence so does g by $g = f - \zeta$. Moreover, by (4.25) and Remark 2.6, we find that

$$\|g\|_{L^\infty(\mathcal{X})} \lesssim \sum_{k\le k'} 2^k \sim 2^{k'} \lesssim \frac{\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}}{\|\chi_{B(x_1,R)}\|_{L^{p(\cdot)}(\mathcal{X})}} \lesssim \frac{\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}}{\|\chi_{B(x_1,16R)}\|_{L^{p(\cdot)}(\mathcal{X})}}.$$

So far, we have proved that $\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}^{-1} g$ is a $(p(\cdot),\infty)$-atom up to a constant multiple.

Next, we deal with ζ. We claim that, for all $x \in \mathcal{X}$, $\zeta(x) \lesssim f^*(x)$. Indeed, for all $x \in \mathcal{X}$, there exists $j \in \mathbb{Z}$ such that $x \in \Omega^j \setminus \Omega^{j+1}$. From this, (4.25) and the fact that, for all $k \ge j+1$, $\operatorname{supp} h_i^k \subset \Omega^k \subset \Omega^{j+1}$, we deduce that

$$|\zeta(x)| \le \sum_{k<k'}\sum_{i\in\mathbb{N}}|h_i^k(x)| \lesssim \sum_{k<j} 2^k \sim 2^j \lesssim f^*(x),$$

so the above claim holds true. By this claim and the fact that $f^* \in L^q(\mathcal{X})$ (when $q=1$, it is proved above, and when $q \in (1,\infty)$, this is a consequence of Remark 3.4 and Lemma 2.5), we conclude that

$$\zeta = \sum_{k>k'}\sum_{i\in\mathbb{N}} h_i^k$$

converges in $L^q(\mathcal{X})$. For any $N \in \mathbb{N}$, let

$$F_N := \{(k,i) : k \in \mathbb{Z},\ k > k',\ i \in \mathbb{N} \text{ and } i + |k| \le N\}$$

and

$$\zeta_N := \sum_{(k,i)\in F_N} h_i^k.$$

Since $\sum_{k>k'}\sum_{i\in\mathbb{N}} h_i^k$ converges in $L^q(\mathcal{X})$, it follows that there exists $N_0\in\mathbb{N}$ large enough such that

$$\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}^{-1}\|\zeta-\zeta_{N_0}\|_{L^q(\mathcal{X})} \le \frac{[\mu(B(x_1,16R))]^{1/q}}{\|\chi_{B(x_1,16R)}\|_{L^{p(\cdot)}(\mathcal{X})}},$$

which, combined with (4.26) and (4.27), further implies that $\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}^{-1}(\zeta-\zeta_{N_0})$ is a $(p(\cdot),q)$-atom. Therefore,

$$\begin{aligned}
f &= g+\zeta_{N_0}+(\zeta-\zeta_{N_0})\\
&= \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}\frac{g}{\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}} + \sum_{(k,i)\in F_{N_0}} \lambda_i^k a_i^k + \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}\frac{\zeta-\zeta_{N_0}}{\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}}
\end{aligned}$$

is a finite linear combination of $(p(\cdot),q)$-atoms. Moreover,

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X})} \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})} + \widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_i^k a_i^k\}_{(k,i)\in F_{N_0}}) \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})},$$

which completes the proof of (i).

To prove (ii), we assume that $f\in H^{p(\cdot),\infty}_{\mathrm{at}}(\mathcal{X})\cap\mathcal{C}(\mathcal{X})$. Obviously, f is a uniformly continuous and bounded function, and hence h_i^k is continuous by its construction. Thus, the fact that $\|f^*\|_{L^\infty(\mathcal{X})}\lesssim\|f\|_{L^\infty(\mathcal{X})}$ (see Remark 3.4) implies that there exists an integer $k''>k'$ such that $\Omega^k=\emptyset$ for all $k>k''$. Therefore,

$$\zeta = \sum_{k'<k<k''}\sum_{i\in\mathbb{N}} h_i^k.$$

Since f is uniformly continuous, it follows that for any $\varepsilon\in(0,\infty)$, there exists $\delta\in(0,\infty)$ such that when $d(x,y)<\delta$, $|f(x)-f(y)|<\varepsilon$. Let

$$\zeta_1^\varepsilon := \sum_{(k,i)\in G_1} h_i^k \quad\text{and}\quad \zeta_2^\varepsilon := \sum_{(k,i)\in G_2} h_i^k,$$

where

$$G_1 := \{(k,i) : 45r_i^k\ge\delta,\ k'<k<k''\}$$

and

$$G_2 := \{(k,i) : 45r_i^k<\delta,\ k'<k<k''\}.$$

Notice that $\{B(x_i^k,r_i^k/4)\}_{i\in\mathbb{N}}$ are disjoint and $\operatorname{supp} h_i^k\subset B(x_i^k,15r_i^k)\subset B(x_1,16R)$. Then the summation in ζ_1^ε is finite, and hence ζ_1^ε is a continuous function by the fact that each h_i^k is continuous.

We claim that $\|\zeta_2^\varepsilon\|_{L^\infty(\mathcal{X})}\lesssim(k''-k')\varepsilon$. Indeed, for each $x\in B(x_i^k,15r_i^k)$ with $(k,i)\in G_2$, there exists $y\in B(x_i^k,45r_i^k)\cap(\Omega_k)^{\complement}$ such that $d(x,y)<45r_i^k<\delta$. Thus,

$$|h_i^k(x)| = |h_i^k(x)-h_i^k(y)| < \varepsilon,$$

which, combined with (4.25), implies that $\|h_i^k\|_{L^\infty(\mathcal{X})}\le\varepsilon$, and hence the above claim holds true. From this claim and the continuity of ζ_1^ε, we deduce that ζ is continuous, and hence $g=f-\zeta$ and $\zeta_2^\varepsilon=\zeta-\zeta_2^\varepsilon$ are also continuous.

Observe that $\zeta_1^\varepsilon = \sum_{(k,i)\in G_1} \lambda_i^k a_i^k$ is a finite linear combination of $(p(\cdot),\infty)$-atoms and

$$\|\zeta_1^\epsilon\|_{H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})} \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

From this and the fact that ζ has the vanishing moment, we find that ζ_2^ε also has the vanishing moment. Moreover, $\operatorname{supp}\zeta_2^\varepsilon \subset B(x_1,16R)$ and $\|\zeta_2^\varepsilon\|_{L^\infty(\mathcal{X})} \lesssim (k''-k')\varepsilon$. Now, choose $\varepsilon_0 \in (0,\infty)$ small enough such that

$$\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}^{-1} \|\zeta_2^{\varepsilon_0}\|_{L^\infty(\mathcal{X})} \le \|\chi_{B(x_1,16R)}\|_{L^{p(\cdot)}(\mathcal{X})}^{-1}.$$

Then

$$\begin{aligned} f &= g+\zeta_1^{\varepsilon_0}+\zeta_2^{\varepsilon_0} \\ &= \|f\|_{H^{*,p(\cdot)}(\mathcal{X})} \frac{g}{\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}} + \sum_{(k,i)\in F_1} \lambda_i^k a_i^k + \|f\|_{H^{*,p(\cdot)}(\mathcal{X})} \frac{\zeta_2^{\varepsilon_0}}{\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}} \end{aligned}$$

is a finite linear combination of $(p(\cdot),\infty)$-atoms, and moreover

$$\|f\|_{H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})} \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})} + \widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_i^k a_i^k\}_{(k,i)\in G_1}) \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

This finishes the proof of (ii) and hence of Theorem 4.24. ■

Proposition 4.26. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$. Then the subset $H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X}) \cap \mathcal{C}(\mathcal{X})$ is dense in $H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})$ under the quasi-norm $\|\cdot\|_{H^{*,p(\cdot)}(\mathcal{X})}$, and hence in $H^{*,p(\cdot)}(\mathcal{X})$.*

Proof. By [33, Theorem 2.6], we choose an $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI, $\{S_k\}_{k\in\mathbb{Z}}$, with bounded support on $\mathcal{X}$ as in Remark 3.8(iii). For any $(p(\cdot),\infty)$-atom a with support $B := B(x_0,r)$ for some $x_0 \in \mathcal{X}$ and $r \in (0,\infty)$, let $a_k := S_k(a)$. Then, from the properties of S_k, it is easy to deduce that $S_k(a)$ is a continuous $(p(\cdot),\infty)$-atom with $\operatorname{supp} S_k(a) \subset B(x_0, r+c2^{-k})$ for some constant c independent of a and k. By the identity approximation property of $\{S_k\}_{k\in\mathbb{Z}}$ (see, for example, [29, Lemma 3.1(v)]), we find that, for all $q \in [1,\infty)$,

$$\lim_{k\to\infty} \|S_k(a)-a\|_{L^q(\mathcal{X})} = 0. \tag{4.28}$$

Now, let f be any element of $H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})$, namely, f has a decomposition

$$f = \sum_{j=1}^N \lambda_j a_j,$$

where $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ and $\{a_j\}_{j=1}^N$ are $(p(\cdot),\infty)$-atoms supported on balls $\{B(x_j,r_j)\}_{j=1}^N$, with $\{x_j\}_{j=1}^N \subset \mathcal{X}$ and $\{r_j\}_{j=1}^N \subset (0,\infty)$, such that

$$\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j=1}^N) \lesssim \|f\|_{H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})}.$$

Then, for any $\varepsilon \in (0,\infty)$, by (4.28), we find that there exists a $K \in \mathbb{N}$ such that, for all $j \in \{1,\dots,N\}$ and $k \in \mathbb{N}$ with $k > K$,

$$\|S_k(a_j)-a_j\|_{L^q(\mathcal{X})} \le \frac{[\mu(B(x_j,2r_j))]^{1/q}}{\|\chi_{B(x_j,2r_j)}\|_{L^{p(\cdot)}(\mathcal{X})}} \varepsilon,$$

which further implies that $\varepsilon^{-1}[S_k(a_j)-a_j]$ is a $(p(\cdot),q)$-atom supported on $B(x_j,2r_j)$. For $k\in\mathbb{N}$, let

$$f_k:=\sum_{j=1}^{N}(\varepsilon\lambda_j)\frac{S_k(a_j)}{\varepsilon}.$$

Obviously, $f_k\in H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})\cap\mathcal{C}(\mathcal{X})$. Moreover, by Theorem 4.24(i), we conclude that

$$\begin{aligned}\|f_k-f\|_{H^{*,p(\cdot)}(\mathcal{X})}&\sim\left\|\sum_{j=1}^{N}(\varepsilon\lambda_j)\frac{S_k(a_j)-a_j}{\varepsilon}\right\|_{H_{\rm at}^{p(\cdot),q}(\mathcal{X})}\\&\lesssim\varepsilon\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_ja_j\}_{j=1}^N)\lesssim\varepsilon\|f\|_{H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})}.\end{aligned}$$

Therefore, $f_k\to f$ in $H^{*,p(\cdot)}(\mathcal{X})$ as $k\to\infty$, and hence $H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})\cap\mathcal{C}(\mathcal{X})$ is dense in $H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})$ in the quasi-norm $\|\cdot\|_{H^{*,p(\cdot)}(\mathcal{X})}$. This, combined with the fact that $H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$, finishes the proof of Proposition 4.26. ■

We end this section by giving another proof of Theorem 4.3(ii) by borrowing some ideas from the proofs of [29, Lemma 4.15 and Theorem 4.16]. To this end, we need the following set-theoretical lemma.

Lemma 4.27. *Let $L\in(0,\infty)$ be fixed and $\{\Omega_m\}_{m=1}^\infty$ a sequence of subsets in $\mathcal{X}$. Define*

$$F:=\sum_{m=2}^{\infty}2^{-m}L\chi_{\Omega_m\setminus(\Omega_1\cup\cdots\cup\Omega_{m-1})}.$$

Then

$$\tfrac12F\chi_{\mathcal{X}\setminus\Omega_1}\le\sum_{m=2}^{\infty}2^{-m}L\chi_{\Omega_m}\le2F\chi_{\mathcal{X}\setminus\Omega_1}.\tag{4.29}$$

Proof. If $x\in\Omega_1$ or $x\notin\Omega_m$ for any $m\ge2$, then there is nothing to prove, since every term in (4.29) becomes 0. Let us suppose otherwise. Then we find the smallest $m_0\ge2$ such that $x\in\Omega_{m_0}\setminus(\Omega_1\cup\cdots\cup\Omega_{m_0-1})$. In this case, we have

$$2^{-m_0}L\le\sum_{m=2}^{\infty}2^{-m}L\chi_{\Omega_m}(x)\le\sum_{m=m_0}^{\infty}2^{-m}L=2^{-m_0+1}L$$

and

$$2^{-m_0}L=2^{-m_0}L\chi_{\Omega_{m_0}\setminus(\Omega_1\cup\cdots\cup\Omega_{m_0-1})}(x)\le F(x)\le\sum_{m=m_0}^{\infty}2^{-m}L=2^{-m_0+1}L.$$

This finishes the proof of Lemma 4.27. ■

Another proof of Theorem 4.3(ii). By Corollary 4.10, without loss of generality, we may assume that $f\in L^\infty(\mathcal{X})\cap H^{*,p(\cdot)}(\mathcal{X})$. We let

$$L:=\|f\|_{L^\infty(\mathcal{X})}\quad\text{and}\quad f_0:=f.$$

We then define $f_1,\dots,f_m$ and $\theta_1,\dots,\theta_m$, inductively. For the time being, let us say that $\theta_1\gg\cdots\gg\theta_m\downarrow0$. First, we define f_1 to be the function g^t in Proposition 4.9 associated to f with $t=\theta_1L$. Proceeding by induction, assume that f_{m-1} is defined. Then let

$$\Omega_m:=\{x\in\mathcal{X}:(f_{m-1})^*(x)>\theta_mL\}.$$

Define f_m to be the function g^t in Proposition 4.9 associated to f_{m-1} with $t = \theta_m L$. Notice that, by Lemma 4.5, each Ω_m has a decomposition

$$\Omega_m = \{x \in \mathcal{X} : (f_{m-1})^*(x) > \theta_m L\} = \bigcup_i B(x_{m,i}, r_{m,i}).$$

Define the partition of unity, $\{\phi_{m,i}\}_i$, as in Lemma 4.6. Then, according to our construction,

$$f_m = f_{m-1} - \sum_i b_{m,i},$$

where $b_{m,i} := (f - \eta_{m_i})\phi_{m,i}$ is as in Proposition 4.9, which, combined with Proposition 4.7, implies that there exists a positive constant K such that

$$f_m^*(x) \le f_{m-1}^*(x) + K\theta_m \sum_{j\in\mathbb{N}} \frac{\mu(B(x_{m,j}, r_{m,j}))}{\mu(B(x_{m,j}, r_{m,j} + d(x_{m,j}, x)))} \left[\frac{r_{m,j}}{r_{m,j} + d(x_{m,j}, x)}\right]^\beta$$

for all $x \in (\Omega_m)^\complement$ and $m \in \mathbb{N}$. Observe that K is independent of $\{f_m\}_{m=1}^\infty$ and $\{\theta_m\}_{m=1}^\infty$.

By the definition of f_m and Proposition 4.9(iv), we know that $\|f_m\|_{L^\infty(\mathcal{X})} \lesssim \theta_m L$, and hence

$$\|b_{m,i}\|_{L^\infty(\mathcal{X})} \le \theta_m DL,$$

where $D \in (1, \infty)$ is a fixed constant which is used later. More precisely, by Lemma 4.6(i),

$$|b_{m,i}(x)| \le \theta_m DL\chi_{B(x_{m,j}, 2r_{m,j})}(x) \tag{4.30}$$

for μ-almost every $x \in \mathcal{X}$. Notice that $f_m - f_{m-1} = -\sum_j b_{m,j}$ belongs to $L^\infty(\mathcal{X})$, thanks to Lemma 4.6(vi).

Since, for all $n \in \mathbb{N}$,

$$f = f_0 = f_1 + \sum_i b_{1,i} = \cdots = f_N + \sum_{m=1}^N \sum_j b_{m,j}$$

and

$$\|f_N\|_{L^\infty(\mathcal{X})} \lesssim \theta_N L,$$

it follows that

$$\left\| f - \sum_{m=1}^N \sum_j b_{m,j} \right\|_{L^\infty(\mathcal{X})} = \|f_N\|_{L^\infty(\mathcal{X})} \lesssim \theta_N L.$$

Therefore, assuming $\theta_N \downarrow 0$, we obtain

$$f = \sum_{m=1}^\infty \sum_j b_{m,j} \quad \text{in } L^\infty(\mathcal{X}),$$

and hence

$$f = \sum_{m=1}^\infty \sum_j b_{m,j} \quad \text{in } (\mathcal{G}_0^\epsilon(\beta, \gamma))'. \tag{4.31}$$

With these observations in mind, let

$$\lambda_{m,j} := \theta_m DL\|\chi_{B(x_{m,j}, 2r_{m,j})}\|_{L^{p(\cdot)}(\mathcal{X})} \quad \text{and} \quad e_{m,j} := (\lambda_{m,j})^{-1} b_{m,j}.$$

Then each $e_{m,j}$ is a $(p(\cdot),\infty)$-atom. Furthermore, by Remark 2.6, Theorem 2.7, (i) and (ii) of Lemma 4.5, and Lemma 4.27, we have

$$\begin{aligned}\mathrm{J}&:=\left\|\left(\sum_{m=1}^{\infty}\sum_{j}\left[\frac{|\lambda_{m,j}|}{\|\chi_{B(x_{m,j},r_{m,j})}\|_{L^{p(\cdot)}(\mathcal{X})}}\chi_{B(x_{m,j},r_{m,j})}\right]^{\underline{p}}\right)^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim\left\|\left[\sum_{m=1}^{\infty}\sum_{j}|\theta_m L\chi_{B(x_{m,j},r_{m,j})}|^{\underline{p}}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\left\|\left[\sum_{m=1}^{\infty}(\theta_m L\chi_{\Omega_m})^{\underline{p}}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim\left\|\theta_1 L\chi_{\Omega_1}+\sum_{m=2}^{\infty}\theta_m L\chi_{\Omega_m\setminus(\Omega_1\cup\cdots\cup\Omega_{m-1})}\right\|_{L^{p(\cdot)}(\mathcal{X})}.\end{aligned}$$

For any $N\in\mathbb{N}$, we define

$$\mathrm{J}_N:=\left\|\theta_1 L\chi_{\Omega_1}+\sum_{m=2}^{N}\theta_m L\chi_{\Omega_m\setminus(\Omega_1\cup\cdots\cup\Omega_{m-1})}\right\|_{L^{p(\cdot)}(\mathcal{X})}.$$

We claim that, for all $N\in\mathbb{N}$,

$$\mathrm{J}_N\le\left(1+\frac{1}{N^2}\right)^{1/\underline{p}}\mathrm{J}_{N-1}\tag{4.32}$$

by choosing $\theta_N\ll\theta_{N-1}$. Once (4.32) is proved, we obtain

$$\mathrm{J}_N\le\mathrm{J}_1\prod_{k=2}^{N}\left(1+\frac{1}{k^2}\right)^{1/\underline{p}}\le\|f^*\|_{L^{p(\cdot)}(\mathcal{X})}\prod_{k=2}^{N}\left(1+\frac{1}{k^2}\right)^{1/\underline{p}}\lesssim\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}$$

because $\prod_{k=2}^{\infty}(1+1/k^2)$ is convergent. Thus,

$$\begin{aligned}\mathrm{J}&\le\left\|\theta_1 L\chi_{\Omega_1}+\sum_{m=2}^{\infty}\theta_m L\chi_{\Omega_m\setminus(\Omega_1\cup\cdots\cup\Omega_{m-1})}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\
&=\varliminf_{N\to\infty}\left\|\theta_1 L\chi_{\Omega_1}+\sum_{m=2}^{N}\theta_m L\chi_{\Omega_m\setminus(\Omega_1\cup\cdots\cup\Omega_{m-1})}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\le\varliminf_{N\to\infty}\mathrm{J}_N\lesssim\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}\end{aligned}$$

and we obtain the desired conclusion.

It remains to prove (4.32). We first notice that, for all $N\in\mathbb{N}$,

$$\mathrm{J}_{N+1}\le\left\|\theta_1 L\chi_{\Omega_1}+\sum_{m=2}^{N}\theta_m L\chi_{\Omega_m\setminus(\Omega_1\cup\cdots\cup\Omega_{m-1})}+\theta_{N+1}L\chi_{\Omega_{N+1}\setminus(\Omega_1\cup\cdots\cup\Omega_N)}\right\|_{L^{p(\cdot)}(\mathcal{X})}.$$

Altogether, we obtain

$$(\mathrm{J}_{N+1})^{\underline{p}}\le(\mathrm{J}_N)^{\underline{p}}+\|L\theta_{N+1}\chi_{\Omega_{N+1}\setminus\Omega_N}\|_{L^{p(\cdot)}(\mathcal{X})}^{\underline{p}}.\tag{4.33}$$

Choose $\{\theta_N\}_{N\in\mathbb{N}}$ such that, for each $N\in\mathbb{N}$, $0<2\theta_{N+1}\le\theta_N$ and

$$\begin{aligned}\|L\theta_{N+1}\chi_{\Omega_{N+1}\setminus\Omega_N}\|_{L^{p(\cdot)}(\mathcal{X})}&\le\left[\frac{1}{(N+1)^2}\right]^{1/\underline{p}}\|\theta_1 L\chi_{\Omega_1}\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\le\left[\frac{1}{(N+1)^2}\right]^{1/\underline{p}}\mathrm{J}_N.\end{aligned}\tag{4.34}$$

Such a choice is possible. Indeed, we have $(f_N)^* \in L^{p(\cdot)}(\mathcal{X})$ and

$$L\theta_{N+1}\chi_{\Omega_{N+1}\setminus\Omega_N} \le (f_N)^*.$$

Thus, the absolute continuity of the $L^{p(\cdot)}(\mathcal{X})$ norm, together with an induction argument, allows us a choice of θ_N as in (4.34). This shows (4.32), thanks to (4.33) and (4.34), and hence finishes the proof of Theorem 4.3(ii). ∎

Before we conclude this section, let us compare the two proofs of Theorem 4.3(ii) presented in this article.

REMARK 4.28. (i) The second proof invokes the absolute continuity of $L^{p(\cdot)}(\mathcal{X})$, while the first one does not. So, it is expected that the first proof can be readily transformed into one when $L^{p(\cdot)}(\mathcal{X})$ is replaced by the Morrey space $\mathcal{M}^u_q(\mathcal{X})$, whose norm is defined by (6.3) below.

(ii) The second proof seems to have an advantage despite the fact that it is available only when one can use the absolute continuity of the norm. Indeed, by adjusting $\{\theta_N\}_{N\in\mathbb{N}}$ in the proof, that is, by replacing $\{\theta_N\}_{N\in\mathbb{N}}$ with a smaller one, one can control the growth speed of $\{J_N\}_{N\in\mathbb{N}}$. This fact may be useful elsewhere.

5. Characterizations in terms of Littlewood–Paley functions

In this section, we establish characterizations of $H^{*,p(\cdot)}(\mathcal{X})$ in terms of the Littlewood–Paley function. The main results of this section are stated in Subsection 5.1, and in Subsection 5.2 we give their proofs.

5.1. Main results. We begin with the following definition, taken from [33, p. 1510].

DEFINITION 5.1. Let $\epsilon_1 \in (0,1]$ and $\epsilon_2 \in (0,\infty)$. A family of bounded linear operators, $\{D_t\}_{t\in(0,\infty)}$, on $L^2(\mathcal{X})$ is called a *Calderón reproducing formula of order* (ϵ_1,ϵ_2) (for short, (ϵ_1,ϵ_2)-CRF) in $L^2(\mathcal{X})$ if, for all $f \in L^2(\mathcal{X})$,

$$f = \int_0^\infty D_t^2(f)\,\frac{dt}{t} \tag{5.1}$$

in $L^2(\mathcal{X})$, and moreover, for all $f \in L^2(\mathcal{X})$ and $x \in \mathcal{X}$,

$$D_t(f)(x) = \int_{\mathcal{X}} D_t(x,y)f(y)\,d\mu(y),$$

where $D_t(\cdot,\cdot)$ is a measurable function from $\mathcal{X}\times\mathcal{X}$ to $\mathbb{C}$ satisfying the following estimates: there exists a positive constant C such that, for all $t \in (0,\infty)$ and all $x, x', y, y' \in \mathcal{X}$ with $d(x,x') \le [t+d(x,y)]/2$,

(A1) $|D_t(x,y)| \le C\dfrac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\dfrac{t}{t+d(x,y)}\right]^{\epsilon_2}$;

(A2)

$$|D_t(x,y)-D_t(x',y)| \le C\left[\frac{d(x,x')}{t+d(x,y)}\right]^{\epsilon_1}\left[\frac{t}{t+d(x,y)}\right]^{\epsilon_2}\frac{1}{V_t(x)+V_t(y)+V(x,y)};$$

(A3) property (A2) still holds true with the roles of x and y interchanged;

(A4) $\displaystyle\int_{\mathcal{X}} D_t(x,z)\,d\mu(z) = 0 = \int_{\mathcal{X}} D_t(z,y)\,d\mu(z).$

In what follows, we define

$$\mathring{\mathcal{G}}(x_1,r,\beta,\gamma) := \left\{ f \in \mathcal{G}(x_1,r,\beta,\gamma) : \int_{\mathcal{X}} f(x)\,d\mu(x) = 0 \right\},$$

and the space $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ to be the completion of $\mathring{\mathcal{G}}(\epsilon,\epsilon)$ in $\mathring{\mathcal{G}}(\beta,\gamma)$ when $\beta,\gamma \in (0,\epsilon)$.

Recall that the *Littlewood–Paley* $\mathcal{S}$*-function* (also called the *Lusin area function*) $\mathcal{S}(f)$ of $f \in L^{p(\cdot)}(\mathcal{X})$ is defined by setting, for all $x \in \mathcal{X}$,

$$\mathcal{S}(f)(x) := \left\{ \int_{\Gamma(x)} |D_t(f)(y)|^2 \frac{d\mu(y)\,dt}{V_t(x)t} \right\}^{1/2}, \tag{5.2}$$

where

$$\Gamma(x) := \{(y,t) \in \mathcal{X} \times (0,\infty) : d(x,y) < t\}.$$

Definition 5.2. Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and $\{D_t\}_{t\in(0,\infty)}$ be an (ϵ_1,ϵ_2)-CRF in $L^2(\mathcal{X})$ as in Definition 5.1. Assume that the parameters ϵ, ϵ_1, ϵ_2, β, γ satisfy $\epsilon_1 \in (0,1]$, $\epsilon_2 \in [\epsilon_1 + n/2, \infty)$, $\epsilon \in (0,\epsilon_1)$ and $\beta,\gamma \in (0,\epsilon)$. Then the *Hardy space* $H^{p(\cdot)}(\mathcal{X})$ *via the Lusin area function* is defined by

$$H^{p(\cdot)}(\mathcal{X}) := \{ f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))' : \mathcal{S}(f) \in L^{p(\cdot)}(\mathcal{X})\},$$

and its quasi-norm is given by $\|f\|_{H^{p(\cdot)}(\mathcal{X})} := \|\mathcal{S}(f)\|_{L^{p(\cdot)}(\mathcal{X})}$.

Let $q \in [1,\infty] \cap (p_+,\infty]$. Then define $\mathring{H}^{p(\cdot),q}_{\rm at}(\mathcal{X})$ in the same way as $H^{p(\cdot),q}_{\rm at}(\mathcal{X})$ with $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ replaced by $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$.

We first establish the atomic characterization of $H^{p(\cdot)}(\mathcal{X})$, which, when $p(\cdot) \equiv p \in (0,1]$, was obtained in [33, Theorem 2.21].

Theorem 5.3. *Let* $p(\cdot) \in C^{\log}_{(n/(n+1),1]}(\mathcal{X})$ *and* $q \in [1,\infty] \cap (p_+,\infty]$. *Then*

$$H^{p(\cdot)}(\mathcal{X}) = \mathring{H}^{p(\cdot),q}_{\rm at}(\mathcal{X})$$

with equivalent quasi-norms.

Comparing $\mathring{H}^{p(\cdot),q}_{\rm at}(\mathcal{X})$ with the atomic Hardy space corresponding to $H^{*,p(\cdot)}(\mathcal{X})$, we have the following conclusion.

Theorem 5.4. *Let* $p(\cdot) \in C^{\log}_{(n/(n+1),1]}(\mathcal{X})$ *and* $q \in [1,\infty] \cap (p_+,\infty]$. *Then*

$$\mathring{H}^{p(\cdot),q}_{\rm at}(\mathcal{X}) = H^{p(\cdot),q}_{\rm at}(\mathcal{X})$$

with equivalent quasi-norms.

As an application of Theorem 5.3, we establish the g_λ^*-function characterization of $H^{p(\cdot)}(\mathcal{X})$. Let $\lambda \in (0,\infty)$. Recall that the *Littlewood–Paley* g_λ^**-function* of $f \in L^{p(\cdot)}(\mathcal{X})$ is defined by setting, for all $x \in \mathcal{X}$,

$$g_\lambda^*(f)(x) := \left\{ \int_0^\infty \int_{\mathcal{X}} \left[\frac{t}{t+d(x,y)}\right]^\lambda |D_t(f)(y)|^2 \frac{d\mu(y)\,dt}{[V_t(x)+V_t(y)]t} \right\}^{1/2}.$$

THEOREM 5.5. *Let ϵ, ϵ_1, β, γ and $p(\cdot)$ be as in Definition 5.2. If $\lambda \in (n + 2n/p_-, \infty)$, then $f \in H^{p(\cdot)}(\mathcal{X})$ if and only if $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ and $g_\lambda^*(f) \in L^{p(\cdot)}(\mathcal{X})$. Moreover,*

$$\|f\|_{H^{p(\cdot)}(\mathcal{X})} \sim \|g_\lambda^*(f)\|_{L^{p(\cdot)}(\mathcal{X})}$$

with implicit positive constants independent of f.

REMARK 5.6. When $p(\cdot) \equiv p \in (0, 1]$, Theorem 5.5 is just [33, Proposition 3.4(ii)].

Let $\epsilon_1 \in (0, 1]$, $\epsilon_2, \epsilon_3 \in (0, \infty)$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. For all $k \in \mathbb{Z}$, let $D_k := S_k - S_{k-1}$. Assume that $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\beta, \gamma \in (0, \epsilon)$. Recall that the *Littlewood–Paley g-function* of $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ is defined by setting, for all $x \in \mathcal{X}$,

$$g(f)(x) := \left\{\sum_{k\in\mathbb{Z}} |D_k(f)(x)|^2\right\}^{1/2};$$

see [34].

THEOREM 5.7. *Let $\epsilon_1 \in (0, 1]$, $\epsilon_2, \epsilon_3 \in (0, \infty)$ and let $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$. Assume that $p(\cdot) \in C^{\log}_{(n/(n+\epsilon),\infty)}(\mathcal{X})$ satisfies $p_- \in (n/(n + \epsilon), 1]$ and $\beta, \gamma \in (n[1/p_- - 1], \epsilon)$. Then $f \in H^{p(\cdot)}(\mathcal{X})$ if and only if $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ and $g(f) \in L^{p(\cdot)}(\mathcal{X})$. Moreover,*

$$C^{-1}\|g(f)\|_{L^{p(\cdot)}(\mathcal{X})} \le \|f\|_{H^{p(\cdot)}(\mathcal{X})} \le C\|g(f)\|_{L^{p(\cdot)}(\mathcal{X})}$$

with C being a positive constant independent of f.

REMARK 5.8. In the case of $p(\cdot) := p$ with $p \in (0, \infty)$, Theorem 5.7 was proved in [34, Theorem 5.16].

5.2. Proofs of main results of Section 5.1. We begin with the proof of Theorem 5.4. To this end, we first establish the following estimate.

LEMMA 5.9. *Let $p_+ \in (0, 1]$ and $\gamma \in [p_+, 1]$. Then there exists a positive constant C such that, for all $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and any sequence $\{B_j\}_{j\in\mathbb{N}}$ of balls in $\mathcal{X}$,*

$$\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^\gamma\Big)^{1/\gamma} \le \widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}). \tag{5.3}$$

Proof. To prove (5.3), let

$$\lambda := \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^\gamma\Big)^{1/\gamma}.$$

Then, by (3.9) and Remark 2.3(iv), we see that

$$\begin{aligned}
\int_{\mathcal{X}} \left\{\sum_{j\in\mathbb{N}} \left[\frac{|\lambda_j|\chi_{B_j}(x)}{\lambda\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\right]^{\underline{p}}\right\}^{p(x)/\underline{p}} d\mu(x) &\ge \int_{\mathcal{X}} \sum_{j\in\mathbb{N}} \left[\frac{|\lambda_j|\chi_{B_j}(x)}{\lambda\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\right]^{p(x)} d\mu(x)\\
&\ge \sum_{j\in\mathbb{N}} \frac{|\lambda_j|^\gamma}{\lambda^\gamma} \int_{\mathcal{X}} \left[\frac{\chi_{B_j}(x)}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\right]^{p(x)} d\mu(x)\\
&= \sum_{j\in\mathbb{N}} \frac{|\lambda_j|^\gamma}{\lambda^\gamma} = 1,
\end{aligned}$$

which implies that (5.3) holds true. This finishes the proof of Lemma 5.9. ∎

Proof of Theorem 5.4. Let $\epsilon \in (0,1)$ and $\beta, \gamma \in (0, \epsilon)$. Observe that

$$(\mathcal{G}_0^\epsilon(\beta,\gamma))' \subset (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'.$$

Then it follows that $H_{\rm at}^{p(\cdot),q}(\mathcal{X}) \subset \mathring{H}_{\rm at}^{p(\cdot),q}(\mathcal{X})$. Thus, to prove this theorem it suffices to show that $\mathring{H}_{\rm at}^{p(\cdot),q}(\mathcal{X}) \subset H_{\rm at}^{p(\cdot),q}(\mathcal{X})$. Let $f \in \mathring{H}_{\rm at}^{p(\cdot),q}(\mathcal{X})$. Then, by Definition 5.2, we know that $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ and there exist sequences $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $\{a_j\}_{j\in\mathbb{N}}$ of $(p(\cdot), q)$-atoms such that $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ and

$$\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}) \lesssim \|f\|_{\mathring{H}_{\rm at}^{p(\cdot),q}(\mathcal{X})},$$

where B_j is the support of a_j for all $j \in \mathbb{N}$. For any $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, let

$$\langle \widetilde{f}, \varphi\rangle := \sum_{j\in\mathbb{N}} \lambda_j \langle a_j, \varphi\rangle.$$

Observe that, for any $j \in \mathbb{N}$, we have $a_j \in H_{\rm at}^{p(\cdot),q}(\mathcal{X})$ and $\|a_j\|_{H_{\rm at}^{p(\cdot),q}(\mathcal{X})} \le 1$. Then, from Theorem 4.3 and Lemma 3.6, we deduce that, for any $j \in \mathbb{N}$,

$$|\langle a_j, \varphi\rangle| \lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} \|a_j\|_{H_{\rm at}^{p(\cdot),q}(\mathcal{X})} \lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)}.$$

This, combined with Lemma 5.9, yields

$$\begin{aligned} |\langle \widetilde{f}, \varphi\rangle| &\lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} \sum_{j\in\mathbb{N}} |\lambda_j| \\ &\lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} \widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}) \lesssim \|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} \|f\|_{\mathring{H}_{\rm at}^{p(\cdot),q}(\mathcal{X})}, \end{aligned}$$

which implies that $\widetilde{f} \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $\widetilde{f} = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Moreover, $\widetilde{f} = f$ on $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$, $\widetilde{f} \in H_{\rm at}^{p(\cdot),q}(\mathcal{X})$ and

$$\|\widetilde{f}\|_{H_{\rm at}^{p(\cdot),q}(\mathcal{X})} \lesssim \|f\|_{\mathring{H}_{\rm at}^{p(\cdot),q}(\mathcal{X})}.$$

Suppose that there exists another extension of f, say $\widetilde{g} \in H_{\rm at}^{p(\cdot),q}(\mathcal{X})$. Then $\widetilde{g} = f$ on $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$. Thus, by [33, Lemma 5.2], $\widetilde{f} - \widetilde{g}$ is a constant, denoted by $\widetilde{C}$. If $\widetilde{C} \neq 0$, then this contradicts the fact that no non-zero constant function belongs to $H_{\rm d}^{p(\cdot)}(\mathcal{X}) = H_{\rm at}^{p(\cdot),q}(\mathcal{X})$. Thus, $\widetilde{C} = 0$, which implies that $\widetilde{f} \in H_{\rm at}^{p(\cdot),q}(\mathcal{X})$ is the unique extension of f. This finishes the proof of Theorem 5.4. ■

To prove Theorem 5.3, we need some general facts on the Lusin area function. Let $\mathcal{D} := \{Q_\tau^k : k \in \mathbb{Z}, \tau \in I_k\}$ be the set of all dyadic cubes as in Lemma 3.13. For all $k \in \mathbb{Z}$, we set

$$\Omega_k := \{x \in \mathcal{X} : \mathcal{S}(f)(x) > 2^k\}.$$

Observe that $\{\Omega_k\}_{k\in\mathbb{Z}}$ is a decreasing family. With this in mind, let

$$\mathcal{D}_k := \left\{Q \in \mathcal{D} : \mu(Q \cap \Omega_k) > \tfrac{1}{2}\mu(Q) \text{ and } \mu(Q \cap \Omega_{k+1}) \le \tfrac{1}{2}\mu(Q)\right\}.$$

Let D be the positive constant in (3.5). For $Q_\tau^k \in \mathcal{D}$, define

$$\widehat{Q}_\tau^k := Q_\tau^k \times (D2^{-k}, D2^{-k+1}].$$

Notice that

$$\mu \otimes \frac{dt}{t}\Big(\mathcal{X} \times (0,\infty) \setminus \bigcup_{Q\in\mathcal{D}} \widehat{Q}\Big) = 0 \tag{5.4}$$

thanks to Lemma 3.13(i). Define the set $\mathcal{D}_k^{\mathrm{mc}}$ of maximal dyadic cubes by

$$\mathcal{D}_k^{\mathrm{mc}} := \{Q \in \mathcal{D}_k : \text{if } \widetilde{Q} \supsetneq Q \text{ and } \widetilde{Q} \in \mathcal{D}, \text{ then } \widetilde{Q} \notin \mathcal{D}_k\}.$$

Then $\mathcal{D}_k^{\mathrm{mc}}$ is the set of all dyadic cubes in $\mathcal{D}_k$, which are maximal with respect to inclusion, so that we discard cubes which are not maximal. Hence, by Lemma 3.13(i), we see that

$$\mu\Big(\mathcal{X} \setminus \bigcup_{k\in\mathbb{Z}} \bigcup_{Q\in\mathcal{D}_k^{\mathrm{mc}}} Q\Big) = 0,$$

and hence (5.4) can be rephrased as

$$\mu \otimes \frac{dt}{t}\Big(\mathcal{X} \times (0,\infty) \setminus \bigcup_{k\in\mathbb{Z}} \bigcup_{Q_k^{\mathrm{mc}}\in\mathcal{D}_k^{\mathrm{mc}}} \bigcup_{Q\in\mathcal{D}_k,\, Q\subset Q_k^{\mathrm{mc}}} \widehat{Q}\Big) = 0.$$

For all $k \in \mathbb{Z}$, let

$$\widetilde{Q}_k^{\mathrm{mc}} := \bigcup_{Q\in\mathcal{D}_k,\, Q\subset Q_k^{\mathrm{mc}}} \widehat{Q}.$$

We invoke the following lemma from [33, Lemma 2.23].

Lemma 5.10. *Let $\{D_t\}_{t\in(0,\infty)}$ be an (ϵ_1,ϵ_2)-CRF in $L^2(\mathcal{X})$ with $\epsilon_1 \in (0,1]$ and $\epsilon_2 \in (\epsilon_1 + n/2, \infty)$, and let $\epsilon \in (0,\epsilon_1)$ and $\beta,\gamma \in (0,\epsilon)$. If $A \in (0,\epsilon_2]$, then for any $f \in (\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma))'$,*

$$f = \sum_{l=0}^{\infty} 2^{-Al} \sum_{k\in\mathbb{Z}} \sum_{Q_k^{\mathrm{mc}}\in\mathcal{D}_k^{\mathrm{mc}}} \int_{\widetilde{Q}_k^{\mathrm{mc}}} \varphi_{2^l t}(\cdot,y) D_t(f)(y) \,\frac{d\mu(y)\,dt}{t}$$

in $(\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma))'$, where $\varphi_{2^l t}(x,y)$ is an adjusted bump function in x associated with the ball $B(y,2^l t)$, which means that there exists a positive constant C such that, for all $x,y \in \mathcal{X}$,

(i) $\operatorname{supp} \varphi_{2^l t}(\cdot,y) \subset B(y,2^l t)$;

(ii) $|\varphi_{2^l t}(x,y)| \le C/V_{2^l t}(y)$;

(iii) *for all $\eta \in (0,\epsilon_1)$,*

$$\|\varphi_{2^l t}(\cdot,y)\|_{\dot{C}^{\eta}(\mathcal{X})} := \sup_{x,y\in\mathcal{X},\, x\neq y} \frac{|f(x)-f(y)|}{[d(x,y)]^{\eta}} \le C(2^l t)^{-1} \frac{1}{V_{2^l t}(y)};$$

(iv) $\displaystyle\int_{\mathcal{X}} \varphi_{2^l t}(x,y)\, d\mu(x) = 0.$

For a fixed cube $Q_k^{\mathrm{mc}} \in \mathcal{D}_k^{\mathrm{mc}}$, we know that there exist $k_0 \in \mathbb{Z}$ and $\beta_0 \in I_{k_0}$ such that $Q_k^{\mathrm{mc}} = Q_{\beta_0}^{k_0}$, and for all $l \in \mathbb{Z}$, we set

$$B_k^{\mathrm{mc},l} := B(z_{\beta_0}^{k_0}, D2^{2+l-k_0}).$$

Now, let

$$\lambda_{Q_k^{\mathrm{mc}}}^l := 2^{-(A+\kappa)l+k} \|\chi_{B_k^{\mathrm{mc},l}}\|_{L^{p(\cdot)}(\mathcal{X})},$$

where A is as in Lemma 5.10, and for all $x \in \mathcal{X}$,

$$a_{Q_k^{\mathrm{mc}}}^l(x) := \frac{2^{l\kappa-k}}{\|\chi_{B_k^{\mathrm{mc},l}}\|_{L^{p(\cdot)}(\mathcal{X})}} \int_{\widetilde{Q}_k^{\mathrm{mc}}} \varphi_{2^l t}(x,y) D_t(f)(y) \,\frac{d\mu(y)\,dt}{t}.$$

Lemma 5.11. *Let $q \in [1,\infty]$, $l \in \mathbb{Z}_+$ and $Q_k^{\mathrm{mc}} = Q_{\beta_0}^{k_0}$ for some $k_0 \in \mathbb{Z}$ and $\beta_0 \in I_{k_0}$. Then, for any $f \in (\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma))'$, $a_{Q_k^{\mathrm{mc}}}^l$ is a $(p(\cdot),q)$-atom supported on $B_k^{\mathrm{mc},l}$ up to a constant multiple.*

To prove Lemma 5.11, we need the following estimate.

LEMMA 5.12. *For all $l \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$,*

$$\left\{\sum_{Q\in\mathcal{D}_k,\, Q\subset Q_k^{\mathrm{mc}}} \int_{\widehat{Q}} |D_t(f)(y)|^2 \,\frac{d\mu(y)\,dt}{t}\right\}^{1/2} \lesssim 2^k 2^{-l\kappa}[\mu(B_k^{\mathrm{mc},l})]^{1/2},$$

where the implicit positive constant is independent of f, k and l.

Proof. Let $Q \in \mathcal{D}_k$. We estimate

$$\mathrm{I} := \int_{Q\setminus\Omega_{k+1}} [\mathcal{S}(f)(x)]^2\,d\mu(x)$$

from below. If we write out the definition of I in full, we obtain

$$\begin{aligned}
\mathrm{I} &= \int_{Q\setminus\Omega_{k+1}} \int_{\Gamma(x)} |D_t(f)(y)|^2 \,\frac{d\mu(y)\,dt}{V_t(x)t}\,d\mu(x)\\
&= \int_{\mathcal{X}}\int_0^\infty \int_{B(y,t)\cap(Q\setminus\Omega_{k+1})} \frac{1}{V_t(x)}|D_t(f)(y)|^2\,d\mu(x)\frac{d\mu(y)\,dt}{t}.
\end{aligned}$$

Since $V_t(x) \sim V_t(y)$ when $d(x,y) \le t$, it follows that

$$\begin{aligned}
\mathrm{I} &\sim \int_{\mathcal{X}}\int_0^\infty \int_{B(y,t)\cap(Q\setminus\Omega_{k+1})} \frac{1}{V_t(y)}|D_t(f)(y)|^2\,d\mu(x)\frac{d\mu(y)\,dt}{t}\\
&\sim \int_{\mathcal{X}}\int_0^\infty \frac{\mu(B(y,t)\cap(Q\setminus\Omega_{k+1}))}{V_t(y)}|D_t(f)(y)|^2\,\frac{d\mu(y)\,dt}{t}\\
&\gtrsim \int_{\widehat{Q}} \frac{\mu(B(y,t)\cap(Q\setminus\Omega_{k+1}))}{V_t(y)}|D_t(f)(y)|^2\,\frac{d\mu(y)\,dt}{t}.
\end{aligned}$$

Notice that, when $(y,t) \in \widehat{Q}$, we have $y \in Q$ and $t \ge \operatorname{diam}(Q)$, and when $Q \in \mathcal{D}_k$, we further have $2\mu(Q\setminus\Omega_{k+1}) \ge \mu(Q)$. Hence, by (2.2), we find that

$$\mathrm{I} \gtrsim \int_{\widehat{Q}} \frac{\mu(Q\setminus\Omega_{k+1})}{V_t(y)}|D_t(f)(y)|^2\,\frac{d\mu(y)\,dt}{t} \gtrsim \int_{\widehat{Q}} |D_t(f)(y)|^2\,\frac{d\mu(y)\,dt}{t}. \tag{5.5}$$

Let us suppose $Q = Q_{\beta_0}^{k_0} \in \mathcal{D}_k$ and $Q \subset Q_k^{\mathrm{mc}}$ for some $k_0 \in \mathbb{Z}$ and $\beta_0 \in I_{k_0}$. Then, from (5.5), we deduce that

$$\begin{aligned}
&\left\{\sum_{Q\in\mathcal{D}_k,\, Q\subset Q_k^{\mathrm{mc}}} \int_{\widehat{Q}} |D_t(f)(x,y)|^2 \,\frac{d\mu(y)\,dt}{t}\right\}^{1/2}\\
&\qquad\lesssim \left\{\sum_{Q\in\mathcal{D}_k,\, Q\subset Q_k^{\mathrm{mc}}} \int_{Q\setminus\Omega_{k+1}} [\mathcal{S}(f)(x)]^2\,d\mu(x)\right\}^{1/2}\\
&\qquad\lesssim 2^k[\mu(Q_k^{\mathrm{mc}})]^{1/2} \lesssim 2^k 2^{-l\kappa}[\mu(B_k^{\mathrm{mc},l})]^{1/2}.
\end{aligned}$$

This finishes the proof of Lemma 5.12. ∎

Proof of Lemma 5.11. Let us first show that $a^l_{Q_k^{\mathrm{mc}}}$ is supported on $B_k^{\mathrm{mc},l}$. If $(y,t) \in \widetilde{Q}_k^{\mathrm{mc}}$, then $(y,t) \in \widehat{Q}$ for some $Q \in \mathcal{D}_k$ such that $Q \subset Q_k^{\mathrm{mc}} := Q_{\beta_0}^{k_0}$ for some $k_0 \in \mathbb{Z}$ and $\beta_0 \in I_{k_0}$.

Consequently, we obtain $y \in Q \subset Q_k^{\mathrm{mc}} = Q_{\beta_0}^{k_0}$ and $t \le D2^{-k_0+1}$. If $x \in B(y, 2^l t)$, then

$$d(x, z_{\beta_0}^{k_0}) \le d(x,y) + d(y, z_{\beta_0}^{k_0}) \le D2^{2+l-k_0},$$

which implies $x \in B_k^{\mathrm{mc},l}$. Hence, $\operatorname{supp} a_{Q_k^{\mathrm{mc},l}} \subset B_k^{\mathrm{mc},l}$ by the support property of $\varphi_{2^l t}(\cdot, y)$.

To show the size condition on $a^l_{Q_k^{\mathrm{mc}}}$, it suffices to estimate $\|a^l_{Q_k^{\mathrm{mc}}}\|_{L^\infty(\mathcal{X})}$. Observe that, when $(y,t) \in \widetilde{Q}_k^{\mathrm{mc}}$, we have $d(y, z_{\beta_0}^{k_0}) \le D2^{-k_0}$. Thus, for all $\xi \in B(z_{\beta_0}^{k_0}, D2^{2+l-k_0})$,

$$d(\xi, y) \le d(\xi, z_{\beta_0}^{k_0}) + d(z_{\beta_0}^{k_0}, y) < D2^{3+l-k_0},$$

namely,

$$B(z_{\beta_0}^{k_0}, D2^{2+l-k_0}) \subset B(y, D2^{3+l-k_0}).$$

From this, (2.2) and the fact that $t \le D2^{-k_0+1}$, we further deduce that, for all $(y,t) \in \widetilde{Q}_k^{\mathrm{mc}}$,

$$\mu(B(z_{\beta_0}^{k_0}, D2^{2+l-k_0})) \le \mu(B(y, D2^{3+l-k_0})) \lesssim (D2^{3-k_0}t^{-1})^n \mu(B(y, 2^l t)) \lesssim \mu(B(y, 2^l t)).$$

Therefore, using condition (ii) in Lemma 5.10, we find that

$$\begin{aligned}\left\{\int_{\widetilde{Q}_k^{\mathrm{mc}}} |\varphi_{2^l t}(x,y)|^2 \,\frac{d\mu(y)\,dt}{t}\right\}^{1/2} &\lesssim \left\{\int_{\widetilde{Q}_k^{\mathrm{mc}}} \frac{1}{[V_{2^l t}(y)]^2}\,\frac{d\mu(y)\,dt}{t}\right\}^{1/2}\\ &\lesssim [\mu(B(z_{\beta_0}^{k_0}, D2^{2+l-k_0}))]^{-1}\left\{\int_{\widetilde{Q}_k^{\mathrm{mc}}} \frac{d\mu(y)\,dt}{t}\right\}^{1/2}\\ &\lesssim [\mu(B(z_{\beta_0}^{k_0}, D2^{2+l-k_0}))]^{-1/2}.\end{aligned}$$

From this, the Hölder inequality and Lemma 5.12, we conclude that, for all $x \in \mathcal{X}$,

$$\begin{aligned}&\left|\int_{\widetilde{Q}_k^{\mathrm{mc}}} \varphi_{2^l t}(x,y) D_t(f)(y)\,\frac{d\mu(y)\,dt}{t}\right|\\ &\quad\le \left\{\int_{\widetilde{Q}_k^{\mathrm{mc}}} |D_t(f)(y)|^2\,\frac{d\mu(y)\,dt}{t}\right\}^{1/2}\left\{\int_{\widetilde{Q}_k^{\mathrm{mc}}} |\varphi_{2^l t}(x,y)|^2\,\frac{d\mu(y)\,dt}{t}\right\}^{1/2} \lesssim 2^k 2^{-l\kappa},\end{aligned}$$

which implies that $\|a^l_{Q_k^{\mathrm{mc}}}\|_{L^\infty(\mathcal{X})} \lesssim 1/\|\chi_{B_k^{\mathrm{mc},l}}\|_{L^{p(\cdot)}(\mathcal{X})}$.

Finally, by the vanishing moment condition on $\varphi_{2^l t}(\cdot, y)$, we see that

$$\int_{\mathcal{X}} a^l_{Q_k^{\mathrm{mc}}}(x)\,d\mu(x) = 0.$$

Thus, $a^l_{Q_k^{\mathrm{mc}}}$ is a $(p(\cdot),\infty)$-atom supported on $B_k^{\mathrm{mc},l}$. This finishes the proof of Lemma 5.11. ■

We now turn to the proof of Theorem 5.3.

Proof of Theorem 5.3. We first show that $\mathring{H}^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X}) \hookrightarrow H^{p(\cdot)}(\mathcal{X})$. Let $f \in \mathring{H}^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})$. Then there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $(p(\cdot),q)$-atoms $\{a_j\}_{j\in\mathbb{N}}$, where, for $j \in \mathbb{N}$, $\operatorname{supp} a_j \subset B_j := B(x_j, r_j)$ for some $x_j \in \mathcal{X}$ and $r_j \in (0,\infty)$, such that

$$\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}) \lesssim \|f\|_{\mathring{H}^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})}$$

and $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma))'$ with ϵ, β, γ as in Definition 5.2.

By the triangle inequality, we have

$$\mathcal{S}(f)\le\sum_{j\in\mathbb{N}}|\lambda_j|\mathcal{S}(a_j)\chi_{2B_j}+\sum_{j\in\mathbb{N}}|\lambda_j|\mathcal{S}(a_j)\chi_{\mathcal{X}\setminus(2B_j)}=:\mathrm{I}+\mathrm{II}.$$

For I, by the boundedness of $\mathcal{S}$ on $L^q(\mathcal{X})$ (see [33, Proposition 2.17]), we obtain

$$\|\mathcal{S}(a_j)\|_{L^q(\mathcal{X})}\lesssim\|a_j\|_{L^q(\mathcal{X})}\lesssim\frac{[\mu(B_j)]^{1/q}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}$$

with $\epsilon_1\in(0,1]$ satisfying $p_-\in\left(\frac{n}{n+\epsilon_1},\infty\right)$, which, together with Proposition 2.11, implies that $\|\mathrm{I}\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\|f\|_{\mathring{H}^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})}$.

For II, by an argument similar to that used in [33, pp. 1521–1522], we see that, for all $x\in\mathcal{X}\setminus(2B_j)$,

$$\mathcal{S}(a_j)(x)\lesssim\frac{\mu(B_j)}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\left[\frac{r_j}{d(x,x_j)}\right]^{\epsilon_1}\frac{1}{V(x,x_j)},$$

which, together with (2.2), implies that, for all $x\in\mathcal{X}\setminus(2B_j)$,

$$\begin{aligned}\mathcal{S}(a_j)(x)&\lesssim\frac{1}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\left[\frac{\mu(B_j)}{\mu(B(x,d(x,x_j)))}\right]^{\epsilon_1/n+1}\\&\lesssim\frac{1}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}[M(\chi_{B_j})(x)]^{\epsilon_1/n+1}.\end{aligned}\tag{5.6}$$

From (5.6), Theorem 2.7 and the fact that $p_-\in(n/(n+\epsilon_1),\infty)$, we deduce that

$$\begin{aligned}\|\mathrm{II}\|_{L^{p(\cdot)}(\mathcal{X})}&\lesssim\left\|\sum_{j\in\mathbb{N}}|\lambda_j|\frac{[M(\chi_{B_j})]^{\epsilon_1/n+1}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\&\lesssim\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}})\lesssim\|f\|_{\mathring{H}^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})}.\end{aligned}$$

This implies $\|\mathcal{S}(f)\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\|f\|_{\mathring{H}^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})}$, and hence $\mathring{H}^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})\hookrightarrow H^{p(\cdot)}(\mathcal{X})$.

Conversely, we prove $H^{p(\cdot)}(\mathcal{X})\hookrightarrow\mathring{H}^{p(\cdot),q}_{\mathrm{at}}(\mathcal{X})$. The proof follows an argument similar to that used in the proof of [33, Theorem 2.21]. According to Lemmas 5.10 and 5.11, we have an expression

$$f=\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{Q_k^{\mathrm{mc}}\in\mathcal{D}_k^{\mathrm{mc}}}\lambda^l_{Q_k^{\mathrm{mc}}}a^l_{Q_k^{\mathrm{mc}}}$$

in $(\mathring{\mathcal{G}}^{\epsilon}_0(\beta,\gamma))'$. It thus remains to establish the norm estimate. Indeed, by the construction, Theorem 2.7 and the fact that $\chi_{B_k^{\mathrm{mc},l}}\lesssim 2^{ln}M(\chi_{Q_k^{\mathrm{mc}}})$, we find that

$$\begin{aligned}\mathrm{J}&:=\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda^l_{Q_k^{\mathrm{mc}}}\}_{l\in\mathbb{Z}_+,\,k\in\mathbb{Z},\,Q_k^{\mathrm{mc}}\in\mathcal{D}_k^{\mathrm{mc}}},\{B_k^{\mathrm{mc},l}\}_{l\in\mathbb{Z}_+,\,k\in\mathbb{Z},\,Q_k^{\mathrm{mc}}\in\mathcal{D}_k^{\mathrm{mc}}})\\&\lesssim\left\|\left(\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{Q_k^{\mathrm{mc}}\in\mathcal{D}_k^{\mathrm{mc}}}[2^{-(A-\kappa)l}2^{-k}\chi_{B_k^{\mathrm{mc},l}}]^{\underline{p}}\right)^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\&\lesssim\left\|\left(\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{Q_k^{\mathrm{mc}}\in\mathcal{D}_k^{\mathrm{mc}}}[2^{-l(A-\kappa-n)}2^{-k}\chi_{Q_k^{\mathrm{mc}}}]^{\underline{p}}\right)^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})},\end{aligned}$$

which, combined with Theorem 2.7 again and the fact that $\chi_{Q_k^{\mathrm{mc}}} \lesssim M(\chi_{Q_k^{\mathrm{mc}}\cap\Omega_k})$ when $Q_k^{\mathrm{mc}} \in \mathcal{D}_k^{\mathrm{mc}}$, further implies that

$$\begin{aligned}
\mathrm{J} &\lesssim \Big\|\Big(\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{Q_k^{\mathrm{mc}}\in\mathcal{D}_k^{\mathrm{mc}}}[2^{-l(A-\kappa-n)}2^{-k}\chi_{Q_k^{\mathrm{mc}}\cap\Omega_k}]^{\underline{p}}\Big)^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim \Big\|\Big(\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}}[2^{-l(A-\kappa-n)}2^{-k}\chi_{\Omega_k}]^{\underline{p}}\Big)^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})} \sim \Big\|\Big(\sum_{k\in\mathbb{Z}}2^{-k\underline{p}}\chi_{\Omega_k}\Big)^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\lesssim \Big\|\Big(\sum_{k\in\mathbb{Z}}2^{-k\underline{p}}\chi_{\Omega_k\setminus\Omega_{k+1}}\Big)^{1/\underline{p}}\Big\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \|\mathcal{S}(f)\|_{L^{p(\cdot)}(\mathcal{X})}.
\end{aligned}$$

This shows that $f \in \mathring{H}_{\mathrm{at}}^{p(\cdot),q}(\mathcal{X})$, and hence finishes the proof of Theorem 5.3. ■

By [34, Theorems 2.6 and 3.10] and an argument similar to that used in the proof of Theorem 5.3, we obtain the following conclusion, the details being omitted. In the case of $p(\cdot)$ being a constant, we refer to [34, Theorems 5.13 and 5.16].

COROLLARY 5.13. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2,\epsilon_3 \in (0,\infty)$ and $\epsilon \in (0,\epsilon_1\wedge\epsilon_2)$. Assume that $p(\cdot) \in C^{\log}_{(n/(n+\epsilon),\infty)}(\mathcal{X})$ satisfies $p_- \in (n/(n+\epsilon),1]$ and $\beta,\gamma \in (n[1/p_- - 1],\epsilon)$. Assume in addition that $q \in [1,\infty]\cap(p_+,\infty]$. Then*

$$\mathring{H}_{\mathrm{at}}^{p(\cdot),q}(\mathcal{X}) = \widetilde{H}^{p(\cdot)}(\mathcal{X})$$

with equivalent quasi-norms, where $\widetilde{H}^{p(\cdot)}(\mathcal{X})$ is defined to be the set of all $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ such that $\|f\|_{\widetilde{H}^{p(\cdot)}(\mathcal{X})} := \|\widetilde{\mathcal{S}}(f)\|_{L^{p(\cdot)}(\mathcal{X})}$ is finite; here, for all $x\in\mathcal{X}$,

$$\widetilde{\mathcal{S}}(f)(x) := \Big\{\sum_{k\in\mathbb{Z}}\int_{d(x,y)<2^{-k}}|D_k(f)(y)|^2\,\frac{d\mu(y)}{V_{2^{-k}}(x)}\Big\}^{1/2}.$$

We now conclude the proof of Theorem 5.5 as follows.

Proof of Theorem 5.5. Obviously, for all $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ with ϵ, β, γ as in Theorem 5.5, we have

$$\|f\|_{H^{p(\cdot)}(\mathcal{X})} \sim \|\mathcal{S}(f)\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \|g_\lambda^*(f)\|_{L^{p(\cdot)}(\mathcal{X})}.$$

Conversely, let $f \in H^{p(\cdot)}(\mathcal{X})$. Then, by Theorem 5.3, f has an expression

$$f = \sum_{j\in\mathbb{N}}\lambda_j a_j \quad \text{in } (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))',$$

where $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $\{a_j\}_{j\in\mathbb{N}}$ are $(p(\cdot),q)$-atoms supported on $\{B_j\}_{j\in\mathbb{N}} := \{B(x_j,r_j) : x_j\in\mathcal{X},\ r_j\in(0,\infty)\}_{j\in\mathbb{N}}$ satisfying

$$\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}}) \lesssim \|f\|_{H^{p(\cdot)}(\mathcal{X})}.$$

Thus,

$$g_\lambda^*(f) \le \sum_{j\in\mathbb{N}}|\lambda_j|g_\lambda^*(a_j)\chi_{2B_j} + \sum_{j\in\mathbb{N}}|\lambda_j|g_\lambda^*(a_j)\chi_{\mathcal{X}\setminus(2B_j)} =: \mathrm{I}+\mathrm{II}.$$

For I, by [33, Proposition 2.17(ii)], we find that

$$\|g_\lambda^*(a_j)\|_{L^q(\mathcal{X})} \lesssim \|a_j\|_{L^q(\mathcal{X})} \lesssim \frac{[\mu(B_j)]^{1/q}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}.$$

From this and Proposition 2.11, we deduce that

$$\|\mathrm{I}\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \|f\|_{H^{p(\cdot)}(\mathcal{X})}.$$

For II, we first observe that, for all $x \in \mathcal{X}$,

$$g_\lambda^*(a_j)(x) \lesssim \sum_{k=0}^{\infty} 2^{k(n-\lambda)/2} \mathcal{S}_{2^k}(a_j)(x). \tag{5.7}$$

By an argument similar to that used in the proof of [33, Proposition 3.4(ii)], we conclude that, for all $x \in \mathcal{X} \setminus (2B_j)$,

$$\mathcal{S}_{2^k}(a_j)(x) \lesssim 2^{k(n+\epsilon_1)} \frac{\mu(B_j)}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}} \left[\frac{r_j}{d(x,x_j)}\right]^{\epsilon_1} \frac{1}{V(x,x_j)},$$

which, combined with (2.2), implies that, for all $x \in \mathcal{X} \setminus (2B_j)$,

$$\begin{aligned} \mathcal{S}_{2^k}(a_j)(x) &\lesssim \frac{2^{k(n+\epsilon_1)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}} \left[\frac{\mu(B_j)}{\mu(B(x, d(x,x_j)))}\right]^{\epsilon_1/n+1} \\ &\lesssim \frac{2^{k(n+\epsilon_1)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}} [M(\chi_{B_j})(x)]^{\epsilon_1/n+1}. \end{aligned}$$

By this, (5.7), Corollary 2.10 and choosing $\epsilon_1 \in (n/p_- - n, 1)$ such that $\lambda > 3n + 2\epsilon_1$, we have

$$\begin{aligned} \|\mathrm{II}\|_{L^{p(\cdot)}(\mathcal{X})} &\lesssim \left\| \sum_{j\in\mathbb{N}} \sum_{k=0}^{\infty} 2^{k(3n/2-\lambda/2+\epsilon_1)} \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}} [M(\chi_{B_j})]^{\epsilon_1/n+1} \right\|_{L^{p(\cdot)}(\mathcal{X})} \\ &\lesssim \widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}) \lesssim \|f\|_{H^{p(\cdot)}(\mathcal{X})}, \end{aligned}$$

which, together with the estimation of I, implies that

$$\|g_\lambda^*(f)\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \|f\|_{H^{p(\cdot)}(\mathcal{X})}.$$

This finishes the proof of Theorem 5.5. ■

By Theorems 4.3, 5.3 and 5.4, we deduce the following conclusion, the details being omitted.

Corollary 5.14. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),1]}(\mathcal{X})$. Then $H^{*,p(\cdot)}(\mathcal{X}) = H^{p(\cdot)}(\mathcal{X})$ with equivalent quasi-norms.*

Finally, we prove Theorem 5.7.

Proof of Theorem 5.7. For all $x \in \mathcal{X}$, we first observe that

$$\begin{aligned} \widetilde{\mathcal{S}}(f)(x) &= \left\{ \sum_{k\in\mathbb{Z}} \sum_{\tau\in I_k} \sum_{v=1}^{N(k,\tau)} \int_{d(x,y)<2^{-k}} |D_k(f)(y)|^2 \chi_{Q_\tau^{k,v}}(x) \frac{d\mu(y)}{V_{2^{-k}}(x)} \right\}^{1/2} \\ &\lesssim \left\{ \sum_{k\in\mathbb{Z}} \sum_{\tau\in I_k} \sum_{v=1}^{N(k,\tau)} \left[\sup_{z\in B(z_\tau^{k,v}, \widetilde{c}2^{-k})} |D_k(f)(z)| \right]^2 \chi_{Q_\tau^{k,v}}(x) \right\}^{1/2} =: \mathrm{I}(x), \end{aligned}$$

where $\widetilde{c}$ is a positive constant such that $B(x, 2^{-k}) \subset B(z_\tau^{k,v}, \widetilde{c}2^{-k})$ for every $x \in Q_\tau^{k,v}$. By an argument similar to that used in the proof of [34, (5.18)], we conclude that

$$\mathrm{I}(x) \lesssim \Big\{\sum_{k\in\mathbb{Z}}\Big[\sum_{\widetilde{k}\in\mathbb{Z}} 2^{-|k-\widetilde{k}|\epsilon'} 2^{[(k\wedge\widetilde{k})-k']n(1-1/r)} \times \Big\{M\Big(\sum_{\widetilde{\tau}\in I_{\widetilde{k}}}\sum_{\widetilde{v}=1}^{N(\widetilde{k},\widetilde{v})} |D_{\widetilde{k}}(f)(y_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}})|^r \chi_{Q_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}}\Big)(x)\Big\}^{1/r}\Big]^2\Big\}^{1/2},$$

where $r \in (n/[n+\epsilon_1'], p_-)$ and $y_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}$ is an arbitrary point in $Q_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}$. From this, the Minkowski inequality and Theorem 2.7, we deduce that

$$\begin{aligned}\|\widetilde{\mathcal{S}}(f)\|_{L^{p(\cdot)}(\mathcal{X})} &\lesssim \Big\|\Big\{\sum_{\widetilde{k}\in\mathbb{Z}}\Big[M\Big(\sum_{\widetilde{\tau}\in I_{\widetilde{k}}}\sum_{\widetilde{v}=1}^{N(\widetilde{k},\widetilde{v})} |D_{\widetilde{k}}(f)(y_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}})|^r \chi_{Q_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}}\Big)\Big]^{2/r}\Big\}^{1/2}\Big\|_{L^{p(\cdot)}(\mathcal{X})} \\ &\lesssim \Big\|\Big\{\sum_{\widetilde{k}\in\mathbb{Z}}\sum_{\widetilde{\tau}\in I_{\widetilde{k}}}\sum_{\widetilde{v}=1}^{N(\widetilde{k},\widetilde{v})} |D_{\widetilde{k}}(f)(y_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}})|^2 \chi_{Q_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}}\Big\}^{1/2}\Big\|_{L^{p(\cdot)}(\mathcal{X})}.\end{aligned}$$

Thus, by the fact that $y_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}$ is an arbitrary point in $Q_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}$, we obtain

$$\begin{aligned}\|\widetilde{\mathcal{S}}(f)\|_{L^{p(\cdot)}(\mathcal{X})} &\lesssim \Big\|\Big\{\sum_{\widetilde{k}\in\mathbb{Z}}\sum_{\widetilde{\tau}\in I_{\widetilde{k}}}\sum_{\widetilde{v}=1}^{N(\widetilde{k},\widetilde{v})} \Big[\inf_{z\in Q_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}} |D_{\widetilde{k}}(f)(z)|\Big]^2 \chi_{Q_{\widetilde{\tau}}^{\widetilde{k},\widetilde{v}}}\Big\}^{1/2}\Big\|_{L^{p(\cdot)}(\mathcal{X})} \\ &\lesssim \Big\|\Big\{\sum_{k\in\mathbb{Z}} |D_k(f)|^2\Big\}^{1/2}\Big\|_{L^{p(\cdot)}(\mathcal{X})} \sim \|g(f)\|_{L^{p(\cdot)}(\mathcal{X})},\end{aligned}$$

which, together with Corollary 5.13, implies that

$$\|f\|_{H^{p(\cdot)}(\mathcal{X})} \lesssim \|g(f)\|_{L^{p(\cdot)}(\mathcal{X})}.$$

Conversely, to prove

$$\|g(f)\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \|f\|_{H^{p(\cdot)}(\mathcal{X})},$$

we only need to use an argument similar to that used in the proof of [34, Theorem 5.13], the details being omitted. The proof of Theorem 5.7 is complete. ■

6. Applications

This section is devoted to giving some applications of the Hardy spaces $H^{*,p(\cdot)}(\mathcal{X})$. More precisely, in Subsection 6.1, we establish Olsen's inequality for fractional integral operators on $H^{*,p(\cdot)}(\mathcal{X})$. Moreover, we consider the boundedness of singular integral operators on $H^{*,p(\cdot)}(\mathcal{X})$ in Subsection 6.2, and that of quasi-Banach valued sublinear operators in Subsection 6.3.

6.1. Fractional integral operators and Olsen's inequality. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI as in Definition 3.7, and $D_k := S_k - S_{k-1}$ for each $k \in \mathbb{Z}$. In this subsection, we are concerned with the fractional integral operator I_α with $\alpha \in (0, n)$, which was originally introduced in [83], and given by setting, for all $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with ϵ, β, γ as in Definition 3.3 and $x \in \mathcal{X}$,

$$I_\alpha(f)(x) := \sum_{k\in\mathbb{Z}} [\mu(B(x,2^{-k}))]^\alpha D_k(f)(x). \tag{6.1}$$

LEMMA 6.1. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and a be a $(p(\cdot),\infty)$-atom supported on $B(z,2^{-l})$ for some $z \in \mathcal{X}$ and $l \in \mathbb{Z}$. Then there exists a positive constant C such that, for all $k \in \mathbb{Z}$ and $x \in \mathcal{X}$,*

$$|D_k(a)(x)| \le C 2^{\epsilon_1(k-(l\vee k))} \frac{1}{\|\chi_{B(z,2^{-l})}\|_{L^{p(\cdot)}(\mathcal{X})}} \chi_{B(z,2^{-(l\wedge k)+2})}(x). \tag{6.2}$$

Proof. When $k \ge l$, (6.2) reads, for all $x \in \mathcal{X}$,

$$|D_k(a)(x)| \le C \frac{1}{\|\chi_{B(z,2^{-l})}\|_{L^{p(\cdot)}(\mathcal{X})}} \chi_{B(z,2^{-l+2})}(x).$$

From the fact that for each $k \in \mathbb{Z}$, $\operatorname{supp} S_k(x,\cdot) \subset B(x,2^{-k})$, we deduce that, for each $k \in \mathbb{Z}$, $\operatorname{supp} D_k(a) \subset B(z,2^{-(l\wedge k)+2})$. Thus, (6.2) is a consequence of the definition of $(p(\cdot),\infty)$-atoms.

Let us suppose instead that $k < l$ and $x \in B(z,2^{-k+2})$. Then (6.2) reads

$$|D_k(a)(x)| \le C \frac{2^{\epsilon_1(k-l)}}{\|\chi_{B(z,2^{-l})}\|_{L^{p(\cdot)}(\mathcal{X})}} \chi_{B(z,2^{-l+2})}(x).$$

The vanishing moment condition on a yields

$$D_k(a)(x) = \int_{\mathcal{X}} [D_k(x,y) - D_k(x,z)] a(y)\, d\mu(y).$$

Observe that, when $y \in B(z,2^{-l})$,

$$2d(y,z) < 2^{1-l} \le 2^{-k} + d(x,y).$$

Then we are in a position to use Definition 3.7(ii) to obtain

$$\begin{aligned}|D_k(x,y) - D_k(x,z)| &\lesssim \left[\frac{d(y,z)}{2^{-k}+d(y,x)}\right]^{\epsilon_1} \left[\frac{2^{-k}}{2^{-k}+d(y,x)}\right]^{\epsilon_2} \frac{1}{V_k(y)+V_k(x)+V(y,x)}\\ &\lesssim \left[\frac{d(y,z)}{2^{-k}+d(y,x)}\right]^{\epsilon_1} \frac{1}{V_k(y)} \sim 2^{\epsilon_1(k-l)} \frac{1}{V_k(y)},\end{aligned}$$

which, together with the fact that, when $y \in B(z,2^{-l})$, $B(z,2^{-l}) \subset B(y,2^{-k})$, implies that (6.2) holds true in this case. This finishes the proof of Lemma 6.1. ∎

As an immediate consequence of Lemma 6.1, we have the following corollary.

COROLLARY 6.2. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ and $\alpha \in (0,n)$. Then there exists a positive constant C such that, for any $(p(\cdot),\infty)$-atom a supported on $B(z,2^{-l})$, with $z \in \mathcal{X}$ and $l \in \mathbb{Z}$, and for any $x \in \mathcal{X}$,*

$$|I_\alpha(a)(x)| \le C \sum_{k=-\infty}^{\infty} 2^{\epsilon_1[k-(l\vee k)]} [\mu(B(z,2^{-k}))]^\alpha \frac{1}{\|\chi_{B(z,2^{-l})}\|_{L^{p(\cdot)}(\mathcal{X})}} \chi_{B(z,2^{-(l\wedge k)+2})}(x).$$

Let $1 \le q \le u < \infty$ be fixed. Recall that the *Morrey space* $\mathcal{M}_q^u(\mathcal{X})$ is defined to be the set of all $g \in L^q_{\mathrm{loc}}(\mathcal{X})$ for which

$$\|g\|_{\mathcal{M}_q^u(\mathcal{X})} := \sup_{z\in\mathcal{X},\, r\in(0,\infty)} [\mu(B(z,r))]^{1/u-1/q}\|g\|_{L^q(B(z,r))} < \infty. \tag{6.3}$$

THEOREM 6.3 (Olsen's inequality). *Let* $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ *and* ϵ, ϵ_1, ϵ_2, ϵ_3 *be as in Definition* 3.7. *Assume that the parameters* α *and* q *satisfy* $0 < \alpha < n$, $1 \le q \le 1/\alpha < \infty$, $q \in (p_+, \infty)$ *and*

$$\epsilon_1 \underline{p} > n. \tag{6.4}$$

If $g \in \mathcal{M}_q^{1/\alpha}(\mathcal{X})$, *then the operator*

$$f \in L^{q,0}_{\mathrm{b}}(\mathcal{X}) \mapsto gI_\alpha(f) \in \mathcal{M}_0(\mathcal{X}),$$

where $\mathcal{M}_0(\mathcal{X})$ *denotes the set of all measurable functions on* $\mathcal{X}$, *extends to a bounded linear operator* L_g *on* $H^{*,p(\cdot)}(\mathcal{X})$, *and the operator norm of* L *satisfies the inequality*

$$\|L_g\|_{H^{*,p(\cdot)}(\mathcal{X})\to L^{p(\cdot)}(\mathcal{X})} \le C\|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})},$$

where C *is a positive constant independent of* g.

Proof. Since by Lemma 4.23, $L^{q,0}_{\mathrm{b}}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$, we only need to prove

$$\|L_g(f)\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\|f\|_{H^{*,p(\cdot)}(\mathcal{X})}$$

for all $f \in L^{q,0}_{\mathrm{b}}(\mathcal{X})$. Thanks to Theorem 4.24, f admits a finite atomic decomposition, namely, there exist $N \in \mathbb{N}$ and $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ and $(p(\cdot), q)$-atoms $\{a_j\}_{j=1}^N$ such that, for $j \in \{1,\dots,N\}$, a_j is supported in $B(x_j, 2^{-l_j})$ for some $x_j \in \mathcal{X}$ and $l_j \in \mathbb{Z}$,

$$f = \sum_{j=1}^N \lambda_j a_j$$

in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and

$$\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j=1}^N, \{B(x_j, 2^{-l_j})\}_{j=1}^N) \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}.$$

By Corollary 6.2 and (2.2), we see that, for all $x \in \mathcal{X}$,

$$\begin{aligned}
|g(x)I_\alpha(f)(x)| &\lesssim \sum_{j=1}^N |\lambda_j| \sum_{k=-\infty}^{l_j} 2^{\epsilon_1(k-l_j)} \frac{[\mu(B(x_j,2^{-k}))]^\alpha}{\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}} |g(x)|\chi_{B(x_j,2^{-k+2})}(x)\\
&\quad + \sum_{j=1}^N |\lambda_j| \sum_{k=l_j+1}^{\infty} \frac{[\mu(B(x_j,2^{-k}))]^\alpha}{\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}} |g(x)|\chi_{B(x_j,2^{-l_j+2})}(x)\\
&\lesssim \sum_{j=1}^N |\lambda_j| \sum_{k=-\infty}^{l_j} 2^{\epsilon_1(k-l_j)} \frac{[\mu(B(x_j,2^{-k}))]^\alpha}{\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}} |g(x)|\chi_{B(x_j,2^{-k+2})}(x)\\
&\quad + \sum_{j=1}^N |\lambda_j| \sum_{k=l_j+1}^{\infty} 2^{-(k-l_j)\alpha\kappa} \frac{[\mu(B(x_j,2^{-l_j}))]^\alpha}{\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}} |g(x)|\chi_{B(x_j,2^{-l_j+2})}(x)\\
&\lesssim \sum_{j=1}^N |\lambda_j| \sum_{k=-\infty}^{l_j} 2^{\epsilon_1(k-l_j)} \frac{[\mu(B(x_j,2^{-k}))]^\alpha}{\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}} |g(x)|\chi_{B(x_j,2^{-k+2})}(x).
\end{aligned}$$

Choose $\kappa^\dagger > 0$ so that $\kappa^\dagger \underline{p} > 1$ and $\epsilon_1 > n\kappa^\dagger$ by using (6.4). For all $x \in \mathcal{X}$, let

$$G_{j,k}(x) := \frac{[\mu(B(x_j, 2^{2-k}))]^\alpha}{\|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\|\chi_{B(x_j,2^{-k+2})}\|_{L^{p(\cdot)}(\mathcal{X})}}|g(x)|\chi_{B(x_j,2^{-k+2})}(x).$$

Then $\operatorname{supp} G_{j,k} \subset B(x_j, 2^{-k+2})$ and

$$\|G_{j,k}\|_{L^q(\mathcal{X})} = \frac{[\mu(B(x_j, 2^{2-k}))]^\alpha \|g\|_{L^q(B(x_j,2^{-k+2}))}}{\|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\|\chi_{B(x_j,2^{-k+2})}\|_{L^{p(\cdot)}(\mathcal{X})}} \le \frac{[\mu(B(x_j, 2^{2-k}))]^{1/q}}{\|\chi_{B(x_j,2^{-k+2})}\|_{L^{p(\cdot)}(\mathcal{X})}}.$$

Observe that, for all $x \in \mathcal{X}$,

$$|g(x)I_\alpha(f)(x)| \lesssim \sum_{j=1}^N |\lambda_j| \sum_{k=-\infty}^{l_j} \|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\|\chi_{B(x_j,2^{-k+2})}\|_{L^{p(\cdot)}(\mathcal{X})} \frac{2^{\epsilon_1(k-l_j)}G_{j,k}(x)}{\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}}.$$

By Proposition 2.11 and Theorem 2.7, we conclude that

$$\begin{aligned}
&\|L_g(f)\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\quad\lesssim \|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\left\|\left\{\sum_{j=1}^N \sum_{k=-\infty}^{l_j}\left[\frac{|\lambda_j|2^{\epsilon_1(k-l_j)}}{\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}}\chi_{B(x_j,2^{2-k})}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\quad\lesssim \|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\left\|\left\{\sum_{j=1}^N \sum_{k=-\infty}^{l_j}\frac{|\lambda_j|^{\underline{p}}2^{\underline{p}(\epsilon_1-n\kappa^\dagger)(k-l_j)}}{[\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}]^{\underline{p}}}[M(\chi_{B(x_j,2^{-l_j})})]^{\kappa^\dagger\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\quad\lesssim \|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\left\|\left\{\sum_{j=1}^N \sum_{k=-\infty}^{l_j}\frac{|\lambda_j|^{\underline{p}}2^{\underline{p}(\epsilon_1-n\kappa^\dagger)(k-l_j)}}{[\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}]^{\underline{p}}}\chi_{B(x_j,2^{-l_j})}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\quad\lesssim \|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\left\|\left\{\sum_{j=1}^N \left[\frac{|\lambda_j|}{\|\chi_{B(x_j,2^{-l_j})}\|_{L^{p(\cdot)}(\mathcal{X})}}\chi_{B(x_j,2^{-l_j})}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathcal{X})}\\
&\quad\lesssim \|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j=1}^N, \{B(x_j, 2^{-l_j})\}_{j=1}^N) \lesssim \|g\|_{\mathcal{M}_q^{1/\alpha}(\mathcal{X})}\|f\|_{H^{*,p(\cdot)}(\mathcal{X})},
\end{aligned}$$

which completes the proof of Theorem 6.3. ∎

In the following remark, we explain why Theorem 6.3 deserves its name and give an example to which we can apply Theorem 6.3.

REMARK 6.4. (i) Recall that the *fractional integral operator* I_α on $\mathbb{R}^n$, with $0 < \alpha < 1$, is defined by setting, for all suitable functions f on $\mathbb{R}^n$ and all $x \in \mathbb{R}^n$,

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-n\alpha}}\,dy.$$

Olsen's inequality is an inequality of the form

$$\|gI_\alpha f\|_Z \le C\|f\|_X\|g\|_Y,$$

where X, Y, Z are suitable quasi-Banach spaces and C is a positive constant independent of f and g. There exists an extensive literature on Olsen inequalities; see [17, 69–71, 73, 75–78] for theoretical aspects, and [24–26] for applications to PDEs.

(ii) The restriction (6.4) forces $n < 1$. We point out that there exists an RD-space such that both n and κ can be taken to be $\log_3 2 < 1$. Indeed, let K be the *Cantor set* on $[0, 1]$

and $\mu := \mathcal{H}^{\log_3 2}$, where $\mathcal{H}^{\log_3 2}$ denotes the $\log_3 2$-dimensional Hausdorff measure. Let

$$X := \bigcup_{l=0}^{\infty} \{3^l x : x \in K\} \subset \mathbb{R}.$$

Then we claim that $(X, |\cdot|, \mu)$ is an RD-space with $n = \kappa = \log_3 2$. More precisely, $\mu(X \cap I) \sim |I|^{\log_3 2}$ for any interval I intersecting X, where the implicit positive constants are independent of I. Indeed, let I be a compact interval contained in $[0, 3^{l_0}]$ for some $l_0 \in \mathbb{Z}_+$. Then, by noticing that $x \in X$ if and only if $3x \in X$, we have

$$\mu(X \cap I) = 2^{l_0} \mu(K \cap [3^{-l_0} I]) \sim 2^{l_0} |3^{-l_0} I|^{\log_3 2} = |I|^{\log_3 2}, \tag{6.5}$$

where the implicit positive constants are independent of I.

To show the equivalence in (6.5), we first observe that

$$\mu(K \cap [3^{-l_0} I]) \le |3^{-l_0} I|^{\log_3 2}$$

from the definition of μ. To see the opposite inequality, we let $J := 3^{-l} I$. We may assume that $m := -\log_3 |J|$ is an integer and $\inf_{x \in J} x = 0$. Then

$$K = \bigcup_{k=1}^{2^m} (K \cap J_k),$$

where each J_k is a translate of J. Thus,

$$\mu(K \cap [3^{-l_0} I]) = \mu(K \cap J) = 2^{-m} \mu(K) = |J|^{\log_3 2} \mu(K) = |3^{-l_0} I|^{\log_3 2} \mu(K).$$

Since $\mu(K)$ is known to be non-zero (see [18]), we obtain the desired claim.

(iii) Let f belong to $L^\infty(\mathcal{X})$ with bounded support. Since, for all $k \in \mathbb{Z}$ and $x \in \mathcal{X}$,

$$|S_k(f)(x)| \lesssim \|f\|_{L^\infty(\mathcal{X})} \frac{\mu(\operatorname{supp} f)}{V_{2^{-k}}(x)},$$

it follows that I_α given by (6.1) has the following expression:

$$\begin{aligned}
I_\alpha(f)(x) &= \lim_{L_1 \to \infty} \sum_{k=-L_1}^{\infty} [\mu(B(x, 2^{-k}))]^\alpha D_k(f)(x) \\
&= \lim_{L_1 \to \infty} \left\{ \lim_{L_2 \to \infty} \sum_{k=-L_1}^{L_2} [\mu(B(x, 2^{-k}))]^\alpha D_k(f)(x) \right\} \\
&= \lim_{L_1 \to \infty} \left\{ \lim_{L_2 \to \infty} \sum_{k=-L_1}^{L_2} [\mu(B(x, 2^{-k}))]^\alpha [S_k(f)(x) - S_{k-1}(f)(x)] \right\} \\
&= \lim_{L_1 \to \infty} \lim_{L_2 \to \infty} \left\{ [\mu(B(x, 2^{L_1}))]^\alpha S_{-L_1}(f)(x) - [\mu(B(x, 2^{-L_2}))]^\alpha S_{L_2 - 1}(f)(x) \right\} \\
&\quad + \lim_{L_1 \to \infty} \lim_{L_2 \to \infty} \sum_{k=-L_1+1}^{L_2} \left\{ [\mu(B(x, 2^{-k}))]^\alpha - [\mu(B(x, 2^{-k-1}))]^\alpha \right\} S_k(f)(x) \\
&= \lim_{L_1 \to \infty} \lim_{L_2 \to \infty} \sum_{k=-L_1+1}^{L_2} \left\{ [\mu(B(x, 2^{-k}))]^\alpha - [\mu(B(x, 2^{-k-1}))]^\alpha \right\} S_k(f)(x).
\end{aligned}$$

If C_2 and κ in (2.2) satisfy $C_2 2^\kappa < 1$, then, in view of (3.3), we find that, for all $f \in L^\infty(\mathcal{X})$

with $f \geq 0$ μ-a.e. and all $x \in \mathcal{X}$,

$$I_\alpha(f)(x) \gtrsim \int_{\mathcal{X}} \sum_{l=-\infty}^{\infty} \frac{\chi_{B(x,2^{-l})}(y)}{[\mu(B(x,2^{-l}))]^{1-\alpha}} f(y)\, d\mu(y) \gtrsim \int_{\mathcal{X}} \frac{f(y)}{[\mu(B(x,d(x,y)))]^{1-\alpha}}\, d\mu(y),$$

where the implicit positive constant is independent of f and x. Therefore, the fractional integral operator I_α is closely related to the fractional integral operator dealt with in [72].

6.2. Singular integral operators. In this subsection, we consider the boundedness of singular integral operators on the space $H^{p(\cdot)}(\mathcal{X})$. To this end, we first recall the definition of singular integral operators studied in [33, 55].

Recall that $\mathcal{C}_{\mathrm{b}}(\mathcal{X})$ denotes the space of all continuous functions on $\mathcal{X}$ with bounded support. For $\eta \in (0, 1]$, let

$$\mathcal{C}_{\mathrm{b}}^\eta(\mathcal{X}) := \left\{ f \in \mathcal{C}_{\mathrm{b}}(\mathcal{X}) : \|f\|_{\mathcal{C}_{\mathrm{b}}^\eta(\mathcal{X})} := \sup_{\substack{x,y\in\mathcal{X}\\ x\neq y}} \frac{|f(x)-f(y)|}{[d(x,y)]^\eta} < \infty \right\}.$$

Assume that T is a bounded linear operator on $L^2(\mathcal{X})$. The operator is said to have a distributional kernel K, which is locally integrable away from the diagonal of $\mathcal{X} \times \mathcal{X}$, if for any $f, g \in \mathcal{C}_{\mathrm{b}}^\eta(\mathcal{X})$ with $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$,

$$\langle Tf, g\rangle := \int_{\mathcal{X}\times\mathcal{X}} g(x)K(x,y)f(y)\, d\mu(y)\, d\mu(x). \tag{6.6}$$

First, we have the following conclusion, which is of independent interest.

Proposition 6.5. *Let $\epsilon_1 \in (0,1]$, $p(\cdot) \in C^{\log}_{(n/(n+\epsilon_1),1]}(\mathcal{X})$ and T be a bounded linear operator on $L^2(\mathcal{X})$ with distributional kernel K as in (6.6). Suppose that there exists a positive constant C such that, for all $x, y, y' \in \mathcal{X}$ with $d(y,y') \leq d(x,y)/2$ and $x \neq y$,*

$$|K(x,y) - K(x,y')| \leq C \frac{[d(y,y')]^{\epsilon_1}}{V(x,y)[d(x,y)]^{\epsilon_1}}. \tag{6.7}$$

Then there exists a positive constant $\widetilde{C}$ such that, for all $(p(\cdot),2)$-atoms a,

$$\|Ta\|_{L^{p(\cdot)}(\mathcal{X})} \leq \widetilde{C}.$$

Proof. Let a be a $(p(\cdot),2)$-atom supported on $B_0 := B(x_0, r_0)$ for some $x_0 \in \mathcal{X}$ and $r_0 \in (0,\infty)$. Then we have

$$\|Ta\|_{L^{p(\cdot)}(\mathcal{X})} \lesssim \|\chi_{2B_0} Ta\|_{L^{p(\cdot)}(\mathcal{X})} + \|\chi_{\mathcal{X}\setminus(2B_0)} Ta\|_{L^{p(\cdot)}(\mathcal{X})} =: \mathrm{I} + \mathrm{II}.$$

For I, by the boundedness of T on $L^2(\mathcal{X})$, we see that

$$\|Ta\|_{L^2(\mathcal{X})} \lesssim \|a\|_{L^2(\mathcal{X})} \lesssim \frac{[\mu(B_0)]^{1/2}}{\|\chi_{B_0}\|_{L^{p(\cdot)}(\mathcal{X})}},$$

which, together with Proposition 2.11, implies that $\mathrm{I} \lesssim 1$. For II, from (6.7) and the vanishing moment condition on a, we deduce that, for all $x \in \mathcal{X} \setminus (2B_0)$,

$$|Ta(x)| \lesssim \frac{1}{\|\chi_{B_0}\|_{L^{p(\cdot)}(\mathcal{X})}} \frac{r_0^{\epsilon_1}\mu(B_0)}{[d(x,x_0)]^{\epsilon_1} V(x,x_0)}.$$

By this and (2.2), we further find that, for all $x \in \mathcal{X} \setminus (2B_0)$,

$$|Ta(x)| \lesssim \frac{1}{\|\chi_{B_0}\|_{L^{p(\cdot)}(\mathcal{X})}} \left[\frac{\mu(B_0)}{\mu(B(x, d(x, x_0)))}\right]^{\epsilon_1/n+1} \lesssim \frac{[M(\chi_{B_0})(x)]^{\epsilon_1/n+1}}{\|\chi_{B_0}\|_{L^{p(\cdot)}(\mathcal{X})}},$$

which, combined with Theorem 2.7, implies that $\mathrm{II} \lesssim 1$. This finishes the proof of Proposition 6.5. ∎

Let T be a bounded linear operator on $L^2(\mathcal{X})$. We say that $T1 = 0$ if T satisfies, for all $g \in \mathcal{C}_{\mathrm{b}}^{\eta}(\mathcal{X})$,

$$\int_{\mathcal{X}} T^* g(x)\, d\mu(x) = 0,$$

where T^* denotes the adjoint operator of T on $L^2(\mathcal{X})$, and that $T^*1 = 0$ if T satisfies, for all $h \in L^2(\mathcal{X})$ with bounded support and $\int_{\mathcal{X}} h(x)\, d\mu(x) = 0$,

$$\int_{\mathcal{X}} Th(x)\, d\mu(x) = 0.$$

THEOREM 6.6. *Let* $\epsilon_1 \in (0, 1]$, $p(\cdot) \in C^{\log}_{(n/(n+\epsilon_1),1]}(\mathcal{X})$ *and* T *be a bounded linear operator on* $L^2(\mathcal{X})$ *with distributional kernel* K *as in* (6.6). *Assume that there exists a positive constant* c *such that the kernel* K *of* T *satisfies the following conditions:*

(i) *for all* $x, y \in \mathcal{X}$ *with* $x \neq y$,

$$|K(x, y)| \le c/V(x, y);$$

(ii) *for all* $x, y, y' \in \mathcal{X}$ *with* $d(y, y') \le d(x, y)/2$ *and* $x \neq y$, (6.7) *is satisfied;*

(iii) *for all* $x, x', y \in \mathcal{X}$ *with* $d(x, x') \le d(x, y)/2$ *and* $x \neq y$,

$$|K(x', y) - K(x, y)| \le c \frac{1}{V(x, y)} \left[\frac{d(x, x')}{d(x, y)}\right]^{\epsilon_1};$$

(iv) *for all* $x, x', y, y' \in \mathcal{X}$ *with* $d(x, x') \le d(x, y)/3$ *and* $d(y, y') \le d(x, y)/3$,

$$|[K(x, y) - K(x', y)] - [K(x, y') - K(x', y')]| \le c \frac{[d(x, x')]^{\epsilon_1} [d(y, y')]^{\epsilon_1}}{V(x, y)[d(x, y)]^{2\epsilon_1}}.$$

If $T1 = 0 = T^*1$, *then* T *extends to a bounded linear operator on* $H^{p(\cdot)}(\mathcal{X})$.

Proof. Let $\epsilon \in (0, \epsilon_1)$ and $\mathring{\mathcal{G}}_{\mathrm{b}}(\widetilde{\beta}, \widetilde{\gamma})$ with $\widetilde{\beta}, \widetilde{\gamma} \in [\epsilon, \epsilon_1)$ be the set of all functions in $\mathring{\mathcal{G}}(\widetilde{\beta}, \widetilde{\gamma})$ with bounded support, namely,

$$\mathring{\mathcal{G}}_{\mathrm{b}}(\widetilde{\beta}, \widetilde{\gamma}) = \mathring{\mathcal{G}}(\widetilde{\beta}, \widetilde{\gamma}) \cap \mathcal{C}_{\mathrm{b}}(\mathcal{X}).$$

Then, from the proof of Theorem 4.3, we deduce that $\mathring{\mathcal{G}}_{\mathrm{b}}(\widetilde{\beta}, \widetilde{\gamma})$ is dense in $H^{p(\cdot)}(\mathcal{X})$ when $\widetilde{\beta}, \widetilde{\gamma} \in [\epsilon, \epsilon_1)$ and $p_- \in (n/[n + \epsilon_1], 1]$.

Let $f \in \mathring{\mathcal{G}}_{\mathrm{b}}(\widetilde{\beta}, \widetilde{\gamma})$. Since $T1 = 0$, it follows from [33, Lemma 2.9] that, for all $g \in \mathring{\mathcal{G}}_0^{\epsilon}(\beta, \gamma)$ with $\beta, \gamma \in (0, \epsilon)$,

$$T^* g \in \mathring{\mathcal{G}}_0^{\epsilon}(\beta, \gamma).$$

From this and Theorem 5.3, we further deduce that

$$\langle Tf, g\rangle = \langle f, T^* g\rangle = \sum_{j \in \mathbb{N}} \lambda_j \langle a_j, T^* g\rangle = \sum_{j \in \mathbb{N}} \lambda_j \langle Ta_j, g\rangle,$$

where $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ and $\{a_j\}_{j\in\mathbb{N}}$ are $(p(\cdot),\infty)$-atoms supported on

$$\{B_j\}_{j\in\mathbb{N}}:=\{B(x_j,r_j):x_j\in\mathcal{X},\ r_j\in(0,\infty)\}_{j\in\mathbb{N}}$$

satisfying $f=\sum_{j\in\mathbb{N}}\lambda_j a_j$ in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ and

$$\widetilde{\mathcal{A}}_{p(\cdot)}(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}})\lesssim\|f\|_{H^{p(\cdot)}(\mathcal{X})}. \tag{6.8}$$

Let $\{D_t\}_{t\in(0,\infty)}$ be an (ϵ_1,ϵ_2)-CRF in $L^2(\mathcal{X})$ with $\epsilon_2\in(0,\infty)$. Then $D_t\in\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ for all $t\in(0,\infty)$, and hence

$$D_t(Tf)=\sum_{j\in\mathbb{N}}\lambda_j D_t(Ta_j)$$

pointwise. From this, we obtain

$$\begin{aligned}\|\mathcal{S}(Tf)\|_{L^{p(\cdot)}(\mathcal{X})}&\lesssim\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|\mathcal{S}(Ta_j)\chi_{2B_j}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\&\quad+\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|\mathcal{S}(Ta_j)\chi_{\mathcal{X}\setminus(2B_j)}\Big\|_{L^{p(\cdot)}(\mathcal{X})}.\end{aligned} \tag{6.9}$$

From [33, Proposition 2.17] and the $L^2(\mathcal{X})$-boundedness of T, we deduce that, for all $j\in\mathbb{N}$,

$$\|\mathcal{S}(Ta_j)\|_{L^2(\mathcal{X})}\lesssim\|a_j\|_{L^2(\mathcal{X})}\lesssim\frac{[\mu(B_j)]^{1/2}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}},$$

which, together with Proposition 2.11 and (6.8), implies that

$$\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|\mathcal{S}(Ta_j)\chi_{2B_j}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\Big\|\sum_{j\in\mathbb{N}}\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\|f\|_{H^{p(\cdot)}(\mathcal{X})}. \tag{6.10}$$

Meanwhile, by the proof of [33, Proposition 3.6], we know that, for all $\epsilon_1'\in(0,\epsilon_1)$, $j\in\mathbb{N}$ and $x\in\mathcal{X}\setminus(2B_j)$,

$$\mathcal{S}(Ta_j)(x)\lesssim\frac{1}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\,\frac{r_j^{\epsilon_1'}\mu(B_j)}{[d(x,x_j)]^{\epsilon_1'}V(x,x_j)},$$

which, combined with (2.2), implies that

$$\mathcal{S}(Ta_j)(x)\lesssim\frac{1}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\left[\frac{\mu(B_j)}{\mu(B(x,d(x,x_j)))}\right]^{\epsilon_1'/n+1}\lesssim\frac{[M(\chi_{B_j})(x)]^{\epsilon_1'/n+1}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}.$$

From this and Theorem 2.7, we further conclude that

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|\mathcal{S}(Ta_j)\chi_{\mathcal{X}\setminus(2B_j)}\Big\|_{L^{p(\cdot)}(\mathcal{X})}&\lesssim\Big\|\sum_{j\in\mathbb{N}}|\lambda_j|\frac{[M(\chi_{B_j})]^{\epsilon_1'/n+1}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\&\lesssim\Big\|\sum_{j\in\mathbb{N}}\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathcal{X})}}\Big\|_{L^{p(\cdot)}(\mathcal{X})}\\&\lesssim\|f\|_{H^{p(\cdot)}(\mathcal{X})},\end{aligned} \tag{6.11}$$

where we choose $\epsilon_1'\in(0,\epsilon_1)$ such that $p_->n/(n+\epsilon_1')$.

Combining (6.9)–(6.11), we obtain

$$\|Tf\|_{H^{p(\cdot)}(\mathcal{X})}\sim\|\mathcal{S}(Tf)\|_{L^{p(\cdot)}(\mathcal{X})}\lesssim\|f\|_{H^{p(\cdot)}(\mathcal{X})},$$

which implies that T is bounded on $H^{p(\cdot)}(\mathcal{X})$. This finishes the proof of Theorem 6.6. ∎

REMARK 6.7. When $p(\cdot) := p$ with a constant $p \in (n/(n+\epsilon_1), 1]$, Theorem 6.6 is just [33, Proposition 3.6].

6.3. Quasi-Banach valued sublinear operators. Recall that a *quasi-Banach space* $\mathcal{B}$ is a complete space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is non-negative, non-degenerate (namely, $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, namely, there exists a constant $K \in [1, \infty)$ such that, for all $f, g \in \mathcal{B}$,

$$\|f + g\|_{\mathcal{B}} \le K(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}});$$

see, for example, [85, 86]. It is easy to see that, when $p(\cdot) \in C^{\log}_{(0,\infty)}(\mathcal{X})$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathcal{X})$ and the variable exponent Hardy space $H^{*,p(\cdot)}(\mathcal{X})$ are quasi-Banach spaces.

DEFINITION 6.8.

(i) Let $\gamma \in (0, 1]$. A quasi-Banach space $\mathcal{B}_\gamma$ with quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ is called a *γ-quasi-Banach space* if there exists a constant $K_1 \in [1, \infty)$ such that, for all $m \in \mathbb{N}$ and $\{f_j\}_{j=1}^m \subset \mathcal{B}_\gamma$,

$$\Big\|\sum_{j=1}^m f_j\Big\|_{\mathcal{B}_\gamma}^\gamma \le K_1 \sum_{j=1}^m \|f_j\|_{\mathcal{B}_\gamma}^\gamma.$$

(ii) For any given γ-quasi-Banach space $\mathcal{B}_\gamma$ with $\gamma \in (0, 1]$ and linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is said to be *$\mathcal{B}_\gamma$-sublinear* if there exists a positive constant K_2 such that, for all $m \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^m \subset \mathbb{C}$ and $\{f_j\}_{j=1}^m \subset \mathcal{Y}$,

$$\Big\|T\Big(\sum_{j=1}^m \lambda_j f_j\Big)\Big\|_{\mathcal{B}_\gamma}^\gamma \le K_2 \sum_{j=1}^m |\lambda_j|^\gamma \|T(f_j)\|_{\mathcal{B}_\gamma}^\gamma \tag{6.12}$$

and, for all f and g in $\mathcal{Y}$,

$$\|T(f) - T(g)\|_{\mathcal{B}_\gamma} \le K_2 \|T(f - g)\|_{\mathcal{B}_\gamma}. \tag{6.13}$$

REMARK 6.9. (i) The γ-quasi-Banach spaces as in Definition 6.8 have been investigated in [45]; in the case of $K_1 = 1$, they were introduced in [85] (see also [5, 86]).

(ii) Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces ℓ^q, $L^q(\mathcal{X})$ and $H^q(\mathcal{X})$ with $q \in (0, 1)$ are typical q-quasi-Banach spaces. Moreover, by the Aoki–Rolewicz theorem (see [3, 65]), any quasi-Banach space is a γ-quasi-Banach space for some $\gamma \in (0, 1)$.

THEOREM 6.10. *Let $p(\cdot) \in C^{\log}_{(n/(n+1),1]}(\mathcal{X})$, $\gamma \in [p_+, 1]$ and $\mathcal{B}_\gamma$ be a γ-quasi-Banach space. Assume that one of the following is satisfied:*

(i) *$q \in (1, \infty)$ and $T : H^{p(\cdot),q}_{\mathrm{fin}}(\mathcal{X}) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator such that*

$$A_1 := \sup\{\|Ta\|_{\mathcal{B}_\gamma} : a \text{ is a } (p(\cdot), q)\text{-atom}\} < \infty;$$

(ii) *$T : H^{p(\cdot),\infty}_{\mathrm{fin}}(\mathcal{X}) \cap \mathcal{C}(\mathcal{X}) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator such that*

$$A_2 := \sup\{\|Ta\|_{\mathcal{B}_\gamma} : a \text{ is a continuous } (p(\cdot), \infty)\text{-atom}\} < \infty.$$

Then T uniquely extends to a bounded $\mathcal{B}_\gamma$-sublinear operator from $H^{*,p}(\mathcal{X})$ to $\mathcal{B}_\gamma$.

Proof. Assume first that (i) holds true. Let $f \in H_{\rm fin}^{p(\cdot),q}(\mathcal{X})$. Then we have $f = \sum_{j=1}^N \lambda_j a_j$, where $N \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ and $\{a_j\}_{j=1}^N$ are $(p(\cdot), q)$-atoms satisfying

$$\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j=1}^N) \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}. \tag{6.14}$$

From (6.14), (6.12), assumption (i) and Lemma 5.9, we deduce that

$$\begin{aligned} \|T(f)\|_{\mathcal{B}_\gamma} &= \Big\|T\Big(\sum_{j=1}^N \lambda_j a_j\Big)\Big\|_{\mathcal{B}_\gamma} \lesssim \Big\{\sum_{j=1}^N |\lambda_j|^\gamma \|T(a_j)\|_{\mathcal{B}_\gamma}^\gamma\Big\}^{1/\gamma} \\ &\lesssim \Big\{\sum_{j=1}^N |\lambda_j|^\gamma\Big\}^{1/\gamma} \lesssim \widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j=1}^N) \lesssim \|f\|_{H^{*,p(\cdot)}(\mathcal{X})}. \end{aligned} \tag{6.15}$$

Now, by the density of $H_{\rm fin}^{p(\cdot),q}(\mathcal{X})$ in $H^{*,p(\cdot)}(\mathcal{X})$, together with a density argument, we deduce that the desired conclusion holds true.

Finally, if assumption (ii) is satisfied, then, by the fact that $H_{\rm fin}^{p(\cdot),\infty}(\mathcal{X}) \cap \mathcal{C}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$ (see Proposition 4.26), we also obtain the desired conclusion in this case. This finishes the proof of Theorem 6.10. ∎

7. Duality of $H^{*,p(\cdot)}(\mathcal{X})$ with $p_+ \in (0,1]$

In this section, we first introduce a kind of BMO spaces corresponding to a function $\phi : \mathcal{X} \times (0,\infty) \to (0,\infty)$ on $\mathcal{X}$, denoted by $\mathrm{BMO}_\phi(\mathcal{X})$. Then we prove that, when $p_+ \in (0,1]$, the dual space of $H^{*,p(\cdot)}(\mathcal{X})$ is $\mathrm{BMO}_\phi(\mathcal{X})$ for a special function ϕ.

We begin with the following definition.

Definition 7.1. For a function $\phi : \mathcal{X} \times (0,\infty) \to (0,\infty)$, the *space* $\mathrm{BMO}_\phi(\mathcal{X})$ is defined to be the set of all $h \in L^1_{\rm loc}(\mathcal{X})$ such that

$$\|h\|_{\mathrm{BMO}_\phi(\mathcal{X})} := \sup_{x\in\mathcal{X},\, r\in(0,\infty)} \frac{1}{\phi(x,r)} \left[\int_{B(x,r)} \left|h(y) - \fint_{B(x,r)} h\right|^2 d\mu(y)\right]^{1/2}$$

is finite, where, for all locally integrable functions f,

$$\fint_{B(x,r)} f := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(z)\, d\mu(z).$$

Theorem 7.2. *Let* $p(\cdot) \in C^{\log}_{(n/(n+1),\infty)}(\mathcal{X})$ *and* $p_+ \in (0,1]$. *For all* $x \in \mathcal{X}$ *and* $r \in (0,\infty)$, *define*

$$\phi(x,r) := \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathcal{X})}}{[\mu(B(x,r))]^{1/2}}.$$

(i) *For each* $h \in \mathrm{BMO}_\phi(\mathcal{X})$, *the mapping*

$$L_h : f \in L_{\rm b}^{2,0}(\mathcal{X}) \mapsto \int_{\mathcal{X}} h(x) f(x)\, dx \tag{7.1}$$

extends to a bounded linear functional on $H^{*,p(\cdot)}(\mathcal{X})$ *such that*

$$\|L_h\|_{(H^{*,p(\cdot)}(\mathcal{X}))^*} \le \|h\|_{\mathrm{BMO}_\varphi(\mathcal{X})}.$$

(ii) *Any bounded linear functional L on $H^{*,p(\cdot)}(\mathcal{X})$ can be realized as above with some function $h \in \mathrm{BMO}_\phi(\mathcal{X})$ and*

$$\|h\|_{\mathrm{BMO}_\phi(\mathcal{X})} \le C\|L\|_{(H^{*,p(\cdot)}(\mathcal{X}))^*}$$

with C being a positive constant independent of L.

Proof. To prove (i), we first show that the functional L_h is well defined on all $(p(\cdot),2)$-atoms a. Indeed, if $\operatorname{supp} a \subset B$ for some ball $B \subset \mathcal{X}$, then, by the vanishing moment of the $(p(\cdot),2)$-atom a and the Hölder inequality, we see that

$$\begin{aligned} |L_h(a)| &= \left|\int_{\mathcal{X}} h(x)a(x)\,d\mu(x)\right| = \left|\int_{\mathcal{X}} \left[h(x) - \fint_B h\right] a(x)\,d\mu(x)\right| \\ &\le [\mu(B)]^{-1/2}\|\chi_B\|_{L^{p(\cdot)}(\mathcal{X})}\|a\|_{L^2(\mathcal{X})}\|h\|_{\mathrm{BMO}_\phi(\mathcal{X})} \le \|h\|_{\mathrm{BMO}_\phi(\mathcal{X})}. \end{aligned} \tag{7.2}$$

Thus, the claim holds true. Moreover, L_h is well defined on $L^{2,0}_{\mathrm{b}}(\mathcal{X})$ by (7.2). Now, for any $f \in L^{2,0}_{\mathrm{b}}(\mathcal{X})$, by Lemma 4.23, we have $f = \sum_{j=1}^N \lambda_j a_j$ almost everywhere on $\mathcal{X}$, where $N \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ and $\{a_j\}_{j=1}^N$ are $(p(\cdot),2)$-atoms satisfying

$$\widetilde{\mathcal{E}}_{p(\cdot)}(\{\lambda_j a_j\}_{j=1}^N) \lesssim \|f\|_{H^{p(\cdot),2}_{\mathrm{fin}}(\mathcal{X})}.$$

From this and Lemmas 2.9 and 4.23, we deduce that

$$\begin{aligned} \left|\int_{\mathcal{X}} h(x)f(x)\,d\mu(x)\right| &\le \sum_{j=1}^N |\lambda_j| \left|\int_{\mathcal{X}} a_j(x)h(x)\,d\mu(x)\right| \le \|h\|_{\mathrm{BMO}_\phi(\mathcal{X})} \sum_{j=1}^N |\lambda_j| \\ &\le \|h\|_{\mathrm{BMO}_\phi(\mathcal{X})}\|f\|_{H^{p(\cdot),2}_{\mathrm{fin}}(\mathcal{X})}. \end{aligned}$$

Using this and the fact that $L^{2,0}_{\mathrm{b}}(\mathcal{X})$ is dense in $H^{*,p(\cdot)}(\mathcal{X})$ (see Lemma 4.23), we conclude that L_h extends to a unique bounded linear functional on $H^{*,p(\cdot)}(\mathcal{X})$ such that

$$\|L_h\|_{(H^{*,p(\cdot)}(\mathcal{X}))^*} \le \|h\|_{\mathrm{BMO}_\varphi(\mathcal{X})},$$

which completes the proof of (i).

Let us now prove (ii). To this end, let L be a bounded linear functional on $H^{*,p(\cdot)}(\mathcal{X})$. Notice that, for any ball $B(x,r)$ of $\mathcal{X}$, with $x \in \mathcal{X}$ and $r \in (0,\infty)$, and $f \in L^2(B(x,r))$,

$$\frac{[\mu(B(x,r))]^{1/2}}{2\|f\|_{L^2(B(x,r))}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathcal{X})}} \left[f - \fint_{B(x,r)} f\right]\chi_{B(x,r)}$$

is a $(p(\cdot),2)$-atom. Then, by Theorem 4.3, the mapping

$$f \in L^2(B(x,r)) \mapsto L\left(\left[f - \fint_{B(x,r)} f\right]\chi_{B(x,r)}\right)$$

is a bounded linear mapping on $L^2(B(x,r))$. Thus, by the self-duality of $L^2(B(x,r))$, we obtain a function $b_{B(x,r)} \in L^2(B(x,r))$ such that, for all $f \in L^2(B(x,r))$,

$$L\left(\left[f - \fint_{B(x,r)} f\right]\chi_{B(x,r)}\right) = \int_{B(x,r)} b_{B(x,r)}(y)f(y)\,d\mu(y) \tag{7.3}$$

and

$$\|b_{B(x,r)}\|_{L^2(\mathcal{X})} \le \|L\|_{(H^{*,p(\cdot)}(\mathcal{X}))^*} \frac{2\|f\|_{L^2(B(x,r))}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathcal{X})}}{[\mu(B(x,r))]^{1/2}}.$$

By choosing $f := b_{B(x,r)}$ in (7.3), we see that

$$\begin{aligned}\|b_{B(x,r)}\|^2_{L^2(B(x,r))} &= L\left(\left[b_{B(x,r)} - \fint_{B(x,r)} b_{B(x,r)}\right]\chi_{B(x,r)}\right)\\ &\lesssim \|L\|_{(H^{*,p(\cdot)}(\mathcal{X}))^*}\phi(x,r)\|b_{B(x,r)}\|_{L^2(B(x,r))}.\end{aligned}$$

Thus,

$$\|b_{B(x,r)}\|_{L^2(B(x,r))} \lesssim \phi(x,r)\|L\|_{(H^{*,p(\cdot)}(\mathcal{X}))^*}. \tag{7.4}$$

Let $B(x,r)$ and $B(\widetilde{x},\widetilde{r})$ be balls in $\mathcal{X}$ such that $B(x,r)$ is contained in $B(\widetilde{x},\widetilde{r})$, where $x,\widetilde{x}\in\mathcal{X}$ and $r,\widetilde{r}\in(0,\infty)$. Then, by (7.3) with $B(x,r)$ replaced by $B(\widetilde{x},\widetilde{r})$ and f replaced by $f\chi_{B(x,r)}$, we find that

$$L\left(\left[f\chi_{B(x,r)} - \frac{1}{\mu(B(\widetilde{x},\widetilde{r}))}\int_{B(x,r)} f(y)\,d\mu(y)\right]\chi_{B(\widetilde{x},\widetilde{r})}\right) = \int_{B(x,r)} b_{B(\widetilde{x},\widetilde{r})}(y)f(y)\,d\mu(y).$$

From this and (7.3), we obtain

$$\begin{aligned}&\int_{B(x,r)} b_{B(\widetilde{x},\widetilde{r})}(y)f(y)\,d\mu(y) - \int_{B(x,r)} b_{B(x,r)}(y)f(y)\,d\mu(y)\\ &\qquad= L\left(\left[\frac{\chi_{B(x,r)}}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,d\mu(y) - \frac{1}{\mu(B(\widetilde{x},\widetilde{r}))}\int_{B(x,r)} f(y)\,d\mu(y)\right]\chi_{B(\widetilde{x},\widetilde{r})}\right)\\ &\qquad= \int_{B(\widetilde{x},\widetilde{r})} b_{B(\widetilde{x},\widetilde{r})}(z)\left[\chi_{B(x,r)}(z)\fint_{B(x,r)} f - \frac{1}{\mu(B(\widetilde{x},\widetilde{r}))}\int_{B(x,r)} f(y)\,d\mu(y)\right]d\mu(z)\\ &\qquad= \int_{B(x,r)} f(y)\left[\fint_{B(x,r)} b_{B(\widetilde{x},\widetilde{r})} - \fint_{B(\widetilde{x},\widetilde{r})} b_{B(\widetilde{x},\widetilde{r})}\right]d\mu(y)\end{aligned}$$

for all $f\in L^2(B(x,r))$. It follows that

$$b_{B(\widetilde{x},\widetilde{r})}(z) = b_{B(x,r)}(z) + \fint_{B(x,r)} b_{B(\widetilde{x},\widetilde{r})} - \fint_{B(\widetilde{x},\widetilde{r})} b_{B(\widetilde{x},\widetilde{r})}$$

for μ-almost every $z\in B(x,r)$. We let

$$c_{B(x,r),B(\widetilde{x},\widetilde{r})} := \fint_{B(x,r)} b_{B(\widetilde{x},\widetilde{r})} - \fint_{B(\widetilde{x},\widetilde{r})} b_{B(\widetilde{x},\widetilde{r})}.$$

Then

$$b_{B(\widetilde{x},\widetilde{r})}(y) = b_{B(x,r)}(y) + c_{B(x,r),B(\widetilde{x},\widetilde{r})} \tag{7.5}$$

for μ-almost every $y\in B(x,r)$ if $B(x,r)$ is contained in $B(\widetilde{x},\widetilde{r})$.

In particular, we can define, for all $y\in\mathcal{X}$,

$$h(y) := b_{B(x_1,R)}(y) - c_{B(x_1,1),B(x_1,R)}$$

as long as $\max\{1,d(x_1,y)\}<R$ despite the ambiguity of such R. Indeed, when $\widetilde{R}>R>\max\{1,d(x_1,y)\}$, for almost every $y\in B(x_1,1)$,

$$b_{B(x_1,\widetilde{R})}(y) = b_{B(x_1,1)}(y) + c_{B(x_1,1),B(x_1,\widetilde{R})}, \qquad b_{B(x_1,R)}(y) = b_{B(x_1,1)}(y) + c_{B(x_1,1),B(x_1,R)}$$

and, for almost every $y\in B(x_1,R)$,

$$b_{B(x_1,\widetilde{R})}(y) = b_{B(x_1,R)}(y) + c_{B(x_1,R),B(x_1,\widetilde{R})}.$$

Therefore, it follows that

$$c_{B(x_1,1),B(x_1,R)} + c_{B(x_1,R),B(x_1,\widetilde{R})} = c_{B(x_1,1),B(x_1,\widetilde{R})},$$

and hence

$$\begin{aligned} b_{B(x_1,\widetilde{R})} - c_{B(x_1,1),B(x_1,\widetilde{R})} - b_{B(x_1,R)} + c_{B(x_1,1),B(x_1,R)} \\ = b_{B(x_1,\widetilde{R})} - b_{B(x_1,R)} - c_{B(x_1,R),B(x_1,\widetilde{R})} = 0 \end{aligned}$$

almost everywhere on $B(x_1, R)$.

By (7.5) and the definition of h, we see that, for each ball $B(x,r)$ of $\mathcal{X}$ and almost every $y \in B(x,r)$,

$$\begin{aligned} h(y) - b_{B(x,r)}(y) &= b_{B(x_1,R)}(y) - c_{B(x_1,1),B(x_1,R)} - [b_{B(x_1,R)}(y) - c_{B(x,r),B(x_1,R)}] \\ &= c_{B(x,r),B(x_1,R)} - c_{B(x_1,1),B(x_1,R)}, \end{aligned}$$

where R is large enough. Thus, for each $B(x,r)$, we know that $h - b_{B(x,r)}$ is just the constant $c_{B(x,r),B(x_1,R)} - c_{B(x_1,1),B(x_1,R)}$, which depends on x and r.

Let us show that h realizes L, or more precisely, let us show that, for any $(p(\cdot),2)$-atom a with support B,

$$L(a) = \int_{\mathcal{X}} h(x)a(x)\,d\mu(x).$$

To this end, choose $R > 1$ so that $B \subset B(x_1,R)$. Then, since the integral of a is zero, we have

$$\int_{\mathcal{X}} a(x)h(x)\,d\mu(x) = \int_B a(x)h(x)\,d\mu(x) = \int_B a(x)b_B(x)\,d\mu(x). \tag{7.6}$$

By (7.3), we find that

$$\int_B a(x)b_B(x)\,d\mu(x) = L\bigg(\bigg[a - \frac{1}{\mu(B)}\int_B a(y)\,d\mu(y)\bigg]\chi_B\bigg) = L(a),$$

which, combined with (7.6), implies that

$$L(a) = \int_{\mathcal{X}} h(x)a(x)\,d\mu(x).$$

On the other hand, observe that, for each ball $B \subset \mathcal{X}$, $h - b_B$ is constant almost everywhere on B. Then, by (7.4), we conclude that, for every ball $B := B(x_B, r_B) \subset \mathcal{X}$, with $x_B \in \mathcal{X}$ and $r_B \in (0,\infty)$,

$$\begin{aligned} \bigg\{\int_B \bigg|h(x) - \fint_B h\bigg|^2 d\mu(x)\bigg\}^{1/2} &= \bigg\{\int_B \bigg|b_B(x) - \fint_B b_B\bigg|^2 d\mu(x)\bigg\}^{1/2} \\ &\lesssim \|b_B\|_{L^2(\mathcal{X})} \lesssim \phi(x_B,r_B)\|L\|_{(H^{*,p(\cdot)}(\mathcal{X}))^*}, \end{aligned}$$

which further implies that

$$\|h\|_{\mathrm{BMO}_\phi(\mathcal{X})} \lesssim \|L\|_{(H^{*,p(\cdot)}(\mathcal{X}))^*}.$$

This finishes the proof of Theorem 7.2. ∎

Remark 7.3. It is still unclear how to obtain a description of $(H^{*,p(\cdot)}(\mathcal{X}))^*$ when $p_+ > 1$ and $p_- \le 1$; see [38].

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