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Traces of functions of L^1_2 Dirichlet spaces on the Carathéodory boundary

by

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Abstract. We prove that any weakly differentiable function with a square integrable gradient can be extended to the Carathéodory boundary of any simply connected planar domain $\Omega \neq \mathbb{R}^2$ up to a set of conformal capacity zero. This result is based on the notion of capacitary boundary associated with the Dirichlet space $L^1_2(\Omega)$.

1. Introduction. Let Ω be a domain in $\mathbb{R}^2$. The trace (extension) problem for univalent analytic functions was first considered by C. Carathéodory [6] in 1913. He introduced the notion of an ideal boundary $\partial_C \Omega$ in terms of so-called prime ends. The Carathéodory prime ends represent a compactification of the planar domain in the relative distance introduced by Lavrentiev [19]. The main result of [6] states: if Ω is a simply connected planar domain whose boundary $\partial\Omega$ is a Jordan curve, then every univalent analytic function f from Ω onto the unit disc $\mathbb{D}$ extends continuously to $\partial\Omega$. A more applicable version of this theorem is the following. Let $g : \mathbb{D} \to \Omega$ be a univalent analytic function. Then g extends continuously onto the boundary if and only if the boundary of Ω is locally connected [38].

Univalent analytic functions $f : \Omega \to \mathbb{D}$ have a square integrable gradient. From this point of view it is natural to consider the trace problem for weakly differentiable functions with $\nabla u \in L_2(\Omega)$.

These functions are elements of the Dirichlet space (a uniform Sobolev space) $L^1_2(\Omega)$, the space of locally integrable functions with square integrable weak gradient $\nabla u \in L_2(\Omega)$, equipped with the seminorm

$$\|u | L^1_2(\Omega)\| = \|\nabla u | L_2(\Omega)\|.$$

By the standard definition, $L^1_2(\Omega)$ functions are defined only up to a set of measure zero, but they can be redefined *quasi-everywhere*, i.e. off

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a set of conformal capacity zero. Indeed, every function $u \in L^1_2(\Omega)$ has a unique quasicontinuous representative $\widetilde{u} \in L^1_2(\Omega)$. Here a function $\widetilde{u}$ is termed *quasicontinuous* if for any $\varepsilon > 0$ there is an open set U_ε of conformal capacity less than ε such that $\widetilde{u}$ is continuous on $\Omega \setminus U_\varepsilon$ (see, for example, [15, 22]). The concept of quasicontinuity can be obviously extended to the closure $\overline{\Omega}$ of Ω.

In this paper we deal only with quasicontinuous representatives of functions $u \in L^1_2(\Omega)$.

One of the main results of the paper is the following:

THEOREM A. *Let $\Omega \subset \mathbb{R}^2$, $\Omega \neq \mathbb{R}^2$, be a simply connected domain which is locally connected at any $x \in \partial\Omega$. Then for every $u \in L^1_2(\Omega)$ there exists a quasicontinuous function $\widetilde{u} : \overline{\Omega} \to \mathbb{R}$ such that $\widetilde{u}|_\Omega = u$.*

REMARK 1.1. The function $\widetilde{u}$ is defined at all points of $\partial\Omega$ except a set of conformal capacity zero (i.e. quasi-everywhere).

The proof is based on extension of weakly differentiable functions with a square integrable gradient to the Carathéodory boundary $\partial_C\Omega$.

THEOREM B. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, $\Omega \neq \mathbb{R}^2$. Then every $u \in L^1_2(\Omega)$ has a quasicontinuous extension $\widetilde{u}$ on the Carathéodory boundary $\partial_C\Omega$. The function $\widetilde{u}$ is defined quasi-everywhere* (*everywhere except on a set of conformal capacity zero*) *on $\partial_C\Omega$.*

The main ingredient of our method is the well-known concept of conformal capacity and the less known concept of conformal capacitary boundary introduced by V. Gol'dshtein and S. K. Vodop'yanov [12] for quasiconformal homeomorphisms.

We prove that in the planar case, "points" of the conformal capacitary boundary coincide with the Carathéodory prime ends. This allows us to consider traces of weakly differentiable functions on the classical Carathéodory boundary $\partial_C\Omega$.

The main properties of the space $L^1_2(\mathbb{D})$ where $\mathbb{D} \subset \mathbb{R}^2$ is the unit disc are well known. The Dirichlet spaces $L^1_2(\Omega)$ are conformal invariants. Therefore the Riemann Mapping Theorem permits us to transfer necessary information about boundary behavior from $L^1_2(\mathbb{D})$ to simply connected domains Ω.

More precisely, we extend the concept of quasicontinuity to a "capacitary" completion of a domain Ω. We construct a conformal capacitary boundary as the completion $\{\widetilde{\Omega}_\rho, \rho\}$ of a metric space $\{\Omega_\rho, \rho\}$ for a conformal capacitary metric ρ (see Section 1). Roughly speaking, an "ideal" capacitary boundary point is a boundary continuum of conformal capacity zero.

Our method allows us to treat the general case of simply connected planar domains $\Omega \subset \mathbb{R}^2$. We prove that any function $u \in L^1_2(\Omega)$ has a quasicontinuous extension onto the conformal capacitary boundary $H_\rho = \widetilde{\Omega}_\rho \setminus \Omega_\rho$. The main result in terms of ideal capacitary boundary is:

THEOREM C. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, $\Omega \neq \mathbb{R}^2$. Then for every $u \in L^1_2(\Omega)$ there exists a quasicontinuous function $\widetilde{u} : \widetilde{\Omega}_\rho \to \mathbb{R}$ defined quasi-everywhere on H_ρ such that $\widetilde{u}|_\Omega = u$.*

REMARK 1.2. The concepts of conformal capacitary metric and conformal capacitary boundary were proposed in [12]. By quasi-invariance of conformal capacity under (quasi)conformal homeomorphisms, any such homeomorphism $\varphi : \Omega \to \Omega'$ is a bi-Lipschitz homeomorphism $\varphi : (\Omega, \rho) \to (\Omega', \rho)$ for the corresponding conformal metrics and can be extended to a homeomorphism $\widetilde{\varphi} : (\widetilde{\Omega}, \rho) \to (\widetilde{\Omega}', \rho)$ of the capacitary completions [12].

There is a vast literature concerning "ideal" boundaries of planar domains in the context of conformal homeomorphisms. We discuss a few such concepts in the last section.

The paper is organized as follows:

The main properties of conformal capacitary metrics are proved in Section 2. The focus is on the local properties of the metrics at boundary points and their dependence on the local topological properties of the boundary. In Section 3 we discuss an analog of the Luzin property for capacitary metrics. We call it the strong Luzin property. We prove this condition for fairly large classes of domains that include extension domains for $L^1_2(\Omega)$. In Section 4 we apply the abstract construction of Section 3 to simply connected planar domains and we prove the main results about extension of $L^1_2(\Omega)$ functions to the capacitary boundary. In Section 5 we give a short historical sketch of the "ideal" boundary concept and its connection to the capacitary boundary.

In the terminology of the theory of Sobolev spaces we solve the classical trace problem for the space $L^1_2(\Omega)$ in the case of simply connected planar domains.

REMARK 1.3. The classical trace problem for Sobolev spaces is of great interest, mainly due to its important applications to boundary value problems for partial differential equations. Boundary value problems can be specified with the help of traces on $\partial\Omega$ of Sobolev functions.

There is an extensive literature devoted to the trace problem for Sobolev spaces: we mention the monographs of P. Grisvard [14], J.-L. Lions and E. Magenes [21], V. G. Maz'ya and S. Poborchi [22], [27], and the papers [1]–[5], [8], [16]–[18], [24]–[23], [31], [32], [40], [42], [43].

For smooth domains the traces of Sobolev functions are in Besov spaces. In the case of Lipschitz domains the traces can also be described in terms of Besov spaces. For arbitrary non-Lipschitz domains the trace problem is open. For cusp type singularities a description of traces can be found in [11] in terms of weighted Sobolev spaces.

2. Conformal capacitary metric. Let Ω be a planar domain and let F_0, F_1 be disjoint compact subsets of Ω. We call the triple $E = (F_0, F_1; \Omega)$ a *condenser*.

The value

$$\operatorname{cap}(E) = \operatorname{cap}(F_0, F_1; \Omega) = \inf \int_{\Omega} |\nabla v|^2 \, dx,$$

where the infimum is taken over all nonnegative functions $v \in C(\Omega) \cap L^1_2(\Omega)$ such that $v = 0$ in a neighborhood of F_0, and $v \geq 1$ in a neighborhood of F_1, is called the *conformal capacity* of E.

For $0 < \operatorname{cap}(F_0, F_1; \Omega) < \infty$ there exists a unique function u_0 (an *extremal function*) such that

$$\operatorname{cap}(F_0, F_1; \Omega) = \int_{\Omega} |\nabla u_0|^2 \, dx.$$

An extremal function is continuous in Ω, monotone in $\Omega \setminus (F_0 \cup F_1)$, equal to zero on F_0 and to one on F_1 [15, 41].

DEFINITION 2.1. A homeomorphism $\varphi : \Omega \to \Omega'$ between planar domains is called K-*quasiconformal* if it preserves orientation, belongs to the Sobolev class $L^1_{2,\mathrm{loc}}(\Omega)$ and satisfies the distortion inequality

$$\max_{|\xi|=1} |D\varphi(x) \cdot \xi| \leq K \min_{|\xi|=1} |D\varphi(x) \cdot \xi| \quad \text{for almost all } x \in \Omega.$$

Infinitesimally, quasiconformal homeomorphisms carry circles to ellipses with eccentricity uniformly bounded by K. If $K = 1$ we recover conformal homeomorphisms, while for $K > 1$ planar quasiconformal mappings need not be smooth. The theory of quasiconformal mappings can be found, for example, in [39].

It is well known that conformal capacity is quasi-invariant under planar quasiconformal homeomorphisms.

2.1. Definition of conformal capacitary metrics. A connected and closed (with respect to Ω) set is called a *continuum*. Fix a continuum F in the domain $\Omega \subset \mathbb{R}^2$ and a compact domain V such that $F \subset V \subset \overline{V} \subset \Omega$, and the boundary ∂V is the image of the unit circle $S(0,1)$ under some quasiconformal homeomorphism of $\mathbb{R}^2$.

DEFINITION 2.2. Choose $x, y \in \Omega \subset \mathbb{R}^2$ and join them by a rectifiable curve $l(x,y) \subset \Omega$. Define the *conformal capacitary distance* between x and y in Ω with respect to the pair (F, V) as

$$\rho_{(F,V)}(x,y) = \inf_{l(x,y)} \{\operatorname{cap}^{1/2}(F, l(x,y) \setminus V; \Omega) + \operatorname{cap}^{1/2}(\partial\Omega, l(x,y) \cap V; \Omega)\}$$

where the infimum is taken over all curves $l(x,y)$ as above.

This definition was first introduced in [12] (see also [41]) where it was proved that $\rho_{(F,V)}(\cdot,\cdot)$ is a metric in Ω and that the topology induced by this metric in $\Omega \subset \mathbb{R}^2$ coincides with the Euclidean topology. This metric is quasi-invariant under quasiconformal homeomorphisms and invariant under conformal ones.

Let $\{\widetilde{\Omega}, \rho_{(F,V)}\}$ be the standard completion of the metric space $\{\Omega, \rho_{(F,V)}\}$ and denote by H_ρ the set $\{\widetilde{\Omega} \setminus \Omega, \rho_{(F,V)}\}$. We call H_ρ the *conformal capacitary boundary* of Ω. The set H_ρ of boundary elements does not depend on the choice of the sets F and V in the definition of ρ [12, 30, 41]. Moreover, the sets of boundary elements corresponding to different choices of (F, V) are homeomorphic under the identity mapping. This justifies the notation H_ρ for the conformal capacitary boundary.

The topological properties of H_ρ were studied in [12, 41]. For the reader's convenience we reproduce here the detailed proof of these properties.

Definition 2.3. For $h \in H_\rho$ we denote by $D(h, \varepsilon)$, $\varepsilon > 0$, the disc about h in the metric $\rho_{(F,V)}$.

Call the set

$$s_h = \bigcap_{\varepsilon>0} \overline{D(h,\varepsilon)} \subset \overline{\mathbb{R}^2}$$

the *realization* (or *impression*) of $h \in H_\rho$.

Recall that a domain Ω is called *locally connected* at $z_0 \in \partial\Omega$ if z_0 has arbitrarily small connected neighborhoods in Ω.

Lemma 2.4. *Let a domain Ω be locally connected at a point $x \in \partial\Omega$ and suppose $x \in s_h$ for some $h \in H_\rho$. Then for every sequence $\{x_m \in \Omega\}$ such that $|x_m - x| \to 0$ we have $\rho_{(F,V)}(x_m, h) \to 0$ (as $m \to \infty$).*

Proof. Since Ω is locally connected at $x \in \partial\Omega$, any two points x_k, x_m can be connected by a geodesic path $l(x_k, x_m)$ whose length tends to zero as $k, m \to \infty$. Without loss of generality we can suppose that $l(x_k, x_m) \cap V = \emptyset$. Hence $\operatorname{cap}(F, l(x_k, x_m); \Omega)$ tends to zero as $k, m \to \infty$, and therefore

$$\lim_{n\to\infty} \rho_{(F,V)}(x_n, h) = 0. \ ■$$

Lemma 2.5. *Let the realization s_h of $h \in H_\rho$ be a single point. Then for every sequence $\{x_m \in \Omega\}$, $\rho_{(F,V)}(x_m, h) \to 0$ implies $|x_m - s_h| \to 0$ (as $m \to \infty$).*

Proof. Suppose that $\rho_{(F,V)}(x_m, h) \to 0$. Because s_h is a single point, we have

$$\operatorname{diam}\big(\overline{D(h,\varepsilon)}\big) = \sup_{x,y \in \overline{D(h,\varepsilon)}} |x - y| \to 0 \quad \text{as } \varepsilon \to 0.$$

The sequence $\{x_m\}$ belongs to the boundary element $h \in H_\rho$ that is a class of equivalent sequences. So

$$|x_m - x_n| \to 0 \quad \text{as } m, n \to \infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence in the Euclidean metric, and consequently $|x_m - s_h| \to 0$ as $m \to \infty$. ■

From these lemmas we will deduce:

THEOREM 2.6. *Let a domain Ω be locally connected at any point $x \in \partial\Omega$. Then the identity mapping $i : \Omega \to \Omega$ can be extended to a homeomorphism $\tilde{i}_{\rho(F,V)} : \overline{\Omega} \to \widetilde{\Omega}_{\rho(F,V)}$ if and only if the realizations s_h of all boundary elements $h \in H_{\rho(F,V)}$ are single points.*

Proof. Suppose that the identity mapping $i : \Omega \to \Omega$ can be extended to a homeomorphism $\tilde{i}_{\rho(F,V)} : \overline{\Omega} \to \widetilde{\Omega}_{\rho(F,V)}$. Then every boundary element $h \in H_{\rho(F,V)}$ coincides with a point $x \in \partial\Omega$ and so has a one-point realization.

Conversely, suppose that all realizations s_h for $h \in H_{\rho(F,V)}$ are single points. Then extending the identity $i : \Omega \to \Omega$ to $\tilde{i}_{\rho(F,V)} : \widetilde{\Omega}_{\rho(F,V)} \to \overline{\Omega}$ by setting $\tilde{i}_{\rho(F,V)}(h) = s_h$ we obtain a one-to-one correspondence $\tilde{i}_{\rho(F,V)} : \widetilde{\Omega}_\rho \to \overline{\Omega}$. Let us check the continuity of $\tilde{i}_{\rho(F,V)}$ and $\tilde{i}^{-1}_{\rho(F,V)}$.

Suppose that $x_k \to x$ in $\overline{\Omega}$. Because the realizations s_h of $h \in H_\rho$ are single points and $x \in s_h$, Lemma 2.4 implies $\rho_{(F,V)}(x_m, h) \to 0$ as $m \to \infty$. Hence $\tilde{i}_\rho$ is continuous.

Suppose $h_k \to h_0$ in $\widetilde{\Omega}_{\rho(F,V)}$. Because Ω is locally connected, the realizations of h_k and h_0 are one-point sets and we can identify h_k and h_0 with their realizations. By Lemma 2.5, $h_k \to h_0$ in Ω. Therefore $\tilde{i}^{-1}_{\rho(F,V)}$ is also continuous. Thus $\tilde{i}_{\rho(F,V)}$ is a homeomorphism. ■

2.2. Conformal capacitary boundary, extension domains and Carathéodory prime ends

DEFINITION 2.7. A domain $\Omega \subset \mathbb{R}^2$ is said to be an L^1_2-*extension domain* if there exists a bounded linear operator $E : L^1_2(\Omega) \to L^1_2(\mathbb{R}^2)$ such that $E(u)|_\Omega = u$ for any $u \in L^1_2(\Omega)$.

We call the operator E an *extension operator*. It is known that a simply connected domain $\Omega \subset \mathbb{R}^2$ is an L^1_2-extension domain if and only if Ω is a quasidisc [13].

Recall that a domain $\Omega \subset \mathbb{R}^2$ is called a *quasidisc* if there exists a quasiconformal homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Omega = \varphi(\mathbb{D})$.

THEOREM 2.8. *If a bounded domain $\Omega \subset \mathbb{R}^2$ is an L^1_2-extension domain then the identity mapping* id $: H_{\rho(F,V)} \to \partial\Omega$ *is a homeomorphism for any capacitary metric $\rho_{(F,V)}$ in Ω.*

Proof. Because Ω is an L_2^1-extension domain, there exists an extension operator
$$E : L_2^1(\Omega) \to L_2^1(\mathbb{R}^2)$$
such that $E(u)|_\Omega = u$ for any $u \in L_2^1(\Omega)$. Hence
$$\frac{1}{\|E\|}\|E(u)\|_{L_2^1(\mathbb{R}^2)} \le \|u\|_{L_2^1(\Omega)} \le \|E(u)\|_{L_2^1(\mathbb{R}^2)}.$$

By the definition of the conformal capacity, for any condenser $(F_0, F_1; \Omega)$ we have
$$\frac{1}{\|E\|^2}\operatorname{cap}(F_0, F_1; \mathbb{R}^2) \le \operatorname{cap}(F_0, F_1; \Omega) \le \operatorname{cap}(F_0, F_1; \mathbb{R}^2).$$
So, by the definition of the conformal capacitary metric, for any $x, y \in \Omega$ and any pair (F, V) we obtain
$$\frac{1}{\|E\|^2}\widehat{\rho}_{(F,V)}(x, y) \le \rho_{(F,V)}(x, y) \le \widehat{\rho}_{(F,V)}(x, y)$$
where $\widehat{\rho}_{(F,V)}$ is the conformal capacitary metric in $\mathbb{R}^2$ and $\rho_{(F,V)}$ is the conformal capacitary metric in Ω. This means that the metric $\rho_{(F,V)}$ is equivalent to $\widehat{\rho}_{(F,V)}$ on Ω. The topology induced by $\widehat{\rho}_{(F,V)}$ on $\mathbb{R}^2$ coincides with the Euclidean topology, and so the topology of $H_{\rho(F,V)}$ coincides with the Euclidean topology of $\partial\Omega$. Because the metrics $\rho_{(F,V)}$ and $\widehat{\rho}_{(F,V)}$ are equivalent on Ω, the theorem is proved. ∎

This theorem gives a simple proof that the sets of boundary elements corresponding to different choices of the pair (F, V) are homeomorphic under the identity mapping.

Corollary 2.9. *If a bounded domain $\Omega \subset \mathbb{R}^2$ is an L_2^1-extension domain then for any two capacitary metrics $\rho_{(F,V)}$ and $\rho_{(F_1,V_1)}$ in Ω the corresponding capacitary boundaries $H_{\rho_{(F,V)}}$ and $H_{\rho_{(F_1,V_1)}}$ are homeomorphic.*

Because any quasidisc is an L_2^1-extension domain we immediately obtain

Corollary 2.10. *Let $\Omega \subset \mathbb{R}^2$ be a quasidisc. Then the capacitary boundary H_ρ of Ω is homeomorphic to its Euclidean boundary $\partial\Omega$.*

The notion of ideal boundary in terms of prime ends was introduced by Carathéodory [6]. The Carthéodory prime ends represent a compactification of the planar domains in the relative distance introduced by Lavrentiev [19]. (A detailed historical sketch can be found in [28].) We prove that the capacitary boundary is homeomorphic to the Carathéodory boundary.

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, $\Omega \neq \mathbb{R}^2$. Then the capacitary boundary H_ρ is homeomorphic to the Carathéodory boundary $\partial_C\Omega$.*

Proof. The Carathéodory boundary $\partial_C \Omega$ is homeomorphic to the Euclidean boundary of the unit disc, $\partial\mathbb{D}$. By Corollary 2.10 the capacitary boundary of the unit disc is also homeomorphic to $\partial\mathbb{D}$. ■

On the base of this theorem we give some examples [7] of boundary elements $h \in H_\rho$ of the conformal capacitary boundary.

Example 2.12. Let

$$X = \{(x,y) : y = 1/3^n \text{ for some } n \geq 1 \text{ and } -1 \leq x \leq 2\},$$
$$Y = \{(x,y) : y = 2/3^n \text{ for some } n \geq 1 \text{ and } -2 \leq x \leq 1\}.$$

Let $\Omega = (-2,2) \times (0,1) \setminus (X \cup Y)$. Then $h = \{(x,0) : -1 \leq x \leq 1\}$ is a boundary element of this domain.

Example 2.13. Let $\Omega = \mathbb{R}^2 \setminus K$, where K is given in polar coordinates by

$$K = \{(r,\theta) : \theta = 2\pi p/2^n \text{ for some integer } n \geq 1 \text{ and some odd integer } p \text{ with } 0 < p < 2^n,\ 0 \leq r \leq 1/2^n\}.$$

A boundary element $h \in H_\rho$ of this domain at the origin is homeomorphic to a Cantor set.

By C. Carathéodory [6] the domain Ω is locally connected at boundary points if and only if every boundary element has a one-point realization. Hence we have the following corollary of Theorem 2.6:

Theorem 2.14. *Let Ω be a simply connected domain locally connected at every point $x \in \partial\Omega$. Then the identity mapping $i : \Omega \to \Omega$ can be extended to a homeomorphism $\tilde{i}_\rho : \widetilde{\Omega}_\rho \to \overline{\Omega}$.*

3. Strong Luzin property for the capacitary metric and boundary values of Sobolev functions. Recall the notion of the conformal capacity of a set $E \subset \Omega$. Let Ω be a domain in $\mathbb{R}^2$, and $F \subset \Omega$ a compact subset. The *conformal capacity* of F is defined by

$$\operatorname{cap}(F;\Omega) = \inf\{\|u|L^1_2(\Omega\|^2 : u \geq 1 \text{ on } F,\ u \in C_0(\Omega)\}.$$

In a similar way we define the conformal capacity of open sets.

For an arbitrary set $E \subset \Omega$ we define the *inner conformal capacity* as

$$\underline{\operatorname{cap}}(E;\Omega) = \sup\{\operatorname{cap}(e;\Omega) : e \subset E \subset \Omega,\ e \text{ is a compact set}\},$$

and the *outer conformal capacity* as

$$\overline{\operatorname{cap}}(E;\Omega) = \inf\{\operatorname{cap}(U;\Omega) : E \subset U \subset \Omega,\ U \text{ is an open set}\}.$$

A set $E \subset \Omega$ is called *conformal capacity measurable* if $\underline{\operatorname{cap}}(E;\Omega) = \overline{\operatorname{cap}}(E;\Omega)$; the value

$$\operatorname{cap}(E;\Omega) = \underline{\operatorname{cap}}(E;\Omega) = \overline{\operatorname{cap}}(E;\Omega)$$

is then called the *conformal capacity* of E.

The classical Luzin theorem asserts that every measurable function is *uniformly* continuous if it is restricted to the complement of an open set of sufficiently small measure. It is reasonable to conjecture that every function $u \in L^1_2(\Omega)$ is *uniformly* continuous if it is restricted to the complement of an open subset of $\Omega \subset \mathbb{R}^2$ of sufficiently small conformal capacity. Unfortunately this conjecture is not valid for arbitrary domains; it only holds under additional conditions on Ω. A weak version of the Luzin theorem is valid for capacity:

THEOREM 3.1 (Weak Luzin theorem for p-capacity [22]). *Let $\Omega \subset \mathbb{R}^2$ be an open set. For any $u \in L^1_2(\Omega)$ and $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset \Omega$ with $\operatorname{cap}(U_\varepsilon; \Omega) < \varepsilon$ such that $u|_{\Omega \setminus U_\varepsilon}$ is continuous.*

We discuss here a strong version of the Luzin property for capacity:

DEFINITION 3.2. A domain $\Omega \subset \mathbb{R}^2$ has the *strong Luzin (F, V)-capacitary property* if for every $u \in L^1_2(\Omega)$ and $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset \Omega$ with conformal capacity less than ε such that the restriction of u to $\Omega \setminus U_\varepsilon$ is uniformly continuous for the conformal capacitary metric $\rho_{(F,V)}$.

This property looks very restrictive, but, in reality, it holds for a large class of domains. We prove in this section that any extension domain has the strong Luzin capacitary property, and in the next section that any quasiconformal homeomorphism preserves this property.

Our main motivation for studying this property is the following result:

THEOREM 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a domain with the strong Luzin (F, V)-capacitary property. Then for every $u \in L^1_2(\Omega)$ there exists a quasicontinuous function $\widetilde{u} : \widetilde{\Omega}_\rho \to \mathbb{R}$ defined quasi-everywhere on H_ρ such that $\widetilde{u}|_\Omega = u$.*

Proof. Because Ω has the strong Luzin (F, V)-capacitary property, for every $\varepsilon > 0$ and $u \in L^1_2(\Omega)$ there exists an open set $U_\varepsilon \subset \Omega$ such that $\operatorname{cap}(U_\varepsilon) < \varepsilon$ and u is uniformly continuous for the conformal capacitary metric on the closed (in Ω) set $\Omega^\varepsilon = \Omega \setminus U_\varepsilon$. Consider the closure $\widetilde{\Omega}^\varepsilon$ of Ω^ε in the complete metric space $(\widetilde{\Omega}_\rho, \rho)$. Since u is uniformly continuous on $(\Omega^\varepsilon_\rho, \rho)$, by the Tietze theorem there exists an extension $\widetilde{u}_\varepsilon$ of u to $\widetilde{\Omega}^\varepsilon$. Set $\widetilde{\Omega}^0 = \bigcup_{\varepsilon > 0} \widetilde{\Omega}^\varepsilon$. Then u has an extension $\widetilde{u}$ to $(\widetilde{\Omega}^0, \rho)$ and $\operatorname{cap}(\widetilde{\Omega}_\rho \setminus \widetilde{\Omega}^0) = 0$ because $\Omega_{\varepsilon_1} \supset \Omega_{\varepsilon_2}$ if $\varepsilon_1 < \varepsilon_2$. Therefore $\widetilde{u}|_{H_\rho}$ is defined quasi-everywhere on H_ρ and represents the boundary value of $u \in L^1_2(\Omega)$ on H_ρ. ∎

The definition of the strong (F, V)-capacitary property depends on the choice of (F, V) in the definition of $\rho_{(F,V)}$. But because sets of capacity zero do not depend on the continuum F, the extension $\widetilde{u}$ does not depend on the pair (F, V), and $\widetilde{u}$ is defined quasi-everywhere on H_ρ in the following sense: *for any $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset \Omega$ such that u is uniformly* *continuous on* $\Omega \setminus U_\varepsilon$, $\operatorname{cap}(\Omega_\rho \setminus U_\varepsilon) < \varepsilon$, *and the continuous extension of* $u : \Omega\setminus U_\varepsilon \to \mathbb{R}$ *to the completion* $(\widetilde{\Omega \setminus U_\varepsilon}, \rho)$ *coincides with* $\widetilde{u}$ *on* $H_\rho \cap (\widetilde{\Omega \setminus U_\varepsilon}, \rho)$.

Combining the previous theorem and Theorem 2.14 we immediately obtain

THEOREM 3.4. *Let* $\Omega \subset \mathbb{R}^2$ *be a domain with the strong Luzin* (F, V)*-capacitary property and locally connected at any boundary point. Then for every* $u \in L^1_2(\Omega)$ *there exists a quasicontinuous function* $\widetilde{u} : \overline{\Omega} \to \mathbb{R}$ *defined quasi-everywhere on* $\partial\Omega$ *such that* $\widetilde{u}|_\Omega = u$.

The strong capacitary property is valid for a large class of domains, namely extension domains. The class of extension domains includes domains with smooth or Lipschitz boundaries (see for example [22]).

THEOREM 3.5. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded* L^1_2*-extension domain. Then* Ω *has the strong Luzin* (F, V)*-capacitary property for every capacitary metric* $\rho_{(F,V)}$.

Proof. Choose $u \in L^1_2(\Omega)$. Because Ω is an extension domain, there exists an extension $\widehat{u} \in L^1_2(\mathbb{R}^2)$ of u. By Theorem 3.1 for any $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset \mathbb{R}^2$ of conformal capacity less than ε such that $\widehat{u}$ is continuous on $\mathbb{R}^2 \setminus U_\varepsilon$. As Ω is bounded, $\widehat{u}|_{\overline{\Omega}\setminus U_\varepsilon}$ is uniformly continuous in the metric $\rho_{(F,V)}$ for any continuum $F \subset \Omega$ and any compact domain V such that $F \subset V \subset \overline{V} \subset \Omega$ and ∂V is the image of the unit circle $S(0, 1)$ under some quasiconformal mapping.

Hence u is uniformly continuous on $\Omega \setminus U_\varepsilon$ in the metric $\rho_{(F,V)}$. By monotonicity of conformal capacity, $\operatorname{cap}(U_\varepsilon \cap \Omega) < \operatorname{cap}(U_\varepsilon) < \varepsilon$. Hence by Theorem 2.8, u is also uniformly continuous for any metric $\rho_{(F,V)}$ in $\Omega\setminus U_\varepsilon$. ∎

Combining Theorems 3.3, 3.5 and 2.8 we obtain

THEOREM 3.6. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded* L^1_2*-extension domain. Then for every* $u \in L^1_2(\Omega)$ *there exists a quasicontinuous function* $\widetilde{u} : \overline{\Omega} \to \mathbb{R}$ *defined quasi-everywhere on* $\partial\Omega$ *such that* $\widetilde{u}|_\Omega = u$.

Theorems 3.3, 3.5 and 2.8 can be easily extended to a more flexible class of so-called quasi-extension domains:

DEFINITION 3.7. A domain $\Omega \subset \mathbb{R}^2$ is said to be an L^1_2-*quasi-extension domain* if for any $\varepsilon > 0$ there exists an open set U_ε of conformal capacity less than ε such that $\Omega \setminus \overline{U}_\varepsilon$ is an L^1_2-extension domain.

Typical examples of such domains are domains with boundary singularities of conformal capacity zero.

THEOREM 3.8. *If a bounded domain* Ω *is an* L^1_2*-quasi-extension domain then the identity mapping* $\operatorname{id} : \partial\Omega \to H_\rho$ *is a homeomorphism.*

Proof. Follows directly from Theorem 2.8 and the countable subadditivity of capacity. ∎

THEOREM 3.9. *Let $\Omega \subset \mathbb{R}^2$ be a bounded L^1_2-quasi-extension domain. Then Ω has the strong Luzin capacitary property for any capacitary metric $\rho_{(F,V)}$.*

Proof. Follows directly from Theorem 2.8 and the countable subadditivity of capacity. ∎

Combining Theorems 3.3, 3.9 and 2.8 we obtain

THEOREM 3.10. *Let $\Omega \subset \mathbb{R}^2$ be a bounded L^1_2-quasi-extension domain. Then for every $u \in L^1_2(\Omega)$ there exists a quasicontinuous function $\widetilde{u} : \overline{\Omega} \to \mathbb{R}$ defined quasi-everywhere on $\partial\Omega$ such that $\widetilde{u}|_\Omega = u$.*

Proof. By Theorem 3.9, Ω has the strong Luzin capacitary property for any metric $\rho_{(F,V)}$. Combining Theorems 3.3 and 2.8 and using the countable subadditivity of capacity we finish the proof. ∎

4. Boundary values of Sobolev functions for simply connected domains. Using the Riemann Mapping Theorem we can prove that any simply connected domain $\Omega \neq \mathbb{R}^2$ has the strong Luzin capacitary property, which permits us to extend previous results to any simply connected domain with nonempty boundary.

The unit disc $\mathbb{D} \subset \mathbb{R}^2$ is an L^1_2-extension domain and has the strong Luzin capacitary property. Recall that the conformal capacity of condensers is a quasi-invariant for quasiconformal homeomorphisms between planar domains. Hence the conformal capacity metric is also a quasi-invariant for quasiconformal homeomorphisms. Moreover, this remark immediately yields

PROPOSITION 4.1 ([12]). *Any quasiconformal homeomorphism $\varphi : \Omega \to \Omega'$ between planar domains induces a quasi-isometry of $\widetilde{\Omega}_\rho$ and $\widetilde{\Omega}'_\rho$.*

COROLLARY 4.2. *Let $\varphi : \mathbb{D} \to \Omega$ be a quasiconformal homeomorphism of the unit disc $\mathbb{D}$ onto a domain $\Omega \subset \mathbb{R}^2$. Then Ω has the strong Luzin capacitary property for any capacitary metric $\rho_{(F,V)}$.*

Proof. Choose $u \in L^1_2(\Omega)$ and $\varepsilon > 0$. We will prove that there exists a set W_ε of small conformal capacity such that u is uniformly continuous on $\Omega \setminus W_\varepsilon$ in any capacitary metric $\rho_{(F,V)}$. Because $\varphi : \mathbb{D} \to \Omega$ is a quasiconformal homeomorphism, the composition $v := u \circ \varphi$ belongs to $L^1_2(\mathbb{D})$ (see, for example, [10]). As $\mathbb{D}$ is an extension domain, it has the strong Luzin capacitary property for the metric $\rho_{(\varphi^{-1}(F),\varphi^{-1}(V))}$. Hence there exists an open set U_ε of conformal capacity less than ε such that $v|_{\mathbb{D}\setminus U_\varepsilon}$ is uniformly continuous in the metric $\rho_{(\varphi^{-1}(F),\varphi^{-1}(V))}$. Conformal capacity is a quasi-invariant for the quasiconformal homeomorphism φ. This means that there exists a constant Q which depends only on the quasiconformal distortion of φ and such that the conformal capacity of $W_\varepsilon := \varphi(U_\varepsilon)$ is less than $Q\varepsilon$. By the previous proposition, φ^{-1} induces a quasi-isometry of $\widetilde{\Omega}_\rho$ and $\widetilde{\mathbb{D}}_\rho$. Therefore $u = v \circ \varphi^{-1}$ is uniformly continuous on $\Omega \setminus W_\varepsilon$ in the metric $\rho_{(F,V)}$. We have proved that Ω has the strong Luzin (F, V)-capacitary property. ■

The previous proposition and Theorem 3.3 immediately yield

Theorem C. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, $\Omega \neq \mathbb{R}^2$. Then for every $u \in L^1_2(\Omega)$ there exists a quasicontinuous function $\widetilde{u} : \widetilde{\Omega}_\rho \to \mathbb{R}$ defined quasi-everywhere on the capacitary boundary H_ρ such that $\widetilde{u}|_\Omega = u$.*

Theorems C and 2.11 immediately imply

Theorem B. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, $\Omega \neq \mathbb{R}^2$. Then for every $u \in L^1_2(\Omega)$ there exists a quasicontinuous extension $\widetilde{u}$ of u on the Carathéodory boundary $\partial_C\Omega$. The function $\widetilde{u}$ is defined quasi-everywhere on $\partial_C\Omega$.*

For simply connected domains locally connected at any boundary point, Theorems B and 2.6 imply

Theorem A. *Let $\Omega \subset \mathbb{R}^2$, $\Omega \neq \mathbb{R}^2$, be a simply connected domain which is locally connected at any $x \in \partial\Omega$. Then for every $u \in L^1_2(\Omega)$ there exists a quasicontinuous function $\widetilde{u} : \overline{\Omega} \to \mathbb{R}$ such that $\widetilde{u}|_\Omega = u$.*

Remark 4.3. The quasicontinuous function $\widetilde{u} : \overline{\Omega} \to \mathbb{R}$ is defined on $\partial\Omega$ up to a set of conformal capacity zero (i.e. quasi-everywhere).

For the reader's convenience we repeat some basic facts about quasidiscs.

Definition 4.4. A domain Ω is called a K-*quasidisc* if it is the image of the unit disc $\mathbb{D}$ under a K-quasiconformal homeomorphism of the plane onto itself.

It is well known that the boundary of any K-quasidisc Ω admits a K^2-quasiconformal reflection, and thus, for example, any conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ can be extended to a K^2-quasiconformal homeomorphism of the whole plane to itself.

Boundaries of quasidiscs are called *quasicircles*. It is known that there are quasicircles for which no segment has finite length. The Hausdorff dimension of quasicircles was first investigated by Gehring and Väisälä (1973) [9], who proved that it can take all values in the interval $[1, 2)$. S. Smirnov proved recently [35] that the Hausdorff dimension of any K-quasicircle is at most $1 + k^2$, where $k = (K-1)/(K+1)$.

Ahlfors's 3-point condition [2] gives a complete geometric characterization: a Jordan curve γ in the plane is a quasicircle if and only if for any two points a, b on γ the (smaller) arc between them has diameter comparable to

$|a-b|$. This condition is easily checked for the snowflake. On the other hand, every quasicircle can be obtained by an explicit snowflake-type construction (see [33]).

Because any quasidisc is an L^1_2-extension domain, we can reformulate the previous results in terms of quasidiscs.

PROPOSITION 4.5. *Let $\Omega \subset \mathbb{R}^2$ be a quasidisc. Then the identity mapping* $\mathrm{id} : H_\rho \to \partial\Omega$ *is a homeomorphism.*

PROPOSITION 4.6. *Let $\Omega \subset \mathbb{R}^2$ be a quasidisc. Then Ω has the strong Luzin capacitary property.*

5. Historical sketch and conclusions. The concept of ideal boundaries is common for geometry and analysis. The Poincaré disc is a model of the hyperbolic plane that provides a geometrical realization of the ideal boundary of the hyperbolic plane with the help of a conformal homeomorphism.

By the Riemann Mapping Theorem any simply connected planar domain $\Omega \neq \mathbb{R}^2$ is conformally equivalent to the unit disc. However, the boundary behavior of plane conformal homeomorphisms cannot be described in terms of Euclidean boundaries but it can be described in terms of ideal boundary elements (prime ends), introduced by C. Carathéodory. By the Carathéodory Theorem any conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ induces a one-to-one correspondence of prime ends.

M. A. Lavrentiev [19] introduced a metric (a relative distance) for prime ends. G. D. Suvorov [37] constructed a counterexample that demonstrates the failure of the triangle inequality for the Lavrentiev relative distance and proposed a more accurate concept of relative distance that satisfies the triangle inequality. In terms of this metric the Carathéodory prime ends are a geometric representation of "ideal" compactification "boundary points". The detailed survey of different conformally invariant intrinsic metrics can be found in the paper of V. M. Miklyukov [28].

For dimension more than two, by the Liouville theorem the class of conformal homeomorphisms coincides with the Möbius transformations. Even for quasiconformal homeomorphisms nothing similar to the Riemann Mapping Theorem holds.

In our opinion two main constructions of a quasiconformally invariant "ideal" boundary were proposed. The first one was in the spirit of Banach algebras. Recall that the Royden algebra $\mathbb{R}(\Omega)$ is a quasiconformal invariant, as proved by M. Nakai [29] for dimension two and by L. G. Lewis [20] for arbitrary dimension. As any Banach algebra, the Royden algebra produces a compactification of Ω and any quasiconformal homeomorphism induces a homeomorphism of such compactifications.

The second construction is the capacitary boundary proposed by V. Gol'dshtein and S. K. Vodop'yanov [12]. Its construction is based on the notion of conformal capacity. Recall that conformal capacity is a quasi-invariant of quasiconformal homeomorphisms. By [12], a quasiconformal homeomorphism can be extended to a homeomorphism of domains with capacitary boundaries.

The Royden compactification does not coincide with the Carathéodory compactification. But for domains finitely connected at the boundary the Carathéodory boundary and the set of components of boundary fibers in the Royden boundary coincide (see e.g. [36, Theorem 7.4]).

The "ideal" elements of the capacitary boundary are Carathéodory prime ends.

A necessary and sufficient condition for existence of continuous traces of $L^1_p(\Omega)$, $p > 2$, was obtained by Shvartsman [34] in terms of quasi-hyperbolic metrics.

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