

On Tate–Shafarevich groups of 1-motives

by

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1. Introduction. Let F be a global field with fixed separable algebraic closure F^s and corresponding absolute Galois group Γ_F . A (*Deligne*) 1-motive over F is a complex in degrees -1 and 0 of commutative F -group schemes $M = [Y \rightarrow G]$, where Y is étale-locally isomorphic to \mathbb{Z}^r for some integer $r \geq 0$, and G is a semiabelian variety over F given as an extension $0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$, where A is an abelian variety over F , and T is an F -torus. The Tate–Shafarevich group in degree i of M , where $i = 1$ or 2 , is by definition

$$(1.1) \quad \text{III}^i(F, M) = \text{Ker} \left[H^i(F, M) \rightarrow \prod_{\text{all } v} H^i(F_v, M) \right],$$

where the indicated map is a natural localization map.

The cohomology groups appearing above are flat (fppf) cohomology groups. However, since Y and G are smooth and commutative F -group schemes, a standard five-lemma argument applied to a certain diagram enables us to identify, when convenient, the fppf cohomology groups $H^i(F, M)$ and the Galois cohomology groups $H^i(\Gamma_F, M(F^s))$. Consequently, the above groups, and all other groups appearing in this paper which are closely related to them (e.g., those in Theorem 1.1 below), can be computed using either flat or Galois cohomology.

It is well-known that the groups (1.1) are objects of great arithmetical interest which are notoriously difficult to study. For example, if $M = A$ is an abelian variety over F , then $\text{III}^1(F, A)$ encodes important arithmetical information about A , but even the finiteness of this group is a major outstanding conjecture in number theory. Regarding the latter group, the author [GA00] and Yu [Yu04] studied the restriction map $\text{res}^1: \text{III}^1(F, A) \rightarrow \text{III}^1(K, A)^F$ associated to a finite Galois extension of number fields K/F with Galois group

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Γ and used that study to relate the orders (assumed finite) of $\mathrm{III}^1(F, A)$ and $\mathrm{III}^1(F, A^\chi)$, where A^χ is a quadratic twist of A . In this paper we extend the methods of [Yu04] to examine, independently of any finiteness hypotheses, the restriction map $\mathrm{res}^i: \mathrm{III}^i(F, M) \rightarrow \mathrm{III}^i(K, M)^\Gamma$ for $i = 1$ and 2 and arbitrary 1-motives M over arbitrary global fields F . We obtain the following finiteness result.

THEOREM 1.1. *Let M be a 1-motive over a global field F and let K/F be a finite Galois extension of global fields with Galois group Γ . Then, for $i = 1$ and 2 , the kernel and cokernel of the restriction map*

$$\mathrm{res}^i: \mathrm{III}^i(F, M) \rightarrow \mathrm{III}^i(K, M)^\Gamma$$

are finite groups annihilated by a power of $[K : F]$.

We should note that when $M = G$ is a semiabelian variety (and thus an algebraic group), the proof of that part of the theorem which refers to the kernel of the indicated map is not difficult and follows from a standard restriction-corestriction argument. However, the fact that M is, in general, only a complex of algebraic groups makes the proof of the theorem rather involved for both the kernel and cokernel of res^i .

The proof of the theorem will show that, in fact, $\mathrm{Ker}(\mathrm{res}^i)$ is annihilated by $[K : F]$ (see Remark 5.3). Regarding the case $i = 1$ of the theorem, we note that both the source and the target of res^1 are finite groups if (as widely expected) $\mathrm{III}^1(K, A)$ is a finite group, where A is the abelian variety part of M . However, when $i = 2$, Jossen [Jos13, §7] has constructed an example of a semiabelian variety G over \mathbb{Q} such that the group $\mathrm{III}^2(\mathbb{Q}, G)$ is infinite.

We briefly indicate the contents of each section. Section 2 consists of preliminaries. Section 3 discusses 1-motives over fields (e.g., global and local fields). Section 4 establishes that the primary components of Tate–Shafarevich groups of 1-motives of global fields in cohomological degrees 1 and 2 are groups of finite cotype. This result is new in cohomological degree 2 over global function fields and is well-known in the remaining cases, although no formal statement to this effect seems to have appeared in print before. Section 5 contains the proof of Theorem 1.1. The Appendix corrects an error which appears in a reference relevant to this paper.

2. Preliminaries. For any abelian group B and positive integer n , we will write B_n for the n -torsion subgroup of B , and B/n for B/nB . If B is finite, its order will be denoted by $|B|$. Let ℓ be a prime number. We will write $B(\ell)$ for the ℓ -primary torsion subgroup of B , and $B^{(\ell)}$ for $\varprojlim_m (B/\ell^m)$.

If B is a topological abelian group, we will write $B^D = \mathrm{Hom}_{\mathrm{cont}}(B, \mathbb{Q}/\mathbb{Z})$ and endow it with the compact-open topology, where \mathbb{Q}/\mathbb{Z} is given the discrete topology. Via Pontryagin duality, the category of profinite abelian

groups is anti-equivalent to the category of discrete torsion abelian groups. Both are abelian categories. Further, if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of discrete torsion abelian groups, then the dual sequence $0 \rightarrow C^D \rightarrow B^D \rightarrow A^D \rightarrow 0$ is an exact sequence of profinite abelian groups.

LEMMA 2.1. *Let ℓ be a prime number and let B be a discrete ℓ -primary torsion abelian group. Then the following conditions are equivalent:*

- (i) *The profinite abelian group B^D is a finitely generated \mathbb{Z}_ℓ -module.*
- (ii) *B is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \oplus A$, where $r \geq 0$ is an integer and A is a finite abelian ℓ -group.*
- (iii) *B_ℓ is a finite group.*

Proof. Assertion (ii) follows from (i) and duality by noting that $\mathbb{Z}_\ell^D = \mathbb{Q}_\ell/\mathbb{Z}_\ell$. The implication (ii) \Rightarrow (iii) is trivial. Finally, if (iii) holds, then $B^D/\ell = (B_\ell)^D$ is finite and (i) holds. ■

If ℓ is a prime number, a discrete ℓ -primary torsion abelian group B which satisfies the equivalent conditions of the previous lemma is said to be of *finite cotype*. If B as above is of finite cotype, then every subgroup and every quotient of B is a group of finite cotype.

LEMMA 2.2. *Let B be a discrete torsion abelian group such that $B(\ell)$ is a group of finite cotype for every prime number ℓ . Then every subgroup and every quotient of B of finite exponent is, in fact, finite.*

Proof. If A is a subgroup of B and n is an integer such that $nA = 0$, then $A \subset B_n$ is finite since B_n is finite. If C is a quotient of B such that $nC = 0$, then $C^D \subset (B^D)_n = \prod_{\ell|n} (B(\ell)^D)_n$, which is finite since each group $B(\ell)^D$ is a finitely generated \mathbb{Z}_ℓ -module. ■

PROPOSITION 2.3. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a pair of morphisms in an abelian category \mathcal{A} . Then there exists an induced exact sequence in \mathcal{A} ,*

$$0 \rightarrow \text{Ker } f \rightarrow \text{Ker}(g \circ f) \rightarrow \text{Ker } g \rightarrow \text{Coker } f \rightarrow \text{Coker}(g \circ f) \rightarrow \text{Coker } g \rightarrow 0.$$

Proof. See, for example, [BP69, Hilfssatz 5.5.2, p. 45]. ■

PROPOSITION 2.4. *If $i > 0$, Γ is a finite group and A is a Γ -module, then $H^i(\Gamma, A)$ is annihilated by $|\Gamma|$. If, in addition, A is finitely generated, then $H^i(\Gamma, A)$ is finite.*

Proof. This is well-known. See, for example, [Ser95, VIII, §2, Corollaries 1 and 2, p. 130]. ■

LEMMA 2.5. *Let Γ be a finite group. If $A \rightarrow B \rightarrow C$ is an exact sequence of Γ -modules, then the homology of the induced complex of abelian groups $A^\Gamma \rightarrow B^\Gamma \rightarrow C^\Gamma$ is annihilated by $|\Gamma|$.*

Proof. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be the given maps. The exact sequence of Γ -modules $0 \rightarrow \text{Ker } f \rightarrow A \rightarrow \text{Im } f \rightarrow 0$ yields an exact sequence of abelian groups

$$(2.1) \quad 0 \rightarrow (\text{Ker } f)^\Gamma \rightarrow A^\Gamma \xrightarrow{h} (\text{Im } f)^\Gamma \rightarrow H^1(\Gamma, \text{Ker } f)$$

and $f^\Gamma: A^\Gamma \rightarrow B^\Gamma$ factors as $A^\Gamma \xrightarrow{h} (\text{Im } f)^\Gamma \hookrightarrow B^\Gamma$. Thus there exist isomorphisms of abelian groups $\text{Ker } f^\Gamma = (\text{Ker } f)^\Gamma$ and $\text{Im } f^\Gamma = \text{Im } h$. Similarly, $\text{Ker } g^\Gamma = (\text{Ker } g)^\Gamma = (\text{Im } f)^\Gamma$. Thus the homology of the complex $A^\Gamma \rightarrow B^\Gamma \rightarrow C^\Gamma$ equals

$$(\text{Ker } g^\Gamma)/\text{Im } f^\Gamma = (\text{Im } f)^\Gamma/\text{Im } h = \text{Coker } h.$$

Since $\text{Coker } h$ injects into $H^1(\Gamma, \text{Ker } f)$ by (2.1) and the latter group is annihilated by $|\Gamma|$ by Proposition 2.4, the lemma follows. ■

We will write \mathbf{Ab} for the category of abelian groups.

LEMMA 2.6. *Let n be a positive integer and let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence in \mathbf{Ab} . Then there exists an induced exact sequence in \mathbf{Ab} ,*

$$0 \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow A/n \rightarrow B/n \rightarrow C/n \rightarrow 0.$$

Proof. Apply the functor $\text{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/n, -)$ to the given short exact sequence using [Wei94, Calculation 3.1.1, p. 66]. ■

Let Γ be a finite group and let \mathcal{C} be the quotient category of \mathbf{Ab} modulo its thick subcategory of groups of finite exponent annihilated by a power of $|\Gamma|$. If $f: A \rightarrow B$ is a homomorphism of abelian groups, then f is a monomorphism (respectively, epimorphism, isomorphism) in \mathcal{C} if, and only if, $\text{Ker } f$ (respectively, $\text{Coker } f$, both $\text{Ker } f$ and $\text{Coker } f$) are groups of finite exponent annihilated by a power of $|\Gamma|$ [Gab62, III, §1, Lemme 2, p. 366].

PROPOSITION 2.7. *Let \mathcal{G} be a profinite group and let \mathcal{N} be an open and normal subgroup of \mathcal{G} such that the quotient group $\Gamma = \mathcal{G}/\mathcal{N}$ is finite. Then, for every continuous \mathcal{G} -module C and every integer $i \geq 1$, the restriction map $\text{res}^i: H^i(\mathcal{G}, C) \rightarrow H^i(\mathcal{N}, C)^\Gamma$ is an isomorphism in \mathcal{C} . In other words, the kernel and cokernel of res^i are groups of finite exponent annihilated by a power of $|\Gamma|$.*

Proof. Denote by $\mathbf{Ab}_{\mathcal{G}}$ the (abelian) category of abelian groups equipped with a continuous \mathcal{G} -action and equivariant maps, and define \mathbf{Ab}_{Γ} similarly. Write $\mathcal{C}_{\mathcal{G}}$ for the localization of the abelian category $\mathbf{Ab}_{\mathcal{G}}$ with respect to the thick subcategory of modules of finite exponent annihilated by a power of $|\Gamma|$, and let \mathcal{C}_{Γ} be the corresponding object associated to \mathbf{Ab}_{Γ} . Consider

the following commutative diagram of categories and functors:

$$\begin{array}{ccccc}
 \mathbf{Ab}_{\mathcal{G}} & \xrightarrow{(-)^{\mathcal{N}}} & \mathbf{Ab}_{\Gamma} & \xrightarrow{(-)^{\Gamma}} & \mathbf{Ab} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}_{\mathcal{G}} & \xrightarrow{(-)^{\mathcal{N}}} & \mathcal{C}_{\Gamma} & \xrightarrow{(-)^{\Gamma}} & \mathcal{C}
 \end{array}$$

The vertical arrows are the canonical localization functors, which are exact. The horizontal compositions are the fixed point functors $(-)^{\mathcal{G}}$. The classical Hochschild–Serre spectral sequence is the Grothendieck spectral sequence of the composite functor $\mathbf{Ab}_{\mathcal{G}} \rightarrow \mathbf{Ab}_{\Gamma} \rightarrow \mathbf{Ab}$. Since the vertical functors are exact, this spectral sequence maps to the Grothendieck spectral sequence of the composite $\mathcal{C}_{\mathcal{G}} \rightarrow \mathcal{C}_{\Gamma} \rightarrow \mathcal{C}$. Now, since the functor $(-)^{\Gamma}: \mathcal{C}_{\Gamma} \rightarrow \mathcal{C}$ is exact by Lemma 2.5, the localized spectral sequence degenerates at the second stage and we obtain isomorphisms in \mathcal{C} ,

$$H^i(\mathcal{G}, C) \simeq H^i(\mathcal{N}, C)^{\Gamma}.$$

The preceding morphism is represented in \mathbf{Ab} by the edge morphism $E^i \rightarrow E_2^{0,i}$, which coincides with the restriction map res^i . The proposition is now clear. ■

3. 1-motives over a field. Let F be a field and let F^{s} be a fixed separable algebraic closure of F . For any subextension L/F of F^{s}/F , we will write Γ_L for $\text{Gal}(F^{\text{s}}/L)$. A (*Deligne*) 1-motive over F is a complex in degrees -1 and 0 of commutative F -group schemes $M = [Y \xrightarrow{u} G]$, where Y is étale-locally isomorphic to \mathbb{Z}^r for some integer $r \geq 0$, G is a semiabelian variety over F given as an extension of an abelian variety A by a torus T ,

$$(3.1) \quad 0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0,$$

and u is a morphism of F -group schemes. We will identify the 1-motive $M = [0 \rightarrow G]$ with G placed in degree 0. The exact sequence of complexes

$$(3.2) \quad 0 \rightarrow [0 \rightarrow G] \xrightarrow{\iota} [Y \rightarrow G] \rightarrow [Y \rightarrow 0] \rightarrow 0$$

induces the following distinguished triangle in the derived category of the category of bounded complexes of abelian sheaves on the small fppf site over $\text{Spec } F$:

$$(3.3) \quad Y \rightarrow G \xrightarrow{\iota} M \rightarrow Y[1].$$

We will write M^* for the 1-motive dual to M . Thus

$$M^* = [Y^* \xrightarrow{u^*} G^*],$$

where Y^* is the group of characters of T , and G^* is an extension of the abelian variety A^* dual to A by an F -torus T^* whose group of characters is Y . The dual 1-motive comes equipped with a duality pairing, i.e., a morphism in

the derived category of fppf sheaves $b: M \otimes^{\mathbf{L}} M^* \rightarrow \mathbb{G}_{m,F}[1]$ which induces isomorphisms $M \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}(M^*, \mathbb{G}_{m,F}[1])$ and $M^* \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}(M, \mathbb{G}_{m,F}[1])$. The pairing b induces pairings of fppf cohomology groups

$$(3.4) \quad H^i(F, M) \times H^j(F, M^*) \xrightarrow{\cup} H^{i+j}(F, M \otimes^{\mathbf{L}} M^*) \xrightarrow{H^{i+j}(b)} H^{i+j+1}(F, \mathbb{G}_m).$$

The above constructions can be extended to any base scheme (in place of a field) and, in particular, the notions of a 1-motive over a Dedekind scheme and its dual are defined. Such objects will briefly appear in Remark 4.1(b) and in the proof of Lemma 4.3. See [HSz05, §1] and [GA09, §3] for more details.

Note that, by (3.3),

$$(3.5) \quad H^{-1}(K, M) = \mathrm{Ker}[H^0(K, Y) \rightarrow H^0(K, G)],$$

which is a finitely generated abelian group.

Now let K/F be a finite Galois subextension of F^s/F with Galois group Γ . For every integer $i \geq -1$, set

$$(3.6) \quad H^i(K/F, M) = \mathrm{Ker}[H^i(F, M) \xrightarrow{\mathrm{res}^i} H^i(K, M)^\Gamma],$$

$$(3.7) \quad C^i(K/F, M) = \mathrm{Coker}[H^i(F, M) \xrightarrow{\mathrm{res}^i} H^i(K, M)^\Gamma].$$

LEMMA 3.1. *For every integer $i \geq 1$, $C^i(K/F, M)$ is a group of finite exponent annihilated by a power of $[K : F]$.*

Proof. We work in the category \mathcal{C} of abelian groups modulo groups of finite exponent annihilated by a power of $|\Gamma|$. By Lemma 2.5, the triangle (3.3) induces a 5-column exact and commutative diagram in \mathcal{C} ,

$$\begin{array}{ccccccc} H^{i-1}(F, Y[1]) & \longrightarrow & H^i(F, G) & \longrightarrow & H^i(F, M) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ H^{i-1}(K, Y[1])^\Gamma & \longrightarrow & H^i(K, G)^\Gamma & \longrightarrow & H^i(K, M)^\Gamma & \longrightarrow & \dots \\ & & & & \longrightarrow & H^i(F, Y[1]) & \longrightarrow & H^{i+1}(F, G) \\ & & & & & \downarrow & & \downarrow \\ & & & & \longrightarrow & H^i(K, Y[1])^\Gamma & \longrightarrow & H^{i+1}(K, G)^\Gamma \end{array}$$

By Proposition 2.7, the first, second, fourth and fifth vertical maps above are isomorphisms in \mathcal{C} . Now the five-lemma completes the proof. ■

In the case of (3.6), the following more precise result holds.

LEMMA 3.2. *For every integer $i \geq -1$, $H^i(K/F, M)$ is a group of finite exponent annihilated by $[K : F]$.*

Proof. We use a restriction-corestriction argument. The Weil restriction $R_{K/F}(Y_K)$ is étale-locally isomorphic to $\mathbb{Z}^{r[K:F]}$. Further, $R_{K/F}(G_K)$ is a semiabelian variety over F by [CGP15, Corollary A.5.4(3), p. 508]. The Weil restriction of M_K relative to K/F is the 1-motive over F given by

$$R_{K/F}(M_K) = [R_{K/F}(Y_K) \xrightarrow{v} R_{K/F}(G_K)],$$

where $v = R_{K/F}(u_K)$. Now, for every F -group scheme H , there exists a canonical norm (or corestriction) morphism $R_{K/F}(H_K) \rightarrow H$ defined as follows: if $N_{K/F}: K \rightarrow F$ is the norm map associated to the Galois extension K/F and X is any F -scheme, then $R_{K/F}(H_K)(X) \rightarrow H(X)$ is the homomorphism $H(X_K) \rightarrow H(X)$ induced by $X \times_{\text{Spec } F} \text{Spec}(N_{K/F}): X \rightarrow X_K$. We let $N: R_{K/F}(M_K) \rightarrow M$ be the morphism of 1-motives whose components are the norm morphisms $R_{K/F}(Y_K) \rightarrow Y$ and $R_{K/F}(G_K) \rightarrow G$ described above.

We observe that there exists a canonical morphism of F -group schemes $Y \rightarrow R_{K/F}(Y_K)$ which corresponds to the identity morphism of Y_K under the isomorphism $\text{Hom}_F(Y, R_{K/F}(Y_K)) \simeq \text{Hom}_K(Y_K, Y_K)$ that characterizes the functor $R_{K/F}$. Similarly, there exists a canonical morphism of F -group schemes $G \rightarrow R_{K/F}(G_K)$. These morphisms are the components of a canonical morphism of 1-motives $j: M \rightarrow R_{K/F}(M_K)$. The composite morphism

$$M \xrightarrow{j} R_{K/F}(M_K) \xrightarrow{N} M$$

is the multiplication-by- $[K:F]$ map on M .

Now, by definition of $R_{K/F}(M_K)$, Shapiro’s lemma in Galois cohomology and a standard five-lemma argument (see the previous proof), for every integer $i \geq -1$ there exists a canonical isomorphism of Galois hypercohomology groups $e^i: H^i(F, R_{K/F}(M_K)) \xrightarrow{\sim} H^i(K, M)$. We define the norm map $N^i: H^i(K, M) \rightarrow H^i(F, M)$ as the composition

$$H^i(K, M) \xrightarrow{(e^i)^{-1}} H^i(F, R_{K/F}(M_K)) \xrightarrow{H^i(N)} H^i(F, M).$$

The restriction map $\text{res}^i: H^i(F, M) \rightarrow H^i(K, M)$ is the composition

$$H^i(F, M) \xrightarrow{H^i(j)} H^i(F, R_{K/F}(M_K)) \xrightarrow{e^i} H^i(K, M),$$

and the composite map

$$H^i(F, M) \xrightarrow{\text{res}^i} H^i(K, M) \xrightarrow{N^i} H^i(F, M)$$

is the multiplication-by- $[K:F]$ map on $H^i(F, M)$. The lemma follows at once. ■

We now assume that F is a global field, i.e., either a finite extension of \mathbb{Q} (the number field case), or finitely generated and of transcendence degree 1 over a finite field of constants (the function field case). We will write p for

the characteristic exponent of F . Thus $p = 1$ if F is a number field, and $p = \text{char } F$ if F is a function field. If v is a prime of F , then F_v will denote the completion of F at v . For each prime v of F , we choose and fix a prime \bar{v} of F^{S} lying above v and write w for the prime of K lying below \bar{v} . The decomposition group of w in Γ will be denoted by Γ_w and identified with $\text{Gal}(K_w/F_v)$.

If v is an archimedean prime of F and i is any integer, $H^i(F_v, M)$ will denote the i th Tate cohomology group of M_{F_v} defined in [HSz05, p. 103]. For every $i \in \mathbb{Z}$, $H^i(F_v, M)$ is a finite 2-torsion group.

LEMMA 3.3. *For every integer $i \geq -1$, $\prod_{\text{all } v} H^i(K_w/F_v, M)$ is a group of finite exponent annihilated by $[K : F]$.*

Proof. The lemma is clear since each group $H^i(K_w/F_v, M)$ as in (3.6) is annihilated by $[K_w : F_v]$ by Lemma 3.2 and $[K_w : F_v]$ divides $[K : F]$ for every prime v of F . ■

4. Tate–Shafarevich groups. For every finite set S of primes of F and integer i , set ⁽¹⁾

$$\text{III}_S^i(F, M) = \text{Ker} \left[H^i(F, M) \rightarrow \prod_{v \notin S} H^i(F_v, M) \right].$$

The Tate–Shafarevich group in degree i of M is the group $\text{III}_0^i(F, M)$. Now let $\text{III}_\omega^i(F, M)$ denote the subgroup of $H^i(F, M)$ of all classes which are locally trivial at all but finitely many primes of F . Thus

$$(4.1) \quad \text{III}_\omega^i(F, M) = \bigcup_S \text{III}_S^i(F, M) \subset H^i(F, M),$$

where the union extends over all finite sets S of primes of F . For $i = 1$ and 2, we will write

$$(4.2) \quad \text{C}_\omega^i(K/F, M) = \text{Coker} [\text{III}_\omega^i(F, M) \xrightarrow{\text{res}_\omega^i} \text{III}_\omega^i(K, M)^\Gamma].$$

REMARK 4.1. (a) If M is an arbitrary 1-motive, then $\text{III}_\omega^2(F, M) = H^2(F, M)$ by [HSz05, p. 117, first paragraph]. Consequently, $\text{C}_\omega^2(K/F, M) = \text{C}^2(K/F, M)$ (see (3.7)).

(b) If $\text{III}^1(F, A)$ is finite, then $\text{III}^1(F, M)$ is finite. Indeed, if $\ell \neq p$, then the finiteness of $\text{III}^1(F, M)(\ell)$ follows from that of $\text{III}^1(F, A)(\ell)$ by [HSz05, Proposition 3.7 and proof of Theorem 4.8]. See also [HSz05, Remark 5.10] and note that the hypothesis of [HSz05, Proposition 3.7] is satisfied, by [Mil06, Remark I.6.14(c), p. 84]. Now, if F is a function field with associated curve C , then [GA09, Lemma 6.5] shows that $\text{III}^1(F, M(p)) =$

⁽¹⁾ We warn the reader against confusing $\text{III}_S^i(F, M)$ with the similarly denoted Tate–Shafarevich group associated to the Galois group of the maximal extension of F which is unramified outside S .

$D^1(U, \mathcal{M})(p)$ for an appropriate affine, open and dense subscheme U of C , where \mathcal{M} is a 1-motive over U whose generic fiber is M and $D^1(U, \mathcal{M}) = \text{Ker}[H^1(U, \mathcal{M}) \rightarrow \bigoplus_{v \notin U} H^1(F_v, M)]$. The finiteness of $D^1(U, \mathcal{M})(p)$ follows from that of $\text{III}^1(F, A)(p)$, as in [HSz05, proof of Proposition 3.7, p. 111]. See [GA09, proof of Lemma 5.4, p. 222].

LEMMA 4.2. *If $i = 1$ or 2 , then $C_\omega^i(K/F, M)$ is a group of finite exponent annihilated by a power of $[K : F]$.*

Proof. By Remark 4.1(a) we have $C_\omega^2(K/F, M) = C^2(K/F, M)$, and the latter group is annihilated by a power of $[K : F]$ by Lemma 3.1. Now assume that $i = 1$, let S be any finite set of primes of F and let S' be the set of primes of K lying above the primes in S . By the semilocal theory of [Cha67, §2.1], there exist canonical isomorphisms

$$\left(\prod_{w' \notin S'} H^1(K_{w'}, M) \right)^F = \left(\prod_{v \notin S} \prod_{w' | v} H^1(K_{w'}, M) \right)^F \simeq \prod_{v \notin S} H^1(K_w, M)^{F_w}$$

where, for every prime $v \notin S$, w is the prime of K lying above v fixed previously. Identifying the preceding groups via the indicated isomorphism, we obtain a canonical exact and commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{III}_S^1(F, M) & \longrightarrow & H^1(F, M) & \longrightarrow & \prod_{v \notin S} H^1(F_v, M) \\ & & \downarrow \text{res}_S^1 & & \downarrow \text{res}^1 & & \downarrow \prod_{v \notin S} \text{res}_v^1 \\ 0 & \longrightarrow & \text{III}_{S'}^1(K, M)^F & \longrightarrow & H^1(K, M)^F & \longrightarrow & \prod_{v \notin S} H^1(K_w, M)^{F_w} \end{array}$$

By Lemma 3.1, the cokernel of the middle vertical map is annihilated by a power of $[K : F]$. On the other hand, by Lemma 3.3, the kernel of the right-hand vertical map above is annihilated by $[K : F]$. We conclude that the cokernel of res_S^1 is annihilated by a power of $[K : F]$ which is independent of the choice of S . The lemma follows. ■

LEMMA 4.3. *Let ℓ be a prime number. Then $\text{III}^i(F, M)(\ell)$ is a group of finite cotype for $i = 1$ and $i = 2$.*

Proof. If $\ell \neq p$, there exists a nonempty open affine subscheme U of either the spectrum of the ring of integers of F (in the number field case) or of the curve C associated to F (in the function field case) such that ℓ is invertible on U , M extends to a 1-motive \mathcal{M} over U and $\text{III}^1(F, M)(\ell) = D^1(U, \mathcal{M})(\ell)$, where $D^i(U, \mathcal{M}) = \text{Ker}[H^i(U, \mathcal{M}) \rightarrow \bigoplus_{v \notin U} H^i(F_v, M)]$. See [HSz05, Remark 5.10 and proof of Theorem 4.8]. The case $i = 1$ and $\ell \neq p$ of the lemma now follows from [HSz05, Lemma 3.2(2)]. For the case $i = 1$ and $\ell = p$, see [GA09, Lemma 6.5].

We now address the case $i = 2$. It is shown in [HSz05, proof of Proposition 4.12, p. 116] that for every sufficiently small set U as above, there exists an exact sequence of torsion [HSz05, Lemma 3.2(1)] abelian groups

$$0 \rightarrow B(U, \mathcal{M}) \rightarrow D^2(U, \mathcal{M}) \rightarrow \text{III}^2(F, M) \rightarrow 0,$$

where $B(U, \mathcal{M})$ is defined as the kernel of the map $D^2(U, \mathcal{M}) \rightarrow \text{III}^2(F, M)$ induced by the canonical map $H^2(U, \mathcal{M}) \rightarrow H^2(F, M)$. Now let $m \geq 1$ be an integer, and apply Lemma 2.6 to the preceding short exact sequence setting $n = \ell^m$ in that lemma. We obtain an exact sequence of abelian groups $D^2(U, \mathcal{M})_{\ell^m} \rightarrow \text{III}^2(F, M)_{\ell^m} \rightarrow B(U, \mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^m$. Taking direct limits over m above and noting that $\varinjlim_m B(U, \mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^m = B(U, \mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell} = 0$ since $B(U, \mathcal{M})$ is torsion and $\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}$ is divisible, we conclude that $\text{III}^2(F, M)(\ell)$ is a quotient of $D^2(U, \mathcal{M})(\ell)$ for any prime ℓ . Now, if $\ell \neq p$ is invertible on U , then $D^2(U, \mathcal{M})(\ell)$ is a group of finite cotyple by [HSz05, Lemma 3.2(2)]. Thus it only remains to check that $D^2(U, \mathcal{M})(p)$ is a group of finite cotyple for any set U as above (in the function field case).

By [GA09, proof of Theorem 5.10, p. 227], there exists a perfect pairing of topological abelian groups

$$D^1(U, T_p(\mathcal{M}^*)) \times D^2(U, \mathcal{M})(p) \rightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

where $D^1(U, T_p(\mathcal{M}^*)) = \varprojlim_m D^1(U, T_{\mathbb{Z}/p^m}(\mathcal{M}^*))$. Thus we are reduced to checking that $D^1(U, T_p(\mathcal{M}^*))$ is a finitely generated \mathbb{Z}_p -module. To this end, we recall from [GA09, proof of Lemma 5.8(b), p. 226, line 3] the exact sequence of abelian groups

$$0 \rightarrow D^{0,(p)}(U, \mathcal{M}^*) \rightarrow D^1(U, T_p(\mathcal{M}^*)) \rightarrow T_p D^1(U, \mathcal{M}^*),$$

where $D^{0,(p)}(U, \mathcal{M}^*) = \text{Ker}[H^0(U, \mathcal{M}^*)^{(p)} \rightarrow \bigoplus_{v \notin U} H^0(F_v, M^*)^{(p)}]$. The right-hand group above is a finitely generated \mathbb{Z}_p -module by Lemma 2.1 since $D^1(U, \mathcal{M}^*)_p$ is finite by [GA09, proof of Lemma 5.4, p. 222]. To show that $D^{0,(p)}(U, \mathcal{M}^*)$ is also a finitely generated \mathbb{Z}_p -module, it suffices to check that $H^0(U, \mathcal{M}^*)$ is a finitely generated abelian group. The corresponding statement for number fields is established in [HSz05, Lemma 3.2(3)] via a devissage argument that also works for function fields. This completes the proof of Lemma 4.3. ■

5. Proof of Theorem 1.1. The main ingredients of the proof are Lemmas 3.3, 4.2 and 4.3.

For $i = 1$ or 2, let

$$(5.1) \quad \lambda^i(F, M) : \text{III}_{\omega}^i(F, M) \rightarrow \bigoplus_{\text{all } v} H^i(F_v, M)$$

be the canonical localization map whose v -component is induced by the restriction map $H^i(F, M) \rightarrow H^i(F_v, M)$. There exists a canonical exact se-

quence of discrete and torsion abelian groups

$$(5.2) \quad 0 \rightarrow \text{III}^i(F, M) \rightarrow \text{III}_\omega^i(F, M) \xrightarrow{\lambda^i} \bigoplus_{\text{all } v} H^i(F_v, M) \rightarrow \mathfrak{C}^i(F, M) \rightarrow 0,$$

where $\lambda^i = \lambda^i(F, M)$ and $\mathfrak{C}^i(F, M) = \text{Coker } \lambda^i(F, M)$. Now set

$$(5.3) \quad \text{III}_\omega^i(K/F, M) = H^i(K/F, M) \cap \text{III}_\omega^i(F, M),$$

where $H^i(K/F, M)$ is the group (3.6) and the intersection takes place inside $H^i(F, M)$.

We define a map

$$(5.4) \quad \lambda_{K/F}^i = \lambda^i(K/F, M): \text{III}_\omega^i(K/F, M) \rightarrow \bigoplus_{\text{all } v} H^i(K_w/F_v, M)$$

and groups

$$(5.5) \quad \text{III}^i(K/F, M) = \text{Ker } \lambda_{K/F}^i,$$

$$(5.6) \quad \mathfrak{C}^i(K/F, M) = \text{Coker } \lambda_{K/F}^i$$

so that the following diagram, whose bottom row is the exact sequence (5.2), is exact and commutative:

$$(5.7) \quad \begin{array}{ccccccc} \text{III}^i(K/F, M) & \hookrightarrow & \text{III}_\omega^i(K/F, M) & \xrightarrow{\lambda_{K/F}^i} & \bigoplus_{\text{all } v} H^i(K_w/F_v, M) & \twoheadrightarrow & \mathfrak{C}^i(K/F, M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{III}^i(F, M) & \hookrightarrow & \text{III}_\omega^i(F, M) & \xrightarrow{\lambda^i} & \bigoplus_{\text{all } v} H^i(F_v, M) & \twoheadrightarrow & \mathfrak{C}^i(F, M) \end{array}$$

LEMMA 5.1. *The groups $\text{III}^i(K/F, M)$ and $\mathfrak{C}^i(K/F, M)$ are annihilated by $[K : F]$.*

Proof. By Lemmas 3.2 and 3.3, respectively, the source (5.3) and the target of the map $\lambda^i(K/F, M)$ in (5.4) are annihilated by $[K : F]$, which immediately yields the assertion. ■

We now observe that the kernel of $\text{res}_\omega^i: \text{III}_\omega^i(F, M) \rightarrow \text{III}_\omega^i(K, M)^F$ equals $H^i(K/F, M) \cap \text{III}_\omega^i(F, M) = \text{III}_\omega^i(K/F, M)$ (see (5.3)). Restricting the preceding map to $\text{III}^i(F, M) \subset \text{III}_\omega^i(F, M)$, we obtain a map

$$(5.8) \quad \widetilde{\text{res}}^i: \text{III}^i(F, M) \rightarrow \text{III}^i(K, M)^F \cap \text{res}_\omega^i(\text{III}_\omega^i(F, M))$$

such that the composite map

$$(5.9) \quad \text{III}^i(F, M) \xrightarrow{\widetilde{\text{res}}^i} \text{III}^i(K, M)^F \cap \text{res}_\omega^i(\text{III}_\omega^i(F, M)) \hookrightarrow \text{III}^i(K, M)^F$$

equals $\text{res}^i: \text{III}^i(F, M) \rightarrow \text{III}^i(K, M)^F$.

Next, we introduce the group

$$(5.10) \quad \mathrm{tr}(\mathrm{III}^i(K, M)^{\Gamma}) = \frac{\mathrm{III}^i(K, M)^{\Gamma}}{\mathrm{III}^i(K, M)^{\Gamma} \cap \mathrm{res}_{\omega}^i(\mathrm{III}_{\omega}^i(F, M))}.$$

Note that

$$\mathrm{tr}(\mathrm{III}^i(\widetilde{K}, M)^{\Gamma}) \simeq \frac{\mathrm{III}^i(K, M)^{\Gamma} + \mathrm{res}_{\omega}^i(\mathrm{III}_{\omega}^i(F, M))}{\mathrm{res}_{\omega}^i(\mathrm{III}_{\omega}^i(F, M))} \subset C_{\omega}^i(K/F, M),$$

where $C_{\omega}^i(K/F, M)$ is given by (4.2). Since $\mathrm{III}^i(K, M)(\ell)$ is a group of finite cotype for every prime ℓ by Lemma 4.3 and $C_{\omega}^i(K/F, M)$ is a group of finite exponent annihilated by a power of $[K : F]$ by Lemma 4.2, (5.10) is a *finite* group annihilated by a power of $[K : F]$.

REMARK 5.2. Our choice of notation in (5.10) is motivated by the fact that if G is a semiabelian variety over F , then

$$\mathrm{tr}(\mathrm{III}^1(K, G)^{\Gamma}) \simeq \mathrm{trans}(\mathrm{III}^1(K, G)^{\Gamma}) \subset H^2(\Gamma, G(K)),$$

where $\mathrm{trans}: H^1(K, G)^{\Gamma} \rightarrow H^2(\Gamma, G(K))$ is the transgression map [Sha72, p. 51].

We now apply Proposition 2.3 to the pair of maps (5.9) and obtain an equality

$$(5.11) \quad \mathrm{Ker}(\mathrm{res}^i) = \mathrm{Ker}(\widetilde{\mathrm{res}}^i)$$

and a canonical exact sequence

$$(5.12) \quad 0 \rightarrow \mathrm{Coker}(\widetilde{\mathrm{res}}^i) \rightarrow \mathrm{Coker}(\mathrm{res}^i) \rightarrow \mathrm{tr}(\mathrm{III}^i(K, M)^{\Gamma}) \rightarrow 0.$$

It is now clear that Theorem 1.1 is true if, and only if, the kernel and cokernel of the map $\widetilde{\mathrm{res}}^i$ in (5.8) are finite groups annihilated by a power of $[K : F]$.

By the semilocal theory of [Cha67, §2.1], there exist isomorphisms of abelian groups

$$(5.13) \quad \left(\bigoplus_{\text{all } w'} H^i(K_{w'}, M) \right)^{\Gamma} \simeq \bigoplus_{\text{all } v} \left(\bigoplus_{w'|v} H^i(K_{w'}, M) \right)^{\Gamma} \\ \simeq \bigoplus_{\text{all } v} H^i(K_w, M)^{\Gamma_w}$$

where, in the last direct sum, for each prime v of F , w is the fixed prime of K lying above v chosen previously. We define a map

$$(5.14) \quad \lambda_K^i: \mathrm{res}_{\omega}^i(\mathrm{III}_{\omega}^i(F, M)) \rightarrow \bigoplus_{\text{all } v} H^i(K_w, M)^{\Gamma_w}$$

by the commutativity of the diagram

$$\begin{array}{ccc}
 \text{III}_\omega^i(K, M)^\Gamma & \xrightarrow{\lambda^i(K, M)^\Gamma} & \left(\bigoplus_{\text{all } w'} H^i(K_{w'}, M) \right)^\Gamma \\
 \uparrow \text{J} & & \downarrow \wr \\
 \text{res}_\omega^i(\text{III}_\omega^i(F, M)) & \xrightarrow{\lambda_K^i} & \bigoplus_{\text{all } v} H^i(K_w, M)^{\Gamma_w}
 \end{array}$$

where $\lambda^i(K, M)$ is the map (5.1) associated to $M_K = [Y_K \rightarrow G_K]$ and the right-hand vertical map is the composition of the isomorphisms (5.13). Note that, since the kernel of $\lambda^i(K, M)$ equals $\text{III}^i(K, M)^\Gamma$ by (5.2) over K , we have

$$(5.15) \quad \text{Ker } \lambda_K^i = \text{III}^i(K, M)^\Gamma \cap \text{res}_\omega^i(\text{III}_\omega^i(F, M)).$$

The maps (5.1), (5.4) and (5.14) fit into an exact and commutative diagram of abelian groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{III}_\omega^i(K/F, M) & \longrightarrow & \text{III}_\omega^i(F, M) & \longrightarrow & \text{res}_\omega^i(\text{III}_\omega^i(F, M)) \\
 & & \downarrow \lambda^i(K/F, M) & & \downarrow \lambda^i(F, M) & & \downarrow \lambda_K^i \\
 0 & \longrightarrow & \bigoplus_{\text{all } v} H^i(K_w/F_v, M) & \longrightarrow & \bigoplus_{\text{all } v} H^i(F_v, M) & \longrightarrow & \bigoplus_{\text{all } v} H^i(K_w, M)^{\Gamma_w}
 \end{array}$$

Now we apply the snake lemma to the preceding diagram, using (5.15) and the definitions (5.5) and (5.6), and obtain an exact sequence of discrete and torsion abelian groups

$$0 \rightarrow \text{III}^i(K/F, M) \rightarrow \text{III}^i(F, M) \xrightarrow{\widetilde{\text{res}}^i} \text{III}^i(K, M)^\Gamma \cap \text{res}(\text{III}_\omega^i(F, M)) \rightarrow \mathfrak{C}^i(K/F, M).$$

Since $\text{III}^i(F, M)(\ell)$ and $\text{III}^i(K, M)(\ell)$ are groups of finite cotype for every prime ℓ by Lemma 4.3, and $\text{III}^i(K/F, M)$ and $\mathfrak{C}^i(K/F, M)$ are groups of finite exponent annihilated by $[K : F]$ by Lemma 5.1, the preceding sequence together with Lemma 2.2 show that both $\text{Ker}(\widetilde{\text{res}}^i)$ and $\text{Coker}(\widetilde{\text{res}}^i)$ are finite groups annihilated by $[K : F]$. The proof of Theorem 1.1 is now complete.

REMARK 5.3. It follows from the above and (5.11) that $\text{Ker}(\text{res}^i)$ is a finite group annihilated by $[K : F]$ for $i = 1$ and $i = 2$. Further, if $M = G$ is a semiabelian variety, then (5.12) and Remark 5.2 show that the cokernel of $\text{res}^1 : \text{III}^1(F, G) \rightarrow \text{III}^1(K, G)^\Gamma$ is annihilated by $[K : F]^2$.

Appendix. Since [GA09] is cited in this paper, we repair here a (fortunately inconsequential) error which appears in [GA09, proof of Proposition 4.10, p. 217, line -8 to line -3]. The notation is that of op. cit.

Recall the element $(\xi_v) \in \bigoplus_{v \notin U} H^i(K_v, N) \times \prod_{v \in U} H^i(\mathcal{O}_v, \mathcal{N})$, where the set U has the property that $D^1(V, \mathcal{N}) = \text{III}^1(K, N)$ for any nonempty open subset V of U . There exists an element $\xi_U \in H^i(U, \mathcal{N})$ such that $\xi_U|_{K_v} = \xi_v$ for all $v \notin U$. The incorrect part of the proof starts with the statement “The assignment $(\xi_v)_{v \notin U} \mapsto \xi_U$ is functorial in U ” and ends with the claim “This shows that $\text{Ker } \gamma_i(K, N) \subset \text{Im } \beta_i(K, N)$ ”. The problem is that the indicated assignment is not even well-defined, since the element $\xi_U \in H^i(U, \mathcal{N})$ is not uniquely determined. Thus, contrary to what is stated in loc. cit., it is not possible to obtain a well-defined element $\xi \in H^i(K, N)$ such that $\beta_i(K, N)(\xi) = (\xi_v)$. This problem can be repaired as follows. The element $\xi_U \in H^i(U, \mathcal{N})$ is well-defined modulo $D^i(U, \mathcal{N})$, and the problem reduces to showing that the elements $\xi_U + D^i(U, \mathcal{N}) \in H^i(U, \mathcal{N})/D^i(U, \mathcal{N})$ can be chosen so that if V is a nonempty open subset of U , then $\xi_U|_V - \xi_V \in D^i(V, \mathcal{N})$. Indeed, if this is proved, then any representative $\xi \in H^i(K, N)$ of the resulting well-defined class in $H^i(K, N)/\text{III}^i(K, N)$ satisfies $\beta_i(K, N)(\xi) = (\xi_v)$ since $\text{III}^i(K, N) = \text{Ker } \beta_i(K, N)$. Now, if $i = 1$ and $v \in U - V$, then $\xi_V|_{K_v} = \xi_v \in H^1(\mathcal{O}_v, \mathcal{N})$, whence $\xi_V \in H^1(U, \mathcal{N}) \subset H^1(V, \mathcal{N})$ by [Čes16, Corollary 4.2 and Remark 4.3]. Since $(\xi_U - \xi_V)|_{K_v} = 0$ for all $v \notin U$, we have $\xi_U - \xi_V \in D^1(U, \mathcal{N}) = \text{III}^1(K, N)$. Consequently $\xi_U|_V - \xi_V \in \text{III}^1(K, N) = D^1(V, \mathcal{N})$, as required. If $i = 2$ and $v \in U - V$, we have $\xi_U|_{K_v} = (\xi_U|_{\mathcal{O}_v})|_{K_v} = \xi_v = 0$ since $H^2(\mathcal{O}_v, \mathcal{N}) = 0$ for all $v \in U$, whence $(\xi_U|_V - \xi_V)|_{K_v} = 0$ for all $v \notin V$, i.e., $\xi_U|_V - \xi_V \in D^2(V, \mathcal{N})$.

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