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Generic distributional chaos and principal measure in linear dynamics

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Abstract. Generic distributional chaos and principal measure in linear dynamics are investigated. Sufficient conditions for a C_0-semigroup of operators on a Fréchet space to be generically distributionally chaotic are provided and applied to concrete examples. Furthermore, the distributionally chaotic dynamics of product operators (product C_0-semigroups, respectively) are considered. It is shown that under certain conditions, the product operator is generically distributionally chaotic if and only if there is a factor operator exhibiting generic distributional chaos. Another interesting finding is that there exist distributionally chaotic (but not hypercyclic) operators whose principal measure could be less than any fixed positive number.

1. Introduction. Chaos in infinite-dimensional linear systems has been widely studied in the past three decades. Especially, the study of hypercyclicity and Devaney chaos for linear operators and C_0-semigroups has became an active research area (see books [6, 16]). In [15], Godefroy and Shapiro introduced the concept of chaos for linear operators in the sense of Devaney [14], which requires the hypercyclic property and the density of periodic points.

Schweizer and Smítal [25] introduced another popular concept of chaos for interval maps (namely, distributional chaos) by considering the dynamics of pairs with some statistical properties. They proved that a continuous self-map of a compact interval exhibits distributional chaos if and only if it has positive topological entropy. Later, the notions of distributional chaos and principal measure were extended to general dynamical systems [23, 24], and especially to the framework of linear dynamics in the last few years. It seems that the first example of a distributionally chaotic operator on a Fréchet space was given by Oprocha [22], where the annihilation operator of a quan-

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tum harmonic oscillator was investigated. Wu and Zhu [30] further proved that the principal measure of the annihilation operator studied in [22] is 1. Since then, distributional chaos for linear operators on infinite-dimensional spaces has been studied by many authors [1,3,5,7–9,17–21,28,29,32]. A systematic investigation of distributional chaos for linear operators on Fréchet spaces was recently conducted by Bernardes et al. [9], where the concept of distributionally irregular vector and a criterion characterizing distributional chaos were introduced.

Very recently, the study of distributionally chaotic dynamics of linear operators has been successfully extended to C_0-semigroups of operators. Distributional chaos of the translation semigroup on weighted L^p spaces was considered in [4], and some sufficient conditions in terms of the weight for distributional chaos were provided. In [2, 13], Conejero et al. considered Devaney chaos and distributional chaos for some examples of C_0-semigroups of operators which are solutions of certain partial differential equations. Based on the results of [9], Albanese et al. [1] further developed the theory of distributional chaos for C_0-semigroups of operators on Banach spaces. An extension of distributional chaos to families of operators (including C_0-semigroups) on Fréchet spaces was also proposed by Conejero et al. [12].

It is known from [9] that an operator acting on a Fréchet space exhibits distributional chaos if and only if it admits a distributionally chaotic pair. However, given a distributionally chaotic operator, it is in general hard to offer a full description of the distributionally scrambled sets and the set of distributionally chaotic pairs, including their size, invariance property, diversity, and algebraic and topological structures. Moreover, it is mentioned in [22] that the exact value of an operator's principal measure is hard to calculate. Following this line of investigation, we further explore distributional chaos for operators and C_0-semigroups in some new aspects.

In the present work, we deal with the notion of generic distributional chaos for C_0-semigroups of operators on Fréchet spaces. This means that the set of all distributionally chaotic pairs for a C_0-semigroup forms a residual set. Obviously, not every distributionally chaotic operator (or C_0-semigroup) has this property. In Section 2, we provide some sufficient conditions for a C_0-semigroup of operators on a Fréchet space to be generically distributionally chaotic. Then we focus on a concrete example of a C_0-semigroup on the space $C([0,\infty))$ of real-valued continuous functions, which is proved to be Devaney chaotic, topologically mixing and generically distributionally chaotic with principal measure 1. It is also shown that there exists an operator T on $C([0,\infty))$ such that λT is Devaney chaotic, topologically mixing and generically distributionally chaotic with principal measure 1 for every constant $\lambda \neq 0$.

In Section 3, we study distributional chaos and the principal measure for product (also called direct sum) operators and product C_0-semigroups. It is shown that if there is a factor operator that is distributionally chaotic, then so is the product operator, and the principal measure of the product operator is no less than that of any factor operator. Generally, this property does not hold for generic distributional chaos. We show that under certain conditions, the product of finitely many operators is generically distributionally chaotic if and only if some factor operator exhibits generic distributional chaos. Analogous results for product C_0-semigroups are also obtained.

It is worth mentioning that the backward shift, which is an important class of operators in linear dynamics, has principal measure 1 if it is distributionally chaotic [20, 27]. And it is easy to see that every distributionally chaotic operator (or C_0-semigroup) on a Banach space (as a special Fréchet space) has principal measure 1. So we may wonder whether there exists a distributionally chaotic operator on a Fréchet space with principal measure less than 1. Indeed, we prove that there are distributionally chaotic operators whose principal measure is less than any given positive number. Moreover, such operators need not be hypercyclic and hence may not be Devaney chaotic.

Throughout this paper, denote $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{R}^+ = (0, \infty)$.

2. Generically distributionally chaotic C_0-semigroups. Let X be an infinite-dimensional Fréchet space, that is, a metrizable, complete and locally convex topological vector space. Let $(\|\cdot\|_k)_{k\in\mathbb{N}}$ be an increasing sequence of seminorms on X, which defines a translation-invariant metric

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x-y\|_i}{1+\|x-y\|_i}, \qquad \forall x, y \in X, \tag{2.1}$$

such that X is complete under the metric $\rho(\cdot,\cdot)$. Throughout this paper, we denote the Fréchet space by $(X, (\|\cdot\|_k)_{k\in\mathbb{N}}, \rho)$ (or simply X) and let $\mathfrak{L}(X)$ be the space of continuous linear operators on X.

Recall that a family $\mathcal{T} = \{T_t\}_{t\geq 0} \subseteq \mathfrak{L}(X)$ is called a *strongly continuous semigroup of linear operators* on X (or simply a C_0-*semigroup of operators*) if

(i) $T_0 = I$ (the identity operator on X);
(ii) $T_t T_s = T_{t+s}$ for all $s, t \geq 0$;
(iii) $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $s, t \geq 0$.

The *infinitesimal generator* A of a C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ on X is defined by

$$Ax = \lim_{t\to 0^+} \frac{T_t x - x}{t}, \qquad x \in D(A),$$

where $D(A) := \big\{x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists}\big\}$ is the domain of A.

Given a C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ on X, $\mathcal{T}$ is said to be *hypercyclic* if there exists a vector $x \in X$ such that $\{T_t x : t \geq 0\}$ is dense in X. Moreover, $\mathcal{T}$ is *topologically mixing* if for any pair U, V of non-empty open subsets of X, there exists $t_0 > 0$ such that $T_t(U) \cap V \neq \emptyset$ for all $t > t_0$. A vector $x_0 \in X$ is called a *periodic point* for $\mathcal{T}$ if there is $t > 0$ such that $T_t x_0 = x_0$. We denote by $\mathrm{Per}(\mathcal{T})$ the set of all periodic points for $\mathcal{T}$. Further, $\mathcal{T}$ is said to be *Devaney chaotic* if $\mathcal{T}$ is hypercyclic and the set $\mathrm{Per}(\mathcal{T})$ is dense in X.

For any $x, y \in X$ and any $t > 0$, define the *distributional function* of x and y to be $\Phi^t_{x,y} : \mathbb{R}^+ \to [0, 1]$, where

$$\Phi^t_{x,y}(\varepsilon) = \frac{1}{t}\mu(\{0 \leq s \leq t : \rho(T_s x, T_s y) < \varepsilon\}), \quad \forall \varepsilon > 0, \tag{2.2}$$

where μ denotes the Lebesgue measure on $\mathbb{R}$ throughout this paper. The *upper* and *lower distributional functions* of x and y are then defined by

$$\Phi^*_{x,y}(\varepsilon) = \limsup_{t\to\infty} \Phi^t_{x,y}(\varepsilon), \quad \Phi_{x,y}(\varepsilon) = \liminf_{t\to\infty} \Phi^t_{x,y}(\varepsilon), \quad \forall \varepsilon > 0, \tag{2.3}$$

respectively.

DEFINITION 2.1. Let $(X, (\|\cdot\|_k)_{k\in\mathbb{N}}, \rho)$ be a Fréchet space. A C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ of operators on X is said to be *distributionally chaotic* if there exist an uncountable subset $S \subset X$ and $\delta > 0$ such that

$$\Phi_{x,y}(\delta) = 0 \quad \text{and} \quad \Phi^*_{x,y}(\varepsilon) = 1, \quad \forall \varepsilon > 0,$$

for any distinct $x, y \in S$. In this case, S is said to be a *distributionally δ-scrambled set* and (x, y) a *distributionally chaotic pair*. In addition, $\mathcal{T}$ is called *densely distributionally chaotic* if the scrambled set S may be chosen to be dense in X; and $\mathcal{T}$ is *generically distributionally chaotic* if the set of all distributionally chaotic pairs is residual in $X \times X$.

Let $A \subset \mathbb{R}^+$ be a Lebesgue measurable set. The *upper density* and *lower density* of A are defined as

$$\overline{\mathrm{Dens}}(A) := \limsup_{t\to\infty} \frac{\mu(A \cap [0, t])}{t}, \quad \underline{\mathrm{Dens}}(A) := \liminf_{t\to\infty} \frac{\mu(A \cap [0, t])}{t},$$

respectively. Then the conditions $\Phi^*_{x,y}(\varepsilon) = 1$, $\Phi_{x,y}(\delta) = 0$ in Definition 2.1 are equivalent to

$$\begin{aligned} &\overline{\mathrm{Dens}}\{s \geq 0 : \rho(T_s(x), T_s(y)) < \varepsilon\} = 1, \\ &\underline{\mathrm{Dens}}\{s \geq 0 : \rho(T_s(x), T_s(y)) < \delta\} = 0, \end{aligned} \tag{2.4}$$

respectively.

Given $M \subset \mathbb{Z}^+$, the *upper density* and *lower density* of M are defined as

$$\overline{\mathrm{dens}}(M) := \limsup_{n\to\infty} \frac{\mathrm{card}(M \cap [0, n-1])}{n},$$
$$\underline{\mathrm{dens}}(M) := \liminf_{n\to\infty} \frac{\mathrm{card}(M \cap [0, n-1])}{n},$$

respectively, where $\mathrm{card}(M)$ denotes the cardinality of the set M.

To measure the degree of chaos for a given dynamical system, the concept of principal measure was introduced for general dynamical systems [24, 25]. For the study of principal measures of certain linear operators, we refer to [22, 27, 30]. Naturally, we can extend this concept to C_0-semigroups of operators on Fréchet spaces.

Definition 2.2. Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup of operators on a Fréchet space X. The *principal measure* $\mu_{\mathrm{p}}(\mathcal{T})$ of $\mathcal{T}$ is defined as follows:

$$\mu_{\mathrm{p}}(\mathcal{T}) = \sup_{x\in X} \frac{1}{\mathrm{diam}(X)} \int_0^\infty (\Phi^*_{x,0}(s) - \Phi_{x,0}(s))\, ds,$$

where $\Phi^*_{x,0}(s)$ and $\Phi_{x,0}(s)$ are the upper and lower distributional functions of x and 0, respectively, as defined in (2.3), and $\mathrm{diam}(X)$ is the diameter of X.

In [8, 9], Peris et al. introduced the notion of distributionally irregular vector for operators on Fréchet spaces in order to characterize distributional chaos. Later, Albanese et al. [1] extended this notion to C_0-semigroups of operators.

Definition 2.3. Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup of operators on a Fréchet space $(X, (\|\cdot\|_k)_{k\in\mathbb{N}}, \rho)$. Let $k \in \mathbb{N}$. A vector $x \in X$ is called *distributionally k-irregular* if:

(i) there exists $A \subset \mathbb{R}^+$ with $\overline{\mathrm{Dens}}(A) = 1$ such that $\lim_{t\to\infty,\, t\in A} T_t x = 0$;
(ii) there exists $B \subset \mathbb{R}^+$ with $\overline{\mathrm{Dens}}(B) = 1$ such that $\lim_{t\to\infty,\, t\in B} \|T_t x\|_k = \infty$.

Moreover, the orbit of x under $\mathcal{T}$ is called *distributionally near zero* if (i) holds, and *distributionally k-unbounded* if (ii) holds.

Albanese et al. [1] proved that a C_0-semigroup of operators on a Banach space is distributionally chaotic if and only if it admits a distributionally chaotic pair. With a similar argument, we show that this result still holds for C_0-semigroups of operators on Fréchet spaces.

Theorem 2.4. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup of operators on a Fréchet space X. Then $\mathcal{T}$ is distributionally chaotic if and only if $\mathcal{T}$ admits a distributionally chaotic pair.*

Proof. Necessity. This is obvious.

Sufficiency. Suppose that $\mathcal{T}$ admits a distributionally chaotic pair (x, y). To show that $\mathcal{T}$ exhibits distributional chaos, it is sufficient to prove the existence of a distributionally chaotic pair for T_{t_0} with some $t_0 > 0$, as shown in [12, Theorem 4.2].

Given $t_0 > 0$, denote by $F^n_{x,y}$ the distributional function of x, y under T_{t_0}, namely,

$$F^n_{x,y}(\varepsilon) = \frac{1}{n}\operatorname{card}\big(\{0 \le i \le n-1 : \rho(T^i_{t_0}x, T^i_{t_0}y) < \varepsilon\}\big), \qquad \forall \varepsilon > 0. \tag{2.5}$$

By using the local equicontinuity of $\mathcal{T}$, one can follow the proof of [33, Theorem 2.1] to show that for any $\varepsilon > 0$, there exists $\varepsilon_1 = \varepsilon_1(\varepsilon) > 0$ such that

$$F^n_{x,y}(\varepsilon_1) \le \Phi^{nt_0}_{x,y}(\varepsilon) \quad \text{and} \quad \Phi^{nt_0}_{x,y}(\varepsilon_1) \le \frac{n+1}{n} F^{n+1}_{xy}(\varepsilon) \tag{2.6}$$

for any $n \in \mathbb{N}$.

Since (x, y) is a distributionally chaotic pair of $\mathcal{T}$, there exists $\varepsilon_0 > 0$ such that

$$\Phi_{x,y}(\varepsilon_0) = 0 \quad \text{and} \quad \Phi^*_{x,y}(\varepsilon) = 1, \quad \forall \varepsilon > 0.$$

It then follows from (2.6) that there is $\varepsilon'_0 > 0$ such that

$$F^n_{x,y}(\varepsilon'_0) \le \Phi^{nt_0}_{x,y}(\varepsilon_0), \qquad \forall n \in \mathbb{N}.$$

Therefore

$$0 \le \liminf_{n\to\infty} F^n_{x,y}(\varepsilon'_0) \le \liminf_{n\to\infty} \Phi^{nt_0}_{x,y}(\varepsilon_0) = 0. \tag{2.7}$$

Moreover, for any $\delta > 0$, there exists $\delta' > 0$ such that

$$\Phi^{nt_0}_{x,y}(\delta') \le \frac{n+1}{n} F^{n+1}_{x,y}(\delta), \qquad \forall n \in \mathbb{N},$$

which implies

$$1 = \limsup_{n\to\infty} \Phi^{nt_0}_{x,y}(\delta') \le \limsup_{n\to\infty} \frac{n+1}{n} F^{n+1}_{x,y}(\delta) \le 1. \tag{2.8}$$

Combining (2.7) and (2.8) shows that (x, y) is a distributionally chaotic pair of T_{t_0}. ■

As is proven in Theorem 2.4, a C_0-semigroup of operators is distributionally chaotic if it admits a distributionally chaotic pair. In the following, we further investigate the set of distributionally chaotic pairs for a C_0-semigroup of operators. We focus on generic distributional chaos.

PROPOSITION 2.5. *Let $\mathcal{T}$ be a C_0-semigroup of operators on X. If $\mathcal{T}$ admits a residual distributionally scrambled set, then $\mathcal{T}$ is generically distributionally chaotic.*

Proof. Let $(\Gamma_i)_{i\geq 1}$ be a family of dense open subsets of X such that (x,y) is a distributionally chaotic pair of $\mathcal{T}$ for any distinct $x,y\in\bigcap_{i\geq 1}\Gamma_i$. Denote

$$\Upsilon_i=\{(x,y)\in X\times X : x,y\in\Gamma_i,\ x\neq y\},\qquad \forall i\geq 1.$$

Then each Υ_i is a dense open subset of $X\times X$. Moreover, $\bigcap_{i\geq 1}\Upsilon_i$ consists of distributionally chaotic pairs of $\mathcal{T}$. ∎

We note that the assumption in Proposition 2.5 is very strong and rules out a large class of C_0-semigroups. Actually, if a C_0-semigroup $\mathcal{T}=\{T_t\}_{t\geq 0}$ admits a residual distributionally scrambled set, then T_1 also admits a residual distributionally scrambled set, as shown in the proof of Theorem 2.4. It then follows from [10, Theorem 34] that every non-zero vector of X should be an irregular vector of T_1. Therefore $T_t x\nrightarrow 0$ (as $t\to\infty$) for every $x\in X\setminus\{0\}$.

Next, we deal with the case that there is a dense subset X_0 of X such that the orbit of each $x\in X_0$ under $\mathcal{T}$ tends to zero. The following lemma is a continuous version of [9, Proposition 8] for operators.

Lemma 2.6. *Let $\mathcal{T}=\{T_t\}_{t\geq 0}$ be a C_0-semigroup of operators on a Fréchet space X. Then the following assertions are equivalent:*

(i) *$\mathcal{T}$ has a distributionally unbounded orbit.*
(ii) *There exist $\delta\in(0,1)$, a sequence $(y_k)_k\subset X$ and a sequence $(M_k)_k$ of positive numbers with $\lim_{k\to\infty}M_k=\infty$ such that*

$$\lim_{k\to\infty}y_k=0\quad\textit{and}\quad\lim_{k\to\infty}M_k^{-1}\mu(\{s\in[0,M_k]:\rho(T_s y_k,0)>\delta\})=1.$$

(iii) *The set of all vectors with distributionally unbounded orbit is residual in X.*

Proof. The implication (iii)⇒(i) is trivial.

(i)⇒(ii). Let y be a vector with distributionally m-unbounded orbit. Then there exists $A\subset\mathbb{R}^+$ with $\overline{\mathrm{Dens}}(A)=1$ such that

$$\lim_{s\to\infty,\,s\in A}\|T_s y\|_m=\infty.$$

Set $y_k=k^{-1}y$ for each $k\in\mathbb{N}$. Then $\lim_{k\to\infty}y_k=0$. For each $k\in\mathbb{N}$ and any $\epsilon>0$,

$$\begin{aligned}\overline{\mathrm{Dens}}(\{s\geq 0:\|T_s y_k\|_m>\epsilon\})&=\overline{\mathrm{Dens}}(\{s\geq 0:\|T_s y\|_m>k\epsilon\})\\&\geq\overline{\mathrm{Dens}}(A)=1.\end{aligned}$$

It follows that

$$\begin{aligned}(2.9)\qquad&\overline{\mathrm{Dens}}(\{s\geq 0:\rho(T_s y_k,0)>2^{-m}\epsilon/(1+\epsilon)\})\\&\qquad\geq\overline{\mathrm{Dens}}(\{s\geq 0:\|T_s y_k\|_m>\epsilon\})=1.\end{aligned}$$

Let $\delta = 2^{-m}\epsilon/(1+\epsilon)$. Then (2.9) implies that there is a sequence $(M_k)_{k\in\mathbb{N}}$ of positive numbers such that

$$M_k^{-1}\mu(\{s \in [0, M_k] : \rho(T_s y_k, 0) > \delta\}) > 1 - k^{-1}.$$

Therefore

$$\lim_{k\to\infty} M_k^{-1}\mu(\{s \in [0, M_k] : \rho(T_s y_k, 0) > \delta\}) = 1.$$

(ii)⇒(iii). First, we show that there exist $\delta_1 > 0$ and $m \in \mathbb{N}$ such that

$$\lim_{k\to\infty} M_k^{-1}\mu(\{s \in [0, M_k] : \|T_s y_k\|_m > \delta_1\}) = 1. \tag{2.10}$$

Indeed, for $\delta > 0$ in (ii), there exists a positive integer i such that δ is in $[2^{-i-1}, 2^{-i})$. Moreover, for any $x \in X$ with

$$\|x\|_j \le \frac{\delta}{2-\delta}, \qquad \forall j \in [1, i+2],$$

we have

$$\rho(x, 0) = \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{\|x\|_k}{1+\|x\|_k} \le \frac{\delta}{2} + \sum_{k=i+3}^{\infty} \frac{1}{2^k}\frac{\|x\|_k}{1+\|x\|_k} \le \frac{\delta}{2} + \frac{1}{2^{i+2}} \le \delta.$$

Therefore, if $\rho(x, 0) > \delta$, then there exists an integer $j \in [1, i+2]$ such that $\|x\|_j > \delta/(2-\delta)$.

Denote $\delta_1 = \delta/(2-\delta)$ and $m = i+2$. Since $(\|\cdot\|_k)_k$ is a sequence of increasing seminorms, this yields

$$\{s \in [0, M_k] : \rho(T_s y_k, 0) > \delta\} \subset \{s \in [0, M_k] : \|T_s y_k\|_m > \delta_1\},$$

which further implies that (2.10) holds.

Second, for each $k \in \mathbb{N}$, we prove that the set

$$Q_k := \big\{x \in X : \exists t > 0 \text{ with } \mu(\{0 \le s \le t : \|T_s x\|_m > k\}) \ge t(1-k^{-1})\big\}$$

is dense in X.

In fact, given any $x \in X$, $\varepsilon > 0$ and a fixed positive integer m_1, since the sequence $(y_k)_k$ tends to zero in X, we can choose $z = y_l$ with l large enough such that

$$\|z\|_{m_1} < C := \frac{\delta_1 \varepsilon}{4k^2}.$$

From (2.10), we can further require z to satisfy

$$\mu(\{s \in [0, M_l] : \|T_s z\|_m > \delta_1\}) \ge M_l\left(1 - \frac{1}{2k}\right).$$

Let $K = \{0, 1, \dots, 2k-1\}$ and

$$z_j := x + \frac{\varepsilon j}{2kC} z, \qquad \forall j \in K.$$

Then $\|z_j - x\|_{m_1} < \varepsilon$. It is sufficient to show that there is $j_0 \in K$ such that $z_{j_0} \in Q_k$.

Denote

$$A := \{s \in [0, M_l] : \|T_s z\|_m > \delta_1\}.$$

Then $\mu(A) \geq M_l\big(1 - \frac{1}{2k}\big)$. Further set

$$B_j := \{s \in [0, M_l] : \|T_s z_j\|_m \leq k\}, \qquad \forall j \in K.$$

Fix $j_1, j_2 \in K$ with $j_1 \neq j_2$. If $s \in B_{j_1} \cap B_{j_2} \cap A$, then

$$\|T_s z_{j_1} - T_s z_{j_2}\|_m = \frac{|j_1 - j_2|\varepsilon \|T_s z\|_m}{2kC} > \frac{\delta_1 \varepsilon}{2kC} = 2k. \tag{2.11}$$

However, the subadditivity of the seminorm indicates that

$$\|T_s z_{j_1} - T_s z_{j_2}\|_m \leq \|T_s z_{j_1}\|_m + \|T_s z_{j_2}\|_m \leq 2k. \tag{2.12}$$

The contradiction between (2.11) and (2.12) implies $B_{j_1} \cap B_{j_2} \cap A = \emptyset$ for any distinct $j_1, j_2 \in K$. Therefore there exists $j_0 \in K$ such that $\mu(B_{j_0} \cap A) \leq \mu(A)/(2k)$, which yields

$$\mu(A - B_{j_0}) \geq \mu(A)(1 - (2k)^{-1}) \geq M_l(1 - (2k)^{-1})^2 \geq M_l(1 - k^{-1}).$$

According to the definition of B_{j_0}, one has

$$\|T_s z_{j_0}\|_m > k, \qquad \forall s \in A - B_{j_0},$$

so $z_{j_0} \in Q_k$. Therefore Q_k is dense in X for each positive integer k.

Let

$$Q'_k := \{x \in X : \exists t > 0, \gamma \in (0,1) \text{ with } \mu(\{0 \leq s \leq t : \|T_s x\|_m > k - \gamma\}) \geq t(1 - k^{-1})\}$$

for each $k \in \mathbb{N}$. Since $Q_k \subset Q'_k$, Q'_k is also dense in X (for each $k \in \mathbb{N}$). We further show that every Q'_k is open in X. Indeed, let $\bar{x} \in Q_k$; then there exist $\bar{t} > 0$ and $\bar{\gamma} \in (0,1)$ such that

$$\mu(\{0 \leq s \leq \bar{t} : \|T_s \bar{x}\|_m > k - \bar{\gamma}\}) \geq \bar{t}(1 - k^{-1}). \tag{2.13}$$

Set $\bar{\epsilon} = (1 - \bar{\gamma})/2$; the local equicontinuity of $\mathcal{T}$ implies that there exists a sufficiently small neighborhood $\mathcal{O}(\bar{x})$ of $\bar{x}$ such that

$$\|T_s \bar{x} - T_s \bar{y}\|_m < \bar{\epsilon}, \qquad \forall s \in [0, \bar{t}], \forall \bar{y} \in \mathcal{O}(\bar{x}). \tag{2.14}$$

If $\|T_s \bar{x}\|_m > k - \bar{\gamma}$ for some $s \in [0, \bar{t}]$, then

$$\|T_s \bar{y}\|_m > \|T_s \bar{x}\|_m - \bar{\epsilon} > k - (1 + \bar{\gamma})/2.$$

Therefore it follows from (2.13) and (2.14) that $\bar{y} \in Q'_k$ for any $\bar{y} \in \mathcal{O}(\bar{x})$. Hence Q'_k is an open subset of X.

Finally, for any fixed $y \in \bigcap_{k=1}^{\infty} Q'_k$, there exists a sequence $(n_k)_k$ of positive numbers tending to infinity and satisfying

$$\mu(R_k) \geq n_k(1 - k^{-1}),$$

where $R_k := \{0 \le s \le n_k : \|T_s y\|_m > k-1\}$. Let $R := \bigcup_{k=1}^{\infty} R_k$. Then $\overline{\mathrm{Dens}}(R) = 1$ and

$$\lim_{s\to\infty,\, s\in R} \|T_s y\|_m = \infty.$$

Therefore the orbit of y is distributionally m-unbounded. Hence we conclude that the set of all vectors with distributionally m-unbounded orbit is residual in X. ∎

THEOREM 2.7. *Let $\mathcal{T} = \{T_t\}_{t\ge 0}$ be a C_0-semigroup of operators on a separable Fréchet space X. Suppose that there exists a dense subset X_0 of X such that $\lim_{t\to\infty} T_t x = 0$ for each $x \in X_0$. Then the following statements are equivalent:*

- (i) *$\mathcal{T}$ is distributionally chaotic.*
- (ii) *$\mathcal{T}$ has a residual set of distributionally k-irregular vectors for some $k \in \mathbb{N}$.*
- (iii) *$\mathcal{T}$ is generically distributionally chaotic.*

Proof. (i)⇒(ii). Assume that $\mathcal{T}$ is distributionally chaotic. Then $\mathcal{T}$ admits a distributionally irregular vector, as shown by Conejero et al. [12]. So we can deduce from Lemma 2.6 that there exists $k \in \mathbb{N}$ such that the set of vectors with distributionally k-unbounded orbit is residual in X. We only need to show that the the set of vectors with orbit distributionally near zero is residual in X.

Indeed, for each $n \in \mathbb{N}$, denote

$$M_n = \big\{x \in X : \exists t > 0,\ \tau \in (0, n^{-1}) \text{ with } \mu(\{s \in [0,t] : \rho(T_s x, 0) < n^{-1} - \tau\}) \ge t(1-n^{-1})\big\}.$$

Then M_n is open and dense in X. Moreover, it is not hard to check that $\bigcap_{n\in\mathbb{N}} M_n$ is a residual set consisting of vectors with orbits distributionally near zero.

(ii)⇒(iii). Let $(D_i)_{i\in\mathbb{N}}$ be a family of dense open subsets of X such that $\bigcap D_i$ consists of distributionally k-irregular vectors for $\mathcal{T}$. For each $j \in \mathbb{N}$, denote $C_j = \{(x,y) \in X \times X : x - y \in D_j\}$. It is not hard to show that C_j is a dense open subset of $X \times X$. So $\bigcap_j C_j$ is residual in $X \times X$. For every $(x,y) \in \bigcap C_j$, it follows that $x - y$ is a distributionally k-irregular vector, which implies that (x,y) is a distributionally chaotic pair. Therefore $\mathcal{T}$ is generically distributionally chaotic.

(iii)⇒(i). This is trivial. ∎

COROLLARY 2.8. *Let $\mathcal{T} = \{T_t\}_{t\ge 0}$ be a C_0-semigroup of operators on a separable Fréchet space X. Suppose that there exists a dense subset X_0 of X such that $\lim_{t\to\infty} \mathcal{T}_t x = 0$ for each $x \in X_0$. If the infinitesimal generator A of $\mathcal{T}$ admits an eigenvalue λ with $\lambda > 0$, then $\mathcal{T}$ is generically distributionally* *chaotic and $\mathcal{T}$ has a dense distributionally δ-scrambled linear manifold for some $\delta > 0$.*

In the rest of this section, we focus on a concrete example of a C_0-semigroup of operators on a non-normable Fréchet space, which is proved to be both generically distributionally chaotic and Devaney chaotic.

Throughout this paper, denote by $Z = C([0,\infty))$ the space of all real continuous functions on the interval $[0,\infty)$. It is known that Z is a separable Fréchet space with the family of seminorms

$$p_k(f) = \sup_{x\in[0,k]} |f(x)|, \qquad \forall f \in Z, \forall k \in \mathbb{N},$$

which induces a translation-invariant metric

$$\rho_1(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(f-g)}{1+p_k(f-g)}.$$

Let $Z_1 \subset Z$ be the set of all non-negative functions. For any given $h \in Z_1$, define $\mathcal{S}_h = \{S_t\}_{t\geq 0} \subset \mathfrak{L}(Z)$ by

$$S_t(f)(x) = e^{\int_x^{x+t} h(s)\,ds} f(x+t), \qquad \forall f \in Z,\ \forall t \geq 0. \tag{2.15}$$

It is not hard to check that $\mathcal{S}_h = \{S_t\}_{t\geq 0}$ is a C_0-semigroup of operators on Z. Note that the chaotic dynamics of this semigroup on some other linear spaces had also been studied, for example in [2, 26, 31].

THEOREM 2.9. *Let $h \in Z_1$. Then the C_0-semigroup $\mathcal{S}_h$ defined in* (2.15) *is generically distributionally chaotic. Moreover, the principal measure of $\mathcal{S}_h$ is equal to* 1.

Proof. We first construct a distributionally irregular vector for $\mathcal{S}_h$. Denote

$$M_0 = -1, \quad M_n = n!, \quad Q_n = (M_n, M_n+1), \quad P_n = [M_{n-1}+1, M_n], \tag{2.16}$$

for all $n \geq 1$. Set

$$A_1 = \bigcup_{k\geq 1}\left[M_{2k}+1, \frac{M_{2k}+1+M_{2k+1}}{2}\right], \qquad A_2 = \bigcup_{k\geq 1}[M_{2k+1}+1, M_{2k+2}-1],$$

and

$$\widetilde{f}(x) = \begin{cases} 0, & x \in P_{2i-1},\ i\geq 1,\\ i, & x\in P_{2i},\ i\geq 1,\\ [\widetilde{f}(M_i+1)-\widetilde{f}(M_i)](x-M_i)+\widetilde{f}(M_i), & x\in Q_i,\ i\geq 1.\end{cases} \tag{2.17}$$

Then it follows that

$$\overline{\mathrm{Dens}}(A_1) = \overline{\mathrm{Dens}}(A_2) = 1$$

and

$$\lim_{s\to\infty,\, s\in A_1} S_s\widetilde{f} = 0, \qquad \lim_{s\to\infty,\, s\in A_2} p_1(S_s\widetilde{f}) = \infty. \tag{2.18}$$

So $\widetilde{f}(x) \in Z$ is indeed a distributionally 1-irregular vector for $\mathcal{S}_h$. It is also easy to check that for each $f \in Z$ with compact support, $S_t f \to 0$ as $t \to \infty$. According to Theorem 2.7, $\mathcal{S}_h$ is generically distributionally chaotic.

From (2.18), we can compute the upper and lower distributional functions of $\widetilde{f}$ and 0 with respect to $\mathcal{S}_h$:

$$\Phi^*_{\widetilde{f},0}(\varepsilon) = 1, \quad \forall \varepsilon > 0,$$
$$\Phi_{\widetilde{f},0}(\varepsilon) = 0, \quad \forall \varepsilon \in (0,1).$$

Then from the definition of principal measure,

$$\mu_{\mathrm{p}}(\mathcal{S}_h) = \sup_{g\in Z}\int_0^1 (\Phi^*_{g,0}(s) - \Phi_{g,0}(s))\,ds \ge \int_0^1 (\Phi^*_{\widetilde{f},0}(s) - \Phi_{\widetilde{f},0}(s))\,ds = 1. \ \blacksquare$$

REMARK 2.10. In the case of $h(x) \equiv 0$, $\mathcal{S}_h$ is actually the translation semigroup. Theorem 2.9 implies that the translation semigroup on Z is generically distributionally chaotic with principal measure 1.

COROLLARY 2.11. *Let $h \in Z_1$. Then every non-trivial operator $S_t \in \mathcal{S}_h = \{S_t\}_{t\ge 0}$ on Z is generically distributionally chaotic with principal measure $\mu_{\mathrm{p}}(S_t) = 1$.*

Proof. The proof is similar to that of Theorem 2.9. Indeed, $\widetilde{f}$ defined in (2.17) is also a distributionally 1-irregular vector for S_t. We omit the details. ∎

THEOREM 2.12. *Let $h \in Z_1$. Then the C_0-semigroup $\mathcal{S}_h = \{S_t\}_{t\ge 0}$ is topologically mixing and Devaney chaotic. Moreover, every operator S_t with $t > 0$ is also topologically mixing and Devaney chaotic.*

Proof. Given non-empty open sets $U, V \subset Z$, there exist $f_1 \in U$ and $\varepsilon > 0$ such that $O(f_1, \varepsilon) \subset U$, where $O(f_1, \varepsilon) = \{h(x) \in Z : \rho_1(f_1, h) < \varepsilon\}$. Then we can find $\varepsilon_1 > 0$ such that

$$U_1 := \{h(x) \in Z : |f_1(x) - h(x)| < \varepsilon_1,\ \forall x \in [0, \varepsilon_1^{-1}]\} \subset O(f_1, \varepsilon).$$

Let $f_2 \in V$. For any $t > \varepsilon_1^{-1}$, set

$$f_3(x) = \begin{cases} f_1(x), & x \in [0, \varepsilon_1^{-1}], \\ \dfrac{e^{-\int_0^t h(s)\,ds} f_2(0) - f_1(\varepsilon_1^{-1})}{t - \varepsilon_1^{-1}}(x - \varepsilon_1^{-1}) + f_1(\varepsilon_1^{-1}), & x \in [\varepsilon_1^{-1}, t], \\ e^{-\int_{x-t}^x h(s)\,ds} f_2(x - t), & x \in [t, \infty). \end{cases} \tag{2.19}$$

Then $f_3 \in U_1 \subset U$ and $S_t(f_3)(x) \equiv f_2(x) \in V$. So $\mathcal{S}_h$ is topologically mixing. In particular, S_s is a topologically mixing operator for any $s > 0$.

Next, we show that for each fixed $s > 0$, the set of periodic points of S_s is dense in Z. Let $t_0 > 0$. For any fixed $f \in Z$ and any $\delta \in (0, 1)$, there exists a non-negative integer i such that $\delta \in [2^{-i-1}, 2^{-i})$. Choose $n_0 \in \mathbb{N}$ such that $n_0 t_0 > i + 1$. Define

$$g(x) = \begin{cases} f(x), & x \in [0, n_0 t_0], \\ \big[e^{-\int_0^{(n_0+1)t_0} h(s)\,ds} f(0) - f(n_0 t_0)\big](x - n_0 t_0) + f(n_0 t_0), & \\ & x \in [n_0 t_0, (n_0+1)t_0], \\ e^{-\int_{x-k(n_0+1)t_0}^{x} h(s)\,ds} g(x - k(n_0+1)t_0), & \\ & x \in [k(n_0+1)t_0, (k+1)(n_0+1)t_0],\ \forall k \in \mathbb{N}^+. \end{cases} \tag{2.20}$$

Then it is easy to see that $g \in Z$ and

$$\rho_1(g, f) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_{1,k}(g-f)}{1 + p_{1,k}(g-f)} = \sum_{k=i+2}^{\infty} \frac{1}{2^k} \frac{p_{1,k}(g-f)}{1 + p_{1,k}(g-f)} < \frac{1}{2^{i+1}} \le \delta.$$

Moreover, this shows that

$$(S_{t_0}^{n_0+1} g)(x) = g(x), \qquad \forall x \in [0, \infty),$$

so $g(x)$ is a periodic point of S_{t_0}. This implies that the set of periodic points of S_{t_0} is dense in Z. So the set of periodic points of $\mathcal{S}_h$ is also dense in Z. Hence $\mathcal{S}_h$ is Devaney chaotic and each non-trivial operator S_{t_0} with $t_0 > 0$ is also Devaney chaotic. ∎

In [11], Bonet asked whether every infinite-dimensional non-normable separable Fréchet space supports a hypercyclic operator T such that λT is hypercyclic for all $\lambda \neq 0$. For the real continuous function space $Z = C([0, \infty))$, we have the following result.

Theorem 2.13. *Let $T : Z \to Z$ be the translation operator defined by $Tf(x) = f(x+1)$. Then for any fixed $\lambda \in \mathbb{R} \setminus \{0\}$, the operator $\widetilde{T} := \lambda T$ on Z is topologically mixing, Devaney chaotic and generically distributionally chaotic with principal measure $\mu_{\mathrm{p}}(\widetilde{T}) = 1$.*

Proof. To show that $\widetilde{T} = \lambda T$ is topologically mixing and Devaney chaotic, we only need to modify the proof of Theorem 2.12 slightly.

Indeed, following the proof of Theorem 2.12, for any positive integer $n > \varepsilon_1^{-1}$, we replace f_3 defined in (2.19) by

$$\widetilde{f_3}(x) = \begin{cases} f_1(x), & x \in [0, \varepsilon_1^{-1}], \\ \dfrac{\frac{1}{\lambda^n} f_2(0) - f_1(\varepsilon_1^{-1})}{n - \varepsilon_1^{-1}}(x - \varepsilon_1^{-1}) + f_1(\varepsilon_1^{-1}), & x \in [\varepsilon_1^{-1}, n], \\ \dfrac{1}{\lambda^n} f_2(x - n), & x \in [n, \infty). \end{cases}$$

Then $\widetilde{f}_3 \in U_1 \subset U$ and $\widetilde{T}^n \widetilde{f}_3 = f_2 \in V$. Thus $\widetilde{T}$ is topologically mixing. To show density of the periodic points of $\widetilde{T}$, it suffices to replace $g(x)$ defined in (2.20) by

(2.21)
$$\widetilde{g}(x) = \begin{cases} f(x), & x \in [0, i+1], \\ \left[\dfrac{1}{\lambda^{i+2}} f(0) - f(i+1)\right](x - i - 1) + f(i+1), & x \in [i+1, i+2], \\ \dfrac{1}{\lambda^{k(i+2)}} g(x - k(i+2)), \quad x \in [k(i+2), (k+1)(i+2)], \ \forall k \in \mathbb{N}, \end{cases}$$

so $\rho_1(\widetilde{g}, f) \le \delta$ and $\widetilde{T}^{i+2}(\widetilde{g}) = \widetilde{g}$. This implies that the set of all periodic points of $\widetilde{T}$ is dense in Z. Therefore $\widetilde{T}$ is Devaney chaotic.

Finally, we construct a distributionally 1-irregular vector for $\widetilde{T}$. Then similar to the proof of Theorem 2.9, one can show that $\widetilde{T}$ is generically distributionally chaotic and $\mu_{\mathrm{p}}(\widetilde{T}) = 1$.

Set
$$B_1 = \bigcup_{k \ge 1} \left\{ j \in \mathbb{N} : M_{2k} + 1 < j < \frac{M_{2k} + 1 + M_{2k+1}}{2} \right\},$$
$$B_2 = \bigcup_{k \ge 1} \{ j \in \mathbb{N} : M_{2k+1} + 1 < j < M_{2k+2} - 1 \},$$

and

(2.22)
$$G(x) = \begin{cases} 0, & x \in P_{2i-1}, \ i \ge 1, \\ i\left(1 + \dfrac{1}{|\lambda|^{M_{2i}}}\right), & x \in P_{2i}, \ i \ge 1, \\ [G(M_i + 1) - G(M_i)](x - M_i) + G(M_i), & x \in Q_i, \ i \ge 1, \end{cases}$$

where $M_i, P_i, Q_i (i \ge 1)$ are defined in (2.16). Then we have
$$\overline{\mathrm{dens}}(B_1) = \overline{\mathrm{dens}}(B_2) = 1,$$

and
$$\lim_{j \to \infty,\, j \in B_1} \widetilde{T}^j G = 0, \qquad \lim_{j \to \infty,\, j \in B_2} p_1(\widetilde{T}^j G) = \infty,$$

so $G \in Z$ is a distributionally 1-irregular vector of $\widetilde{T}$. ■

3. Distributional chaos and principal measure for product operators and C_0-semigroups. In this section, we investigate the distributionally chaotic dynamics for product operators and product C_0-semigroups. We also compare the degree of chaos of a product operator with that of its factor operators from the viewpoint of principal measure.

3.1. Distributional chaos for product operators. Let

$$(X_i, (\|\cdot\|_{i,k})_{k\in\mathbb{N}}, \rho_i) \quad (i=1,2)$$

be infinite-dimensional separable complex Fréchet spaces. Then their Cartesian product $X = X_1 \times X_2$ is also a Fréchet space with the topology induced by the family of seminorms $\{p_k\}_{k\in\mathbb{N}}$ where

$$p_k(x,y) = \|x\|_{1,k} + \|y\|_{2,k}, \quad \forall k \in \mathbb{N}. \tag{3.1}$$

Define the metric ρ on X by

$$\rho((x_1,x_2),(y_1,y_2)) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_k(x_1-y_1, x_2-y_2)}{1+p_k(x_1-y_1,x_2-y_2)} \tag{3.2}$$

for all $(x_1,x_2),(y_1,y_2) \in X$; then ρ is translation-invariant and

$$\rho_i(x_i,y_i) \le \rho((x_1,x_2),(y_1,y_2)) \le \rho_1(x_1,y_1) + \rho_2(x_2,y_2). \tag{3.3}$$

Suppose $T_i : X_i \to X_i$ $(i=1,2)$ are continuous linear operators. The product operator T of T_1 and T_2 on X is defined by

$$T(x_1,x_2) = (T_1x_1, T_2x_2), \quad \forall (x_1,x_2) \in X. \tag{3.4}$$

It is easy to check that T is a continuous linear operator on X. In this case, each T_i is called a *factor operator*.

THEOREM 3.1. *Let $T_i : X_i \to X_i$ be continuous linear operators on Fréchet spaces X_i $(i=1,2)$. Assume that T_1 is distributionally chaotic. Then the product operator T of T_1 and T_2 is distributionally chaotic. Moreover, $\mu_{\mathrm{p}}(T) \ge \mu_{\mathrm{p}}(T_1)$.*

Proof. Assume that $(x,y) \in X_1 \times X_1$ is a distributionally chaotic pair of T_1. Then $((x,0),(y,0)) \in X \times X$ is also a distributionally chaotic pair of T. So T exhibits distributional chaos.

Let $F^*_{x,y}(s)$, $F_{x,y}(s)$ $[\widehat{F}^*_{u,v}(s), \widehat{F}_{u,v}(s)]$ be the upper and lower distributional functions of (x,y) $[(u,v)]$ with respect to T_1 $[T]$. For any $\epsilon > 0$, there exists a pair $(x,y) \in X_1 \times X_1$ such that

$$\int_0^1 (F^*_{x,y}(s) - F_{x,y}(s))\, ds > \mu_{\mathrm{p}}(T_1) - \epsilon.$$

Setting $\bar{u} = (x,0) \in X$ and $\bar{v} = (y,0) \in X$, we obtain

$$\int_0^1 (\widehat{F}^*_{\bar{u},\bar{v}}(s) - \widehat{F}_{\bar{u},\bar{v}}(s))\, ds = \int_0^1 (F^*_{x,y}(s) - F_{x,y}(s))\, ds > \mu_{\mathrm{p}}(T_1) - \epsilon,$$

which implies $\mu_{\mathrm{p}}(T) \ge \mu_{\mathrm{p}}(T_1)$. ∎

REMARK 3.2. It is not hard to show that Theorem 3.1 holds for the product of any finitely many operators. Namely, for any fixed positive integer $n \ge 2$, let T_i be a continuous linear operator on a Fréchet space X_i

$(i = 1, \dots, n)$ and T be the product of T_i $(i = 1, \dots, n)$. If T_j exhibits distributional chaos for some $j \in \{1, \dots, n\}$, then T is also distributionally chaotic with $\mu_{\mathrm{p}}(T) \geq \mu_{\mathrm{p}}(T_j)$.

It is worth mentioning that the conclusion of Theorem 3.1 does not hold for the property of generic distributional chaos (or dense distributional chaos), as shown by the following example.

Example 3.3. Let T_1 be a generically (or densely) distributionally chaotic operator on a Fréchet space X_1. Denote $T = T_1 \times aI$ on $X_1 \times X_1$, where a is a non-zero constant and I is the identity operator on X_1. When $|a| \geq 1$, it is easy to see that T is distributionally chaotic but not generically (or densely) distributionally chaotic. Indeed, any distributionally chaotic pair of T should have the form $((x_1, y), (x_2, y))$ for some $x_1, x_2, y \in X_1$. Following this line, it can be shown that even the product of a generically distributionally chaotic operator and a distributionally chaotic operator cannot be generically distributionally chaotic.

Conversely, if the product operator T is distributionally chaotic, then how about the factor operators? The following is a partial answer in the Banach space case.

Theorem 3.4. *Let T_i be continuous linear operators on Banach spaces $(X_i, \|\cdot\|_i)$ $(i = 1, 2)$. Suppose that for each $i \in \{1, 2\}$, there exists a dense subset $Y_i \subset X_i$ such that*

$$T_i^k x \to 0 \quad (\text{as } k \to \infty), \ \forall x \in Y_i.$$

Then the following assertions are equivalent:

(i) *The product operator T of T_1 and T_2 exhibits distributional chaos.*
(ii) *There exists $j \in \{1, 2\}$ such that T_j exhibits generic distributional chaos.*

Proof. (i)$\Rightarrow$(ii). Assume that the product operator T is distributionally chaotic on $X = X_1 \times X_2$. Then T admits a distributionally irregular vector $x = (x_1, x_2) \in X$. So there exists $A \subset \mathbb{N}$ with $\overline{\mathrm{dens}}(A) = 1$ such that

$$\lim_{k \in A} \|T^k x\| = \lim_{k \in A} (\|T_1^k x_1\|_1 + \|T_2^k x_2\|_2) = \infty.$$

Denote

$$A_1 = \{k \in A : \|T_1^k x_1\|_1 > \|T_2^k x_2\|_2\}, \qquad A_2 = A \setminus A_1.$$

Then

$$\overline{\mathrm{dens}}(A_1) + \overline{\mathrm{dens}}(A_2) \geq \overline{\mathrm{dens}}(A) = 1,$$

which implies that there exists $j \in \{1, 2\}$ satisfying $\overline{\mathrm{dens}}(A_j) > 0$ and

$$\lim_{k \in A_j} 2\|T_j^k x_j\|_j \geq \lim_{k \in A_j} \|T^k x\| = \infty.$$

According to [9, Proposition 8], there is a residual subset of X_j whose elements have distributionally unbounded orbit under T_j. Then the hypothesis of this theorem implies that T_j is generically distributionally chaotic.

(ii)$\Rightarrow$(i). This is obvious. ∎

Unfortunately, it is still difficult to prove the converse of Theorem 3.1. On the other hand, for the property of distributional chaos of type 2 (DC-2, a property weaker than distributional chaos), a sufficient and necessary condition for the product operator to be DC-2 can be obtained. For this purpose, we first recall some notions related to DC-2 (see [28]).

Let T be an operator on a Fréchet space X. For any $x, y \in X$ and $n \in \mathbb{N}$, denote

$$F^n_{x,y}(\varepsilon) = \frac{1}{n}\operatorname{card}(\{0 \le i \le n-1 : \rho(T^i x, T^i y) < \varepsilon\}), \qquad \forall \varepsilon > 0.$$

The upper and lower density functions of x, y with respect to T are then defined as

$$F^*_{x,y}(\varepsilon) = \limsup_{n\to\infty} F^n_{x,y}(\varepsilon), \qquad \forall \varepsilon > 0,$$
$$F_{x,y}(\varepsilon) = \liminf_{n\to\infty} F^n_{x,y}(\varepsilon), \qquad \forall \varepsilon > 0,$$

respectively. A pair $(x, y) \in X \times X$ is called *distributionally chaotic of type 2* for T if

$$F^*_{x,y}(\varepsilon) \equiv 1, \quad \text{and} \quad F_{x,y}(\varepsilon) < F^*_{x,y}(\varepsilon) \quad \text{for all } \varepsilon \text{ in an interval.} \tag{3.5}$$

Note that (3.5) is equivalent to

$$F^*_{x,y}(\varepsilon) \equiv 1, \quad \text{and} \quad F_{x,y}(t_0) < 1 \quad \text{for some } t_0 > 0.$$

An operator T on X is said to be *distributionally chaotic of type 2* (DC-2, for short) if there is an uncountable set $\Gamma \subset X$ such that for any distinct $x, y \in \Gamma$, (x, y) is a distributionally chaotic pair of type 2. In this case, Γ is called a *distributionally scrambled set of type 2* for T.

Clearly, T is DC-2 if and only if T admits a distributionally chaotic pair of type 2, because if (x, y) is a distributionally chaotic pair of type 2, then $\{\lambda(x - y) : \lambda \in \mathbb{R}\}$ is obviously an uncountable distributionally scrambled set of type 2 for T.

PROPOSITION 3.5. *Let $T_i : X_i \to X_i$ be continuous linear operators on Fréchet spaces X_i $(i = 1, 2)$. Then the product operator $T = T_1 \times T_2$ on $X = X_1 \times X_2$ is DC-2 if and only if there exists $j \in \{1, 2\}$ such that T_j is DC-2.*

Proof. Sufficiency. This is similar to the case of distributional chaos (see Theorem 3.1).

Necessity. Suppose that $((x_1,x_2),(y_1,y_2)) \in X\times X$ is a distributionally chaotic pair of type 2 for T. Then

$$\overline{\mathrm{dens}}(\{n\in\mathbb{N} : \rho(T^n(x_1,x_2),T^n(y_1,y_2))<\varepsilon\})=1, \quad \forall\varepsilon>0, \tag{3.6}$$

$$\overline{\mathrm{dens}}(\{n\in\mathbb{N} : \rho(T^n(x_1,x_2),T^n(y_1,y_2))\ge\varepsilon_0\})>0 \tag{3.7}$$

for some $\varepsilon_0>0$.

If $\rho(T^n(x_1,x_2),T^n(y_1,y_2))<\varepsilon$ for some $n\in\mathbb{N}$, then

$$\rho_i(T_i^n x_i, T_1^n y_i)<\varepsilon, \quad i=1,2.$$

So it follows from (3.6) that for every $i\in\{1,2\}$,

$$\overline{\mathrm{dens}}(\{n\in\mathbb{N} : \rho(T_i^n x_i,T_i^n y_i))<\varepsilon\})=1, \quad \forall\varepsilon>0. \tag{3.8}$$

On the other hand, if $\rho(T^m(x_1,x_2),T^m(y_1,y_2))\ge\varepsilon_0$ for some $m\in\mathbb{N}$, then either

$$\rho_1(T_1^m x_1,T_1^m y_1)\ge\varepsilon_0/2$$

or

$$\rho_2(T_2^m x_2,T_2^m y_2)\ge\varepsilon_0/2,$$

which implies

$$\{n\in\mathbb{N} : \rho(T^n(x_1,x_2),T^n(y_1,y_2))\ge\varepsilon_0\} \subset \bigcup_i \{n\in\mathbb{N} : \rho(T_i^n x_i,T_i^n y_i))\ge\varepsilon_0/2\}.$$

Therefore it follows from (3.7) that there exists $j\in\{1,2\}$ such that

$$\overline{\mathrm{dens}}(\{n\in\mathbb{N} : \rho(T_j^n x_j,T_j^n y_j))\ge\varepsilon_0/2\})>0. \tag{3.9}$$

Combining (3.8) and (3.9) shows that (x_j,y_j) is a distributionally chaotic pair of type 2 for T_j. Hence T_j is DC-2. ∎

Next, we extend the above results for product operators to product C_0-semigroups. Let $\mathcal{T}_i=\{T_{t,i}\}_{t\ge0}$ be a C_0-semigroup of operators on a Fréchet space X_i $(i=1,2)$. Let X be the Cartesian product of X_1 and X_2. The product semigroup $\mathcal{T}=\{T_t\}_{t\ge0}$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ is defined by

$$\begin{aligned} &T_t : X\to X, \quad \forall t\ge0,\\ &T_t(x_1,x_2):=(T_{t,1}x_1,T_{t,2}x_2), \quad \forall(x_1,x_2)\in X. \end{aligned} \tag{3.10}$$

It is known that $\mathcal{T}=\{T_t\}_{t\ge0}$ is a C_0-semigroup of operators on X.

Theorem 3.6. *Let $\mathcal{T}_i=\{T_{t,i}\}_{t\ge0}$ be C_0-semigroups of operators on Fréchet spaces X_i $(i=1,2)$. Let $\mathcal{T}=\{T_t\}_{t\ge0}$ be the product semigroup of $\mathcal{T}_1$ and $\mathcal{T}_2$. Suppose that $\mathcal{T}_1$ is distributionally chaotic. Then so is $\mathcal{T}$. Moreover, $\mu_{\mathrm{p}}(\mathcal{T})\ge\mu_{\mathrm{p}}(\mathcal{T}_1)$.*

Proof. This proof is similar to that of Theorem 3.1, and we omit it. ∎

Based on the study of product operators and C_0-semigroups, one can easily construct C_0-semigroups (or operators) on certain Fréchet spaces that are distributionally chaotic but not Devaney chaotic.

COROLLARY 3.7. *Let X_1, X_2 be Fréchet spaces. Assume $\mathcal{T}_1 = \{T_{t,1}\}_{t\geq 0}$ is a distributionally chaotic C_0-semigroup of operators on X_1, and let I be the identity C_0-semigroup on X_2. Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be the product C_0-semigroup of $\mathcal{T}_1$ and I. Then*

(i) *$\mathcal{T}$ is distributionally chaotic;*
(ii) *$\mathcal{T}$ is not Denavey chaotic.*

THEOREM 3.8. *Let $\mathcal{T}_i = \{T_{i,t}\}_{t\geq 0}$ be C_0-semigroups of operators on Banach spaces X_i $(i = 1, 2)$. Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be the product semigroup of $\mathcal{T}_1$ and $\mathcal{T}_2$. Suppose that for each $j \in \{1, 2\}$, there exists a dense subset $X_{j,0} \subset X_j$ such that $\lim_{t\to\infty} T_{j,t}x = 0$ for all $x \in X_{j,0}$. Then $\mathcal{T}$ is distributionally chaotic if and only if $\mathcal{T}_l$ is generically distributionally chaotic for some $l \in \{1, 2\}$.*

Proof. According to [1, Theorem 3.1], $\mathcal{T}$ is distributionally chaotic if and only if T_{t_0} exhibits distributional chaos for some $t_0 > 0$. The hypothesis of the theorem and Theorem 3.4 imply that T_{t_0} is distributionally chaotic if and only if T_{l,t_0} is generically distributionally chaotic for some $l \in \{1, 2\}$, which is further equivalent to $\mathcal{T}_l$ being generically distributionally chaotic for some $l \in \{1, 2\}$. ∎

As an application of Theorem 3.8, one can consider the complexification of a C_0-semigroup of operators on some real Banach space. Let $\mathcal{T}_1$ be a C_0-semigroup of operators on a real Banach space X_1. Then its complexification $\widetilde{\mathcal{T}_1}$ can be considered as the product semigroup $\mathcal{T}_1 \times \mathcal{T}_1$ on $X_1 \times X_1$. If $\mathcal{T}_1$ satisfies the hypothesis of Theorem 3.8, then $\mathcal{T}_1$ is distributionally chaotic if and only if its complexification $\widetilde{\mathcal{T}_1}$ is.

3.2. Further exploration of principal measure. In this subsection, we further study the principal measures of distributionally chaotic operators and C_0-semigroups.

In Theorem 2.9, we proved that the C_0-semigroup $\mathcal{S}_h$ has principal measure 1 by showing that it admits a distributionally 1-irregular vector. In terms of distributionally irregular vectors, the following proposition provides a more general result.

PROPOSITION 3.9. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup of operators on a Fréchet space $(X, (\|\cdot\|_k)_{k\in\mathbb{N}}, \rho)$. Assume that $\mathcal{T}$ admits a distributionally i-irregular vector x. Then $\mu_{\mathrm{p}}(\mathcal{T}) \geq 2^{1-i}$.*

Proof. From the definition of distributionally irregular vector, there exist $A, B \subset \mathbb{R}^+$ with $\overline{\mathrm{Dens}}(A) = \overline{\mathrm{Dens}}(B) = 1$ such that

$$\lim_{t\to\infty,\, t\in A} T_t x = 0, \qquad \lim_{t\to\infty,\, t\in B} \|T_t x\|_i = \infty. \tag{3.11}$$

Given $s \in (0,1)$, one can find a positive number N_1 such that $\rho(T_t x, 0) < s$ for all $t \in A \setminus (A \cap [0, N_1])$, so

$$\{t > 0 : \rho(T_t x, 0) < s\} \supset \big\{t > 0 : t \in A \setminus (A \cap [0, N_1])\big\},$$

which further implies that $\overline{\mathrm{Dens}}(\{t > 0 : \rho(T_t x, 0) < s\}) = 1$. Therefore $\Phi^*_{x,0}(s) \equiv 1$ for all $s \in (0,1)$.

On the other hand, we show that $\Phi_{x,0}(s) = 0$ for every $s \in (0, 2^{1-i})$. Indeed, given $s_0 \in (0, 2^{1-i})$, there is $M > 0$ such that

$$\frac{M}{1+M} 2^{i-1} > s_0.$$

From (3.11), there exists $N_2 > 0$ such that for each $t \in B \setminus (B \cap [0, N_2])$, we have $\|T_t x\|_i > M$, so

$$\begin{aligned}\rho(T_t x, 0) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|T_t x\|_k}{1 + \|T_t x\|_k} \geq \sum_{k=i}^{\infty} \frac{1}{2^k} \frac{\|T_t x\|_k}{1 + \|T_t x\|_k} \\ &\geq \frac{\|T_t x\|_i}{1 + \|T_t x\|_i} 2^{i-1} > \frac{M}{1+M} 2^{i-1} > s_0.\end{aligned}$$

Therefore $\underline{\mathrm{Dens}}(\{t > 0 : \rho(T_t x, 0) < s_0\}) = 0$, which means that $\Phi_{x,0}(s_0) = 0$. According to the definition of principal measure, we conclude that

$$\mu_{\mathrm{p}}(\mathcal{T}) = \sup_{y\in X} \int_0^1 (\Phi^*_{y,0}(s) - \Phi_{y,0}(s))\, ds \geq \int_0^{2^{1-i}} (\Phi^*_{x,0}(s) - \Phi_{x,0}(s))\, ds = 2^{1-i}. \quad \blacksquare$$

When the Fréchet space $(X, (\|\cdot\|_k)_{k\in\mathbb{N}}, \rho)$ under consideration is just a Banach space $(X, \|\cdot\|)$, that is, $\|\cdot\|_k = \|\cdot\|$ for every $k \in \mathbb{N}$ and $\rho(x,y) = \|x-y\|/(1+\|x-y\|)$, every distributionally chaotic C_0-semigroup of operators has principal measure 1.

Remark 3.10. An analogue of Proposition 3.9 for linear operators on Fréchet spaces can be obtained. In particular, every distributionally chaotic linear operator on a Banach space (as a special Fréchet space) has principal measure 1.

Note that in the literature one can find examples of operators on Fréchet spaces with principal measure 1, for instance the annihilation operator of the unforced quantum oscillator [22, 30] and the weighted shift on Köthe sequence spaces [27]. We may wonder whether the principal measure of a distributionally chaotic operator (or C_0-semigroup) on a Fréchet space could be less than one. We obtain the following result.

THEOREM 3.11. *For any $\varepsilon > 0$, there exists a continuous linear operator T acting on a Fréchet space such that T is distributionally chaotic and $\mu_{\mathrm{p}}(T) < \varepsilon$.*

Proof. Let $r \in \mathbb{N}$. Define an operator $T : Z \to Z$ by

$$(Tf)(x) = \begin{cases} 0, & x \in [0, r], \\ f(r+2) \cdot (x - r), & x \in [r, r+1], \\ f(x+1), & x \in [r+1, \infty), \end{cases} \tag{3.12}$$

for $f \in Z = C([0,\infty))$. It is easy to see that T is a continuous linear operator. Moreover, for each $n \in \mathbb{N}$,

$$(T^n f)(x) = \begin{cases} 0, & x \in [0, r], \\ f(r+n+1) \cdot (x - r), & x \in [r, r+1], \\ f(x+n), & x \in [r+1, \infty). \end{cases}$$

We show that T is distributionally chaotic. Recall the function $\widetilde{f} \in Z$ defined in (2.17) and set $M_n = n!$ for each $n \in \mathbb{N}$. Denote

$$\begin{aligned} \overline{B_1} &= \bigcup_{k\geq 1} \left\{ n \in \mathbb{N} : M_{2k} - r \leq n \leq \frac{M_{2k} + 1 + M_{2k+1}}{2} \right\}, \\ \overline{B_2} &= \bigcup_{k\geq 1} \left\{ n \in \mathbb{N} : M_{2k+1} - r \leq n \leq M_{2k+2} - 1 - r \right\}. \end{aligned} \tag{3.13}$$

Then $\overline{\mathrm{dens}}(\overline{B_1}) = \overline{\mathrm{dens}}(\overline{B_2}) = 1$. Furthermore,

$$\lim_{n\to\infty,\, n\in\overline{B_1}} T^n \widetilde{f} = 0 \quad \text{and} \quad \lim_{n\to\infty,\, n\in\overline{B_2}} p_{r+1}(T^n \widetilde{f}) = \infty,$$

so $\widetilde{f}$ is a distributionally $(r+1)$-irregular vector of T. According to [9, Theorem 12], T is distributionally chaotic. Moreover, it follows from Proposition 3.9 that $\mu_{\mathrm{p}}(T) \geq 1/2^r$.

On the other hand, for any $f, g \in Z$ and each $n \in \mathbb{N}$,

$$\begin{aligned} \rho_1(T^n f, T^n g) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(T^n f - T^n g)}{1 + p_k(T^n f - T^n g)} \\ &= \sum_{k=1+r}^{\infty} \frac{1}{2^k} \frac{p_k(T^n f - T^n g)}{1 + p_k(T^n f - T^n g)} \\ &< \sum_{k=1+r}^{\infty} \frac{1}{2^k} = \frac{1}{2^r}. \end{aligned}$$

Consequently,

$$F^*_{f,g}(s) = F_{f,g}(s) = 1, \quad \forall s > 1/2^r,$$

which implies

$$\mu_{\mathrm{p}}(T) = \sup_{f\in X_1} \int_0^1 (F^*_{f,0}(s) - F_{f,0}(s))\, ds \le 1/2^r.$$

Hence we conclude that $\mu_{\mathrm{p}}(T) = 1/2^r$.

For any $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that $1/2^{r_0} < \varepsilon$. Thus one can find a distributionally chaotic operator T on Z with $\mu_{\mathrm{p}}(T) < \varepsilon$ by choosing $r > r_0$ in (3.12). ∎

Remark 3.12. We note that the operator T defined in (3.12) is not hypercyclic, and hence not Devaney chaotic either.

For the operator T defined in (3.12), it follows from Theorem 3.1 and the proof of Theorem 3.11 that $\mu_{\mathrm{p}}(T\times T) = \mu_{\mathrm{p}}(T)$. However, it is still unknown whether there exists a distributionally chaotic operator T' with $\mu_{\mathrm{p}}(T') < 1$ but $\mu_{\mathrm{p}}(T'\times T') = 1$.

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