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On weak-star convergence in product Hardy spaces on spaces of homogeneous type

by

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Abstract. A classical theorem of Jones and Journé on weak-star convergence in the Hardy space H^1 was generalized to the multiparameter setting by Pipher and Treil. We prove the analogous result when the underlying space is a product space of homogeneous type. The main tools we use are from recent work by Chen, Li and Ward (2013) and by Han, Li and Ward (2014).

1. Introduction. In this paper we extend to the setting of product Hardy spaces H^1 on spaces of homogeneous type the result that almost-everywhere convergence of a sequence of uniformly bounded H^1 functions implies weak-star convergence. See [PT] for the history of this result and its connections with commutators, singular integral operators, Riesz transforms, BMO, div-curl lemmas, and the theory of compensated compactness in partial differential equations.

Our main result is the following.

THEOREM 1.1. *Suppose that a sequence* $\{f_k\} \subset H^1(X_1 \times \cdots \times X_n)$ *satisfies* $\|f_k\|_{H^1} \le 1$ *for all* k *and* $f_k(x) \to f(x)$ *for* μ*-almost every* $x \in X_1 \times \cdots \times X_n$. *Then* $f \in H^1(X_1 \times \cdots \times X_n)$, $\|f\|_{H^1} \le 1$, *and for all* $\phi \in \mathrm{VMO}(X_1 \times \cdots \times X_n)$,

$$\int_{X_1\times\cdots\times X_n} f_k(x)\phi(x)\,d\mu(x) \to \int_{X_1\times\cdots\times X_n} f(x)\phi(x)\,d\mu(x). \tag{1.1}$$

To extend the Calderón–Zygmund singular integral operator theory to a more general setting, in the early 1970s Coifman and Weiss introduced spaces of homogeneous type. As Meyer remarked in his preface to [DH],

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> One is amazed by the dramatic changes that occurred in analysis during the twentieth century.... After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis.

We say that (X, d, μ) is a *space of homogeneous type in the sense of Coifman and Weiss* if d is a quasi-metric on X and μ is a nonzero measure satisfying the doubling condition. To be more precise, let us begin by recalling these spaces. A *quasi-metric* d on a set X is a function $d : X \times X \to [0, \infty)$ satisfying

(1) $d(x, y) = d(y, x) \geq 0$ for all x, $y \in X$,
(2) $d(x, y) = 0$ if and only if $x = y$, and
(3) the quasi-triangle inequality: there exists a constant $A_0 \in [1, \infty)$ such that for all x, y and $z \in X$,

$$d(x, y) \leq A_0[d(x, z) + d(z, y)]. \tag{1.2}$$

We define the quasi-metric ball by $B(x, r) := \{y \in X : d(x, y) < r\}$ for $x \in X$ and $r > 0$. Note that the quasi-metric, in contrast to a metric, may not be Hölder regular, and quasi-metric balls may not be open. In this paper, we assume that

(4) given a neighborhood N of a point x there is an $\epsilon > 0$ such that the sphere $\{y \in X : d(x, y) \leq \epsilon\}$ with center at x is contained in N, and
(5) the sphere $\{y \in X : d(x, y) \leq r\}$ is measurable, and the measure $\mu(\{y \in X : d(x, y) \leq r\})$ is a continuous function of r for each x.

We say that a nonzero measure μ satisfies the *doubling condition* if there is a constant C_μ such that for all $x \in X$ and all $r > 0$,

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty. \tag{1.3}$$

As noted by the reviewer of [PT] in Mathematical Reviews, since H^1 is not reflexive, the fact that H^1 is the dual of VMO does not lead to a functional-analytic proof of Theorem 1.1 using known methods.

The paper is organized as follows. In Section 2 we present some background about spaces of homogeneous type. In Section 3 we prove the one-parameter version of our result, and in Section 4 we prove the product version.

2. Preliminaries. We recall the ingredients and tools that we will use below to prove Theorem 1.1, namely systems of dyadic cubes, the orthonormal basis and wavelet expansion of Auscher and Hytönen [AH], the spaces of test functions and of distributions, the definitions from [HLW] (using these spaces) of H^1, BMO and VMO on product spaces of homogeneous type, and the duality relations between them. See [HLW] for a full account of this material.

2.1. Systems of dyadic cubes in a doubling quasi-metric space. Let X be a set equipped with a quasi-metric d and a doubling measure μ; in particular, (X, d, μ) is a space of homogeneous type. As shown in [HK], building on [Chr], there exists a dyadic decomposition for X: There exist positive absolute constants c_1, C_1 and $0 < \delta < 1$ such that we can construct a set $\{x_\alpha^k\}_{k,\alpha}$ of points and families $\{Q_\alpha^k\}_{k,\alpha}$ of sets in X satisfying the following properties:

$$\text{if } \ell \le k, \text{ then either } Q_\alpha^k \subset Q_\beta^\ell \text{ or } Q_\alpha^k \cap Q_\beta^\ell = \emptyset; \tag{2.1}$$

$$Q_\alpha^k \cap Q_\beta^k = \emptyset \text{ for every } k \in \mathbb{Z} \text{ and } \alpha \neq \beta; \tag{2.2}$$

$$X = \bigcup_\alpha Q_\alpha^k \text{ for every } k \in \mathbb{Z}; \tag{2.3}$$

$$B(x_\alpha^k, c_1\delta^k) \subset Q_\alpha^k \subset B(x_\alpha^k, C_1\delta^k); \tag{2.4}$$

$$\text{if } \ell \le k \text{ and } Q_\alpha^k \subset Q_\beta^\ell, \text{ then } B(x_\alpha^k, C_1\delta^k) \subset B(x_\beta^\ell, C_1\delta^\ell). \tag{2.5}$$

Here for each $k \in \mathbb{Z}$, α runs over an appropriate index set. We call the set Q_α^k a *dyadic cube* and x_α^k the *center* of the cube. Also, k is called the *level* of this cube. We denote the collection of dyadic cubes at level k by $\mathcal{D}^k$, and the collection of all dyadic cubes by $\mathcal{D}$. When $Q_\alpha^k \subset Q_\beta^{k-1}$, we say Q_α^k is a *child* of Q_β^{k-1} and Q_β^{k-1} is the *parent* of Q_α^k. Because X is a space of homogeneous type, there is a uniform constant $\mathcal{N}$ such that each cube $Q \in \mathcal{D}$ has at most $\mathcal{N}$ children.

2.2. Orthonormal basis and wavelet expansion. We recall the orthonormal basis and wavelet expansion of $L^2(X)$ due to Auscher and Hytönen [AH]. To state their result, we first recall the set of *reference dyadic points* x_α^k. Let δ be a fixed small positive parameter (for example, as pointed out in [AH, Section 2.2], it suffices to take $\delta \le 10^{-3}A_0^{-10}$). For $k = 0$, let $\mathscr{X}^0 := \{x_\alpha^0\}_\alpha$ be a maximal collection of 1-separated points in X. Inductively, for $k \in \mathbb{Z}_+$, let $\mathscr{X}^k := \{x_\alpha^k\} \supseteq \mathscr{X}^{k-1}$ and $\mathscr{X}^{-k} := \{x_\alpha^{-k}\} \subseteq \mathscr{X}^{-(k-1)}$ be δ^k- and δ^{-k}-separated collections in $\mathscr{X}^{k-1}$ and $\mathscr{X}^{-(k-1)}$, respectively.

Lemma 2.1 in [AH] shows that, for all $k \in \mathbb{Z}$ and $x \in X$, the reference dyadic points satisfy

$$d(x_\alpha^k, x_\beta^k) \ge \delta^k \ (\alpha \neq \beta), \qquad d(x, \mathscr{X}^k) = \min_\alpha d(x, x_\alpha^k) < 2A_0\delta^k. \tag{2.6}$$

Now let $c_0 := 1$, $C_0 := 2A_0$ and $\delta \le 10^{-3}A_0^{-10}$. Then there exists a set $\{Q_\alpha^k\}_{k\in\mathbb{Z},\alpha\in\mathscr{X}^k}$ of half-open dyadic cubes associated with the reference dyadic points $\{x_\alpha^k\}_{k\in\mathbb{Z},\,\alpha\in\mathscr{X}^k}$. We consider x_α^k as the *center* of Q_α^k. We also identify with $\mathscr{X}^k$ the set of indices α corresponding to $x_\alpha^k \in \mathscr{X}^k$.

Note that $\mathscr{X}^k \subseteq \mathscr{X}^{k+1}$ for $k \in \mathbb{Z}$, so that every x_α^k is also a point of the form x_β^{k+1}, and thus of all the finer levels. We denote $\mathscr{Y}^k := \mathscr{X}^{k+1} \backslash \mathscr{X}^k$, and relabel the points $\{x_\alpha^k\}_\alpha$ that belong to $\mathscr{Y}^k$ as $\{y_\alpha^k\}_\alpha$.

THEOREM 2.1 ([AH, Theorem 7.1]). *Let (X, d, μ) be a space of homogeneous type with quasi-triangle constant A_0, and let $a := (1 + 2\log_2 A_0)^{-1}$. There exists an orthonormal basis ψ_α^k, $k \in \mathbb{Z}$, of $L^2(X)$, having exponential decay*

$$(2.7)\qquad |\psi_\alpha^k(x)| \le \frac{C}{\sqrt{\mu(B(y_\alpha^k, \delta^k))}} \exp\bigl(-\nu(\delta^{-k} d(y_\alpha^k, x))^a\bigr),$$

Hölder regularity

$$(2.8)\qquad |\psi_\alpha^k(x) - \psi_\alpha^k(y)| \le \frac{C}{\sqrt{\mu(B(y_\alpha^k, \delta^k))}} \left(\frac{d(x,y)}{\delta^k}\right)^\eta \exp\bigl(-\nu(\delta^{-k} d(y_\alpha^k, x))^a\bigr)$$

for some $\eta \in (0,1]$ and for $d(x,y) \le \delta^k$, and the cancellation property

$$(2.9)\qquad \int_X \psi_\alpha^k(x)\, d\mu(x) = 0, \qquad k \in \mathbb{Z},\ y_\alpha^k \in \mathscr{Y}^k.$$

Moreover,

$$(2.10)\qquad f(x) = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathscr{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$$

in the sense of $L^2(X)$.

Here δ is a fixed small parameter, say $\delta \le \frac{1}{1000} A_0^{-10}$, and $\nu > 0$ and $C < \infty$ are constants independent of k, α, x and y_α^k. In what follows, we also refer to the functions ψ_α^k as wavelets.

2.3. Spaces of test functions and distributions. We refer the reader to [HLW, Definitions 3.9 and 3.10 and surrounding discussion] for the definitions of the space $\mathring{G}$ of product test functions and its dual space $(\mathring{G})'$ of product distributions on the product space $X_1 \times X_2$. In [HLW], $\mathring{G}$ is denoted by $\mathring{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$, and $(\mathring{G})'$ is denoted by $\mathring{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)'$, where the β_i and γ_i are parameters that quantify the size and smoothness of the test functions, and $\beta_i \in (0, \eta_i)$ where η_i is the regularity exponent from Theorem 2.1. (In fact, in [HLW] the theory is developed for $\beta_i \in (0, \eta_i]$, but for simplicity here we only use $\beta_i \in (0, \eta_i)$ since that is all we need.) We note that the one-parameter scaled Auscher–Hytönen wavelets $\psi_\alpha^k(x)/\sqrt{\mu(B(y_\alpha^k, \delta^k))}$ are test functions, and that their tensor products $\psi_{\alpha_1}^{k_1}(x)\psi_{\alpha_2}^{k_2}(y)(\mu_1(B(y_{\alpha_1}^{k_1}, \delta_1^{k_1}))\mu_2(B(y_{\alpha_2}^{k_2}, \delta_2^{k_2})))^{-1/2}$ are product test functions in $\mathring{G}$ for all $\beta_i \in (0, \eta_i]$ and all $\gamma_i > 0$, for $i = 1, 2$. These facts follow from the theory in [HLW], specifically Definition 3.1 and the discussion after it, Theorem 3.3, and Definitions 3.9 and 3.10 and the discussion between them.

We have the following version of the reproducing formula in the product setting $X_1 \times X_2$.

THEOREM 2.2 ([HLW]). *The reproducing formula*

$$f(x_1,x_2)=\sum_{k_1}\sum_{\alpha_1\in\mathscr{Y}^{k_1}}\sum_{k_2}\sum_{\alpha_2\in\mathscr{Y}^{k_2}}\langle f,\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle\psi_{\alpha_1}^{k_1}(x_1)\psi_{\alpha_2}^{k_2}(x_2) \tag{2.11}$$

holds in both $\mathring{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ *and* $(\mathring{G}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ *with* $0<\beta_i<\eta_i$ *and* $\gamma_i<\eta_i$ *for* $i=1,2$.

We recall from [HLW] the definitions of the Hardy space $H^1(X_1\times X_2)$, the bounded mean oscillation space $\mathrm{BMO}(X_1\times X_2)$, and the vanishing mean oscillation space $\mathrm{VMO}(X_1\times X_2)$.

DEFINITION 2.3 ([HLW]). The *product Hardy space* H^1 is defined by

$$H^1(X_1\times X_2):=\{f\in(\mathring{G})':S(f)\in L^1(X_1\times X_2)\},$$

where $S(f)$ is the *product Littlewood–Paley square function* defined as

$$S(f)(x_1,x_2):=\Big\{\sum_{k_1}\sum_{\alpha_1\in\mathscr{Y}^{k_1}}\sum_{k_2}\sum_{\alpha_2\in\mathscr{Y}^{k_2}}|\langle\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2},f\rangle\widetilde{\chi}_{Q_{\alpha_1}^{k_1}}(x_1)\widetilde{\chi}_{Q_{\alpha_2}^{k_2}}(x_2)|^2\Big\}^{1/2},$$

where $\widetilde{\chi}_{Q_{\alpha_i}^{k_i}}(x_i):=\chi_{Q_{\alpha_i}^{k_i}}(x_i)\mu_i(Q_{\alpha_i}^{k_i})^{-1/2}$ and $\chi_{Q_{\alpha_i}^{k_i}}(x_i)$ is the indicator function of the dyadic cube $Q_{\alpha_i}^{k_i}$ for $i=1,2$.

For $f\in H^1(X_1\times X_2)$, we define $\|f\|_{H^1(X_1\times X_2)}:=\|S(f)\|_{L^1(X_1\times X_2)}$.

DEFINITION 2.4 ([HLW]). We define the *product BMO space* as

$$\mathrm{BMO}(X_1\times X_2):=\{f\in(\mathring{G})':\mathcal{C}_1(f)<L^\infty\},$$

with

$$\mathcal{C}_1(f):=\sup_{\Omega}\bigg\{\frac{1}{\mu(\Omega)}\sum_{k_1,k_2\in\mathbb{Z},\alpha_1\in\mathscr{Y}^{k_1},\alpha_2\in\mathscr{Y}^{k_2},R=Q_{\alpha_1}^{k_1}\times Q_{\alpha_2}^{k_2}\subset\Omega}|\langle\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2},f\rangle|^2\bigg\}^{1/2},$$

where Ω runs over all open sets in $X_1\times X_2$ with finite measures.

DEFINITION 2.5 ([HLW]). Let

$$E(\Omega):=\bigg\{\frac{1}{\mu(\Omega)}\sum_{k_1,k_2\in\mathbb{Z},\,\alpha_1\in\mathscr{Y}^{k_1},\,\alpha_2\in\mathscr{Y}^{k_2},\,Q_{\alpha_1}^{k_1}\times Q_{\alpha_2}^{k_2}\subset\Omega}|\langle\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2},f\rangle|^2\bigg\}^{1/2}$$

We define the *product vanishing mean oscillation space* $\mathrm{VMO}(X_1\times X_2)$ as the subspace of $\mathrm{BMO}(X_1\times X_2)$ consisting of those $f\in\mathrm{BMO}(X_1\times X_2)$ satisfying the three properties:

(a) $\displaystyle\lim_{\delta\to0}\sup_{\Omega:\,\mu(\Omega)<\delta}E(\Omega)=0$,

(b) $\displaystyle\lim_{N\to\infty}\sup_{\Omega:\,\mathrm{diam}(\Omega)>N}E(\Omega)=0$,

(c) $\lim_{N\to\infty} \sup_{\Omega:\,\Omega\subset(B(x_1,N)\times B(x_2,N))^c} E(\Omega) = 0$, where x_1 and x_2 are any fixed points in X_1 and X_2, respectively.

THEOREM 2.6 ([HLW]). *The following duality results hold:*

$$(H^1(X_1\times X_2))' = \mathrm{BMO}(X_1\times X_2), \qquad (\mathrm{VMO}(X_1\times X_2))' = H^1(X_1\times X_2).$$

3. Proof of Theorem 1.1 for one parameter. We note that just recently, a weak convergence result in the one-parameter setting was provided in [HKy]. However, for the convenience of proving our main result in the product setting, we also provide our proof here. To prove Theorem 1.1, paralleling the Euclidean one-parameter case, we will make use of several properties of the A_p classes on spaces of homogeneous type. These properties are collected in 3.1–3.5 below.

Let (X,d,μ) be a space of homogeneous type. A nonnegative locally integrable function $\omega : X\to\mathbb{R}$ is said to belong to $A_p(X)$, $1<p<\infty$, if

$$\sup_B \left(\frac{1}{\mu(B)}\int_B \omega(x)\,d\mu(x)\right)\left(\frac{1}{\mu(B)}\int_B \omega(x)^{-1/(p-1)}\,d\mu(x)\right)^{p-1} < \infty,$$

and to $A_1(X)$ if

$$\sup_B \left(\frac{1}{\mu(B)}\int_B \omega(x)\,d\mu(x)\right)\left(\operatorname*{ess\,sup}_{x\in B}\omega(x)^{-1}\right) < \infty.$$

LEMMA 3.1 ([C2, Lemma 4]). *Let $\omega\in A_p$, $1\le p<\infty$. There exists a constant $C>0$ such that, for any subset E of B,*

$$\left(\frac{\mu(E)}{\mu(B)}\right)^p \le C\,\frac{\int_E \omega(x)\,d\mu(x)}{\int_B \omega(x)\,d\mu(x)}.$$

The *centered Hardy–Littlewood maximal operator M* with respect to the measure μ is defined by

$$Mf(x) := \sup_{r>0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)|\,d\mu(y).$$

THEOREM 3.2 ([C2, Theorem 3]). *If $\omega\in A_1$, then M is of ω-weak type $(1,1)$ with respect to μ, that is, there exists a constant $C>0$ such that, for all $\lambda>0$ and all $f\in L^1_\omega(d\mu)$,*

$$\int_{\{x\in X\,:\,Mf(x)>\lambda\}} \omega(x)\,d\mu(x) \le \frac{C}{\lambda}\int_X |f(x)|\omega(x)\,d\mu(x).$$

Similarly, the *uncentered Hardy–Littlewood maximal operator M* with respect to the measure μ is defined by

$$\widetilde{M}f(x) := \sup_{B\ni x}\frac{1}{\mu(B)}\int_B |f(y)|\,d\mu(y).$$

LEMMA 3.3. *The weight ω is in A_1 if and only if there is a constant $C > 0$ such that*

$$M\omega(x) \le C\omega(x) \quad \textit{for } \mu\textit{-almost every } x \in X.$$

Proof. Suppose that there is a constant $C > 0$ such that $\widetilde{M}\omega(x) \le C\omega(x)$ μ-almost everywhere. Since $\widetilde{M}\omega(x)$ is equivalent to $M\omega(x)$, it is clear that

$$\frac{1}{\mu(B)} \int_B \omega(y)\, d\mu(y) \le C\omega(x) \quad \text{for } \mu\text{-almost every } x \in B.$$

Hence $\omega \in A_1$. Conversely, Theorem 3.2 shows that there exists $C > 0$ such that, for any $\lambda > 0$ and $f \in L^1_\omega$,

$$\int_{\{x\in X\,:\,\widetilde{M}f(x)>\lambda\}} \omega(x)\, d\mu(x) \le \frac{C}{\lambda} \int_X |f(x)|\omega(x)\, d\mu(x).$$

Suppose $x \in B_1 \subset B_2$. Let $f = \chi_{B_1}$ and $z \in B_2$. Then

$$\widetilde{M}f(z) \ge \frac{1}{\mu(B_2)} \int_{B_2} f(y)\, d\mu(y) = \frac{\mu(B_1)}{\mu(B_2)}.$$

The above inequality shows that $B_2 \subset \{x : \widetilde{M}f(x) \ge \mu(B_1)/\mu(B_2)\}$. Hence,

$$\int_{B_2} \omega(x)\, d\mu(x) \le \int_{\{x\,:\,\widetilde{M}f(x)\ge\mu(B_1)/\mu(B_2)\}} \omega(x)\, d\mu(x) \le C\frac{\mu(B_2)}{\mu(B_1)} \int_{B_1} \omega(x)\, d\mu(x).$$

By Lebesgue's differentiation theorem, the lemma follows. ■

We will need the following generalization to spaces of homogeneous type of one direction of a well-known result of Coifman and Rochberg [CR].

LEMMA 3.4. *Let $f \in L^1_{\rm loc}(X)$ be such that $Mf(x) < \infty$ μ-almost everywhere. Then $(Mf)^\delta \in A_1$ for $0 \le \delta < 1$.*

Proof. By Lemma 3.3, it suffices to show that there exists a constant C such that, for any B and μ-almost every $x \in B$,

$$\frac{1}{\mu(B)} \int_B (\widetilde{M}f)^\delta\, d\mu \le C(\widetilde{M}f(x))^\delta.$$

Fix $B = B(x_0, t_0)$ and decompose f as $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$ with $2B = B(x_0, 2t_0)$. Then $\widetilde{M}f(y) \le \widetilde{M}f_1(y) + \widetilde{M}f_2(y)$ and

$$(\widetilde{M}f(y))^\delta \le (\widetilde{M}f_1(y))^\delta + (\widetilde{M}f_2(y))^\delta \quad \text{for } 0 \le \delta < 1.$$

Since $\widetilde{M}$ is weak $(1,1)$ with respect to the measure μ, Kolmogorov's inequality shows that

$$\frac{1}{\mu(B)}\int_B(\widetilde{M}f_1(y))^\delta\,d\mu(y)\le\frac{C}{\mu(B)}\mu(B)^{1-\delta}\|f_1\|_{L^1}^\delta$$
$$\le C\biggl(\frac{1}{\mu(B)}\int_{2B}f\,d\mu\biggr)^\delta\le C(\widetilde{M}f(x))^\delta.$$

Now we estimate $\widetilde{M}f_2$. Given $y\in B$, for any $B(y_0,R)$ that contains y, we have $B\subset B(y_0,A_0^2\max\{t_0,R\})$. If $R<t_0$, then $B(y_0,t_0)\cap B(x_0,t_0)\ne\emptyset$ and hence $B(y_0,t_0)\subset B(x_0,A_0^2t_0)$, which gives $B(y_0,t_0/(2A_0^2))\subset B(x_0,2t_0)$. Then the inequality $\int_{B(y_0,R)}|f_2|\,d\mu>0$ implies $R>t_0/(2A_0^2)$, which yields $B\subset B(y_0,2A_0^4R)$ when $R<t_0$. It is clear that $B\subset B(y_0,2A_0^4R)$ when $R\ge t_0$. Thus,

$$\frac{1}{\mu(B(y_0,R))}\int_{B(y_0,R)}|f_2|\le\frac{C}{\mu(B(y_0,2A_0^4R))}\int_{B(y_0,2A_0^4R)}|f_2|\,d\mu\le C\widetilde{M}f(x),$$

so that $\widetilde{M}f_2(y)\le C\widetilde{M}f(x)$ for all $y\in B$. Therefore,

$$\frac{1}{\mu(B)}\int_B(\widetilde{M}f_2(y))^\delta\,d\mu(y)\le C(\widetilde{M}f(x))^\delta.\ \blacksquare$$

LEMMA 3.5. *If* $\omega\in A_2(X)$, *then* $\log\omega\in\mathrm{BMO}(X)$.

We omit the proof of Lemma 3.5, which echoes the Euclidean version (see for example [D]).

We are ready to show the main result in the one-parameter case. We follow the proof in [JJ].

Proof of Theorem 1.1 for one parameter. Since $H^1(X)$ is a subspace of $L^1(X)$, it follows from Fatou's lemma that $f\in L^1(X)$. To show (1.1) for all $\phi\in\mathrm{VMO}(X)$, by density it suffices to consider $\phi\in\mathring{G}(\beta,\gamma)$. Fix $\delta\in(0,1/(2A_0))$ and pick $\eta>0$ such that $\eta\exp(\delta^{-1})\le\delta C_\mu^{\log_2\delta}$ and $\int_E|f|\,d\mu\le\delta$ whenever $\mu(E)\le C\eta\exp(\delta^{-1})$. Now choose k large enough so that

$$\mu(E_k):=\mu(\{x\in X:|f_k(x)-f(x)|>\eta\})\le\eta.$$

We construct a bump function $\tau(x)$ on X as follows. Define

$$\tau(x):=\max\{0,1+\delta\log(M\chi_{E_k})(x)\}.$$

It is clear that $0\le\tau(x)\le1$ and $\tau\equiv1$ μ-almost everywhere on E_k. Also, $\|\tau\|_{\mathrm{BMO}(X)}\le2\delta\|\log(M\chi_{E_k})^{1/2}\|_{\mathrm{BMO}(X)}\le C\delta$ due to Lemmas 3.4 and 3.5. By the weak $(1,1)$ estimate for M with respect to μ,

$$\mu(\mathrm{supp}(\tau))\le C\mu(E_k)\exp(\delta^{-1})\le C\eta\exp(\delta^{-1}).$$

Consequently,

$$\int_{\operatorname{supp}(\tau)} |f|\,d\mu \le \delta.$$

We now write

$$\begin{aligned}\Big|\int_X (f-f_k)\phi\,d\mu\Big| &\le \Big|\int_X (f-f_k)\phi(1-\tau)\,d\mu\Big| + \Big|\int_X (f-f_k)\phi\tau\,d\mu\Big| \\ &\le \eta\|\phi\|_{L^1(d\mu)} + \int_{\operatorname{supp}(\tau)} |f|\,d\mu + \Big|\int_X f_k\phi\tau\,d\mu\Big| \\ &\le \delta+\delta+\Big|\int_X f_k\phi\tau\,d\mu\Big|.\end{aligned}$$

The proof of (1.1) will therefore be established provided we verify

$$\|\phi\tau\|_{\mathrm{BMO}(X)} \le C\delta. \tag{3.1}$$

Suppose $B=B(y_0,r_0)$ with $r_0<\delta$. The Hölder regularity of ϕ gives

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |\phi\tau-(\phi\tau)_B|\,d\mu &\le \frac{2}{\mu(B)}\int_B |\phi\tau-\phi_B\tau_B|\,d\mu \\ &\le \frac{2}{\mu(B)}\int_B |\phi\tau-\phi_B\tau|\,d\mu + \frac{2|\phi_B|}{\mu(B)}\int_B |\tau-\tau_B|\,d\mu \\ &\le C\delta^\beta + 2\|\phi\|_{L^\infty}\|\tau\|_{\mathrm{BMO}(X)} \le C(\delta^\beta+\delta).\end{aligned}$$

For $r_0>\delta$ and $B(y_0,\delta)\cap B(x_0,\delta^{-1})=\emptyset$, the size condition of ϕ yields

$$\frac{1}{\mu(B)}\int_B |\phi\tau-(\phi\tau)_B|\,d\mu \le \frac{2}{\mu(B)}\int_B |\phi\tau|\,d\mu \le C\delta^\gamma.$$

For $r_0>\delta$ and $B(y_0,\delta)\cap B(x_0,\delta^{-1})\neq\emptyset$, we get $B(y_0,\delta^{-1})\subset B(x_0,A_0\delta^{-1})$, and hence $\mu(B(x_0,\delta^{-1}))\le \mu(B(y_0,A_0\delta^{-1}))$. The doubling condition shows that

$$\mu(B(y_0,A_0\delta^{-1})) \le C_\mu^{\log_2(A_0\delta^{-2})}\mu(B(y_0,\delta)).$$

Thus,

$$\frac{1}{\mu(B)} \le \frac{C_\mu^{\log_2(A_0\delta^{-2})}}{\mu(B(y_0,A_0\delta^{-1}))} \le \frac{C_\mu^{\log_2(A_0\delta^{-2})}}{\mu(B(x_0,\delta^{-1}))} \le \frac{C_\mu^{\log_2(A_0\delta^{-2})}}{V_1(x_0)},$$

and so

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |\phi\tau-(\phi\tau)_B|\,d\mu &\le \frac{2}{\mu(B)}\int_B |\phi\tau|\,d\mu \le \frac{2C_\mu^{\log_2(A_0\delta^{-2})}}{V_1(x_0)}\mu(\operatorname{supp}(\tau)) \\ &\le \frac{2C_\mu^{\log_2(A_0\delta^{-2})}}{V_1(x_0)}\eta\exp(\delta^{-1}) \le C\delta.\end{aligned}$$

Therefore,

$$\frac{1}{\mu(B)}\int_B |\phi\tau - (\phi\tau)_B|\,d\mu \le C\delta \tag{3.2}$$

and (3.1) follows. By weak-star compactness of the ball in H^1, there exists a $g \in H^1$ with $\|g\|_{H^1} \le 1$ and a subsequence $\{f_{k_l}\}_{l\in\mathbb{N}}$ such that $\{f_{k_l}\}_{l\in\mathbb{N}}$ weak-star converges to g. By (1.1), we have $\int f\phi = \int g\phi$ for all $\phi \in \mathring{G}(\beta,\gamma)$, and hence $f = g \in H^1$. ∎

4. Proof of Theorem 1.1 in the product case. We begin by recalling several key tools we will use to pass from the product Euclidean setting to the setting of product spaces of homogeneous type. These tools are the random dyadic lattices, the dyadic product BMO space, the averaging theorem relating the dyadic and continuous product BMO spaces, several properties of product *bmo* ("little BMO"), and the construction of a product bump function $\tau(x_1,x_2)$ on $X_1 \times X_2$. Then we prove Theorem 1.1 for product spaces of homogeneous type.

In [HK, Theorem 5.1] Hytönen and Kairema constructed random dyadic lattices on spaces of homogeneous type, extending an earlier result of Nazarov, Treil and Volberg [NTV]. Specifically, there exists a probability space $(\Omega,\mathbb{P})$ such that for each $\omega \in \Omega$ there is an associated dyadic lattice $\mathcal{D}(\omega) = \{Q^k_\alpha(\omega)\}_{k,\alpha}$ related to dyadic points $\{x^k_\alpha(\omega)\}_{k,\alpha}$ with the properties (2.1)–(2.5) above, and the following *smallness property* holds: there exist absolute constants $C,\eta > 0$ such that

$$\mathbb{P}(\{\omega\in\Omega : x, x^* \text{ are not in the same cube } Q \in \mathcal{D}^k(\omega)\}) \le C\biggl(\frac{\rho(x,x^*)}{\delta^k}\biggr)^{\eta}$$

for all $x, x^* \in X$, where $\mathcal{D}^k(\omega)$ is the set of all dyadic cubes at level k in $\mathcal{D}(\omega)$.

Fix $\omega \in \Omega$. For a cube $Q \in \mathcal{D}(\omega)$, let $\mathrm{ch}(Q)$ denote the set of all children of $Q \in \mathcal{D}(\omega)$. From (2.1) and (2.2), we know that $Q = \bigcup_{I\in\mathrm{ch}(Q)} I$. For a cube $Q \in \mathcal{D}(\omega)$, define the *averaging operator* E^ω_Q by

$$E^\omega_Q f = E^{\mathcal{D}(\omega)}_Q f := \Bigl(\fint_Q f\,d\mu\Bigr)\chi_Q,$$

where as usual $\fint_Q f\,d\mu = \mu(Q)^{-1}\int_Q f\,d\mu$ and χ_Q is the characteristic function of Q. (We reserve the more usual name of *expectation operator* for the expectation $\mathbb{E}_\omega$ over random dyadic lattices, defined below.) Define the *difference operator* Δ^ω_Q by

$$\Delta^\omega_Q f = \Delta^{\mathcal{D}(\omega)}_Q f := \Bigl(\sum_{J\in\mathrm{ch}(Q)} E^\omega_J f\Bigr) - E^\omega_Q f.$$

For convenience, we sometimes write E_Q and Δ_Q instead of E_Q^ω and Δ_Q^ω. Note that for every $x \in X$, at each level k there exists exactly one cube $Q^k(x) \in \mathcal{D}^k(\omega)$ such that $x \in Q^k(x)$. So for each $k \in \mathbb{Z}$ we can define

$$E_k f(x) := \sum_\alpha E_{Q_\alpha^k} f(x) = E_{Q^k(x)} f(x),$$
$$\Delta_k f(x) := \sum_\alpha \Delta_{Q_\alpha^k} f(x) = E_{k+1} f(x) - E_k f(x).$$

For $j = 1, 2$, let $(\Omega_j, \mathbb{P}_j)$ be a probability space for (X_j, ρ_j, μ_j) such that for each $\omega_j \in \Omega_j$ there is an associated dyadic lattice $\mathcal{D}_j(\omega_j)$ satisfying properties (2.1)–(2.6). We define the dyadic product BMO$(X_1 \times X_2)$ space via the difference operator. Let $\Delta^\omega := \Delta_{Q_1^{\omega_1}}^{\omega_1} \Delta_{Q_2^{\omega_2}}^{\omega_2}$ where $Q_1^{\omega_1} \in \mathcal{D}_1(\omega_1)$ and $Q_2^{\omega_2} \in \mathcal{D}_2(\omega_2)$. Let R^ω denote the rectangle $Q_1^{\omega_1} \times Q_2^{\omega_2}$.

Definition 4.1. Let $f^\omega(x) = f^{(\omega_1,\omega_2)}(x_1, x_2)$ be a locally integrable function on $X_1 \times X_2$. We say that f^ω belongs to the *dyadic product bounded mean oscillation space* $\mathrm{BMO}_{\omega_1,\omega_2} := \mathrm{BMO}_{\mathcal{D}_1(\omega_1)\times\mathcal{D}_2(\omega_2)}(X_1 \times X_2)$ if there exists a constant $C > 0$ such that for every open set $\mathcal{A} \subset X_1 \times X_2$,

$$\frac{1}{\mu(\mathcal{A})} \sum_{R^\omega \subset \mathcal{A}} \int_{\widetilde{X}} |\Delta^\omega f^\omega|^2 \, d\mu \le C^2.$$

We define the dyadic product BMO norm $\|f^\omega\|_{\mathrm{BMO}_{\omega_1,\omega_2}}$ of f^ω to be the infimum of C such that the inequality above holds.

Theorem 4.2 ([CLW]). *Let (X_1, d_1, μ_1) and (X_2, d_2, μ_2) be spaces of homogeneous type. For $j = 1, 2$, let $(\Omega_j, \mathbb{P}_j)$ be a probability space, and $\{\mathcal{D}(\omega_j)\}_{\omega_j\in\Omega_j}$ a collection of random dyadic lattices on X_j such that properties* (2.1)–(2.6) *hold. Let $\{f^\omega\}$, $\omega := (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, be a family of functions with $f^\omega \in \mathrm{BMO}_{\mathcal{D}(\omega_1)\times\mathcal{D}(\omega_2)}(X_1 \times X_2)$ for each $\omega \in \Omega_1 \times \Omega_2$, such that*

- (i) *$\omega \mapsto f^\omega$ is measurable, and*
- (ii) *$\|f^\omega\|_{\mathrm{BMO}_{\mathcal{D}(\omega_1)\times\mathcal{D}(\omega_2)}(X_1\times X_2)} \le C_d$ for some constant C_d independent of ω.*

Then the function f defined by the expectation

$$f(x) := \mathbb{E}_\omega f^\omega(x)$$

belongs to $\mathrm{BMO}(X_1 \times X_2)$, *and* $\|\mathbb{E}_\omega f^\omega\|_{\mathrm{BMO}(X_1\times X_2)} \le CC_d$.

Definition 4.3. A real-valued function $f \in L^1_{\mathrm{loc}}(X_1 \times X_2)$ is in the space *bmo*$(X_1 \times X_2)$ (called "little BMO" in the literature) if its *bmo* norm is finite:

$$\|f\|_{bmo(X_1\times X_2)} := \sup_R \int_R |f(x_1, x_2) - f_R| \, d\mu_1(x_1) \, d\mu_2(x_2) < \infty. \tag{4.1}$$

Lemma 4.4. *If* $f, g \in bmo$, *then* $\max\{f,g\} \in bmo$.

Lemma 4.5. *Suppose* Ω *is an open set in* $X_1 \times X_2$ *with finite measure. Let* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *be dyadic cubes in* X_1 *and* X_2, *respectively. Then*

$$\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\, R\subset\Omega} \|\Delta_{Q_1\times Q_2} f\|_2^2 \le \int_{X_1} \sum_{Q_2\in\mathcal{D}_2(\omega_2)} \|\Delta_{Q_2} f(x_1,\cdot)\|_2^2\, d\mu_1(x_1).$$

Proof. Let

$$\tilde f(x_1,\cdot) = \sum_{Q_2\in\mathcal{D}_2(\omega_2)} \Delta_{Q_2} f(x_1,\cdot) \quad \text{for } x_1\in X_1.$$

Then $\Delta_{Q_2} f(x_1,\cdot) = \Delta_{Q_2}\tilde f(x_1,\cdot)$. Since $\Delta_{Q_1\times Q_2} = \Delta_{Q_2}\otimes\Delta_{Q_2}$, we get

$$\Delta_{Q_1\times Q_2} f = \Delta_{Q_1\times Q_2}\tilde f,$$

and so

$$\begin{aligned}\sum \|\Delta_{Q_1\times Q_2} f\|_2^2 &= \sum \|\Delta_{Q_1\times Q_2}\tilde f\|_2^2 \\ &\le \|\tilde f\|_2^2 = \int_{X_1} \Big\| \sum_{Q_2\in\mathcal{D}_2(\omega_2)} \Delta_{Q_2} f(x_1,\cdot)\Big\|_2^2\, d\mu_1(x_1) \\ &= \int_{X_1} \sum_{Q_2\in\mathcal{D}_2(\omega_2)} \|\Delta_{Q_2} f(x_1,\cdot)\|_2^2\, d\mu_1(x_1). \ \blacksquare\end{aligned}$$

Lemma 4.6. *Suppose* $\phi\in \mathring{G}(\beta_1,\beta_2,\gamma_1,\gamma_2)$ *and* b *is a bounded function with* $\|b\|_\infty \le 1$. *Then, for all* $\alpha\in(0,1)$, *for each open* $\Omega\subset X_1\times X_2$, *and for each rectangle* $R = Q_1\times Q_2 \in \mathcal{D}_1(\omega_1)\otimes\mathcal{D}_2(\omega_2)$, *we have*

$$\sum_{R\subset\Omega,\,\mathrm{diam}(R)\le\alpha} \|\Delta_R(\phi b)\|_2^2 \le C(\|b\|_{bmo}+\alpha)\,\mu(\Omega). \tag{4.2}$$

Proof. The proof is by iteration. For one parameter, it suffices to prove (4.2) for $\Omega = Q_0$, where Q_0 is a dyadic cube in X_1. Without loss of generality we may assume that $\mathrm{diam}(Q_0)\le\alpha$. Then

$$\begin{aligned}\sum_{Q\subset Q_0} \|\Delta_Q(\phi b)\|_2^2 &= \int_{Q_0} |\phi b(x) - (\phi b)_{Q_0}|^2\, d\mu(x) \\ &\le 2\int_{Q_0} |\phi b(x) - (\phi)_{Q_0}(b)_{Q_0}|^2\, d\mu(x) \\ &\le 2\int_{Q_0} |\phi b(x) - \phi(x)(b)_{Q_0}|^2\, d\mu(x) \\ &\quad + 2\int_{Q_0} |\phi(x)(b)_{Q_0} - (\phi)_{Q_0}(b)_{Q_0}|^2\, d\mu(x) \\ &\le C(\|b\|_{bmo}^2+\alpha)\,\mu(\Omega)\end{aligned}$$

by inequality (3.2). Applying Lemma 4.5, we obtain

$$\begin{aligned}\sum_{Q_1\in\mathcal{D}_1(\omega_1),\,Q_2\in\mathcal{D}_2(\omega_2)} &\|\varDelta_{Q_1\times Q_2} f\|^2 \\ &\le \int_{X_1} \sum_{Q_2\in\mathcal{D}_2(\omega_2)} \|\varDelta_{Q_2} f(x_1,\cdot)\|^2\,d\mu_1(x_1) \\ &\quad + \int_{X_2} \sum_{Q_1\in\mathcal{D}_1(\omega_1)} \|\varDelta_{Q_1} f(\cdot,x_2)\|^2\,d\mu_2(x_2) \\ &\le C(\|b\|_{bmo}^2+\alpha)\int_{X_1} \mu_2(\{x_2:(x_1,x_2)\in\Omega\})\,d\mu_1(x_1) \\ &\quad + C(\|b\|_{bmo}^2+\alpha)\int_{X_2} \mu_1(\{x_2:(x_1,x_2)\in\Omega\})\,d\mu_2(x_2) \\ &\le 2C(\|b\|_{bmo}^2+\alpha)\,\mu(\Omega). \quad\blacksquare\end{aligned}$$

Next we construct a bump function $\tau(x_1,x_2)$ in the product setting.

LEMMA 4.7. *Let E be a subset of $X_1\times X_2$ with finite measure, and let $\delta\in(0,1)$ be a given parameter. Then there exists a function $\tau\in bmo$ such that $\tau\equiv 1$ on E, $\|\tau\|_{bmo}<C_1\delta$, and $\mu(\operatorname{supp}(\tau))<C_2e^{2/\delta}\mu(E)$, where C_1 and C_2 are some absolute constants.*

Proof. Let M_s be the strong maximal function, in which the averages are taken over arbitrary rectangles in $X_1\times X_2$. A weight w is in $A_1(X_1\times X_2)$ if there exists a constant C such that $M_sw(x)\le Cw(x)$ for μ-almost every $x\in X_1\times X_2$.

We define the following A_1 weight, with $M_s^{(k)}$ denoting the k-fold iteration of the strong maximal function:

$$m(x_1,x_2)=K^{-1}\sum_{k=0}^{\infty} c^k M_s^{(k)}\chi_E(x_1,x_2),$$

where $K=\sum_k c^k$ and $c>0$ is chosen to ensure the convergence of the series. Then $\|m\|_2\le C\|\chi_E\|_2=C\mu(E)^{1/2}$. Observe that $m=1$ μ-almost everywhere on E, and $m\le 1$ μ-almost everywhere outside E.

Define the function

$$\tau(x_1,x_2):=\max\{0,1+\delta\log m(x_1,x_2)\}.$$

Then $\tau\in bmo$, and $\tau=1$ μ-almost everywhere on E. By Lemma 4.4 and the fact that $\log w\in bmo$ for every A_1 weight w, which is proved exactly as in the one-parameter Euclidean setting, we have $\|\tau\|_{bmo}\le C\delta$.

The estimate for the size of the support of τ follows from Chebyshev's theorem and the estimate $\|m\|_2\le C\mu(E)^{1/2}$. ∎

We are ready to prove our main result for product spaces of homogeneous type. We follow the lines of the product Euclidean proof from [PT].

Proof of Theorem 1.1 in the product case. First note that $\mathring{G}(\beta_1,\beta_2,\gamma_1,\gamma_2)$ is dense in $\mathrm{VMO}(X_1\times X_2)$. To prove the theorem, it suffices to show (1.1) for all $\phi\in\mathring{G}(\beta_1,\beta_2,\gamma_1,\gamma_2)$.

Next, note that as shown in [HLPW], $H^1(X_1\times X_2)$ is a subspace of $L^1(X_1\times X_2)$. Thus, since $f_n\to f$ a.e., and $\|f_n\|_{H^1(X_1\times X_2)}\le 1$, by Fatou's lemma we have $f\in L^1(X_1\times X_2)$ with $\|f\|_{L^1(X_1\times X_2)}\le 1$.

Fix $\delta\in(0,1/(2A_0))$ and pick $\eta>0$ such that $\eta\exp(2/\delta)\le\delta C_\mu^{\log_2\delta}$ and $\int_E|f|\,d\mu\le\delta$ whenever $\mu(E)\le C_2\eta\exp(2/\delta)$, where C_2 is as in Lemma 4.7. Now choose K_0 large enough such that when $k>K_0$,

$$\mu(E_k):=\mu\big(\{(x_1,x_2)\in X_1\times X_2:|f_k(x_1,x_2)-f(x_1,x_2)|>\eta\}\big)\le\eta.$$

Define

$$\tau(x_1,x_2)=\max\{0,1+\delta\log m(x_1,x_2)\},$$

where $m(x_1,x_2)=K^{-1}\sum_{\ell=0}^{\infty}c^\ell M_s^{(\ell)}\chi_{E_k}(x_1,x_2)$ as defined in Lemma 4.7. It is clear that $0\le\tau(x_1,x_2)\le 1$ and $\tau=1$ μ-almost everywhere on E_k. By Lemma 4.7, we have $\tau\in bmo$ with $\|\tau\|_{bmo}\le C_2\delta$ and

$$\int_{\mathrm{supp}(\tau)}|f|\,d\mu\le\delta.$$

For every $k>K_0$, we now write

$$\int_{X_1\times X_2}(f-f_k)\phi\,d\mu=\int_{X_1\times X_2}(f-f_k)\phi(1-\tau)\,d\mu+\int_{X_1\times X_2}(f-f_k)\phi\tau\,d\mu.$$

Note that $\tau=1$ μ-almost everywhere on E_k. In the complement of E_k we have $|f-f_k|<\eta$. Thus the first integral on the right-hand side of the above equality is bounded by $\eta\|\phi\|_{L^1(X_1\times X_2)}$, which is in turn less than δ if η is sufficiently small. Further, the second integral is bounded by

$$\int_{\mathrm{supp}(\tau)}|f\phi|\,d\mu+\Big|\int_{X_1\times X_2}f_k\phi\tau\,d\mu\Big|\le\delta+\Big|\int_{X_1\times X_2}f_k\phi\tau\,d\mu\Big|.$$

The proof of (1.1) will therefore be established provided we verify

$$\|\phi\tau\|_{\mathrm{BMO}(X_1\times X_2)}\le C\delta. \tag{4.3}$$

We will show (4.3) by first proving that the dyadic BMO norm of $\phi\tau$ has the required estimate, and then by using Theorem 4.2.

For every open set $\mathcal{A}\subset X_1\times X_2$ with finite measure and $x\in\mathcal{A}$, there exists a constant $r(x)<\delta/(3A_0)$ such that $B(x,r(x))\subset\mathcal{A}$, and so

$$\mathcal{A}=\bigcup_{x\in\mathcal{A}}B(x,r(x)).$$

By [C2, Lemma 3], there exists a countable subfamily of disjoint spheres $B(x_i, r(x_i))$ such that each sphere $B(x, r(x))$, $x \in \mathcal{A}$, is contained in $B(x_i, 3A_0 r(x_i))$ for some $i \in \mathbb{N}$. Hence,

$$\int_{\mathcal{A}} |\phi\tau|^2 \, d\mu \le \sum_{i=1}^{\infty} \int_{B(x_i, 3A_0 r(x_i))} |\phi\tau|^2 \, d\mu.$$

Since $3A_0 r(x_i) < \delta$, we use Lemma 4.6 to get

$$\begin{aligned}\int_{B(x_i, 3A_0 r(x_i))} |\phi\tau|^2 \, d\mu &= \sum_{R \subset B(x_i, 3A_0 r(x_i))} \|\Delta_R(\phi b)\|_2^2 \\ &\le C(\|\tau\|_{bmo} + \delta)\mu(B(x_i, 3A_0 r(x_i))).\end{aligned}$$

Therefore,

$$\int_{\mathcal{A}} |\phi\tau|^2 \, d\mu(x) \le \sum_{i=1}^{\infty} \int_{B(x_i, 3A_0 r(x_i))} |\phi\tau|^2 \, d\mu(x) \le C\delta \sum_i \mu(B(x_i, 3A_0 r(x_i))).$$

Since $\mu(B(x_i, 3A_0 r(x_i))) \le C\mu(B(x_i, r(x_i)))$ and $\{B(x_i, r(x_i))\}_{i \in \mathbb{N}}$ are disjoint, we have

$$\int_{\mathcal{A}} |\phi\tau|^2 \, d\mu(x) \le \sum_i \mu(B(x_i, 3A_0 r(x_i))) \le C\mu(\mathcal{A}).$$

This completes the proof of (4.3). ∎

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