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On subspaces of invariant vectors

by

Tatiana Shulman (Warszawa)

Abstract. Let X_π be the subspace of fixed vectors for a uniformly bounded representation π of a group G on a Banach space X. We study the problem of the existence and uniqueness of a subspace Y that complements X_π in X. Similar questions for G-invariant complement to X_π are considered. We prove that every non-amenable discrete group G has a representation with non-complemented X_π and find some conditions that provide a G-invariant complement. A special attention is given to representations on $C(K)$ that arise from an action of G on a metric compact K.

Introduction. The subspaces of vectors which are invariant under group representations have recently attracted renewed attention because of their use in the Banach space version of Kazhdan's property (T) (see [1], [10]). In the Hilbert space case, arguments used for studying property (T) rely heavily on the existence of orthogonal complements of subspaces (of invariant vectors). In the Banach space setting, the lack of orthogonality immediately causes difficulties. It is not even clear if the subspace of invariant vectors is always complemented, as mentioned in [10]. In fact, the existence of a complementing subspace allows one to reduce a representation to a triangular form, with two representations on the diagonal and a 1-cocycle in the corner, so for concrete groups cohomological techniques can be used for a further study of the representation.

Therefore, a complement to the subspace of invariant vectors is a natural object of interest. If a Banach space is super-reflexive, then for any uniformly bounded representation, the subspace of invariant vectors is complemented [1]. Moreover in this case there is a complement which is invariant under the representation. Namely it is proved in [1] that if π is a uniformly bounded representation of a group G on a super-reflexive space X, then

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X decomposes into the sum

$$X = X_\pi \oplus \mathrm{Ann}(X^*_{\bar\pi}),$$

where X_π is the subspace of invariant vectors, $X^*_{\bar\pi}$ is the subspace of vectors invariant under the dual representation $\bar\pi$ and $\mathrm{Ann}(X^*_{\bar\pi})$ is its preannihilator. It is easy to check that $\mathrm{Ann}(X^*_{\bar\pi})$ is G-invariant.

Of course, when X_π is finite-dimensional, as is often the case in ergodic theory, it is complemented. However, in this note we show that the subspace of invariant vectors need not be complemented. What is more, in Section 1 we prove that each non-amenable group admits an isometric representation such that the subspace of invariant vectors is not complemented (Theorem 1).

In Section 2 we study the decomposition $X = X_\pi \oplus \mathrm{Ann}(X^*_{\bar\pi})$, and more generally, the question of the existence and uniqueness of an invariant complement to X_π. In fact, the decomposition is strongly connected with abstract ergodic theorems and is often called the ergodic decomposition. It was first considered by Yosida [14] (and independently by Kakutani) for power-bounded operators T on reflexive spaces. It was shown that

$$X = X_T \oplus \overline{(1-T)X}.$$

Duality considerations were introduced by Heyneman [7]. In fact, he proved that

$$\mathrm{Ann}(X^*_{\bar\pi}) = \overline{\mathrm{span}\{(1-\pi(g))X \mid g \in G\}}.$$

Eberlein [5] studied bounded semigroups of operators with conditionally weakly compact orbits and called such semigroups *weakly almost periodic*. The generalization of Yosida's theorem to continuous group representations is due to Ryll-Nardzewski [12, Theorem 5]; it essentially says that for a weakly almost periodic representation π, there is a projection M on the subspace of invariant vectors which commutes with π and is such that for any $x \in X$, Mx is in the closed convex hull of the orbit of x (the full formulation of the theorem is given in Section 2).

It is easy to deduce from this theorem that the decomposition $X = X_\pi \oplus \mathrm{Ann}(X^*_{\bar\pi})$ holds for strongly continuous representations of compact groups (Corollary 6) and for uniformly bounded representations on reflexive Banach spaces (Corollary 5), which yields [1, Prop. 2.6], where such decomposition is obtained on super-reflexive Banach spaces (recall that super-reflexivity implies reflexivity). In these cases, if the representation is isometric, then the corresponding projection onto X_π has norm 1. Though in general a G-invariant complement need not be unique (Example 13), in the cases above it is unique. We also show that for any uniformly bounded representation of an amenable group, the subspaces X_π and $\mathrm{Ann}(X^*_{\bar\pi})$ have trivial intersection (Theorem 10). For non-amenable groups this is not true in general [1, Ex. 2.29].

In Sections 3 and 4 we focus on representations induced by group actions on compact metric spaces. Though for such representations the decomposition $X = X_\pi \oplus \mathrm{Ann}(X^*_\pi)$ need not hold in general, it is shown that it does hold if the action is nice, namely Lyapunov stable (Theorem 17). Lyapunov stable actions were introduced in [8]. It was shown there that if an action is Lyapunov stable, then there is a conditional expectation on the subspace (actually, subalgebra) of invariant functions. In Theorem 17 we give a new proof of this fact. Moreover, we construct a conditional expectation commuting with the representation and show that such an expectation is unique. Along the way we give a short proof of the assertion in [8] on the uniqueness of invariant measures. In Section 4 we introduce lower semicontinuous actions, a class of actions wider than the Lyapunov stable ones. We show that for lower semicontinuous actions the subspace of invariant functions is complemented.

1. The subspace of invariant vectors need not be complemented. Let G be a group and X be a Banach space. By a *representation* of G on X we will mean a homomorphism from G into the group $B_{\mathrm{inv}}(X)$ of bounded invertible operators on X. A representation π is *isometric* if $\pi(g)$ is an isometry for each $g \in G$, and *uniformly bounded* if $\sup_{g\in G} \|\pi(g)\| < \infty$.

Below, for any family $S \subseteq B(H)$ of operators on a Hilbert space, we denote by S' its *commutant*, that is $S' = \{T \in B(H) \mid TA = AT$ for any $A \in S\}$. Recall that the *group von Neumann algebra* $L(G)$ of a group G is

$$L(G) = \{\lambda(g) \mid g \in G\}'',$$

where λ is the left regular representation of G.

THEOREM 1. *Any discrete non-amenable group admits an isometric representation such that the set of invariant vectors is not complemented.*

Proof. Let G be a discrete non-amenable group, $H = l_2(G)$ and $\lambda : G \to B(H)$ the left regular representation. Define a representation $\tilde{\lambda} : G \to B(H \otimes H)$ by

$$\tilde{\lambda}(g) = \lambda(g) \otimes \mathrm{Id},$$

where Id denotes the identity operator. Let $X = B(H \otimes H)$. Define a representation $\pi : G \to B_{\mathrm{inv}}(X)$ by

$$\pi(g)x = \tilde{\lambda}(g)x\tilde{\lambda}(g)^{-1}, \qquad x \in X.$$

Let N be the set of π-invariant vectors. Then

$$N = \{x \in X \mid \tilde{\lambda}(g)x\tilde{\lambda}(g)^{-1} = x, \forall g \in G\} \tag{1}$$
$$= \{\tilde{\lambda}(g) \mid g \in G\}' = \{\lambda(g) \otimes \mathrm{Id} \mid g \in G\}'$$
$$= \{T \otimes \mathrm{Id} \mid T \in L(G)\}' = (L(G) \otimes \mathrm{Id})'.$$

As is well known, a von Neumann algebra is injective if and only if its commutant is injective [4]. Since G in non-amenable, $L(G)$ is not injective [2]. Since injectivity is preserved by $*$-isomorphisms, $L(G) \otimes \mathrm{Id}$ is not injective either and we conclude that N is not injective. Since

$$(L(G) \otimes \mathrm{Id})' = L(G)' \otimes B(H),$$

we have

$$M_2(N) \cong N,$$

where $M_2(N)$ is the algebra of 2×2-matrices over N. By [6, Corollary 4.6], if N were complemented then there would exist a completely bounded projection from $B(H \otimes H)$ onto N, and this is equivalent to injectivity of N (see [3, Theorem 3.1] and [11]). Thus N is not complemented. ∎

Question. *Does there exist a group which admits a uniformly bounded representation on a separable Banach space such that the set of invariant vectors is not complemented?*

Question. *Does there exist an amenable group which admits a uniformly bounded representation such that the set of invariant vectors is not complemented?*

2. On the decomposition $X = X_\pi \oplus \mathrm{Ann}(X^*_{\bar{\pi}})$. For a representation π, one can define the adjoint representation $\bar{\pi}$ of G on the dual space X^* by

$$(\bar{\pi}(g)f)(x) = f(\pi(g^{-1})x), \quad x \in X,\ f \in X^*.$$

It is easy to see that if π is uniformly bounded or isometric, then $\bar{\pi}$ is uniformly bounded or isometric respectively.

For a subspace $Y \subseteq X$, we denote by $\mathrm{Ann}(Y)$ its annihilator in X^*, that is,

$$\mathrm{Ann}(Y) = \{f \in X^* \mid f(x) = 0 \text{ for each } x \in Y\}.$$

For a subspace $Y \subseteq X^*$, its *preannihilator* in X will also be denoted by $\mathrm{Ann}(Y)$:

$$\mathrm{Ann}(Y) = \{x \in X \mid f(x) = 0 \text{ for each } f \in Y\},$$

since it will always be clear from the context what we mean.

Let X_π be the subspace of π-invariant vectors,

$$X_\pi = \{x \in X \mid \pi(g)x = x \text{ for all } g \in G\}.$$

PROPOSITION 2. *If X_π has a $\pi(G)$-invariant complement Y, then $Y \supseteq \mathrm{Ann}(X^*_\pi)$.*

Proof. Let P be the projection onto X_π parallel to Y. Since Y is $\pi(G)$-invariant, $[P, \pi(g)] = 0$ for all $g \in G$. Hence, for any $x \in X$,

$$P(\pi(g)x - x) = \pi(g)Px - Px = 0.$$

Thus $\pi(g)x - x \in Y$. Let $f \in X^*$ with $f|_Y = 0$. Then $f(\pi(g)x - x) = 0$ for all $g \in G$ and $x \in X$, that is, $f \in X^*_\pi$. Hence $\mathrm{Ann}(Y) \subseteq X^*_\pi$, whence $Y \supseteq \mathrm{Ann}(X^*_\pi)$. ∎

We will use the following theorem of Ryll-Nardzewski [12].

THEOREM 3 (Ryll-Nardzewski [12]). *If G is an equicontinuous group of endomorphisms of a locally convex linear space X and if $O_G(x)$ denotes the closed convex hull of all vectors of the form Tx where T runs over G, then*

(i) *if $O_G(x)$ is weakly compact, then there exists exactly one G-invariant element in $O_G(x)$ (it will be denoted by Mx);*
(ii) *the set X_0 of all vectors $x \in X$ such that $O_G(x)$ is weakly compact forms a closed linear subspace of X;*
(iii) *the operator M (defined in (i)) has the following properties on X_0: it is linear, continuous and $TM = MT = M^2 = M$.*

PROPOSITION 4. *For any uniformly bounded representation π such that all the orbits are weakly precompact,*

$$X = X_\pi \oplus \mathrm{Ann}(X^*_\pi).$$

The corresponding projection onto X_π has norm at most $\sup_{g\in G} \|\pi(g)\|$.

Proof. Let M be the projection of Theorem 3. Since for each $x \in X$, Mx is in the closed convex hull of the orbit of x, it is easy to see that $(1 - M)X \subseteq \mathrm{Ann}(X^*_\pi)$. Since M commutes with π, $(1 - M)X$ is a $\pi(G)$-invariant subspace, and by Proposition 2, $(1 - M)X \supseteq \mathrm{Ann}(X^*_\pi)$. Thus $(1 - M)X = \mathrm{Ann}(X^*_\pi)$ and we obtain

$$X = MX \oplus (1 - M)X = X_\pi \oplus \mathrm{Ann}(X^*_\pi).$$

Since for each $x \in X$, Mx is in the closed convex hull of the orbit of x, we conclude that $\|Mx\| \le \sup_{g\in G} \|\pi(g)x\|$ and the last statement follows. ∎

COROLLARY 5. *For any uniformly bounded representation π on a reflexive Banach space X,*

$$X = X_\pi \oplus \mathrm{Ann}(X^*_\pi).$$

The corresponding projection onto X_π has norm at most $\sup_{g\in G} \|\pi(g)\|$.

Corollary 6. *For any strongly continuous representation π of a compact group on a Banach space X,*

$$X = X_\pi \oplus \mathrm{Ann}(X^*_{\bar\pi}).$$

The corresponding projection onto X_π has norm at most $\sup_{g\in G}\|\pi(g)\|$.

Proof. Since the group is compact and the representation is strongly continuous, all the orbits are compact. Hence, by the uniform boundedness principle, π is uniformly bounded. The statement now follows from Proposition 4. ∎

Remark 7. As was pointed out to me by one of the referees, under the assumptions of Corollary 6 one can construct the relevant projection on X_π as the average $x \mapsto \int_G \pi(g)x\,dg$ with respect to the Haar measure.

Remark 8. It follows from Proposition 2 that in Corollaries 5 and 6 the space X_π has a unique $\pi(G)$-invariant complement. In general a $\pi(G)$-invariant complement need not be unique, as shown by Example 13 in Section 3.

Lemma 9. *Let π be a representation of a group G on a Banach space X. Suppose that for any $f \in X^*$, the $*$-weakly closed convex hull $E_w(f)$ of the $\bar\pi$-orbit $O(f)$ of f contains a $\bar\pi$-invariant vector. Then*

$$X^* = X^*_{\bar\pi} + \mathrm{Ann}(X_\pi).$$

Proof. By assumption, for every $f \in X^*$, there is an invariant functional $f_0 \in E_w(f)$. Let $x \in X_\pi$. Then all functionals from $O(f)$ take the same value at x as f. Hence the same is true for all functionals in $E_w(f)$, and thus $f(x) = f_0(x)$. Therefore $f - f_0 \in \mathrm{Ann}(X_\pi)$, and so $X^* = X^*_{\bar\pi} + \mathrm{Ann}(X_\pi)$. ∎

Theorem 10. *If π is a uniformly bounded representation of an amenable group on a Banach space X then*

(i) $X^* = X^*_{\bar\pi} + \mathrm{Ann}(X_\pi)$,
(ii) $X_\pi \cap \mathrm{Ann}(X^*_{\bar\pi}) = \{0\}$.

Proof. (i) For each $f \in X^*$, the $*$-weakly closed convex span $E_w(f)$ of $O(f)$ is $*$-weakly compact. Since all the operators $\bar\pi(g)$, $g \in G$, are $*$-weakly continuous, it follows from amenability that $E_w(f)$ contains a fixed point of $\bar\pi$. Now by Lemma 9 we conclude that $X^* = X^*_{\bar\pi} + \mathrm{Ann}(X_\pi)$.

(ii) It is easy to see that $X^*_{\bar\pi} + \mathrm{Ann}(X_\pi)$ annihilates $X_\pi \cap \mathrm{Ann}(X^*_{\bar\pi})$. ∎

Remark 11. Note that the decomposition in (i) is not necessarily direct. For example if $X = l_1(\mathbb{Z})$, $G = \mathbb{Z}$ and $\pi(n)f(k) = f(n+k)$ then $X_\pi = 0$, $\mathrm{Ann}(X_\pi) = X^*$, $X^*_{\bar\pi}$ is the space of constant sequences, and hence

$$\mathrm{Ann}(X_\pi) \cap X^*_{\bar\pi} \neq \{0\}.$$

Also in this example $X_\pi + \mathrm{Ann}(X^*_{\bar\pi})$ is a closed subspace of X and $X \neq X_\pi + \mathrm{Ann}(X^*_{\bar\pi})$.

Remark 12. (i) A necessary condition for $X_\pi \oplus \mathrm{Ann}(X_\pi^*)$ to be dense in X is that X_π separates X_π^*. In particular, the separation is necessary for the ergodic decomposition $X = X_\pi \oplus \mathrm{Ann}(X_\pi^*)$.

(ii) Theorem 1.7 of [9] yields the following result: Let π be a bounded continuous representation of a locally compact amenable group G in a Banach space X. The ergodic decomposition $X = X_\pi \oplus \mathrm{Ann}(X_\pi^*)$ holds if and only if X_π separates X_π^*. These conditions are satisfied if and only if the closed convex hull of every orbit contains an invariant vector.

3. Lyapunov stable actions. Let K be a compact metric space and suppose a group G act continuously on K.

Define a representation π of G on $C(K)$ by

$$\pi(g)\phi(x) = \phi(g^{-1}x).$$

It is easy to see that π is isometric. The following example shows that the decomposition

$$C(K) = C(K)_\pi \oplus \mathrm{Ann}(C(K)_\pi^*)$$

does not hold in general, even when the group is abelian. It also shows that a $\pi(G)$-invariant complement need not be unique.

Example 13. Let $K = [0,1]$ and define a $\mathbb{Z}$-action α on K as

$$\alpha(n)(x) = x^{2^n}.$$

Then as above we define a representation π of $\mathbb{Z}$ on $C(K)$ by

$$\pi(n)\phi(x) = \phi(\alpha(-n)x).$$

Let us show that $C(K)_\pi$ is the subspace of constant functions. For each $x \in [0,1)$, $0 \in \overline{O(x)}$. Hence for $\phi \in C(K)_\pi$ and each $x \in [0,1)$, $\phi(x) = \phi(0)$. Thus $\phi = \mathrm{const}$. We will show now that $\mathrm{Ann}(C(K)_\pi^*) \subseteq C_0(0,1)$. Define $h_i \in C(K)^*$, $i = 1,2$, by $h_1(\phi) = \phi(0)$, $h_2(\phi) = \phi(1)$, for any $\phi \in C(K)$. It is easy to see that h_i, $i = 1,2$, are constant on orbits of functions in $C(K)$, and so $h_i \in C(K)_\pi^*$, $i = 1,2$. Hence

$$\mathrm{Ann}(C(K)_\pi^*) \subseteq \mathrm{Ann}(h_1) \cap \mathrm{Ann}(h_2) = C_0(0,1] \cap C_0[0,1) = C_0(0,1).$$

Thus $\mathrm{Ann}(C(K)_\pi^*)$ does not complement $C(K)_\pi$. However $C(K)_\pi$ has $\pi(G)$-invariant complements $C_0(0,1] = \{\phi \in C(K) \mid \phi(0) = 0\}$ and $C_0[0,1) = \{\phi \in C(K) \mid \phi(1) = 0\}$ (and many others).

However, we will show that if an action is nice enough (namely, Lyapunov stable) then the decomposition

$$C(K) = C(K)_\pi \oplus \mathrm{Ann}(C(K)_\pi^*)$$

does hold.

Definition 14 ([8]). An action of G on K is *Lyapunov stable* if for any $\epsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $d(x, y) < \delta$ implies $d(gx, gy) < \varepsilon$ for all $g \in G$.

Remark 15. The original definition in [8] was different: for any $x \in K$ and $\epsilon > 0$ there must exist $\delta = \delta(x, \varepsilon) > 0$ such that $d(x, y) < \delta$ implies $d(gx, gy) < \varepsilon$ for all $g \in G$. But a standard compactness argument shows that for compact K the two definitions coincide. Indeed, for each $x \in K$, let $U_x = \{y \in K : d(y, x) < \delta(x, \varepsilon/2)/2\}$ and choose a finite subcovering $U_{x_1}, \ldots, U_{x_n}$. Let $\delta = \min_i \delta(x_i, \varepsilon/2)/2$, $i = 1, \ldots, n$. If $d(x, y) \leq \delta$, and one takes i with $x \in U_{x_i}$, then $d(y, x_i) \leq \delta + \delta(x_i, \varepsilon/2)/2 \leq \delta(x_i, \varepsilon/2)$. It follows that $d(gx, gx_i) \leq \varepsilon/2$ and $d(gy, gx_i) \leq \varepsilon/2$, whence $d(gx, gy) \leq \varepsilon$.

Usually in the literature Lyapunov stable actions are called equicontinuous, but in [8] the authors use the term “equicontinuous” for a different class of actions.

Below, π will always be the representation induced by some group action on a compact metric space K.

Lemma 16. *Let an action of G on K be Lyapunov stable and π be as above. Then for any $\phi \in C(K)$, its orbit $O(\phi)$ is precompact.*

Proof. It is easy to see that Lyapunov stability implies that for any $\phi \in C(K)$, $O(\phi)$ is an equicontinuous family of functions. Since $O(\phi)$ is bounded, it is precompact by the Arzelà–Ascoli theorem. ∎

Theorem 17. *Let an action of G on K be Lyapunov stable and π be as above. Then*

$$C(K) = C(K)_\pi \oplus \operatorname{Ann}(C(K)^*_\pi).$$

The corresponding projection onto $C(K)_\pi$ is a conditional expectation.

Proof. Lemma 16 and Proposition 4 imply that the decomposition $C(K) = C(K)_\pi \oplus \operatorname{Ann}(C(K)^*_\pi)$ holds, and since the representation is isometric, the corresponding projection onto $C(K)_\pi$ has norm 1. Since $C(K)$ and $C(K)_\pi$ are C^*-algebras, by [13] it is a conditional expectation. ∎

Now we obtain a short proof of an assertion in [8] on the uniqueness of invariant measures. Our proof also implies the existence of an invariant measure.

Corollary 18 ([8, Lemma 6.1]). *Suppose a group G acts on a compact metric space K in such a way that the orbit of some point $a \in K$ is dense in K. If the action is Lyapunov stable, then K carries no more than one invariant regular probability measure.*

Proof. Since the orbit of $a \in K$ is dense in K, the only invariant functions are constants, so $C(K)_\pi = \mathbb{C}$. By Theorem 17 this implies that $\operatorname{codim}(\operatorname{Ann}(C(K)^*_\pi)) = 1$. Hence $\dim C(K)^*_\pi = 1$. The latter means that there is exactly one invariant regular measure on K, because regular measures are in one-to-one correspondence with points of $C(K)^*$ by the Riesz theorem. ∎

4. Lower semicontinuous actions. Now we will show that for actions more general than Lyapunov stable ones, namely for lower semicontinuous actions, the subspace $C(K)_\pi$ is complemented.

Let K be a compact metric space and M be a partition of K into closed subsets (parts). For $x \in K$, let $M(x)$ denote the part which contains x. According to the standard definitions, M is called *lower semicontinuous* if $\{x \in K \mid M(x) \cap U \neq \emptyset\}$ is an open set in K, for every open set U in K.

If $P \subseteq C(K)$ is a subspace, then the *P-partition* of K is the partition associated with the following equivalence relation R:

$$(x, y) \in R \text{ if and only if } p(x) = p(y) \text{ for every } p \in P.$$

Definition 19. We will say that an action is *lower semicontinuous* if the corresponding $C(K)_\pi$-partition is lower semicontinuous.

We do not know any example of an action that is not lower semicontinuous. Let us show that this class of actions contains all Lyapunov stable actions.

We are going to use the following result from [8].

Theorem 20 ([8, Lemma 3.1]). *For a Lyapunov stable action, any two orbits are either separated from each other, or have the same closure. The quotient space of closures of orbits is Hausdorff.*

The following corollary shows that for Lyapunov stable actions the partition into closures of orbits and the $C(K)_\pi$-partition are the same.

Corollary 21. *Suppose we have a Lyapunov stable action and let R be the equivalence relation defining the $C(K)_\pi$-partition of K. Then $(x, y) \in R$ if and only if $\overline{O(x)} = \overline{O(y)}$.*

Proof. Since the functions in $C(K)_\pi$ are those which are constant on orbits, the "if" part follows.

To prove the "only if" part, assume that $\overline{O(x)} \neq \overline{O(y)}$. Then by Urysohn's lemma and Theorem 20, there is a continuous function ψ on the orbit space K/s such that $\psi(\overline{O(x)}) \neq \psi(\overline{O(y)})$. Define $\phi \in C(K)$ by $\phi(x) = \psi(\overline{O(x)})$. Then $\phi \in C(K)_\pi$ and $\phi(x) \neq \phi(y)$, hence $(x, y) \notin R$. ∎

Theorem 22. *Lyapunov stable actions are lower semicontinuous.*

Proof. For any open $U \subseteq K$ we need to check that the set

$$E = \{x \in K \mid (x,u) \in R \text{ for some } u \in U\}$$

is open. By Corollary 21 and Theorem 20,

$$\begin{aligned} E &= \{x \mid \overline{O(x)} = \overline{O(u)} \text{ for some } u \in U\} \\ &= \{x \mid \overline{O(x)} \cap U \neq \emptyset\} = \{x \mid O(x) \cap U \neq \emptyset\} = \bigcup_{g\in G} gU, \end{aligned}$$

which is obviously open. ∎

An easy example of a lower semicontinuous action which is not Lyapunov stable, is the action from Example 13. Indeed, for this action $C(K)_\pi$ is the subspace of constant functions, and hence the $C(K)_\pi$-partition consists of only one member, the whole interval $[0,1]$, which implies that the action is lower semicontinuous. On the other hand, this action is not Lyapunov stable because the ergodic decomposition fails for it.

For any subspace P of $C(K)$, let

$$K(P) = \bigcup\{K' \subseteq K \mid K' \text{ is a member of the } P\text{-partition of } K, \text{ and } K' \text{ contains more than one point of } K\} \tag{2}$$

(in other words $K(P)$ is the complement of the union of all one-element parts). According to [15], P has a *lower semicontinuous quotient* if the restriction of the P-partition to $\overline{K(P)}$ is lower semicontinuous.

LEMMA 23. *Let $P \subseteq C(K)$ be a subspace such that the P-partition of K is lower semicontinuous. Then P has a lower semicontinuous quotient.*

Proof. It suffices to show that a subpartition of a lower semicontinuous partition is lower semicontinuous. Let M be a lower semicontinuous partition and M_0 be its subpartition. Let K_0 be the closure of the union of all members of M_0. Suppose that U is open in K_0. We need to show that $\{x \in K_0 \mid M_0(x) \cap U \neq \emptyset\}$ is open in K_0. Since $U \cup \{K \setminus K_0\}$ is open in K (because its complement is $K_0 \setminus U$) and M is lower semicontinuous, the set

$$\{x \in K \mid M(x) \cap (U \cup \{K \setminus K_0\}) \neq \emptyset\}$$

is open, whence

$$\{x \in K_0 \mid M_0(x) \cap U \neq \emptyset\} = \{x \in K \mid M(x) \cap (U \cup \{K \setminus K_0\}) \neq \emptyset\} \cap K_0$$

is open in K_0. ∎

PROPOSITION 24. *Suppose a G-action on K is lower semicontinuous. Then $C(K)_\pi$ is complemented.*

Proof. Obviously $C(K)_\pi$ is a C^*-subalgebra of $C(K)$, and hence is isomorphic to $C(Z)$ for some Hausdorff space Z. The statement now follows from Lemma 23 and [15, Th. 4]. ∎

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Tatiana Shulman
Institute of Mathematics
Polish Academy of Sciences
00-656 Warszawa, Poland
E-mail: tshulman@impan.pl