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Uniformly rigid models for rigid actions

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Abstract. We show that any ergodic nonperiodic rigid system can be topologically realized by a uniformly rigid and (topologically) weak mixing topological dynamical system.

1. Introduction. A fundamental problem in ergodic theory and topological dynamics is the one of recurrence. In this paper we are interested in the relation in the measurable and topological context of a special strong form of recurrence, called rigidity. The main result states that any ergodic rigid system can be topologically realized in a uniformly rigid and topologically weakly mixing system.

A measure preserving system $(X, \mathcal{X}, \mu, T)$ is *rigid* if there exists an increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that T^{n_i} converges to the identity in the strong operator topology. This means that for any $f \in L^2(\mu)$, we have $\|f - f \circ T^{n_i}\|_2 \to 0$ as $i \to \infty$. This is also equivalent to saying that $\mu(A \cap T^{n_i}A)$ converges to $\mu(A)$ for any measurable set A. Usually one refers to $(n_i)_{i\in\mathbb{N}}$ as a *rigidity sequence* of $(X, \mathcal{X}, \mu, T)$. Very recently, nice results have been found about what sequences can be rigidity sequences for weakly mixing systems [3, 6, 7].

Topological analogues of rigidity were introduced by Glasner and Maon [9]. A topological dynamical system (X, T) is (*topologically*) *rigid* if there exists an increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $T^{n_i}x$ converges to x as $i \to \infty$ for every $x \in X$ (i.e. T^{n_i} converges pointwise to the identity). A topological dynamical system is *uniformly rigid* if $\sup_{x\in X} d(x, T^{n_i}x) \to 0$ as $i \to \infty$, i.e. T^{n_i} converges uniformly to the identity map. It is clear that uniform rigidity implies rigidity but the converse is not true even for minimal systems [9, 14]. By the Lebesgue dominated convergence theorem, if (X, T) is topologically rigid then $(X, \mathcal{B}(X), \mu, T)$ is rigid in the measurable setting for any invariant

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measure μ, where $\mathcal{B}(X)$ is the Borel σ-algebra. So, as one could expect, topological rigidity is a much stronger notion than measurable rigidity. We refer to [10] for an expanded discussion on rigidity in the measurable and topological setting. However, we will show that there is no real difference from the measurable point of view. Our main result states that any ergodic rigid system can be topologically realized in a uniformly rigid system.

Let $(X, \mathcal{X}, \mu, T)$ be an ergodic dynamical system. We say that $(\hat{X}, \mathcal{B}(\hat{X}), \hat{\mu}, \hat{T})$ is a *topological model* (or just a *model*) for $(X, \mathcal{X}, \mu, T)$ if $(\hat{X}, \hat{T})$ is a topological system, $\hat{\mu}$ is an invariant Borel probability measure on $\hat{X}$, and the systems $(X, \mathcal{X}, \mu, T)$ and $(\hat{X}, \mathcal{B}(\hat{X}), \hat{\mu}, \hat{T})$ are measure-theoretically isomorphic. In this case, one also says that $(X, \mathcal{X}, \mu, T)$ can be (topologically) realized by $(\hat{X}, \hat{T})$.

THEOREM 1.1. *Let $(X, \mathcal{X}, \mu, T)$ be a nonperiodic ergodic invertible measure preserving system, rigid for the sequence $(n_i)_{i\in\mathbb{N}}$. Then there exists a topological model $(\widehat{X}, \widehat{T})$ for $(X, \mathcal{X}, \mu, T)$ which is uniformly rigid for a subsequence of $(n_i)_{i\in\mathbb{N}}$. Moreover, $(\widehat{X}, \widehat{T})$ can be taken topologically weak mixing.*

Letting $\mathcal{A}$ be the algebra of continuous functions on $\widehat{X}$ we deduce

COROLLARY 1.2. *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure preserving system, rigid for the sequence $(n_i)_{i\in\mathbb{N}}$. Then there exists a subsequence $(n'_i)_{i\in\mathbb{N}}$ of $(n_i)_{i\in\mathbb{N}}$ and a separable subalgebra $\mathcal{A} \subset L^\infty(\mu)$ which is dense in $L^2(\mu)$ such that $\|f - f \circ T^{n'_i}\|_\infty \to 0$ for any $f \in \mathcal{A}$.*

In [9] this result is attributed to Weiss but the proof has not been published.

A sequence $(n_i)_{i\in\mathbb{N}}$ is called a *rigidity sequence* if there exists a measure preserving system for which $(n_i)_{i\in\mathbb{N}}$ is a rigidity sequence. Since in Theorem 1.1 we get a subsequence of the original sequence, a natural question arises:

PROBLEM 1.3. *Give conditions for $(n_i)_{i\in\mathbb{N}}$ to be a uniform rigidity sequence for a nonperiodic topologically weakly mixing dynamical system. Is there a sequence $(n_i)_{i\in\mathbb{N}}$ which is a rigidity sequence in the measurable category, but it is not a uniform rigidity sequence in the topological category?*

2. Preliminaries

2.1. Measurable and topological systems. A *measure preserving system* is a 4-tuple $(X, \mathcal{X}, \mu, T)$ where $(X, \mathcal{X}, \mu)$ is a probability space and T is a measurable measure preserving transformation on X. In this paper, we assume that T is invertible and both T and T^{-1} are measure preserving transformations. It is *ergodic* if every invariant set has measure 0 or 1. For an ergodic system, either the space X consists of a finite set of points on which μ is equidistributed, or the measure μ is atomless. In the former case the system is called *periodic*, and in the latter *nonperiodic*.

A *topological dynamical system* is a pair (X, T) where X is a compact metric space and $T\colon X \to X$ is a homeomorphism. It is said to be *transitive* when there is a point $x \in X$ whose orbit $\{T^n x : n \in \mathbb{Z}\}$ is dense in X. It is *minimal* if each point has a dense orbit. A topological dynamical system is *weakly mixing* if the Cartesian product system $(X \times X, T \times T)$ is transitive. This is equivalent to the condition that for any nonempty open sets A, B, C, D, there exists $n \in \mathbb{Z}$ such that $A \cap T^{-n}B \neq \emptyset$ and $C \cap T^{-n}D \neq \emptyset$. A topological dynamical system is (*strongly*) *mixing* if for any nonempty open sets A, B there exists $M \in \mathbb{N}$ such that $A \cap T^{-n}B \neq \emptyset$ for any $n \in \mathbb{Z}$ with $|n| \geq M$.

By the Krylov–Bogolyubov theorem, any topological dynamical system (X, T) admits a nonempty convex set of invariant probability measures, which is denoted by $M(X, T)$. The extremal points of $M(X, T)$ are the ergodic measures.

A deep link between measure preserving systems and topological dynamical systems is the Jewett–Krieger Theorem [13, 15], which asserts that any ergodic nonperiodic measure preserving system is measurably isomorphic to a uniquely ergodic topological dynamical system (X, T), meaning that (X, T) has only one invariant measure (which is ergodic). Many generalization of the Jewett–Krieger Theorem in different contexts have been found [16, 19], and very recently several applications have been given to the pointwise convergence of different ergodic averages [4, 5, 12] and to the construction of interesting examples in topological dynamics [17]. All these recent results show that topological dynamical systems can help to understand purely ergodic problems.

2.2. Partitions. Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system. A *partition* α of X is a family of disjoint measurable subsets of X whose union is X. Let α and β be two partitions of $(X, \mathcal{X}, \mu, T)$. We say that α *refines* β, denoted by $\alpha \succ \beta$ or $\beta \prec \alpha$, if each element of β is a union of elements of α. The relation $\alpha \succ \beta$ is equivalent to $\sigma(\beta) \subseteq \sigma(\alpha)$, where $\sigma(\mathcal{A})$ is the σ-algebra generated by the family $\mathcal{A}$.

Let α and β be two partitions. Their *join* is the partition $\alpha \vee \beta = \{A \cap B : A \in \alpha,\ B \in \beta\}$ and one can extend this definition naturally to a finite number of partitions. For $m \leq n$, define

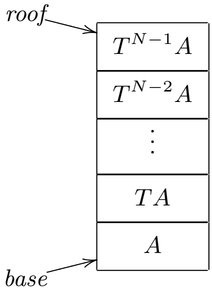
$$\alpha_m^n = \bigvee_{i=m}^{n} T^{-i}\alpha = T^{-m}\alpha \vee T^{-(m+1)}\alpha \vee \cdots \vee T^{-n}\alpha,$$

where $T^{-i}\alpha = \{T^{-i}A : A \in \alpha\}$.

2.3. Rohklin towers. Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure preserving system and let A be a measurable set. If $N \in \mathbb{N}$ and the sets $A, TA, \dots, T^{N-1}A$ are pairwise disjoint, then the array

$$\mathfrak{c} = \{A, TA, \dots, T^{N-1}A\}$$

is called a *column* or *Rohklin tower* with base A and height N. We usually refer to the sets T^iA, $i = 0, \dots, N-1$, as the *levels* of the column. The levels A and $T^{N-1}A$ are called the *base* and the *roof* respectively.



A set $\mathfrak{t}$ is called a *tower* if it is a disjoint union of columns

$$\mathfrak{c}_\mathfrak{i} = \{A^i, TA^i, \dots, T^{N_i-1}A^i\}, \quad i = 1, \dots, l.$$

The union $\bigcup_{i=1}^{l} A^i$ of the bases is the *base* of $\mathfrak{t}$, and the union $\bigcup_{i=1}^{l} T^{N_i-1}A^i$ of the roofs is the *roof* of $\mathfrak{t}$.

2.4. Kakutani–Rokhlin towers. For an ergodic system $(X, \mathcal{X}, \mu, T)$, let $B \in \mathcal{X}$ be a set of positive measure. Then it is clear that $\bigcup_{n\geq 0} T^nB = X$ (mod μ). Define the *return time function* $r_B : B \to \mathbb{N} \cup \{\infty\}$ by

$$r_B(x) = \min\{n \geq 1 : T^nx \in B\}$$

when this minimum is finite, and $r_B(x) = \infty$ otherwise. Let $B_k = \{x \in B : r_B(x) = k\}$ and note that by Poincaré's recurrence theorem, B_∞ is a null set. Let $\mathfrak{c}_k$ be the column $\{B_k, TB_k, \dots, T^{k-1}B_k\}$. We call the tower

$$\mathfrak{t} = \mathfrak{t}(B) = \{\mathfrak{c}_k : k = 1, 2, \dots\}$$

the *Kakutani tower* over B. If the Kakutani tower over B has finitely many columns (i.e. the function r_B is bounded), we say that B has a *finite height* and we call the Kakutani tower over B a *Kakutani-Rokhlin tower* or a *K-R tower*. The number $\max r_B$ is called the *height* of B or the *height* of the K-R tower.

We will need the following useful lemma (see [8, 20, 21] for a proof), which is a special case of the Alpern lemma [1].

Lemma 2.1. *Let $(X, \mathcal{X}, \mu, T)$ be a nonperiodic ergodic system. For any positive integers N_1, N_2 with $\gcd(N_1, N_2) = 1$, there exists a set C of finite height such that the K-R tower $\mathfrak{t}(C)$ satisfies* $\operatorname{range} r_C = \{N_1, N_2\}$.

2.5. Refining a tower according to a partition. Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system. Let $\mathfrak{t}$ be a tower with columns $\{\mathfrak{c}_k : k \in K\}$ (K is finite or countable) and base $B = \bigcup_{k \in K} B_k \subseteq X$. Given a partition (finite or countable) α of X, we define an equivalence relation on B as follows: $x \sim y$ iff x and y are in the same base B_k and for every $0 \le j < N_k$, $T^j x$ and $T^j y$ are in the same element of α, i.e. x and y have the same (α, N_k)-*name*. Now we consider each equivalence class $B_{k,\mathbf{a}}$, with $\mathbf{a}$ an (α, N_k)-name, as a base of the column $\mathfrak{c}_{k,\mathbf{a}} = \{B_{k,\mathbf{a}}, TB_{k,\mathbf{a}}, \dots, T^{N_k - 1} B_{k,\mathbf{a}}\}$ and say that the resulting tower $\mathfrak{t}_\alpha = \{\mathfrak{c}_{k,\mathbf{a}} : \mathbf{a} \in \alpha^{N_k}, k \in K\}$ is the *tower* $\mathfrak{t}$ *refined according to* α. We usually refer to the columns of the refined tower as *pure columns*.

2.6. Symbolic dynamics. Let Σ be a set. Let $\Omega = \Sigma^{\mathbb{Z}}$ be the set of all sequences $\omega = \dots \omega_{-1}\omega_0\omega_1 \dots = (\omega_n)_{n \in \mathbb{Z}}$, $\omega_n \in \Sigma$, $n \in \mathbb{Z}$, endowed with the product topology. The shift map $\sigma : \Omega \to \Omega$ is defined by $(\sigma\omega)_n = \omega_{n+1}$ for all $n \in \mathbb{Z}$. The pair (Ω, σ) is called the *full shift* over Σ. Any subsystem (closed and invariant subset) of (Ω, σ) is called a *subshift*.

Each element of $\Sigma^* = \bigcup_{k \ge 1} \Sigma^k$ is called a *word* or a *block* (over Σ). If $A = a_1 \dots a_n$, we use $|A| = n$ to denote its length. If $\omega = \dots \omega_{-1}\omega_0\omega_1 \dots \in \Omega$ and $a \le b \in \mathbb{Z}$, then $\omega[a, b] =: \omega_a \omega_{a+1} \cdots \omega_b$ is the $(b-a+1)$-word occurring in ω starting at place a and ending at place b. Similarly we define $A[a, b]$ when A is a word. A word A *appears* in the word B if there are some $a \le b$ such that $B[a, b] = A$.

For $n \in \mathbb{N}$ and words $A_1, \dots, A_n$, we denote by $A_1 \dots A_n$ the concatenation of $A_1, \dots, A_n$. When $A_1 = \dots = A_n = A$ denote $A_1 \dots A_n$ by A^n. If (X, σ) is a subshift, let $[i] = [i]_X = \{\omega \in X : \omega(0) = i\}$ for $i \in \Sigma$, and $[A] = [A]_X = \{\omega \in X : \omega_0\omega_1 \dots \omega_{(|A|-1)} = A\}$ for any word A.

2.7. Symbolic representation. Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure preserving system. Given a measurable function $f\colon X \to \Sigma \subseteq [0,1]$, one can define the *itinerary homomorphism* f^∞ from X to $\Omega := [0,1]^{\mathbb{Z}}$ by $f^\infty(x) = \omega$, where

$$\omega_n = f(T^n x).$$

The distribution of the stochastic process $(f^\infty)_*(\mu)$ (defined by $(f^\infty)_*(\mu)(A) = \mu((f^\infty)^{-1}(A))$ for each Borel $A \subset [0,1]^{\mathbb{Z}}$) is denoted by $\rho(X, f)$, and we call it the *representation measure of* (X, T) *given* by f. When the system $(X, \mathcal{X}, \mu, T)$ under consideration is fixed, we just write ρ instead of $\rho(X, f)$ for convenience.

Let

$$X_f = \operatorname{supp}((f^\infty)_*(\mu)) = \operatorname{supp}(\rho).$$

Then we get a homomorphism $f^\infty\colon (X, \mathcal{X}, \mu, T) \to (X_f, \mathcal{X}_f, \rho, \sigma)$, called the *representation* of the process (X, f).

An important case is when we consider a finite partition $\alpha = \{A_j\}_{j\in\Sigma}$ (we assume $\mu(A_j) > 0$ for all j). Here $\Sigma \subset [0,1]$ is a subset of real numbers. We think of the partition α as the function f_α defined as $f_\alpha(x) = j$ if $x \in A_j$. Equivalently, when f has finitely many values $\{a_1,\ldots,a_k\}$, we can think of f as the function given by the partition $\alpha = \{A_j\}_{j\in\Sigma}$ where $A_j = f^{-1}(a_j)$. Let (X,α) denote the representation (X, f_α) and call it the *symbolic representation* given by the partition α.

This will not be a model for $(X,\mathcal{X},\mu,T)$ unless $\bigvee_{i=-\infty}^{\infty} T^{-i}\alpha = \mathcal{X}$ modulo null sets.

2.8. Copying names. An important way to produce partitions (equivalently, finite valued functions) is by copying or painting names on towers.

If $\mathfrak{c} = \{T^jB\}_{j=0}^{N-1}$ is a column and $\mathbf{a} \in \Sigma^N$ then *copying the name* $\mathbf{a}$ *on the column* $\mathfrak{c}$ means that on $\bigcup_{j=0}^{N-1} T^jB$ we define a partition (may be not on the whole space) by letting

$$A_k = \bigcup\{T^jB : \mathbf{a}_j = k\}, \quad k \in \Sigma.$$

If there is a tower $\mathfrak{t}$ with q columns $\mathfrak{c}_i = \{T^jB_i\}_{j=0}^{N_i-1}$, and q names $\mathbf{a}(i) \in \Sigma^{N_i}, i = 1,\ldots,q$, then copying these names on $\mathfrak{t}$ means we copy each name $\mathbf{a}(i)$ on column $\mathfrak{c}_i$, i.e. we define a partition by

$$A_k = \bigcup\{T^jB_i : \mathbf{a}(i)_j = k,\ i = 1,\ldots,q\}, \quad k \in \Sigma.$$

These partitions can be extended to a partition $\alpha = \{A_{a_1},\ldots,A_{a_l}\}$ of the whole space by assigning, for example, the value a_1 to the rest of the space.

3. Proof of Theorem 1.1. In this section we prove Theorem 1.1. For the sake of clarity, we divide the proof into two steps. First, we prove that we can realize an ergodic rigid system in a uniformly rigid topological dynamical system, and then we show how to add the (topologically) weakly mixing condition.

Let $(X,\mathcal{X},\mu,T)$ be an ergodic rigid system with rigidity sequence $(n_i)_{i\in\mathbb{N}}$. We start by considering a special topological model: we may assume, by [16], that (X,T) is a minimal (strongly) mixing subshift (in fact, since a rigid system has zero entropy, we may consider a subshift over two symbols [20, 11], but we do not need this property).

3.1. Proof strategy. First, it is worth noting that a model given by a finite partition does not suit our purposes, as the following remark shows:

Remark 3.1. Let (X,σ) be a nonperiodic (equivalently infinite) subshift. Then (X,σ) is not rigid.

Proof. It is well-known that infinite symbolic systems always have a forward asymptotic pair (see [2, Chapter 1] for example), i.e. there exist

$\omega, \omega' \in X$ such that $\omega_0 \neq \omega'_0$ and $\omega_n = \omega'_n$ for all $n \geq 1$. If (X, σ) is rigid for the sequence $(n_i)_{i \in \mathbb{N}}$ then $\sigma^{n_i} \omega \to \omega$ and $\sigma^{n_i} \omega' \to \omega'$, which implies that $\omega_0 = \omega_{n_i} = \omega'_{n_i} = \omega'_0$, a contradiction. ∎

The proof of Theorem 1.1 relies on the idea of building a topological model for an ergodic system using itineraries of a given function. This idea was already used in [11, 20] to find special models for systems with zero entropy. Let $f \colon X \to [0,1]$ be a measurable function. Recall that the *itinerary function* $f^\infty \colon X \to [0,1]^{\mathbb{Z}}$ is

$$f^\infty(x) = (\ldots, f(T^{-2}x), f(T^{-1}x), f(x), f(Tx), f(T^2x), \ldots),$$

and that the topological system associated to f is the support of the measure $(f^\infty)_*(\mu)$ in $[0,1]^{\mathbb{Z}}$ endowed with the shift action.

The function $f \colon X \to [0,1]$ *generates for T* if the σ-algebra generated by the functions $f \circ T^n$, $n \in \mathbb{Z}$, is all of $\mathcal{X}$ (modulo null sets). This is equivalent to there being a set of full measure A on which the itinerary function f^∞ is injective (see [18, Chapter 1] for a reference). Thus, when f generates for T, the itinerary function f^∞ is an isomorphism between $(X, \mathcal{X}, \mu, T)$ and $(X_f, \mathcal{X}_f, \rho, \sigma)$.

The general strategy consists in finding a sequence $(f_i)_{i \in \mathbb{N}}$ of functions, where f_{i+1} and f_i differ on a set of small measure, so that there exists a pointwise limit function f. The functions f_i are required suitable properties so that the corresponding topological system associated to f has the properties we are looking for.

Each f_i will generate for T, and we will guarantee that f generates for T by controlling the speed of convergence of f_i to f. The f_i's will be continuous and each will take only finitely many values, so we may identify them with finite partitions α_i of X into clopen sets, where $f_i : X \to \{a_1, \ldots, a_{m_i}\} \subseteq [0,1]$ and $\alpha_i = \{A_1, \ldots, A_{m_i}\}$ with $A_j = f_i^{-1}(a_j)$.

In our case, the condition we need is that any function f_i be close to uniformly rigid. We introduce the following definition.

Definition 3.2. We say that $f \colon X \to [0,1]$ is *ϵ-good at n* if

$$\|f - f \circ T^n\|_\infty < \epsilon.$$

Here $\|\cdot\|_\infty$ stands for the essential supremum norm. Of course if f is continuous this coincides with the supremum norm.

Let $(K_i)_{i \in \mathbb{N}}$ be a sequence of positive integers such that $\sum_{i=1}^\infty 1/K_i < \infty$.

Our goal is to build a sequence $(f_i)_{i \in \mathbb{N}}$ of generating continuous functions and a subsequence $(n'_i)_{i \in \mathbb{N}}$ of $(n_i)_{i \in \mathbb{N}}$ such that

$$\text{(3.1)} \qquad f_i \text{ is } \Big(\sum_{l=j}^{i} 1/K_l\Big)\text{-good at } n'_j \text{ for any } j \leq i \text{ and } \mu(\{f_i \neq f_{i+1}\}) < r_i,$$

where r_i goes to 0 fast enough (for instance $r_i = 2^{-i}$). In this case, we say that the sequence $(f_i)_{i\in\mathbb{N}}$ is *good for the sequence* $(K_i)_{i\in\mathbb{N}}$.

We will also require that the cardinality of the image of f_{i+1} be strictly larger than the one for f_i. This guarantees that the pointwise limit of f_i is well defined and also generates for T. To see that $f = \lim f_i$ generates for T, note that since the functions f_i are generating, there exists a set A of full measure where all f_i^∞ are injective (see [18, Chapter 1] for example). The Borel–Cantelli lemma ensures that in a set B of full measure, $x \in B$ implies that $f^\infty(x) = f_i^\infty(x)$ for some $i \in \mathbb{N}$. So if $x, y \in A \cap B$ and $f^\infty(x) = f^\infty(y)$, then there exist i, j such that $f_i^\infty(x) = f_j^\infty(y)$. We can assume $j > i$, since $i = j$ is not possible by the injectivity of f_i^∞ in A. There is an open subset of X where each value of f_j is different from all values of f_i (recall that the functions are continuous). The minimality of (X, T) implies that $f_i(T^n x) \neq f_j(T^n y)$ for some n. This shows that f^∞ is injective on a set of full measure, and so f generates for T.

3.2. Some facts. Our proof is based on modifying a tall enough tower. We do so by taking averages between given portions of a subcolumn. We formalize this idea with the next definition.

Let $A = a_1 \dots a_n$ and $B = b_1 \dots b_n$ be two blocks and $\lambda \in \mathbb{R}$. Write $\lambda A = (\lambda a_1) \dots (\lambda a_n)$ and $A \pm B = (a_1 \pm b_1) \dots (a_n \pm b_n)$.

Definition 3.3. Let $A = a_1 \dots a_n, B = b_1 \dots b_n \in [0,1]^n$ and $K \in \mathbb{N}$. We say that $C = c_1 \dots c_{(K+1)n}$ is a *transition* from A to B in K steps if C is the concatenation of the blocks $A + \frac{j}{K}(B - A)$ for $j = 0, \dots, K$.

Remark 3.4. A and B represent two given subcolumns of length n, and C represents a subcolumn of length $(K+1)n$ where the first n and last n levels are A and B respectively.

Lemma 3.5. *Let $A = a_1 \dots a_n, B = b_1 \dots b_n \in [0,1]^n$ and let $C = c_1 \dots c_{(K+1)n} \in [0,1]^{(K+1)n}$ be the transition from A to B in K steps. Then for any $l = 1, \dots, Kn$ we have*

$$|c_l - c_{l+n}| \leq 1/K.$$

Remark 3.6. This lemma shows that if K is large enough then we have a "smooth" K-step transition between two blocks of the same length, which will be useful to ensure rigidity.

Proof of Lemma 3.5. There exist $j \leq K - 1$ and $1 \leq p \leq n$ such that

$$c_l = \frac{K-j}{K} a_p + \frac{j}{K} b_p \quad \text{and} \quad c_{l+n} = \frac{K-j-1}{K} a_p + \frac{j+1}{K} b_p.$$

Thus $c_l - c_{l+n} = (a_p - b_p)/K$, and the result follows. ∎

The next lemma shows that if two blocks have a similar top and bottom, then when performing a transition between them, the top of a block and the bottom of the consecutive one have a "smooth" transition. This will be useful in order to get the first property in (3.1).

LEMMA 3.7. *Let* $A = a_1 \dots a_n, B = b_1 \dots b_n \in [0,1]^n$ *and let* $C = c_1 \dots c_{(K+1)n} \in [0,1]^{(K+1)n}$ *be the transition from* A *to* B *in* K *steps. Let* $n/2 \geq p \geq l \geq 0$. *If* $|a_{n-p+l} - a_l| \leq \delta$ *and* $|b_{n-p+l} - b_l| \leq \delta$ *(i.e.* A *and* B *have similar top and bottom) then for every* $j = 0, \dots, K-1$ *we have* $|c_{jn+n-p+l} - c_{(j+1)n+l}| \leq \delta + 1/K$.

REMARK 3.8. We think of the term $c_{jn+n-p+l}$ as some level close to the top of the block $A + \frac{j}{K}(B-A)$, while $c_{(j+1)n+l}$ is a level close to the bottom of the block $A + \frac{j+1}{K}(B-A)$.

Proof. By definition we have

$$\begin{aligned} c_{jn+n-p+l} - c_{(j+1)n+l} &= a_{n-p+l} + \frac{j}{K}(b_{n-p+l} - a_{n-p+l}) - a_l - \frac{j+1}{K}(b_l - a_l) \\ &= \frac{K-j}{K}(a_{n-p+l} - a_l) + \frac{j}{K}(b_{n-p+l} - b_l) - \frac{b_l - a_l}{K}, \end{aligned}$$

and the result follows. ∎

LEMMA 3.9. *Let* $(X, \mathcal{X}, \mu, T)$ *be a measure preserving rigid system. Then for each* $k \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T^k)$ *is also rigid.*

Proof. Let $(n_i)_{i\in\mathbb{N}}$ be a rigidity sequence for T and let $f \in L^2(\mu)$. We have $\|f - f \circ T^{n_i k}\|_2 \leq \sum_{j=0}^{k-1} \|f \circ T^{n_i j} - f \circ T^{n_i(j+1)}\|_2 = k\|f - f \circ T^{n_i}\|_2 \to 0$. We conclude that $(n_i)_{i\in\mathbb{N}}$ is also a rigidity sequence for T^k. ∎

3.3. Proof of Theorem 1.1: Getting a uniformly rigid model. We now proceed to prove Theorem 1.1. Recall that we assume that (X,T) is a minimal (strongly) mixing subshift and we consider a sequence $(r_i)_{i\in\mathbb{N}}$ of positive numbers converging to 0 fast enough (for instance $r_i = 2^{-i}$).

Let $(K_i)_{i\in\mathbb{N}}$ be an increasing sequence of positive integers such that $\sum 1/K_i < \infty$. For simplicity we assume $K_0 = 1$. We construct the sequence $(f_i)_{i\in\mathbb{N}}$ of functions good for $(K_i)_{i\in\mathbb{N}}$ inductively.

Let $\alpha_0 = \{A_1, \dots, A_{m_0}\}$ be a clopen generator for T, and $a_1, \dots, a_{m_0}$ be real numbers in $[0,1]$. Let $f_0 : X \to \{a_1, \dots, a_{m_0}\} \subseteq [0,1]$ be such that $A_j = f_i^{-1}(a_j)$ for $1 \leq j \leq m_0$. It is a continuous function and since $K_0 = 1$, we see that f_0 trivially satisfies the properties we require for any $n'_0 \in (n_k)_{k\in\mathbb{N}}$ (we consider values in $[0,1]$). To illustrate our method and make the proof clearer we show how to obtain f_1 from f_0.

STEP 1. Let α_0 denote the partition associated to the different values of f_0 (i.e. α_0 is the canonical partition at the origin). Consider the integer K_1 and the positive number r_1. Since f_0 has finitely many values, there exists a constant $c_0 > 0$ such that $|f_0(x) - f_0(y)| \leq c_0$ implies $f_0(x) = f_0(y)$.

For $k \in \mathbb{N}$, consider the set

$$A_{0,k} := \{x \in X : |f_0(x) - f_0(T^{ln_k}x)| > c_0 \text{ for some } l \in [1, 2K_1] \cap \mathbb{N}\}.$$

Since, by Lemma 3.9, the transformations $T, T^2, \ldots, T^{2K_1}$ are rigid for the sequence $(n_k)_{k\in\mathbb{N}}$, the measure of $A_{0,k}$ goes to 0 as $k \to \infty$. By our choice of c_0, the condition $x \in A_{0,k}^c$ implies that $f_0(x) = f_0(T^{n_k}x) = \cdots = f_0(T^{2K_1 n_k}x)$.

We pick n_{k_1} such that the measure of A_{0,k_1} is smaller than $r_1/(4K_1)$ and we set $A_0 = A_{0,k_1}$ and $n'_1 = n_{k_1}$.

We can use Lemma 2.1 to build a large Kakutani–Rokhlin tower of heights H_1 and $H_1 + 1$ (and with a clopen base). We then refine this column according to the α_0-names. We can assume that H_1 has the form $2K_1 n'_1 N_1 + n'_1$, where $1/N_1 \leq r_1/6$. We can subdivide every pure column into N_1 subcolumns of length $2K_1 n'_1$, from bottom to top. We call these subcolumns *principal*. The remaining n'_1 levels are called the *top*. For convenience, for those columns whose height is $H_1 + 1$ we add the top level to the top (so the top has n'_1 or $n'_1 + 1$ levels). Similarly, the first n'_1 levels are the *bottom* of the column (see Figure 1).

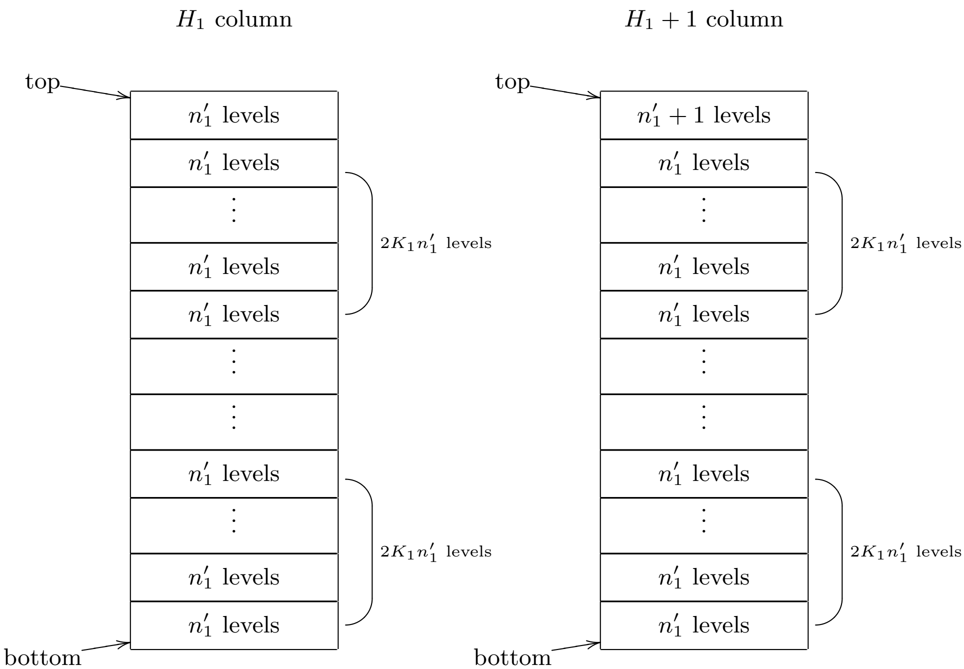


Fig. 1. Principal subcolumns and top of a tower

Our aim is to modify f_0 to f_1 such that $|f_1(x) - f_1(T^{n'_1}x)| < 1/K_1$ for every $x \in X$. Translated to columns, this means that the difference of levels at distance n_1' is smaller than $1/K_1$. Since c_0 is small enough, in many cases two such levels are equal, but there is a small portion where this does not happen. We fix this problem by allowing the levels to take more values between 0 and 1. Now we explain how to do this. Let us consider two consecutive principal subcolumns and consider the first n_1' levels of each of them. We remark that if one level is in A_0^c (meaning that the corresponding set of this level is a subset of A_0^c), then it is constant in the ln_1'-levels above it for $l = 1, \dots, 2K_1$. Indeed, this property characterizes belonging to A_0: a level that is in A_0 will change its value in some of the levels ln_1', $l = 1, \dots, 2K_1$, above it. We correct these values as follows:

STEP 1-I: *Modification of the top and the bottom.* We change the values of the top and the bottom of any pure column putting 0's, i.e. we paint (recall Section 2.8) the bottom and top with the 0 symbol on each level. This step is to ensure that the transition from one pure column to another one is $1/K_1$-good at n_1'. We may lose the property that f_0 is a generating function, but we fix this later at the end of the next step.

STEP 1-II: *Modification inside a pure column.* Consider two consecutive principal subcolumns and look at the first n_1' levels of the first one and the first n_1' levels of the second. Perform a transition in $2K_1$ steps between these two subcolumns. Lemma 3.5 ensures that all levels of the first principal subcolumn become $1/K_1$-good at n_1'.

This of course may change the $2K_1 - 1$ remaining levels of the first principal subcolumn, but in fact not many of them are modified: among the first n_1' levels, those belonging to A_0^c remain unchanged in their n_1' translations. Recall that this follows from the fact that if $x \in A_0^c$ then $|f_0(x) - f_0(T^{ln_1'}x)| \le c_0$ for all $l = 1, \dots, 2K_1$, which implies that $f_0(x) = f_0(T^{n_1'}x) = \cdots = f_0(T^{2K_1 n_1'}x)$.

On the other hand, we remark that for any level in A_0 we change at most $2K_1 - 1$ levels, so the number of levels we have changed in the first principal subcolumn is at most

$$(2K_1 - 1)\#(\text{levels in } A_0 \text{ in the first } n_1' \text{ levels}).$$

We repeat this process for all principal subcolumns, remarking that in the last one we perform the transition using the top (which has zeros). Therefore, any level is $1/K_1$-good for n_1'. It remains to show that we have modified f_0 on a small set.

For the first and last principal subcolumn and the top n_1' levels we may change all levels, which number no more than $4K_1 n_1' + n_1'$. For any other principal subcolumn we do not change more than

$$(2K_1 - 1)\#(\text{levels in } A_0 \text{ in the first } n_1' \text{ levels})$$

levels. Therefore, in any pure column we change at most

$$(4K_1 + 1)n'_1 + (2K_1 - 1)\#(\text{levels in } A_0)$$

levels (here the number of levels in A_0 is an upper bound for the number of levels we may find in the first n'_1 levels of the principal subcolumns).

Therefore, we modified any pure column in a proportion of at most

$$\frac{(4K_1 + 1)n'_1 + 1 + (2K_1 - 1)\#(\text{levels in } A_0)}{N_1 2K_1 n'_1 + n'_1},$$

and therefore we have changed f_0 on a set of measure smaller than

$$3/N_1 + (2K_1 - 1)\mu(A_0),$$

which is less than r_1 by our assumptions. Since all levels are clopen sets, we have built a continuous function f_1 (with finitely many values) whose associated partition α_1 is close to α_0 in the partition metric. The function f_1 is $1/K_1$-good at n'_1 and $1/K_0 + 1/K_1$-good at n'_0 (this last condition is trivial in this case).

We then make sure that all pure columns are different, modifying the first level of each by much less than all the constants involved, i.e. we paint (recall Section 2.8) the first level of each pure column with a different, but very small value. Recall that the definition of being good involves a strict inequality, so we have enough freedom to achieve this without violating the "good" property.

By making all pure columns different, we achieve that the sets defined by α_0 names of length H_1 are unions of different α_1 names of length H_1, which implies that α_1 is also a generating partition.

We remark that we have to perform the modification in the order given above. We need to perform transitions of blocks after modifications of the top and bottom, in order to correct the lack of rigidity we may have introduced.

Step $i+1$. The general case, obtaining f_{i+1} from f_i, is similar, but when trying to secure $1/K_{i+1}$-goodness at n'_{i+1} we have to be careful not to spoil the previous good conditions (at this step topological mixing will help us).

Suppose we are given f_i and $n'_1, \ldots, n'_i$ such that f_i is $(\sum_{l=j}^{i} 1/K_l)$-good at n'_j for $j \leq i$. We now show how to find n'_{i+1} and build f_{i+1} with the corresponding properties.

Since f_i takes finitely many values, there exists $c_i > 0$ such that the inequality $|f_i(x) - f_i(y)| \leq c_i$ implies that $f_i(x) = f_i(y)$.

Since (X, T) is topologically mixing, there exists $L_i \geq n'_i$ such that any couple of itineraries of length n'_i can be joined by an itinerary of any length greater than or equal to L_i.

Consider the set

$$A_{i,k} = \{x : |f_i(x) - f_i(T^{ln_k}x)| > c_i \text{ for some } l = 1, \dots, 2K_{i+1}\}.$$

Since $T, T^2, \dots, T^{2K_{i+1}}$ are rigid, for large enough k_{i+1} the measure of $A_{i,k_{i+1}}$ is smaller than $r_{i+1}/(6K_{i+1})$, and of course we can also require that $2L_i/n_{k_{i+1}} \leq r_i/3$.

Set $A_i = A_{i,k_{i+1}}$ and $n'_{i+1} = n_{k_{i+1}}$ as above. We remark that $x \in A_i^c$ implies that the values $f_i(x), f_i(T^{n'_{i+1}}x), \dots, f_i(T^{2K_{i+1}n'_{i+1}}x)$ are all equal.

We then use Lemma 2.1 to construct a tower with heights H_{i+1} and $H_{i+1} + 1$ and we can assume that $H_{i+1} = N_{i+1}2K_{i+1}n'_{i+1} + n'_{i+1}$, where $1/N_{i+1} \leq r_{i+1}/9$. Similarly to the first step, we subdivide every pure column into N_{i+1} subcolumns of length $2K_{i+1}n'_{i+1}$, from bottom to top, and we call these subcolumns *principal*. The remaining n'_{i+1} levels are called the *top*. Again, for those columns whose height is $H_{i+1} + 1$ we add the top level to the top. The first n'_{i+1} levels are the *bottom* of the column.

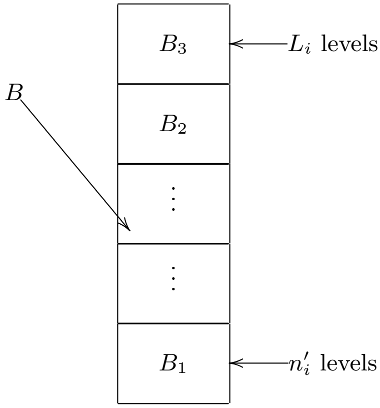
Refine the columns according to the names given by the partition α_i (associated to f_i). Pick a pure column and modify it according to the following steps:

STEP $(i+1)$-I: *Modification of the bottom and the top.* When we are close to the top of a column, we do not know where the point will lie after n'_{i+1} levels, so we will modify the bottoms and the tops of the columns so that the transitions satisfy the good conditions. To achieve this, we first modify the top and bottom of any pure column by putting 0's, i.e. we paint those levels with 0.

STEP $(i+1)$-II: *Guarantee not to spoil anything.* We may continue similarly to Step 1-II, i.e. performing transitions between blocks. Unfortunately, this does not suffice since by doing so we may violate the conditions of being good for the previous steps. More precisely, the function f_i is $(\sum_{l=j}^{i} 1/K_l)$-good at n'_j for any $j \leq i$, but if we perform transitions we may lose this property, especially on the levels close to the bottom and top of the blocks we concatenate. In order to keep this property when performing transitions, we need to ensure that Lemma 3.7 can be applied. To do so, we make use of mixing and we proceed as follows.

Pick a pure column and consider a principal subcolumn (different from the one at the bottom, whose n'_{i+1} first levels are modified in Step $(i+1)$-I). Let $B = a_1 \dots a_{n'_{i+1}}$ be the block in $[0,1]^{n'_{i+1}}$ corresponding to the values of its first n'_{i+1} levels. Let $B_1 = a_1 \dots a_{n'_i}$ and $B_2 = a_{n'_{i+1}-n'_i-L_i+1} \dots a_{n'_{i+1}-L_i} \in [0,1]^{n'_i}$. Since we assume that (X,T) is topologically mixing, we can find $B_3 \in [0,1]^{L_i}$ such that $B_2B_3B_1$ is a valid itinerary of f_i. We then replace the top L_i levels of B by B_3, obtaining a block B'. Since $B_2B_3B_1$ is a valid itinerary for f_i, we have

$$|B'_{n'_{i+1}-n'_j+k} - B'_k| \leq \sum_{l=j}^{i} \frac{1}{K_l} \quad \text{for any } j \leq i \text{ and any } k \leq n'_j.$$



Step $(i+1)$-III: *Modification inside a pure column.* We are now ready to perform transitions.

Consider two consecutive principal subcolumns modified according to Steps $(i+1)$-I and $(i+1)$-II and perform a transition between the first n'_{i+1} levels of these subcolumns. We recall that a level among the first $n'_{i+1}-L_i$ levels of a principal subcolumn (so not modified in Steps $(i+1)$-I or $(i+1)$-II) is in A_i^c if and only if it is constant in the ln'_{i+1} levels above it for $l = 1, \ldots, 2K_{i+1}$. This means that the transition will not change the values of these levels. Lemma 3.5 guarantees the precision $1/K_{i+1}$ we are looking for. The modifications we made in Step $(i+1)$-II and Lemma 3.7 also ensure that the properties for $j \leq i+1$ are also respected (here we add some error term, given by $1/K_{i+1}$, but this value is small since we assume that the series is convergent).

Again we modify the first level of each pure column by a small quantity so that all pure columns are different. The small quantity is chosen in order to keep the good properties of f_i (defined by a strict inequality).

It remains to show that we have changed f_i in a set of small measure. For any principal subcolumn (different from the ones at the bottom and top), we change at most

$$(2K_{i+1}-1)\big(L_i + \#(\text{levels in } A_i \text{ among the first } n'_{i+1} \text{ levels})\big)$$

levels. We may change all levels from the first and last principal subcolumns and the top (n_{i+1} or $n_{i+1}+1$) levels. Therefore, in a pure column the number of levels we change is at most

$$4K_{i+1}(n'_{i+1}+1) + 1 + (2K_{i+1}-1)\big(N_{i+1}L_i + \#(\text{levels in } A_i)\big),$$

and thus we have modified any pure column in a proportion smaller than

$$\frac{4K_{i+1}n'_{i+1}+1+(2K_{i+1}-1)(N_{i+1}L_i+\#(\text{levels in } A_i))}{2N_{i+1}K_{i+1}n'_{i+1}+n'_{i+1}}.$$

We deduce that the set we modified has measure at most

$$\frac{3}{N_{i+1}}+\frac{2L_i}{n'_{i+1}}+(2K_{i+1}-1)\mu(A_i),$$

and this value is smaller than r_{i+1} by our assumptions. So, we have built f_{i+1} which is continuous, generates for T and $\mu(\{f_{i+1}\neq f_i\})<r_{i+1}$.

We now consider the function f, the pointwise limit of the sequence $(f_i)_{i\in\mathbb{N}}$.

Claim. *$\|f-f\circ T^{n'_i}\|_\infty\to 0$ as $i\to\infty$.*

Let X' be a set of full measure where f_i converges to f. Let $\epsilon>0$ and let $j\in\mathbb{N}$ be such that $\sum_{i\geq j}1/K_i\leq\epsilon/3$. Let $x\in X'$ and $i\geq j$. We can find $\bar{i}\geq i$ such that $|f_{\bar{i}}(x)-f(x)|\leq\epsilon/3$ and $|f_{\bar{i}}(T^{n'_i}x)-f(T^{n_i}x)|\leq\epsilon/3$. Then, since $f_{\bar{i}}$ is $\sum_{j=i}^{\bar{i}}1/K_j$-good for n'_i, we get

$$\begin{aligned}|f(x)-f(T^{n'_i}x)| &\leq |f(x)-f_{\bar{i}}(x)|+|f_{\bar{i}}(x)-f_{\bar{i}}(T^{n'_i}x)|+|f_{\bar{i}}(T^{n'_i}x)-f(T^{n'_i}x)|\\ &\leq \epsilon/3+\sum_{j=i}^{\bar{i}}\frac{1}{K_j}+\epsilon/3\leq\epsilon.\end{aligned}$$

Since x and $i\geq j$ are arbitrary, we get the conclusion.

Now it remains to prove:

Claim. *The corresponding model $(X_f,\sigma)=(\operatorname{supp} f^\infty\mu,\sigma)$ is uniformly rigid for $(n'_i)_{i\in\mathbb{N}}$.*

Let $\epsilon>0$ and $M\in\mathbb{N}$ be such that if $\omega,\omega'\in[0,1]^{\mathbb{Z}}$ satisfy $|\omega_l-\omega'_l|\leq\epsilon/8$ for any $|l|\leq M$ then $d(\omega,\omega')\leq\epsilon/4$, where d is a metric on X_f. Let j be such that $\|f-f\circ T^{n'_i}\|_\infty\leq\epsilon/2$ for any $i\geq j$. Let $\omega\in Y$ and $i\geq j$. We can pick x such that $\omega'=f^\infty(x)$ satisfies $|\omega_l-\omega'_l|\leq\epsilon/4$ for any $|l|\leq M+n'_i$ and $|\omega'_{n'_i+p}-\omega'_p|\leq\epsilon/2$ for any $p\in\mathbb{Z}$. We deduce that

$$\begin{aligned}d(\sigma^{n'_i}\omega,\omega) &\leq d(\sigma^{n'_i}\omega,\sigma^{n'_i}\omega')+d(\sigma^{n'_i}\omega',\omega')+d(\omega,\omega')\\ &\leq \epsilon/4+\epsilon/2+\epsilon/4=\epsilon.\end{aligned}$$

Since this holds for any $i\geq j$ and $\omega\in Y$, we find that (Y,σ) is uniformly rigid with rigidity sequence $(n'_i)_{i\in\mathbb{N}}$.

3.4. Proof of Theorem 1.1: Adding the weakly mixing condition. We modify our construction of the previous section so that the resulting system is (topologically) weakly mixing. Notice that though we assume that (X,T) is mixing, this does not guarantee that (X_f,σ) is weakly mixing.

To make the representation (X_f, σ) weakly mixing, one needs to add the following condition: for all nonempty open sets A, B, C, D there exists n such that $\sigma^n A \cap B \neq \emptyset$ and $\sigma^n C \cap D \neq \emptyset$. This can be guaranteed by the following property of α (recall that α is the partition corresponding to f): For each $m \geq 0$ and $E_1, F_1, E_2, F_2 \in \bigvee_{j=0}^{m-1} T^{-j}\alpha$, there is some s such that

$$\mu \times \mu\big((T \times T)^s (E_1 \times F_1) \cap (E_2 \times F_2)\big) > 0.$$

To this end, we need a similar property in each α_i. The strategy in this section consists in securing this property gradually by modifying the bottom of a single pure column at each step (the ones described in the previous section) in such a way that we keep the rigidity property.

Now we give the details. Let $\{\alpha_i\}_{i=0}^{\infty}$ be the partitions of the previous section.

Lemma 3.10. *One can achieve the following properties of the partitions $\{\alpha_i\}_{i=0}^{\infty}$: there are a sequence $\{s_i\}_{i=0}^{\infty}$ of positive integers and sequences $\{r_i\}_{i=0}^{\infty}, \{e_i\}_{i=0}^{\infty}$ of positive numbers with $r_{i+1} < \min\{r_i/2, e_i^2/4i\}$ such that for all $i \geq 1$:*

- $(1)_i$ $d(\alpha_i, \alpha_{i+1}) = \mu(\{f_i \neq f_{i+1}\}) < r_{i+1}$.
- $(2)_i$ *Let $\bigvee_{j=0}^{i-1} T^{-j}\alpha_i = \{U_1^i, \dots, U_{\eta_i}^i\}$ with U_j^i nontrivial. Then there is a subset $\{U_1^{i+1}, \dots, U_{\eta_i}^{i+1}\} \subset \bigvee_{j=0}^{i-1} T^{-j}\alpha_{i+1}$ such that the α_i-name of U_h^i and the α_{i+1}-name of U_h^{i+1} are the same, for all $1 \leq h \leq \eta_i$.*
- $(3)_i$ *For all $E_1, F_1, E_2, F_2 \in \{U_1^{i+1}, \dots, U_{\eta_i}^{i+1}\}$ as in $(2)_i$, one has*

$$\mu \times \mu\big((T \times T)^{s_{i+1}} (E_1 \times F_1) \cap (E_2 \times F_2)\big) \geq e_{i+1}^2 > 0.$$

Proof. Assume inductively that we have constructed partitions $\{\alpha_i\}_{i=0}^{n}$, a sequence $\{s_i\}_{i=0}^{n}$ of positive integers and sequences $\{r_i\}_{i=0}^{n}, \{e_i\}_{i=0}^{n}$ of positive numbers with $r_{i+1} < \min\{r_i/2, e_i^2/4i\}$ for each $i \leq n-1$. Let $\alpha_i = \{A_1^i, \dots, A_{m_i}^i\}$ for $1 \leq i \leq n$. Let $f_i \colon X \to \{a_1, \dots, a_{m_i}\} \subseteq [0,1]$ be such that $A_j^i = f_i^{-1}(a_j)$.

The sequence $\{\alpha_i\}_{i=0}^{n}$ has properties $(1)_i$–$(3)_i$ for $0 \leq i \leq n-1$.

Now we make the induction step. First we need to define a word ω_n which contains all pairs of names of nontrivial elements in $\bigvee_{i=0}^{n-1} T^{-i}\alpha_n$. We do it as follows.

Let $\bigvee_{j=0}^{n-1} T^{-j}\alpha_n = \{U_1^n, \dots, U_{\eta_n}^n\}$ with U_i^n nontrivial. Let B_t be the name of U_t^n for each $1 \leq t \leq \eta_n$. Then $W_n = \{B_1, \dots, B_{\eta_n}\} \subset \{a_1, \dots, a_{m_n}\}^n$ is the set of all names of nontrivial elements of $\bigvee_{i=0}^{n-1} T^{-i}\alpha_n$. Since (X, T) is topologically mixing, there exists L_n such that any couple of W_n can be joined by an itinerary of length greater than L_n.

Now fix a large number $s_{n+1} > L_n + n$, and construct the word ω_n as follows: For each pair $(j_1, j_2) \in \{1, \dots, \eta_n\}^2$, make sure that words B_{j_1} and B_{j_2} appear in ω_n, and the distance between them is s_{n+1}.

Let $\mathfrak{t}$ be the tower in Step $n+1$ of the previous section. Refine $\mathfrak{t}$ according to α_n, and choose one column $\mathfrak{c}_{n+1}$ of the resulting tower. Let the base of $\mathfrak{c}_{n+1}$ be C_{n+1}. Let $e_{n+1} = \mu(C_{n+1})$. Now we adjust $\mathfrak{c}_{n+1}$ as follows. Copy the name ω_n on some place close to the bottom of the column $\mathfrak{c}_{n+1}$, for instance to the bottom of the second principal subcolumn. We consider towers of level $n+1$ such that the bottom is large enough with respect to the length of ω_n so we can apply Steps I, II and III described in the previous section. This process keeps the good properties related to uniform rigidity.

As in the previous section, we get a new function f_{n+1} and a corresponding partition α_{n+1}, and we can ensure that

$$d(\alpha_n, \alpha_{n+1}) = \mu(\{f_n \neq f_{n+1}\}) < r_{n+1} < \min\left\{\frac{r_n}{2}, \frac{e_n^2}{4n}\right\}.$$

By the construction of α_{n+1}, there is a subset $\{U_1^{n+1}, \ldots, U_{\eta_n}^{n+1}\} \subset \bigvee_{j=0}^{n-1} T^{-j}\alpha_{n+1}$ such that the α_n-name of U_h^n and the α_{n+1}-name of U_h^{n+1} are the same, for all $1 \leq h \leq \eta_n$.

Let $D_{i_1}, D_{i_2}, D_{j_1}, D_{j_2} \in \{U_1^{n+1}, \ldots, U_{\eta_n}^{n+1}\}$, and let their respective names be B_{i_1}, B_{i_2}, $B_{j_1}, B_{j_2} \in W_n$, where $1 \leq i_1, i_2, j_1, j_2 \leq \eta_n$. Then by the definition of ω_n, the pairs (B_{i_1}, B_{j_1}) and (B_{i_2}, B_{j_2}) appear in the word ω_n. Let p be the position of B_{i_1} in the column $\mathfrak{c}_{n+1}$ and let r be the distance from the position of B_{i_1} to the position of B_{i_2}. Then

$$\begin{aligned} T^{p-1}C_{n+1} &\subset D_{i_1}, & T^{p-1+s_{n+1}}C_{n+1} &\subset D_{j_1}, \\ T^{p-1+r}C_{n+1} &\subset D_{i_2}, & T^{p-1+r+s_{n+1}}C_{n+1} &\subset D_{j_2}. \end{aligned}$$

It follows that

$$\begin{aligned} T^{p-1}C_{n+1} \times T^{p-1+r}C_{n+1} &\subset (D_{i_1} \cap T^{-s_{n+1}}D_{j_1}) \times (D_{i_2} \cap T^{-s_{n+1}}D_{j_2}) \\ &= (D_{i_1} \times D_{i_2}) \cap (T \times T)^{-s_{n+1}}(D_{j_1} \times D_{j_2}). \end{aligned}$$

Hence

$$\begin{aligned} &\mu \times \mu\big((D_{i_1} \times D_{i_2}) \cap (T \times T)^{-s_{n+1}}(D_{j_1} \times D_{j_2})\big) \\ &\qquad \geq \mu \times \mu(T^{p-1}C_{n+1} \times T^{p-1+r}C_{n+1}) \geq e_{n+1}^2 > 0. \end{aligned}$$

Thus $(1)_n$–$(3)_n$ hold. The proof is complete. ■

Recall that α is the partition corresponding to f.

Proposition 3.11. *The representation (X_f, σ) is also weakly mixing.*

Proof. We show that for nonempty open sets A, B, C, D there exists n such that $\sigma^n A \cap B$ and $\sigma^n C \cap D$ are nonempty. This is guaranteed by the following property: For each $m \geq 0$ and $E_1, F_1, E_2, F_2 \in \bigvee_{j=0}^{m-1} T^{-j}\alpha$, there is some s such that

$$\mu \times \mu\big((T \times T)^s(E_1 \times F_1) \cap (E_2 \times F_2)\big) > 0.$$

We follow the notation of Lemma 3.10. By the definition of α and Lemma 3.10, there is some $t > m$ large enough such that there are $E_1', F_1', E_2', F_2' \in \bigvee_{j=0}^{m-1} T^{-j}\alpha_t$ that have the same names as E_1, F_1, E_2, F_2 respectively. Choose $C_1', D_1', C_2', D_2' \in \{U_1^t, \ldots, U_{\eta_{t-1}}^t\} \subset \bigvee_{j=0}^{t-2} T^{-j}\alpha_t$ such that $C_1' \subset E_1'$, $D_1' \subset F_1'$, $C_2' \subset E_2'$, $D_2' \subset F_2'$. Then there are $C_1 \subset E_1$, $D_1 \subset F_1$, $C_2 \subset E_2$, $D_2 \subset F_2$ in $\bigvee_{j=0}^{t-2} T^{-j}\alpha$ that have the same names as C_1', D_1', C_2', D_2' respectively.

By Lemma 3.10(3),

$$\mu \times \mu\big((T \times T)^{s_t}(C_1' \times D_1') \cap (C_2' \times D_2')\big) \geq e_t^2.$$

Then from $d(\bigvee_{j=0}^{t-1} T^{-j}\alpha_t, \bigvee_{j=0}^{t-1} T^{-j}\alpha) \leq td(\alpha_t, \alpha) < t\sum_{j=t+1}^{\infty} r_j$, one has

$$\begin{aligned}
\mu \times \mu\big(&(T \times T)^{s_t}(C_1 \times D_1) \cap (C_2 \times D_2)\big) \\
&\geq \mu \times \mu\big((T \times T)^{s_t}(C_1' \times D_1') \cap (C_2' \times D_2')\big) - t\sum_{j=t+1}^{\infty} r_j \\
&\geq e_t^2 - t\sum_{j=t+1}^{\infty} r_j \geq e_t^2 - t\left(r_{t+1} + \frac{r_{t+1}}{2} + \frac{r_{t+1}}{2^2} + \cdots\right) \\
&\geq e_t^2 - tr_{t+1}\sum_{j=0}^{\infty} \frac{1}{2^j} \geq e_t^2 - 2tr_{t+1} \geq e_t^2/2 > 0.
\end{aligned}$$

In particular,

$$\begin{aligned}
\mu \times \mu\big(&(T \times T)^{s_t}(E_1 \times F_1) \cap (E_2 \times F_2)\big) \\
&> \mu \times \mu\big((T \times T)^{s_t}(C_1 \times D_1) \cap (C_2 \times D_2)\big) > 0. \qquad \blacksquare
\end{aligned}$$

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