STUDIA MATHEMATICA 236 (1) (2017)

ODE for L^p norms

by

Jarno Talponen (Joensuu)

Abstract. In this paper we relate the geometry of Banach spaces to the theory of differential equations, apparently in a new way. We will construct Banach function space norms arising as weak solutions to ordinary differential equations (ODE) of the first order. This provides as a special case a new way of defining varying exponent L^p spaces, different from the Musielak–Orlicz type approach. We explain heuristically how the definition of the norm by means of a particular ODE is justified. The resulting class of spaces includes the classical L^p spaces as a special case. A noteworthy detail regarding our $L^{p(\cdot)}$ norms is that they satisfy Hölder's inequality (properly).

1. Introduction. In this paper we introduce a novel way of defining function space norms by means of weak solutions to ordinary differential equations (ODE). This provides a new perspective for looking at varying exponent L^p spaces.

The classical Birnbaum–Orlicz norms were defined in the 1930's, and since then there have been various generalizations of these norms in several directions. Notable examples of norms and spaces carry the names of Besov, Lizorkin, Lorentz, Luxemburg, Musielak, Nakano, Orlicz, Triebel, Zygmund (see e.g. [BO], [L], [BY], [Mu]). These norms have recently been applied to other areas of mathematics as well as to some real-world problems (see e.g. [DHR], [RR02]). Roughly speaking, these norms can be viewed as belonging to a family of derivatives of the Minkowski functional. This kind of approach leads to several varying exponent $L^{p(\cdot)}$ type constructions, e.g. for sequence spaces, Lebesgue spaces, Hardy spaces and Sobolev spaces. There is a vast literature on these topics (see [Ko], [LT], [NS] and [RR91] for samples and further references). There are also other ways of looking at the varying exponent L^p spaces, such as the Marcinkiewicz space (see [Mar]).

2010 *Mathematics Subject Classification*: Primary 46E30, 34A12; Secondary 46B10, 31B10.
Key words and phrases: Banach function space, Orlicz space, Nakano space, L^p space, variable exponent, differential equation, ODE, weak solution.
Received 8 March 2016; revised 11 August 2016.
Published online 25 November 2016.

DOI: 10.4064/sm8561-8-2016

Let us recall that the general Nakano or Musielak–Orlicz type norms are defined as follows:

$$\|\!|f|\!\|=\inf\Big\{\lambda>0\colon \int_{\Omega}\phi(|f(t)|/\lambda,t)\,dm(t)\le 1\Big\}.$$

Here ϕ is a positive function satisfying suitable structural conditions. For instance, $\phi(s,t)=s^{p(t)}$, or $\psi(s,t)=s^{p(t)}/p(t)$, $1\le p(\cdot)<\infty$, produces a norm that can be seen as a varying exponent L^p norm. In the latter case we use the name *Nakano norm* (cf. [JKL]); these norms turn out to be of particular importance in this paper.

In contrast, the basic form of the norm that will be introduced here differs considerably from the above-mentioned norms in the sense that it does not arise as a derivation of the Minkowski functional, and it does not apply any norming set of functionals either. In some cases the classes of spaces introduced here do not coincide as sets with any of the classes mentioned above for a given $p\colon[0,1]\to[1,\infty)$ measurable. This is due to obstructions that will become obvious shortly. However, roughly speaking, the norms studied here are equivalent to Nakano norms (see Proposition 3.3).

The Musielak–Orlicz norms enjoy the attractive property of being rearrangement invariant in the sense that for a measure-preserving transformation $T\colon\Omega\to\Omega$ such that $\psi(|f(x)|,x)=\psi(|f\circ T(x)|,T(x))$ for a.e. $x\in\Omega$ we have $\|\!|f\circ T|\!\|=\|\!|f|\!\|$. However, one may argue that the rearrangement invariance and the apparent simplicity of the definition of the norm come at a cost. Namely, the definition of the norm is opaque in the sense that it involves an infimum with an integral inequality having rather complicated interdependencies at the binding surface of the feasible set. For instance, by looking at the definition of the norm it is difficult to decide how adding 1_Δ, $\Delta\subsetneq\Omega$ measurable, $m(\Delta)>0$, to f contributes to the norm, even if Δ is in some sense conveniently placed. Here 1_Δ is the characteristic function of the set Δ.

The 'virtues and vices' of the norms to be introduced are mirror images of the ones mentioned above. The ODE-driven norms considered here, in comparison, will typically not be rearrangement invariant in the above sense, and in particular they do not reduce isometrically to the above Nakano norms (see an example after Proposition 3.3, cf. an example in [T]). On the other hand, our norms will be 'localized' in the sense that one can analyze the (infinitesimal) contribution of a single coordinate to the norm, a built-in feature of the construction. To make a point, it is possible to compute these norms by solving the defining ODE numerically for continuous functions f and p. (It is, of course, also possible to compute the above infimum numerically, but we stress that the methods needed to solve our first order ODEs are linear in nature and elementary.) Thus, our approach to the definition of varying exponent L^p space norms is rather inductive than global.

Next we will discuss the motivating ideas behind the ODE-driven norms. The author [T] studied varying exponent $\ell^{p(\cdot)}$ spaces formed in the following naïve fashion. As usual, we denote by $\mathrm{X}\oplus_p\mathrm{Y}$ the direct sum of Banach spaces X and Y with the norm given by

$$\|(x,y)\|^p_{\mathrm{X}\oplus_p\mathrm{Y}} = \|x\|^p_{\mathrm{X}} + \|y\|^p_{\mathrm{Y}}, \qquad x\in\mathrm{X},\, y\in\mathrm{Y},\, 1\le p<\infty.$$

Let $p\colon \mathbb{N}\to[1,\infty)$ be a 'varying exponent'. Define first a 2-dimensional Banach space by $\mathbb{R}\oplus_{p(1)}\mathbb{R}$, then a 3-dimensional one $(\mathbb{R}\oplus_{p(1)}\mathbb{R})\oplus_{p(2)}\mathbb{R}$ and proceed recursively to obtain n-dimensional spaces

$$(\ldots((\mathbb{R}\oplus_{p(1)}\mathbb{R})\oplus_{p(2)}\mathbb{R})\oplus_{p(3)}\ldots)\oplus_{p(n-1)}\mathbb{R};$$

finally, by taking the inverse limit, this yields a space which can be written formally as

$$\ldots(\ldots((\mathbb{R}\oplus_{p(1)}\mathbb{R})\oplus_{p(2)}\mathbb{R})\oplus_{p(3)}\ldots)\oplus_{p(n)}\mathbb{R})\oplus_{p(n+1)}\ldots.$$

This space is normed by taking a limit of seminorms corresponding to the n-dimensional spaces above. The recursive construction of the spaces can be regarded trivial at each step, but the end result may exhibit some peculiar properties, depending on the selection of the sequence $(p(n))_{n\in\mathbb{N}}$ (see [T]). For instance, it provides an *easy* example of a separable Banach space X with a 1-unconditional basis such that X contains all spaces ℓ^p, $1\le p<\infty$, almost isometrically. In any case, this appears a rather natural way of constructing Banach sequence spaces and seems to have been first discovered by A. Sobczyk and J. W. Tukey (1) (see [S, p. 96], cf. Kalton et al. [ACK, Ka]).

The main aim of this paper is to study a 'continuous version' of the above class of sequence spaces $\ell^{p(\cdot)}$, thus a space of suitable functions $f\colon[0,1]\to\mathbb{R}$, instead of sequences. The idea is somewhat similar here: knowing the norm of f up to a coordinate $0<t<1$, i.e. $\|1_{[0,t]}f\|$, and knowing the value $|f(t^+)|$, is sufficient information to predict the accumulation of the norm right after t, i.e. knowing $\|1_{[0,t+dt]}f\|$. For example, if $f(r)=0$ for $t<r<s$, then we should have $\|1_{[0,t]}f\| = \|1_{[0,s]}f\|$, and if $|f(t^+)|>0$, then $\|1_{[0,t]}f\| < \|1_{[0,s]}f\|$, and so on. This intuitive description of the accumulation of the norm is captured by a suitable ODE in such a way that its weak solution, $\varphi_f\colon[0,1]\to[0,\infty)$, shall represent the norm as follows:

$$\varphi_f(t) = \|1_{[0,t]}f\|, \tag{1.1}$$

(1) The author presented some of the ideas in this paper in the following function space related meetings in 2014: Edwardsville, Albacete, Mekrijarvi; in 2015: Kitakyushu, Delhi. The author is grateful to L. Maligranda for bringing the above historical account to the author's attention in Kitakyushu (see also [Mal]).

so that in particular $\varphi_f(0) = 0$ and $\varphi_f(1) = \|f\|$. Equation (1.1) neatly outlines the overall strategy implemented at the beginning of the paper. The basic idea in accomplishing this and the heuristic motivation appear shortly (see Section 1.2). Differential equations have been previously studied in connection with varying exponent spaces and Sobolev spaces (see e.g. [DR]) but apparently not in the same vein as they arise here.

The required mathematical machinery in this paper is classical, and there is no apparent reason why this alternative approach could not have been experimented with much earlier. Also, our approach does not lead to excessively technical considerations, so hopefully it is accessible to a wide range of analysts.

1.1. Preliminaries and auxiliary results. We will usually consider the unit interval $[0, 1]$ endowed with the Lebesgue measure m. Here almost every (a.e.) means m-a.e., unless otherwise specified. Denote by L^0 the space of Lebesgue-to-Borel measurable functions on the unit interval. We denote by $\ell^0(\mathbb{N})$ the vector space of sequences of real numbers with pointwise operations. We refer to [CL], [FH$^+$], [LT] and [Ru] for suitable background information.

We will mainly study here varying exponent L^p spaces with ODE-determined norm, denoted by $L^{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$, respectively. The author considers these notations intuitive, even though in the literature the Nakano spaces and norms sometimes bear such notations. Therefore, when Nakano norms are considered, they will be explicitly specified and denoted by $|\!|\!|\cdot|\!|\!|$ to clearly distinguish them.

We will study Carathéodory's weak formulation of ODEs, that is, in the sense of Picard type integrals, where solutions are required to be only absolutely continuous. This means that, given an ODE

$$\varphi(0) = x_0, \qquad \varphi'(t) = \Theta(\varphi(t), t) \quad \text{for a.e. } t \in [0, 1],$$

we call φ a *weak solution in the sense of Carathéodory* if φ is absolutely continuous, $t \mapsto \Theta(\varphi(t), t)$ is measurable and

$$\varphi(T) = x_0 + \int_0^T \Theta(\varphi(t), t)\, dt$$

for all $T \in [0, 1]$, where the integral is the Lebesgue integral. In what follows, we will refer to Carathéodory's solutions simply as solutions.

Whenever we make a statement about a derivative we implicitly state that it exists. We will write $F \leq G$, involving elements of L^0, if $F(t) \leq G(t)$ for a.e. $t \in [0, 1]$. We denote the characteristic function or indicator function of a set A by 1_A defined by $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise.

LEMMA 1.1. *Suppose that $\varphi, \psi \in C[0,1]$ are absolutely continuous such that $\varphi(0) \le \psi(0)$ and*

$$\varphi(t) \ge \psi(t) \;\Rightarrow\; \varphi'(t) \le \psi'(t) \qquad a.e.$$

Then $\varphi \le \psi$.

Proof. Observe that

$$\varphi'(t) \le (\min(\varphi,\psi))' \quad \text{for a.e. } t \in [0,1]. \blacksquare$$

We will frequently calculate terms of the form $(a^p + b^p)^{1/p}$ where $a, b \ge 0$ and $1 \le p < \infty$. We adopt from [T] the shorthand notation

$$a \boxplus_p b = (a^p + b^p)^{1/p}.$$

This defines a commutative semigroup on $\mathbb{R}_+$, in particular, the associativity

$$a \boxplus_p (b \boxplus_p c) = (a \boxplus_p b) \boxplus_p c$$

is useful.

The space $\ell^{p(\cdot)} \subset \ell^0$, $p\colon \mathbb{N} \to [1,\infty)$, consists of those elements (x_n) such that the following limit of a non-decreasing sequence exists and is finite:

$$\lim_{n\to\infty} (\ldots(((|x_1| \boxplus_{p(1)} |x_2|) \boxplus_{p(2)} |x_3|) \boxplus_{p(3)} |x_4|) \boxplus_{p(4)} \ldots \boxplus_{p(n-1)} |x_n|) \boxplus_{p(n)} |x_{n+1}|$$

and the above limit becomes the norm of the space (see [T]).

1.2. Arriving at the varying exponent L^p norm ODE. Let us 'derive' heuristically our basic differential equation for varying exponent L^p norm. As mentioned in the introduction, we wish to extend the varying exponent $\ell^{p(\cdot)}$ norm in the sense of [T] to a continuous setting. Although our motivation involves the above sequence spaces, we are only required to look at simple structures $\mathrm{X} \oplus_p \mathrm{Y}$ one at a time due to the infinitesimal nature of the enterprise.

We will assume a Platonist approach to developing the definition of the varying exponent norms here. Thus we wish to find a function space norm following the gist of $\ell^{p(\cdot)}$ space norms. This leads to thought experiments on the right behavior of the function $t \mapsto \|1_{[0,t]} f\|$. In a sense, the resulting ODE will be a very robust one, and this allows us to write arguments in this paper in a concise fashion, not paying much attention to the general theory of the ODEs involved.

Suppose that we have a varying exponent, i.e. a measurable function $p\colon [0,1] \to [1,\infty)$, and $f\colon [0,1] \to \mathbb{R}$ is another measurable function, a possible candidate to lie in the function space. We wish to arrange matters in such a way that we have an absolutely continuous non-decreasing function $\varphi_f\colon [0,1] \to [0,\infty)$ such that

$$\varphi_f(t) = \|1_{[0,t]} f\|, \quad 0 \le t \le 1,$$

so $\varphi_f(0) = 0$ and $\varphi_f(1) = \|f\| < \infty$.

For example, in the classical case of L^p spaces with a constant function $f = \mathbf{1}$ and $p = 1, 2, \infty$ we have

$$\varphi_{f,1}(t) = t, \quad \varphi_{f,2}(t) = \sqrt{t}, \quad \varphi_{f,\infty}(t) = 1_{(0,1]}(t),$$

respectively. Here the p-norms are 1 but the profiles differ considerably. The first two solutions are absolutely continuous and the last one is not even continuous.

We will study Carathéodory's weak formulation of ODEs. It is convenient to work with absolutely continuous solutions, since this way we may apply to the solutions such usual tools as Fatou's lemma and Lebesgue's convergence theorems (sometimes implicitly). We are only interested in Banach lattice norms, therefore φ_f is always non-decreasing here. In fact, we will require a mildly modified version of Carathéodory's weak formulation, tailor-made to our setting.

We are aiming at a recursive-like formula for φ_f, in a similar spirit to [T], so suppose that we have defined the function φ_f up to the interval $[0, t_0]$. Then we are not interested in the values of f and p on $[0, t_0)$, a Markovian type condition. Suppose, as a thought experiment, that f and p are constant on an interval $[t_0, t_0 + \varDelta]$ where $\varDelta > 0$. Then we should have

$$\begin{aligned} \varphi(t_0 + \varDelta) &= (\varphi(t_0)^{p(t_0)} + \varDelta |f(t_0)|^{p(t_0)})^{1/p(t_0)}, \\ &= \varphi(t_0) \boxplus_{p(t_0)} \varDelta^{1/p(t_0)} |f(t_0)| \end{aligned} \tag{1.2}$$

analogous to the $\ell^{p(\cdot)}$ construction, and actually to the usual L^p norm formula, since

$$\Big(\int_0^{t_0+\varDelta} |f(s)|^p \, dm(s)\Big)^{1/p} = \Big(\int_0^{t_0} |f(s)|^p \, dm(s)\Big)^{1/p} \boxplus_p \Big(\int_{t_0}^{t_0+\varDelta} |f(t_0)|^p \, dm(s)\Big)^{1/p}$$

where the right-most term is $\varDelta^{1/p(t_0)}|f(t_0)|$. Thus, by differentiating (1.2) we find a natural candidate for the norm-determining differential equation:

$$\frac{d^+}{d\varDelta} \varphi(t_0 + \varDelta)\Big|_{\varDelta=0} = \frac{|f(t_0)|^{p(t_0)}}{p(t_0)} \varphi(t_0)^{1-p(t_0)}. \tag{1.3}$$

Here $\frac{d^+}{d\varDelta}$ denotes the right-sided derivative and we set $\varDelta = 0$, because we are interested in 'infinitesimal' increments around t_0. So, the above equation is right if f and φ are constant on the interval $[t_0, t_0 + \varDelta]$, but the equation does *not* concern the values of f, φ and p *beyond* t_0.

In formulating the differential equation we do not require f or p to be continuous anywhere, but motivated by Lusin's theorem and related considerations we will use the above formula in any case and aim to define φ by

$$(1.4)\qquad \varphi(0)=0,\quad \varphi'(t)=\frac{|f(t)|^{p(t)}}{p(t)}\varphi(t)^{1-p(t)}\quad \text{for a.e. } t\in[0,1].$$

Looking at this ODE it becomes evident that if there is a solution φ_f corresponding to f, then there is also a solution φ_{cf} corresponding to cf for any constant $c\in\mathbb{R}$, and moreover the functional $f\mapsto\varphi_f(1)$ is positively homogenous (up to uniqueness of solutions).

This formulation has the drawback that $0^{1-p(t)}$ is not defined. Also, it has a trivial solution $\varphi\equiv 0$, regardless of the values of f, if we use the convention $0^0=0$ and $p\equiv 1$. Moreover, following this idea it is possible to construct other degenerate solutions such that φ vanishes on $[0,t]$ for any $0<t<1$. The behavior of the solutions is difficult to anticipate in the case where $\varphi(t)$ is small and $p(t)$ is large.

To fix these issues, we will consider *stabilized* solutions to the above initial value problem. Namely, we will use initial values $\varphi(0)=x_0>0$ and to correct the error incurred we let $x_0\searrow 0$. It turns out that the corresponding unique solutions φ_{x_0} decreasingly converge pointwise to φ which again satisfies the same ODE. So, this procedure yields a unique solution φ which we will formulate, by a slight abuse of notation, as

$$(1.5)\qquad \varphi(0)=0^+,\quad \varphi'(t)=\frac{|f(t)|^{p(t)}}{p(t)}\varphi(t)^{1-p(t)}\quad \text{for a.e. } t\in[0,1].$$

There is more to the above procedure than merely picking a maximal solution; it turns out that in many situations it is convenient to look at positive-initial-value solutions first. By using Lebesgue's monotone convergence theorem and Lemma 1.1 one easily verifies that if for each $x_0>0$ there is φ_{x_0}, a solution to (1.4), except with initial value $\varphi_{x_0}(0)=x_0$, then there is φ, a unique solution to (1.4), such that $\varphi_{x_0}\searrow\varphi$ uniformly and $\varphi'_{x_0}\nearrow\varphi'$ in L^1 as $x_0\searrow 0$. This unique solution is referred to by (1.5). Moreover, if the 0^+-initial value solution exists, then for each positive initial value $x_0>0$ the corresponding solution exists by suitable Picard iteration and Lemma 1.1.

We define the varying exponent class $L^{p(\cdot)}\subset L^0$ (ODE-determined) as the set of those functions $f\in L^0$ such that φ_f exists as an absolutely continuous solution to (1.5) and $\varphi_f(1)<\infty$. In many cases, but not always, the class becomes a linear space. In such a case the norm can be defined as $f\mapsto\varphi_f(1)$.

Warning 1. Even if the class $L^{p(\cdot)}$ fails to be a linear space, we sometimes write $\|f\|_{L^{p(\cdot)}}=\|f\|_{p(\cdot)}:=\varphi_f(1)$ where φ_f is the solution to (1.5).

Warning 2. As explained above, the class $L^{p(\cdot)}$ and the mapping $f\mapsto\|f\|_{p(\cdot)}$ may differ from the Nakano space and the corresponding norm which are often denoted by the same symbols in the literature.

The above ODE is a separable one for a constant $p(\cdot) \equiv p$, $1 \le p < \infty$, and solving it (see (2.3)) yields $\varphi_f(1)^p = \int_0^1 |f(t)|^p \, dt$, compatible with the classical definition of the L^p norm. If $p(\cdot)$ is locally bounded and $|f(t)|^{p(t)}$ is locally integrable, then Picard iteration performed locally yields a unique solution for each initial value $\varphi(0) = a > 0$, possibly with $\varphi(s) \to \infty$ as $s \nearrow r$ for some $0 < r \le 1$.

2. Constructions of ODE-determined $L^{p(\cdot)}$ spaces. In this section we will study only spaces of the type $L^{p(\cdot)}$ with $p\colon [0,1] \to [1,\infty)$ measurable. Some of the unbounded functions $p(\cdot)$ actually produce a class of functions, rather than a linear space (see Example 3.4). We will first restrict our considerations to those $L^{p(\cdot)}$ classes which are Banach spaces (see Theorem 3.7 below). The norms of these spaces were described in the introductory part.

2.1. Transcending from discrete to continuous state. We will traverse from varying exponent sequence spaces to such function spaces through an intermediate notion which we call simple seminorm. Let us first define a very simple seminorm by the formula

$$|f|_{p,\mu} = \Big(\int |f|^p \, d\mu\Big)^{1/p}$$

where μ is a restricted Lebesgue measure with $\operatorname{supp}(\mu) \subset [0,1]$. Let us consider such measures μ_i with $\max \operatorname{supp}(\mu_i) \le \min \operatorname{supp}(\mu_{i+1})$, $1 \le i \le n-1$, and 'exponent constants' $p_i \in [1,\infty)$. Then we may define a composite seminorm as follows:

$$\begin{aligned}
(2.1)\qquad &\|f\|_{(\ldots(L^{p_1}(\mu_1)\oplus_{r_2}L^{p_2}(\mu_2))\oplus_{r_3}\cdots\oplus_{r_{n-1}}L^{p_{n-1}}(\mu_{n-1}))\oplus_{r_n}L^{p_n}(\mu_n)}\\
&:= (\ldots(|f|_{p_1,\mu_1} \boxplus_{r_2} |f|_{p_2,\mu_2}) \boxplus_{r_3} \cdots \boxplus_{r_{n-1}} |f|_{p_{n-1},\mu_{n-1}}) \boxplus_{r_n} |f|_{p_n,\mu_n}\\
&= (\ldots(\ldots(((|f|^{r_2}_{p_1,\mu_1} + |f|^{r_2}_{p_2,\mu_2})^{r_3/r_2} + |f|^{r_3}_{p_3,\mu_3})^{r_4/r_3}\\
&\qquad + \cdots + |f|^{r_{n-1}}_{p_{n-1},\mu_{n-1}})^{\frac{r_n}{r_{n-1}}} + |f|^{r_n}_{p_n,\mu_n})^{1/r_n}
\end{aligned}$$

where $\max \operatorname{supp}(\mu_i) \le \min \operatorname{supp}(\mu_{i+1})$, $r_{i+1} \ge p_{i+1}$. We will frequently consider simple seminorms

$$\|f\|_N = \|f\|_{(\ldots(L^{p_1}(\mu_1)\oplus_{r_2}L^{p_2}(\mu_2))\oplus_{r_3}\ldots)\oplus_{r_n}L^{p_n}(\mu_n)}$$

and denote this collection by $\mathcal{N}$. Thus N is a name for a seminorm which is recursively defined in (2.1). Let us say that a seminorm is of *standard form* if $r_{i+1} = p_{i+1}$ for all $i \in \{1,\ldots,n-1\}$. In this case we may, in a sense, extend each element N of $\mathcal{N}$ to an $L^{\tilde{p}(\cdot)}$ norm by setting $\tilde{p}_N(t) = p_i$ for $t \in \operatorname{supp}(\mu_i)$ and $\tilde{p}_N(t) = 1$ otherwise (and this extension is unique). If $\bigcup_i \operatorname{supp}(\mu_i) = [0,1]$ then the corresponding standard form norm $N \in \mathcal{N}$ satisfies $\|f\|_N = \|f\|_{\tilde{p}(\cdot)}$ by Lemma 2.1 below.

Observe that the seminorms are decreasing in r's and increasing in p's. Define point intervals $[p_t, r_t]$ as follows: $p_t = p_{i+1}$ and $r_t = r_{i+1}$ on the support of μ_{i+1}. We write $N \preceq p(\cdot)$ whenever $p(t) \in [p_t, r_t]$ for all t such that the interval is defined.

We define a partial order on $\mathcal{N}$ by setting $N \preceq M$ if the following conditions hold:

1. A partition given by the supports of the measures corresponding to N is refined by the supports of the measures corresponding to M:
$$\forall \mu_{N,i}\ \exists \mu_{M,j_1^{(i)}}, \ldots, \mu_{M,j_m^{(i)}} \qquad \mathrm{supp}(\mu_{N,i}) = \bigcup_{1\le k\le m} \mathrm{supp}(\mu_{M,j_k^{(i)}}).$$
2. $[p_{M,t}, r_{M,t}] \subset [p_{N,t}, r_{N,t}]$ for each t such that the left hand interval is defined.

This leads to the definition of a varying exponent L^p norm in a natural way as a limit from below.

Actually, to simplify considerations we will consider simple seminorms of standard form. For these seminorms we define $N \le M$ if the union of the supports of M includes that of N and moreover $\tilde{p}_N \le \tilde{p}_M$. This is again a directed poset. We define $\mathcal{N}_{\le p(\cdot)}$ to be the collection of simple seminorms N of standard form such that $\tilde{p}_N \le p(\cdot)$. The sought-after norms can be defined by applying one of the above orders, but here we will concentrate on the latter.

Let us define a functional as follows:

$$\rho(f) := \limsup_N \|f\|_N \tag{2.2}$$

where the lim sup is taken along $N \in \mathcal{N}_{\le p(\cdot)}$ such that $\tilde{p}_N \to p$ in measure, i.e.

$$\rho(f) = \inf_{K\in\mathcal{N}_{\le p(\cdot)}} \ \sup_{N\in\mathcal{N}_{\le p(\cdot)},\, K\le N} \|f\|_N.$$

By thinking of the basic properties of lim sup and the simple seminorms, we observe that the functions $f \in L^0$ with $\rho(f) < \infty$ form a linear space and ρ is a seminorm on it. We call this space $\widetilde{L}^{p(\cdot)}$ and it turns out that this seminorm is in fact a norm when we identify functions in the usual way, i.e. according to a.e. coincidence.

We will connect the above limiting process of seminorms to ODEs. In doing this we are required to use initial values for the ODEs, and for seminorms as well. Although this procedure, strictly speaking, destroys the seminorm property, we may modify the composite seminorms in such a way that the resulting functions have an initial value in a natural way. Namely, we begin the recursive construction by using $L^1(\delta_0) \oplus_1 L^{p_1}(\mu_1)$ as the first term, in place of $L^{p_1}(\mu_1)$, where δ_0 is the Dirac delta at 0. Then $|f(0)|$ serves as the 'initial value of the seminorm'.

LEMMA 2.1. *Let $\rho(f) < \infty$. Suppose that there are compact subsets $C_i \subset [0,1]$, $1 \le i \le n$, $\max C_i \le \min C_{i+1}$ such that $p|_{C_i} \equiv p_i \in [1,\infty)$. Assume additionally that $f = 1_{\bigcup_i C_i} f$ and $p|_{[0,1]\setminus \bigcup_i C_i} \equiv 1$. Then the mapping $\varphi\colon [0,1] \to \mathbb{R}$ given by $\varphi(t) = \rho(1_{[0,t]} f)$ is absolutely continuous and satisfies*

$$\varphi(0) = 0, \qquad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0,1].$$

Proof. First we observe the analogous fact on an interval with a constant p by studying the following differential equation:

$$\varphi(a) = c, \qquad \varphi'(t) = \frac{|f(t)|^p}{p} \varphi(t)^{1-p} \quad \text{for a.e. } t \in [a,b] \subset [0,1].$$

We use the separability of the above differential equation and the absolute continuity of φ to obtain

$$\int_a^b p\varphi'(t)\varphi(t)^{p-1}\,dt = \varphi(t)^p|_{t=a}^b = \int_a^b |f(t)|^p\,dt. \tag{2.3}$$

Indeed, we see immediately that φ defined in the formulation of the lemma is absolutely continuous in this special case. The above calculation considered in backward order also shows that in the constant p case, φ arises as a solution to the above differential equation on that interval.

From this we obtain the analogous compact subset $C \subset [0,1]$ case by passing to a function of the type $1_C f$. It is clear that the resulting φ is again absolutely continuous and the derivative is

$$\varphi'(t) = \frac{|1_C(t) f(t)|^p}{p} \varphi(t)^{1-p} = 1_C(t) \frac{|f(t)|^p}{p} \varphi(t)^{1-p} \qquad \text{a.e.}$$

This way we easily see that the simple seminorm accumulation functions

$$t \mapsto \|1_{[0,t]} f\|_{(\ldots(L^{p_1}(\mu_1)\oplus_{p_2} L^{p_2}(\mu_2))\oplus_{p_3}\ldots)\oplus_{p_n} L^{p_n}(\mu_n)} \tag{2.4}$$

can be seen as solutions to

$$\varphi(0) = 0, \qquad \varphi'(t) = \frac{1_{\bigcup_i \operatorname{supp}(\mu_i)}(t)|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0,1]$$

where $p(t) = p_i$ for $t \in C_i = \operatorname{supp}(\mu_i)$ and $\varphi'(t) = 0$ for $t \in [0,1] \setminus \bigcup_i C_i$. Indeed, for $x_i = \max C_i$ in (2.4) we obtain an ODE

$$\varphi(x_i) = \|1_{[0,x_i]} f\|, \qquad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in C_{i+1}$$

by induction. Note that the sup in the lim sup in (2.2) is actually attained in this simple case with $p(\cdot)$ essentially piecewise constant on $\bigcup_i C_i$. ∎

Given a measurable function $p\colon [0,1] \to [1,\infty)$, by Lusin's theorem there is for each $\epsilon > 0$ a compact set $C \subset [0,1]$ with $m([0,1] \setminus C) < \epsilon$ such that $p|_C$ is continuous, thus uniformly continuous and bounded.

Thus we can find a sequence of compact subsets $C_m \subset [0,1]$ as above with $m(C_m) \to 1$, and by taking finite unions of such sets we may assume that the sequence is increasing. Next we assume for technical reasons that all the seminorms have a fixed positive initial value component, say $x_0 1_{\{0\}} \in L^1(\delta_0)$ with $x_0 > 0$. We may construct by a diagonal argument a sequence (N_n) of simple seminorms of standard form (but with the added initial value) such that $\tilde{p}_{N_n} \to p$ in measure and for every $f \in L^\infty$ and $m \in \mathbb{N}$ we have

$$\begin{aligned}\frac{d}{dt}\|1_{[0,t]\cap C_m} f\|_{N_n} &= \frac{|1_{C_m}(t) f(t)|^{\tilde{p}_{N_n}(t)}}{\tilde{p}_{N_n}(t)} N_n(1_{[0,t]\cap C_m} f)^{1-\tilde{p}_{N_n}(t)} \\ &\to \frac{|1_{C_m}(t) f(t)|^{p(t)}}{p(t)} \limsup_{n\to\infty} N_n(1_{[0,t]\cap C_m} f)^{1-p(t)}\end{aligned}$$

in measure (more precisely, in $L^0(C_m)$) as $n \to \infty$. Using the initial value a ensures that $N_n(1_{[0,t]\cap C_m} f)^{1-p(t)}$ is uniformly bounded on C_m and we are also using the fact that $|f(t)|^{\tilde{p}_{N_n}(t)}$ is uniformly bounded on C_m. Note that these observations imply that $\limsup_{n\to\infty} N_n(1_{[0,t]\cap C_m} f)$ is absolutely continuous. Thus, the above shows that $\lim_{n\to\infty} N_n(1_{[0,t]\cap C_m} f)$ exists for each t and is absolutely continuous on t. In the same vein, thinking of the definition of ρ, still with the same initial value component in all the seminorms, we observe that $\rho(1_{[0,t]\cap C_m} f) = \lim_{n\to\infty} N_n(1_{[0,t]\cap C_m} f)$. In particular, $t \mapsto \rho(1_{[0,t]\cap C_m} f)$ is absolutely continuous and

$$\frac{d}{dt}\rho(1_{[0,t]\cap C_m} f) = \frac{|1_{C_m}(t) f(t)|^{p(t)}}{p(t)} \rho(1_{[0,t]\cap C_m} f)^{1-p(t)} \tag{2.5}$$

exists a.e. on C_m. If we let $x_0 \searrow 0$, the above terms increase and converge to a value for a.e. $t \in C_m$, so that Lebesgue's monotone convergence theorem yields a solution to the same ODE with initial value 0. According to $m(\bigcup_m C_m) = 1$ and hence the positivity of (2.5) in a positive measure set, ρ is a norm, instead of merely being a seminorm.

We will denote by $\widetilde{L}^{p(\cdot)}$ the normed space of functions $f \in L^0$ with $\rho(f) < \infty$. We next collect and refine some findings obtained so far.

Proposition 2.2. *Given a measurable function $p\colon [0,1] \to [1,\infty)$, the class $\widetilde{L}^{p(\cdot)}$ is a Banach space with the usual pointwise linear operations and the corresponding norm ρ defined above.*

Proof. We have already established that $\widetilde{L}^{p(\cdot)}$ endowed with the functional $\|\cdot\|_{\widetilde{L}^{p(\cdot)}} := \rho$ is a normed space. To prove completeness we will pass to an equivalent norm,

$$\sup_{N\in\mathcal{N}_{\leq p(\cdot)}} \|f\|_N. \tag{2.6}$$

Indeed, clearly $\sup_N \|f\|_N \geq \rho(f)$ and by using Lemma 2.1 and Proposition 3.2 we get an opposite inequality with a multiplicative constant from the latter result. Therefore these norms are equivalent, and it is clear that (2.6) defines a complete norm on $\widetilde{L}^{p(\cdot)}$. Hence ρ is complete as well. ■

THEOREM 2.3. *Let $f \in L^0$ and $p\colon [0,1] \to [1,\infty)$ measurable. The following conditions are equivalent:*

(1) $f \in L^{p(\cdot)}$,
(2) $f \in \widetilde{L}^{p(\cdot)}$ *and the mapping $t \mapsto \rho(1_{[0,t]}f)$ is absolutely continuous.*

Moreover, in both (equivalent) cases we have $\varphi_f(1) = \rho(f)$.

Proof. Assume that $f \in L^0$, $\rho(f) < \infty$ and $t \mapsto \rho(1_{[0,t]}f)$ is absolutely continuous, as in the second condition. To show the first condition, without loss of generality we may assume that the sequence C_m considered previously is such that $p|_{C_m}$ and $f|_{C_m}$ are continuous for every m. By a diagonal argument we may choose a sequence of simple seminorms N_n of standard form such that $N_n(1_{[0,t]}f) \to \rho(1_{[0,t]}f)$ uniformly on t and $\tilde{p}_{N_n} \to p$ in measure as $n \to \infty$. Thus

$$\frac{d}{dt}N_n(1_{[0,t]}f) \to \frac{|f(t)|^{p(t)}}{p(t)}\rho(1_{[0,t]}f)^{1-p(t)}$$

in $L^0(C_m)$ as $n \to \infty$. Using the assumed absolute continuity and the boundedness of f and p on C_m, we obtain

$$\begin{aligned}\rho(1_{[0,T]}f) - \rho(1_{[0,S]}f) &= \int_S^T \frac{d}{dt}\rho(1_{[0,t]}f)\,dt = \sup_m \int_{[S,T]\cap C_m} \frac{d}{dt}\rho(1_{[0,t]}f)\,dt \\ &= \sup_m \int_{[S,T]\cap C_m} \frac{|f(t)|^{p(t)}}{p(t)}\rho(1_{[0,t]}f)^{1-p(t)}\,dt = \int_S^T \frac{|f(t)|^{p(t)}}{p(t)}\rho(1_{[0,t]}f)^{1-p(t)}\,dt.\end{aligned}$$

Strictly speaking, we are also required to control the term $\rho(1_{[0,t]}f)^{1-p(t)}$ which need not be bounded. However, it can be made bounded by using a positive initial value and then letting the initial value tend to zero, as before. Now, $\rho(1_{[0,t]}f)$ is clearly the required solution to the norm-determining ODE. The last part of the statement follows immediately.

The other direction becomes apparent later when we investigate estimates for the norms by means of differential equations (see Proposition 3.2). ■

REMARK 2.4. According to the previous result the functional $f \mapsto \varphi_f(1) =: \|f\|_{p(\cdot)}$ satisfies the triangle inequality.

3. Inequalities. Given a measurable function $p\colon [0,1] \to (1,\infty)$ defined a.e., we denote its pointwise Hölder conjugate by $p^*\colon [0,1] \to (1,\infty)$ (defined a.e.), that is,

$$\frac{1}{p(t)} + \frac{1}{p^*(t)} = 1 \quad \text{for a.e. } t \in [0,1].$$

PROPOSITION 3.1 (Hölder). *Suppose that $f \in L^{p(\cdot)}$ and $g \in L^{p^*(\cdot)}$ with $1 < p(t) < \infty$ for a.e. t. Then they satisfy Hölder's inequality:*

$$\int_0^1 |f(t)g(t)|\,dt \le \|f\|_{p(\cdot)}\|g\|_{q^*(\cdot)}.$$

Although the function classes here need not be linear spaces, we still use the norm notation above (instead of $\varphi_f(1)$ etc.); this is to establish a clear connection with the classical case.

Proof. By using the Hölder inequality for classical L^p and ℓ^p spaces, we obtain by induction an analogous statement for spaces of the type

$$(\ldots(L^{p_1}(\mu_1) \oplus_{p_2} L^{p_2}(\mu_2)) \oplus_{p_3} \ldots) \oplus_{p_n} L^{p_n}(\mu_n)$$

considered above. That is, if we write $\mu(A) = \sum_{i=1}^n \mu_i(A)$ and $f, g \in L^\infty(\mu)$, we have

$$\int |fg|\,d\mu \le \|f\|_{(\ldots(L^{p_1}(\mu_1)\oplus_{p_2}L^{p_2}(\mu_2))\oplus_{p_3}\ldots)\oplus_{p_n}L^{p_n}(\mu_n)} \cdot \|g\|_{(\ldots(L^{p_1^*}(\mu_1)\oplus_{p_2^*}L^{p_2^*}(\mu_2))\oplus_{p_3^*}\ldots)\oplus_{p_n^*}L^{p_n^*}(\mu_n)}.$$

This inequality passes to the limit by an approximation argument similar to a previous one. Namely, by Lusin's theorem we pick an increasing sequence of compact subset $C_n \subset [0,1]$, $n \in \mathbb{N}$, such that $p|_{C_n}$ and $q|_{C_n}$ are continuous. Note that by the compactness of the subset and the continuity of the exponent we have $1 < \min_{t\in C_n} p(t) \le \max_{t\in C_n} p(t) < \infty$, and similarly for $q(\cdot)$. Then, as in the proof of Theorem 2.3, for any sequence of simple seminorms N_k of standard form with $\tilde{p}_{N_k} \to p(\cdot)$ in measure as $k \to \infty$ we have $N_k(1_{C_m} f) \to \|1_{C_m} f\|_{p(\cdot)}$ as $k \to \infty$. By essentially the same argument we also see that if N_k^* are the dual simple seminorms of standard form, obtained by replacing all the exponents with their respective conjugates, then $\tilde{p}_{N_k^*} \to p^*(\cdot)$ in measure as $k \to \infty$ and we have $N_k^*(1_{C_m} g) \to \|1_{C_m} g\|_{p^*(\cdot)}$ as $k \to \infty$. This shows that

$$\int_{C_m} |fg|\,dt \le \|1_{C_m} f\|_{p(\cdot)}\|1_{C_m} g\|_{p^*(\cdot)}.$$

Letting $m \to \infty$ yields the claimed inequality for L^∞ functions. Finally, by approximating the given functions $f \in L^{p(\cdot)}$ and $g \in L^{p^*(\cdot)}$ with functions of the form $1_D f, 1_D g \in L^\infty$ in measure and using the absolute continuity of the solutions φ_f and φ_g we obtain the statement. ∎

Going back to simple seminorms of standard form, note that if $p(\cdot) \equiv p_1$ on $[0, t_0)$ and $p(\cdot) \equiv p_2$ on $[t_0, 1]$, and $f \in L^{p(\cdot)}$, then we have $\|f\|_{L^{p(\cdot)}} = \|f\|_{L^{p_1}(\mu_1) \oplus_{p_2} L^{p_2}(\mu_2)}$ where $\mathrm{supp}(\mu_1) = [0, t_0]$ and $\mathrm{supp}(\mu_2) = [t_0, 1]$ (see Lemma 2.1).

It is easy to see that if $p_2 = 1$ then letting $p_1 \nearrow \infty$, $t_0 \searrow 0$ we obtain $\|\mathbf{1}\|_{p(\cdot)} \nearrow 2$. This is perhaps surprising, since always $\|\mathbf{1}\|_p = 1$ in the constant p case. We may also alter the above example as follows: letting $f_{t_0} \equiv 1/t_0$ on $[0, t_0)$ and $f_{t_0} \equiv 1$ on $[t_0, 1]$ with $p_1 = 1$ and $p_2 \nearrow \infty$ and $t_0 \to 0^+$ yields $\|f_{t_0}\|_{p(\cdot)} \to 1$, whereas $\|f_{t_0}\|_1 \to 2$.

We suspect that the above examples are characteristic in the sense that

$$\tfrac{1}{2}\|f\|_1 \leq \|f\|_{p(\cdot)} \leq 2\|f\|_\infty$$

always holds (so that constant 2 is the best possible according to the above examples). We leave this open problem for future research.

In any case, the above inequalities hold with other constants in place of 2. Namely, suppose that $\varphi_f(t_0) = \|f\|_\infty$. Then

$$\varphi(t_0) = \|f\|_\infty, \qquad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t_0 \leq t \leq 1$$

yields

$$\varphi'(t) \leq \varphi(t) \qquad \text{for a.e. } t_0 \leq t \leq 1.$$

Observe that $\varphi(1) < y(1)$ where y is the solution to $y' = y$ with $y(0) = \|f\|_\infty$, that is, $y(t) = \|f\|_\infty e^t$.

Let $a \in (1, 2)$ be the solution to $a^a = e$. Then b^x/x is increasing in $x \geq 1$ for all $b > a$.

Proposition 3.2. *The following inequalities hold whenever defined:*

(1) $\frac{1}{1+a}\|1_{p(\cdot) \geq p} f\|_p \leq \|1_{p(\cdot) \geq p} f\|_{p(\cdot)}$,
(2) $\frac{1}{1+ae}\|1_{p_1(\cdot) \leq p_2(\cdot)} f\|_{p_1(\cdot)} \leq \|1_{p_1(\cdot) \leq p_2(\cdot)} f\|_{p_2(\cdot)}$,
(3) $\|f\|_{p(\cdot)} < e\|f\|_\infty$.

Proof. The last inequality was already proved, and we will verify the middle inequality which is the most complicated one.

Suppose that $p_1(\cdot) \leq p_2(\cdot)$ and $f \in L^{p_1(\cdot)}$, with $\|f\|_{p_1(\cdot)} = 1 + ae$.

We wish to exclude the case where $\varphi_{p_2(\cdot),f}(1) < 1$, so suppose that $\varphi_{p_2(\cdot),f}(1) \leq 1$. Let $[r, 1]$ be the maximal interval on which $\varphi_{p_2(\cdot),f}(t) \leq \varphi_{p_1(\cdot),f}(t)$ (and $\varphi_{p_2(\cdot),f}(t) \leq 1$). Let $A \subset [0, 1]$ be the set where $|f(t)| > a$. On $[r, 1] \cap A$ we have

$$\frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} \geq \frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)}.$$

Indeed, in this set we have

$$\frac{|f(t)|^{p_2(t)}}{p_2(t)} \geq \frac{|f(t)|^{p_1(t)}}{p_1(t)}$$

and

$$\varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} \geq \varphi_{p_2(\cdot),f}(t)^{1-p_1(t)} \geq \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)}.$$

Thus

$$\begin{aligned}
&\varphi_{p_1(\cdot),f}(1) - \varphi_{p_2(\cdot),f}(1) \\
&\quad = \int_r^1 \left(\frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} - \frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} \right) dt \\
&\quad = \int_{[r,1]\cap A} \left(\frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} - \frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} \right) dt \\
&\qquad + \int_{[r,1]\setminus A} \left(\frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} - \frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} \right) dt \\
&\quad \leq \int_{[r,1]\setminus A} \left(\frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} - \frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} \right) dt \\
&\quad \leq \int_{[r,1]\setminus A} \frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)}\, dt \\
&\quad \leq \|1_{[r,1]\setminus A} f\|_{p_1(\cdot)} \leq \|a1_{[r,1]\setminus A}\|_{p_1(\cdot)} \leq \|a1_{[0,1]}\|_{p_1(\cdot)} \leq e\|a1_{[0,1]}\|_\infty = ae.
\end{aligned}$$

Thus $\varphi_{p_2(\cdot),f}(1) \geq (1+ae) - ae = 1$. ■

The following fact connects the investigated varying exponent norm with the Nakano $L^{p(\cdot)}$ norms

$$|||g|||_{p(\cdot)} := \inf\left\{ \lambda > 0 \colon \int \frac{1}{p(t)} \left(\frac{|g(t)|}{\lambda} \right)^{p(t)} dt \leq 1 \right\}.$$

Proposition 3.3. *Let $p \in L^0$, $p(\cdot) \geq 1$, and $f \in L^{p(\cdot)}$ (ODE-determined). Then*

$$|||f|||_{p(\cdot)} \leq \|f\|_{p(\cdot)} \leq 2|||f|||_{p(\cdot)}.$$

Proof. To prove the left-hand estimate it suffices to check that if $\lambda = \|f\|_{p(\cdot)}$, then

$$(*) \qquad \int_0^1 \frac{1}{p(t)} \left(\frac{|f(t)|}{\lambda} \right)^{p(t)} dt \leq 1.$$

So, suppose that $0 < \varphi_f(1) = \lambda$; then

$$\varphi'_f(t) \geq \frac{|f(t)|^{p(t)}}{p(t)} \lambda^{1-p(t)}$$

(with strict inequality in a set of positive measure if $f \neq 0$), so that

$$\lambda = \varphi_f(1) \geq \int_0^1 \frac{|f(t)|^{p(t)}}{p(t)} \lambda^{1-p(t)}\, dt = \int_0^1 \lambda \frac{1}{p(t)} \left(\frac{|f(t)|}{\lambda}\right)^{p(t)} dt.$$

We will show that this yields $(*)$. We may restrict to the case $|||f|||_{p(\cdot)} = 1$ by the positive homogeneity of the norms. If $\|f\|_{p(\cdot)} \leq 1$ then we have the claim, so assume that $0 < t_0 < 1$ is such that $\varphi_f(t_0) = 1$. Then

$$\varphi'_f(t) \leq \frac{|f(t)|^{p(t)}}{p(t)} \quad \text{for a.e. } t \in [t_0, 1].$$

Thus

$$\varphi_f(1) \leq 1 + \int_{t_0}^1 \frac{|f(t)|^{p(t)}}{p(t)}\, dt \leq 1 + \int_0^1 \frac{|f(t)|^{p(t)}}{p(t)}\, dt = 1 + |||f|||_{p(\cdot)} = 2. \ \blacksquare$$

Thus the above Nakano norms are equivalent to the ODE-driven norms considered here. However, these norms do not coincide in general. For example, if $p_1(\cdot)$ is 1 on $[0, 1/2)$ and 2 on $[1/2, 1]$ and $p_2(\cdot)$ is defined in the opposite way, then $|||f|||_{p_1(\cdot)} = |||f|||_{p_2(\cdot)}$ in the case of Nakano norms (or Musielak–Orlicz norms) for any f with $f(s) = f(1/2 + s)$ for $0 \leq s < 1/2$. The same rearrangement invariance condition does not hold for our $\|\cdot\|_{p(\cdot)}$-norms. Indeed, for the constant function $\mathbf{1}$ we have

$$\|\mathbf{1}\|_{p_1(\cdot)} = \sqrt{\left(\frac{1}{2}\right)^2 + \frac{1}{2}} = \frac{\sqrt{3}}{2} \approx 0.866 < 1.207 \approx \frac{1}{\sqrt{2}} + \frac{1}{2} = \|\mathbf{1}\|_{p_2(\cdot)}.$$

We can use the above ideas to construct counterexamples as well.

Example 3.4. Let $p\colon [0, 1] \to [1, \infty)$ be a measurable function defined by

$$\frac{1}{p(t)} \left(\frac{2}{3}\right)^{1-p(t)} = \frac{1}{t - 1/2}$$

if $t \in (1/2, 1]$, and $p(t) = 1$ on $[0, 1/2]$. Then the constant function $f = \mathbf{1}$ is *not* in $L^{p(\cdot)}$.

Assuming the contrary, clearly $\varphi_f(1/2)$ would be $1/2$. Suppose that $t_0 > 1/2$ is such that $\varphi_f(t) \leq 2/3$ for $1/2 \leq t \leq t_0$. Then we would have

$$\frac{2}{3} - \frac{1}{2} \geq \varphi_f(t_0) - \varphi_f\left(\frac{1}{2}\right) = \int_{1/2}^{t_0} \varphi'_f(t)\, dt \geq \int_{1/2}^{t_0} \frac{1}{t - 1/2}\, dt = \infty,$$

contradicting the assumption that f was in the class, that is, had an absolutely continuous solution φ_f. However, if we allow initial values $\varphi_f(0) = x_0 \geq 1/2$, then we have nice corresponding solutions.

Also note that $1_{[0,1/2]} + 1_{[0,1]} \in L^{p(\cdot)}$. This means that in general the $L^{p(\cdot)}$ class need not be an ideal in the sense of Banach lattice theory, i.e. $g \in L^{p(\cdot)}$, $f \in L^0$, $|f| \leq g$, does *not* imply $f \in L^{p(\cdot)}$.

In the above example, we have $(1_{[0,1/2]} + 1_{[0,1]}), 1_{[0,1/2]} \in L^{p(\cdot)}$ and $(1_{[0,1/2]} + 1_{[0,1]}) - 1_{[0,1/2]} = 1_{[0,1]} \notin L^{p(\cdot)}$. This shows that for some $p(\cdot)$ the class $L^{p(\cdot)}$ fails to be a linear space. This example is a manifestation of the principle that the higher the value of φ, the more stable the differential equation becomes, ceteris paribus.

3.1. The essentially bounded exponent case. Let us take a look at the nice case where $\overline{p} := \operatorname{ess\,sup}_t p(t) < \infty$, as it turns out that the corresponding spaces have less pathological properties.

We have observed previously that $L^{p(\cdot)}$ classes need not have the ideal property in general. However, in the case $\overline{p} < \infty$ the conditions $g \in L^{p(\cdot)}$, $|f| \leq g$ do imply that also $f \in L^{p(\cdot)}$. This follows immediately from the following observation.

PROPOSITION 3.5. *Suppose that $\overline{p} < \infty$, $g \in L^{p(\cdot)}$, $|f| \leq |g|$, and $0 < x_0 < 1$ is a given initial value. Then $f \in L^{p(\cdot)}$ and*

$$|\varphi'_{f,x_0}| \leq |\varphi'_{g,x_0}| \left(\frac{x_0}{\varphi_{g,x_0}(1)} \right)^{1-\overline{p}}.$$

Proof. Consider a simple seminorm N applied to f, g and with the above initial value: $\phi(t) = \|1_{[0,t]} f\|_N$ and $\psi(t) = \|1_{[0,t]} g\|_N$. Clearly $\phi \leq \psi$. Denote by $p(\cdot)$ the corresponding piecewise constant exponent. According to Lemma 2.1 we may differentiate ϕ and ψ a.e. We obtain

$$\phi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \phi(t)^{1-p(t)}, \qquad \psi'(t) = \frac{|g(t)|^{p(t)}}{p(t)} \psi(t)^{1-p(t)},$$

so

$$\frac{\phi'(t)}{\psi'(t)} \leq \left(\frac{\phi(t)}{\psi(t)} \right)^{1-p(t)} \leq \left(\frac{\phi(t)}{\psi(t)} \right)^{1-\overline{p}} \leq \left(\frac{x_0}{\psi(1)} \right)^{1-\overline{p}}.$$

The existence of the solution for f follows from Theorem 3.7 below. ∎

In particular we remain within the class if we restrict supports.

PROPOSITION 3.6. *Let $\overline{p} < \infty$, $f \in L^{p(\cdot)}$ and $A_n \subset [0,1]$ a sequence of measurable subsets such that $m(A_n) \to 0$ as $n \to \infty$. Then $\|1_{A_n} f\|_{p(\cdot)} \to 0$ as $n \to \infty$.*

Proof. Fix $\epsilon > 0$. We claim that given an initial value $x_0 = \epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\varphi_{1_{A_n} f, x_0}(1) < 2\epsilon, \quad n \geq n_0. \tag{3.1}$$

This clearly yields the statement of the lemma.

The absolute continuity of the solution φ_{f,x_0} implies that

$$\int_{A_n} \varphi'_{f,x_0}(t)\, dt \to 0, \quad n \to \infty.$$

This observation together with Proposition 3.5 yields (3.1). ∎

Then $L^\infty \subset L^{p(\cdot)}$ is dense by the triangle inequality. For a general measurable exponent $p\colon [0,1] \to [1,\infty)$ we define a natural Banach subspace

$$L_0^{p(\cdot)} := \overline{\bigcup_{n\in\mathbb{N}} \{1_{p(\cdot)\leq n} f \colon f \in \widetilde{L}^{p(\cdot)}\}} \subset \widetilde{L}^{p(\cdot)}.$$

THEOREM 3.7. *For a general measurable exponent $p(\cdot)$ the above Banach space satisfies $L_0^{p(\cdot)} \subset L^{p(\cdot)}$. In particular, in the case $\overline{p} < \infty$, we have $L^{p(\cdot)} = \widetilde{L}^{p(\cdot)}$, and consequently $L^{p(\cdot)}$ is a Banach space.*

Proof. First we will verify the latter part of the statement, so assume that $\overline{p} < \infty$. Let $f \in \widetilde{L}^{p(\cdot)}$. Similarly to the above, let $D_i \subset [0,1]$, $i \in \mathbb{N}$, be measurable compact subsets such that both $f|_{D_i}$ and $p|_{D_i}$ are continuous and $m(D_i) \to 1$ as $i \to \infty$. Set

$$\psi_i(t) := \|1_{[0,t]\cap D_i} f\|_{\widetilde{L}^{p(\cdot)}}$$

where we consider the functions with a common initial value $0 < x_0 < 1$. We find that each ψ_i is absolutely continuous and

$$\psi_i' = \frac{|1_{D_i} f|^{p(t)}}{p(t)} \psi_i(t)^{1-p(t)} \qquad \text{a.e.}$$

Note that

$$\psi_i(t) \nearrow \psi(t) := \|1_{[0,t]} f\|_{\widetilde{L}^{p(\cdot)}}$$

for $t \in [0,1]$ as $i \to \infty$ by the absolute continuity of simple seminorms.

Using the positivity of the initial value x_0 and $\overline{p} < \infty$, by studying the derivatives ψ_i' we obtain

$$\psi(1)^{1-\overline{p}} \int_r^t \frac{|f(s)|^{p(s)}}{p(s)}\, ds \leq \psi(t) - \psi(r) \leq x_0^{1-\overline{p}} \int_r^t \frac{|f(s)|^{p(s)}}{p(s)}\, ds. \tag{3.2}$$

We conclude that ψ is absolutely continuous and $|f|^{p(\cdot)}/p(\cdot) \in L^1$.

According to Dini's theorem, $\psi_i \to \psi$ converges uniformly on $[0,1]$. Moreover, by again using the positivity of the initial value and $\overline{p} < \infty$ we get

$\psi_i(t)^{1-p(t)} \to \psi(t)^{1-p(t)}$ in L^∞-norm as $i \to \infty$. Thus

$$\psi_i' \to \frac{|f|^{p(t)}}{p(t)} \psi^{1-p(t)}$$

in L^1, and hence

$$\psi(T) = x_0 + \int_0^T \frac{|f(t)|^{p(t)}}{p(t)} \psi(t)^{1-p(t)}\, dt, \quad T \in [0,1].$$

This shows that ψ is a solution witnessing that $f \in L^{p(\cdot)}$, since x_0 was arbitrary.

To verify the first part of the statement, fix $f \in L_0^{p(\cdot)}$; we aim to show that $f \in L^{p(\cdot)}$, i.e. there is a solution φ_f. Let $f_n = 1_{p(\cdot)\le n} f$ for $n \in \mathbb{N}$. Clearly $f_n \nearrow f$ a.e. as $n \to \infty$. As above, it follows from the triangle inequality that $\varphi_{f_n} \nearrow \phi$ uniformly for a suitable ϕ. We consider all the solutions with a common positive initial value $x_0 > 0$.

For each $k \in \mathbb{N}$ and $\varepsilon > 0$ there exist by Egorov's theorem a set $D \subset \{t \in [0,1] \colon p(t) \le k\}$ such that $m(\{t \in [0,1] \colon p(t) \le k\} \setminus D) < \varepsilon$ and

$$\frac{|f_n(t)|^{p(t)}}{p(t)} \varphi_{f_n}(t)^{1-p(t)} \to \frac{|f(t)|^{p(t)}}{p(t)} \phi(t)^{1-p(t)}$$

uniformly on D as $n \to \infty$. Thus

$$\int_D \frac{|f_n(t)|^{p(t)}}{p(t)} \varphi_{f_n}(t)^{1-p(t)}\, dt \to \int_D \frac{|f(t)|^{p(t)}}{p(t)} \phi(t)^{1-p(t)}\, dt.$$

Since ε was arbitrary, Proposition 3.6 yields

$$\int_{p(t)\le k} \frac{|f_n(t)|^{p(t)}}{p(t)} \varphi_{f_n}(t)^{1-p(t)}\, dt \to \int_{p(t)\le k} \frac{|f(t)|^{p(t)}}{p(t)} \phi(t)^{1-p(t)}\, dt$$

for each $k \in \mathbb{N}$. Since $\|f - 1_{p(\cdot)\le n}\, f\|_{\widetilde{L}^{p(\cdot)}} \to 0$, we see that

$$\lim_{k\to\infty} \int_{p(t)\ge k} \frac{|f(t)|^{p(t)}}{p(t)}\, dt = 0.$$

Hence

$$\int_0^T \frac{|f_n(t)|^{p(t)}}{p(t)} \varphi_{f_n}(t)^{1-p(t)}\, dt \to \int_0^T \frac{|f(t)|^{p(t)}}{p(t)} \phi(t)^{1-p(t)}\, dt, \quad T \in [0,1].$$

Taking into account that $\varphi_{f_n} \to \phi$ uniformly, we see that

$$\phi(T) = x_0 + \int_0^T \frac{|f(t)|^{p(t)}}{p(t)} \phi(t)^{1-p(t)}\, dt, \quad T \in [0,1]. \quad \blacksquare$$

The above argument (recall (3.2)) yields the following fact.

Proposition 3.8. *If $\overline{p} < \infty$, then $f \in L^0$ is in $L^{p(\cdot)}$ if and only if*

$$\int_0^1 \frac{|f(t)|^{p(t)}}{p(t)}\, dt < \infty. \ \blacksquare$$

The $L^{p(\cdot)}$ space construction here can be generalized to a multidimensional setting $L^{p(\cdot)}(\Omega)$ with domains $\Omega \subset \mathbb{R}^n$, $n > 1$. There appear to be several ways to accomplish this. For example, in some cases Ω can be conveniently decomposed into level sets of $p(\cdot)$; then taking the L^p norms relative to each level set and using the approach here to aggregate the L^p norms yields an $L^{p(\cdot)}(\Omega)$ norm. We leave this for future research.

Acknowledgments. This work has been financially supported by research grants from the Väisälä foundation and the Finnish Cultural Foundation and the Academy of Finland Project # 268009.

References

[ACK] G. Androulakis, C. Cazacu and N. Kalton, *Twisted sums, Fenchel–Orlicz spaces and property (M)*, Houston J. Math. 24 (1998), 105–126.

[BY] I. Ben Yaacov, *Modular functionals and perturbations of Nakano spaces*, J. Logic Anal. 1 (2009), paper 1, 42 pp.

[BO] Z. Birnbaum und W. Orlicz, *Über die Verallgemeinerung des Begriffes der zueinander Konjugierten Potenzen*, Studia Math. 3 (1931), 1–67.

[CL] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[DHR] L. Diening, P. Hästö and S. Roudenko, *Function spaces of variable smoothness and integrability*, J. Funct. Anal. 256 (2009), 1731–1768.

[DR] L. Diening and M. Růžička, *Calderón–Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. 563 (2003), 197–220.

[FH+] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucia, J. Pelant and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books Math./Ouvrages Math. SMC 8, Springer, New York, 2001.

[JKL] J. E. Jamison, A. Kamińska and P.-K. Lin, *Isometries of Musielak–Orlicz spaces II*, Studia Math. 104 (1993), 75–89.

[Ka] N. Kalton, *Extension of linear operators and Lipschitz maps into $C(K)$-spaces*, New York J. Math. 13 (2007), 317–381.

[Ko] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. 41 (1991), 592–618.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I and II* , Classics in Math., Springer, 1996.

[L] W. Luxemburg, *Banach function spaces*, thesis, T.U. Delft, 1955.

[Mal] L. Maligranda, *Hidegoro Nakano (1909-1974)—on the centenary of his birth*, in: Banach and Function Spaces III (Kitakyushu, 2009), M. Kato et al. (eds.), Yokohama Publ., 2011, 99–171.

[Mar] J. Marcinkiewicz, *Sur l'interpolation d'opérations*, C. R. Acad. Sci. Paris 208 (1939), 1272–1273.

[Mu] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer, Berlin, 1983.

[NS] E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. 262 (2012), 3665–3748.

[RR91] M. Rao and Z. Ren, *Theory of Orlicz Spaces*, Monogr. Textbooks Pure Appl. Math. 146, Dekker, New York, 1991.

[RR02] M. Rao and Z. Ren, *Applications of Orlicz Spaces*, CRC Press, 2002.

[Ru] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1987.

[S] A. Sobczyk, *Projections in Minkowski and Banach spaces*, Duke Math. J. 8 (1941), 78–106.

[T] J. Talponen, *A natural class of sequential Banach spaces*, Bull. Polish Acad. Sci. Math. 59 (2011), 185–196.

Jarno Talponen
Department of Physics and Mathematics
University of Eastern Finland
Box 111
FI-80101 Joensuu, Finland
E-mail: talponen@iki.fi