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Rigged modules II: multipliers and duality

by

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Abstract. In a previous paper with Kashyap (2011) we generalized the theory of W^*-modules to the setting of modules over nonselfadjoint dual operator algebras on a Hilbert space, obtaining the class of weak* rigged modules. The present paper and its contemporaneous predecessor comprise the sequel which we promised at that time would be forthcoming. We give many new results about rigged and weak* rigged modules and their tensor products, such as an Eilenberg–Watts type theorem.

1. Introduction. *Rigged modules* over a (nonselfadjoint) operator algebra are the generalization from [1, 12] of the important class of modules over C^*-algebras known as *Hilbert C^*-modules*. A *W^*-module* is a Hilbert C^*-module over a von Neumann algebra which is 'selfdual' (see e.g. [16, 3]), or equivalently which has a Banach space predual (a result of Zettl, see e.g. [11, Corollary 3.5] for a proof). The *weak* rigged* or *w^*-rigged modules*, introduced in [7] (see also [9]; or [11, Section 5] for an earlier variant), are a generalization of W^*-modules to the setting of modules over a (nonselfadjoint) dual operator algebra. By the latter term we mean a unital weak* closed algebra of operators on a Hilbert space.

In [7] we generalized basic aspects of the theory of W^*-modules, and this may also be seen as the weak* variant of the theory of rigged modules from [2] (see also [12]). The present paper and its contemporaneous predecessor comprise the sequel which we promised at the time of [7] would be forthcoming. In the present paper we discuss rigged modules and 'correspondences' using the concept of the operator space left multiplier algebra of Y in the sense of [4, Section 4.5]. We also discuss a connection between rigged and weak* rigged modules, the exterior tensor product, orthogonally

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complemented submodules, and other topics such as an Eilenberg–Watts type theorem characterizing functors between categories of rigged or weak* rigged modules.

In the course of this work we noticed several things that were missed, or not stated (or proved), or which could be simplified, from the time the earlier work on rigged and weak* rigged modules was done. We take the opportunity to correct/present/simplify these things here.

Turning to background, we will use the notation from [6, 7, 15, 8], and perspectives from [1]. We will assume that the reader is familiar with basic notions from operator space theory which may be found in any current text on that subject. The reader may consult [10] as a reference for any other unexplained terms here. We assume that the reader is familiar with basic Banach space and operator space duality principles such as the Krein–Šmulian theorem. We often abbreviate 'weak*' to 'w^*'. A *right dual operator* M-module is a nondegenerate M-module Y, which is also a dual operator space, such that the module action is completely contractive and separately weak* continuous. We use standard notation for module mapping spaces; e.g. $CB(X,N)_N$ (resp. $CB^\sigma(X,N)_N$) are the completely bounded (resp. and weak* continuous) right N-module maps from X to N. We often use the *normal module Haagerup tensor product* $Y \otimes^{\sigma h}_M Z$, and its universal property from [14], which loosely says that it linearizes completely contractive M-balanced separately weak* continuous bilinear maps (*balanced* means that $u(xa, y) = u(x, ay)$ for $a \in M$). We assume that the reader is familiar with the notation and facts about this tensor product from [6, Section 2]. Although we shall not use it here, in passing we remark that the module tensor product facts in that section work even without assuming all of the constituents of the definition of Y being a dual operator M-module, so long as it is a dual operator space and an M-module. For any operator space X we write $C_n(X)$ for the column space of $n \times 1$ matrices with entries in X, with its canonical norm from operator space theory.

Definition 1.1 ([7]). Suppose that Y is a dual operator space and a right module over a dual operator algebra M. Suppose that there exists a net $(n(\alpha))$ of positive integers, and w^*-continuous completely contractive M-module maps $\phi_\alpha : Y \to C_{n(\alpha)}(M)$ and $\psi_\alpha : C_{n(\alpha)}(M) \to Y$, with $\psi_\alpha(\phi_\alpha(y))$ converging to y in the weak* topology on Y for all $y \in Y$. Then we say that Y is a *right w^*-rigged module* (or *right weak* rigged module*) over M.

We remark that the fact that w^*-rigged modules are dual operator modules seems not to have been proved in the development in [7, Section 2 and the start of Section 3] but seemingly assumed in the proof. We give a proof of this early on in [8].

As in [7, p. 348], the operator space structure of a w^*-rigged module Y over M is determined by $\|[y_{ij}]\|_{M_n(Y)} = \sup_\alpha \|[\phi_\alpha(y_{ij})]\|$ for $[y_{ij}] \in M_n(Y)$.

The *rigged modules* of [1] may be defined similarly to Definition 1.1, but with the words 'dual' and 'w^*-continuous' removed, and the weak* topology replaced by the norm topology, and M now an approximately unital operator algebra. This simpler reformulation of the definition of a rigged module, and its equivalence with the definitions in [1], may be found in [5, Section 3]. The operator space structure of a rigged module Y over M is determined by the same formula as at the end of the last paragraph, but for the appropriate ϕ_α in this case. See [1] for details.

We say that w^*-rigged modules are *unitarily isomorphic* if there exists a completely isometric surjective weak* homeomorphic module map between them. Similarly for rigged modules, with of course 'weak* homeomorphic' dropped.

Every right w^*-rigged module (resp. right rigged module) Y over M gives rise to a canonical left w^*-rigged (resp. left rigged module) M-module $\tilde{Y}$, and a pairing $(\cdot,\cdot) : \tilde{Y} \times Y \to M$ (see [7, 1]). Indeed in the w^*-rigged case, $\tilde{Y}$ turns out to be completely isometric to $CB^\sigma(Y, M)_M$ as dual operator M-modules, together with its canonical pairing with Y. Also, $\tilde{\tilde{Y}} = Y$. The morphisms between w^*-rigged M-modules are the *adjointable* M-module maps [7, 8], which turn out to coincide with the weak* continuous completely bounded M-module maps (see [7, Proposition 3.4]). We write $\mathbb{B}(Z, W)$ for the weak* continuous completely bounded M-module maps from a w^*-rigged M-module Z into a dual operator M-module W, with as usual $\mathbb{B}(Z) = \mathbb{B}(Z, Z)$. We also use this notation for the *adjointable maps* between rigged modules [1]. We write $\mathbb{K}(Y)_A$ for the *compact adjointable* right A-module maps on a right rigged A-module Y, that is, the closure of the span of the maps on Y of the form $y \mapsto y'(x, y)$ for some $y' \in Y$ and $x \in \tilde{Y}$ (see [1]).

2. Rigged modules, multipliers, correspondences, and duality. The following important facts about rigged modules do not appear to be in the literature:

LEMMA 2.1. *If Y is a rigged module over an operator algebra A, viewed as an operator space, and if $\mathcal{M}_\ell(Y)$ is the operator space left multiplier algebra of Y in the sense of* [4, Section 4.5]*, then $\mathcal{M}_\ell(Y) = CB(Y)_A$ completely isometrically isomorphically. This also equals the left multiplier algebra of $\mathbb{K}(Y)_A$, where the latter is the compact adjointable maps on Y.*

Proof. It is known (see e.g. [1, Theorem 3.6]) that $\mathbb{K}(Y)_A$ is a left ideal in $CB(Y)_A$. This gives a map $CB(Y)_A \to LM(\mathbb{K}(Y)_A)$. Conversely, since Y is a left operator $\mathbb{K}(Y)_A$-module (by the same cited theorem), it is a left operator $LM(\mathbb{K}(Y)_A)$-module by [10, 3.1.11]. Hence we obtain a completely contractive homomorphism $LM(\mathbb{K}(Y)_A) \to CB(Y)_A$. It is easy to argue that these maps are mutual inverses, so that $CB(Y)_A \cong LM(\mathbb{K}(Y)_A)$ completely isometrically isomorphically. (This argument may have originally been due to Paulsen.)

By facts in the theory of operator space multipliers (see e.g. [10, Theorem 4.5.5]), the 'identity map' is a completely contractive homomorphism $\mathcal{M}_\ell(Y) \to CB(Y)$. This maps into $CB(Y)_A$, since for example right multiplication by $a \in A$ is easily seen to be a right operator space multiplier, and left and right operator space multipliers commute (see [10, 4.5.6]). From [1] we know that $CB(Y)_A$ is an operator algebra. Also, by the last paragraph, Y is a left operator $CB(Y)_A$-module (with the canonical action). By the operator space multiplier theory (see e.g. [10, Theorem 4.6.2(1) and (2)]) there exists a completely contractive homomorphism $\pi : CB(Y)_A \to \mathcal{M}_\ell(Y)$ with $\pi(T)(y) = T(y)$ for all $y \in Y$ and $T \in CB(Y)_A$. That is, $\pi(T) = T$. Thus $CB(Y)_A = \mathcal{M}_\ell(Y)$. ∎

The last result should have many consequences. In the remainder of this section we give several.

Corollary 2.2. *For any orthogonally complemented* (*in the sense of* [1, Section 7]) *submodule W of a rigged module Y over an operator algebra A, there is a unique contractive linear projection from Y onto W. The right M-summands in the sense of* [4] (*see also* [10, Sections 4.5 and 4.8]) *in such Y are precisely the orthogonally complemented submodules of Y.*

Proof. The orthogonal projections in $\mathcal{M}_\ell(Y)$, which by the previous result are the completely contractive idempotents in $CB(Y)_M$, are the left M-projections on Y by [4], and the right M-summands are their ranges. These ranges are just the orthogonally complemented submodules.

The first assertion is a general fact about right M-summands of an operator space from [4]. ∎

An early prototype of the w^*-rigged modules appeared in [11, Section 5]. We now connect these two notions. Some examples of the modules characterized here may be found e.g. in [11, p. 405]. For example every W^*-module (defined in the first paragraph of our paper) satisfies these conditions.

Theorem 2.3. *Let Y be a rigged module over a dual operator algebra M. Suppose that Y has a predual operator space, and that $(x, \cdot)$ is weak* continuous for all $x \in \tilde{Y}$. Then Y is a w^*-rigged module, and Y is self-dual* (*that is, $CB(Y, M)_M \cong \tilde{Y}$ via the canonical map*), *and $CB(Y)_M = CB^\sigma(Y)_M = \mathbb{B}(Y)_M$. Thus Y belongs to the class of modules considered in* [11, Lemma 5.1 and Corollaries 5.2 and 5.5], *and therefore satisfies all the conclusions of those results.*

Proof. By Lemma 2.1 and because left multipliers on a dual space are known to be weak* continuous [10, Theorem 4.7.1], we have

$$CB(Y)_M = \mathcal{M}_\ell(Y) = CB^\sigma(Y)_M.$$

Given a bounded net $m_t \to m$ weak* in M, suppose that a subnet ym_{t_ν} converges to y' weak* in Y. Then (x, ym_{t_ν}) converges to (x, y') for all $x \in \tilde{Y}$. However it also converges to $(x, y)m$, and so $(x, y' - ym) = 0$. It follows that $y' = ym$, so that by topology $ym_t \to ym$ weak*. So the map $m \mapsto ym$ is weak* continuous by the Krein–Šmulian theorem.

It follows that in the definition of Y being a rigged module (below Definition 1.1) we may assume that the maps ϕ_α, ϕ_α are weak* continuous. Indeed in [1] the 'coordinates' of ϕ_α are usually assumed to be of the form $(x, \cdot)$ for $x \in \tilde{Y}$, hence are weak* continuous. For ψ_α this follows from the fact proved in the last paragraph. So Y is a w^*-rigged module.

Applying the relation at the end of the first paragraph of the proof to the direct column sum $Y \oplus^c M$ (see [7]) we have $CB(Y \oplus^c M)_M$ equal to $CB^\sigma(Y \oplus^c M)_M$, from which it is clear that

$$CB(Y, M)_M = CB^\sigma(Y, M)_M \cong \tilde{Y}.$$

So Y is selfdual. The other conclusions are easy. ■

The last theorem may be viewed as a 'nonselfadjoint variant' of the result of Zettl mentioned in the first lines of the paper.

Remark. The condition in the theorem that $(x, \cdot)$ is weak* continuous may be automatic, although to get this one may need to assume that $b \mapsto yb$ is weak* continuous on M for each $y \in Y$. We were able to show without this $(x, \cdot)$ condition that $Y \otimes_{hM} H^c$ and $Y \otimes^{\sigma h}_M H^c$ are Hilbert spaces, and if these two spaces coincide then the conclusions of the theorem hold. We were also able to prove the theorem with the weak* continuity assumption on $(x, \cdot)$ replaced by the weak* continuity of $b \mapsto yb$ condition, if M acts faithfully on the right on Y (that is, there is a unique $b \in M$ with $Yb = 0$). To see this, let $f \in CB(Y, M)_M$, let $y_t \to y$ be a bounded weak* convergent net in Y, and let $y_0 \in Y$ be fixed. By the first paragraph of the proof the map $y \mapsto y_0 f(y)$ is weak* continuous on Y, so $y_0 f(y_t) \to y_0 f(y)$. Suppose that we have a weak* convergent subnet $f(y_{t_\nu}) \to b$ in M. Then $y_0 f(y_{t_\nu}) \to y_0 b$ weak*. Thus $y_0 b = y_0 f_k(y)$ for all $y_0 \in Y$, and we deduce that $b = f_k(y)$. By topology $f(y_t) \to f_k(y)$ weak*. Hence by the Krein–Šmulian theorem f is weak* continuous. It follows as in the proof that we may assume that the maps ϕ_α, ϕ_α are weak* continuous, and Y is w^*-rigged. Hence $CB(Y, M)_M = CB^\sigma(Y, M)_M \cong \tilde{Y}$, so Y is selfdual, and we may continue as before.

Recall that an *approximately unital operator algebra* is one which has a contractive approximate identity.

THEOREM 2.4. *Suppose that A, B are approximately unital operator algebras, and that Y is a right rigged B-module which is a nondegenerate left A-module via a homomorphism $\theta : A \to \mathbb{B}(Y)_B = M(\mathbb{K}(Y)_B)$. Then with this action Y is a left operator A-module if and only if θ is completely contractive. If these hold then θ is essential in the sense of* [1, pp. 400–401]. *In particular, there is a contractive approximate identity (e_t) for A with $e_t y \to y$ and $x e_t \to x$ for all $y \in Y$ and $x \in \tilde{Y}$.*

Proof. The first assertion may be deduced for example from Lemma 2.1 and the fact that the left operator A-module actions on Y are in bijective correspondence with the completely contractive homomorphisms into $\mathcal{M}_\ell(Y)$ which give a nondegenerate left module action on Y. One direction of this follows from e.g. [10, Theorem 4.6.2(1) and (2)]. The other direction follows from [10, 3.1.12] and the fact that any operator space Y is a left operator $\mathcal{M}_\ell(Y)$-module (by [10, Theorem 4.5.5]).

Viewing $M(\mathbb{K}(Y)_B) \subset (\mathbb{K}(Y)_B)^{**}$, we find that θ extends uniquely to a completely contractive homomorphism $\tilde{\theta} : A^{**} \to (\mathbb{K}(Y)_B)^{**}$ by [10, 2.5.5]. Since $\theta(e_t)z \to z$ for all $z \in \mathbb{K}(Y)_B$ and every contractive approximate identity (e_t) of A, it follows that any weak* limit point η of $(\theta(e_t))$ satisfies $\eta z = z$. So η is a left identity for $(\mathbb{K}(Y)_B)^{**}$, hence equals the identity 1 for that algebra (see [10, Proposition 2.5.8]). So $\theta(e_t) \to 1$ weak*, by topology. Then $z\theta(e_t) \to z$ weak* in $\mathbb{K}(Y)_B^{**}$ for $z \in \mathbb{K}(Y)_B$, and hence weakly in $\mathbb{K}(Y)_B$ (note that $\mathbb{K}(Y)_B$ is an ideal in $\mathbb{B}(Y)_B$). By Mazur's theorem, taking convex combinations we get a norm bounded net satisfying [1, Proposition 6.2(2)]. So θ is essential. The last assertion follows from [1, Proposition 6.3]. ∎

A bimodule satisfying the conditions in the last result will be called a (right) *A-B-correspondence*. The last theorem shows that the original definition in [1, Proposition 6.3] can be substantially simplified.

The interior tensor product of right rigged modules from [1, pp. 400–401] is simply the module Haagerup tensor product (see [12, 10]) of a right A-rigged module and a right A-B-correspondence. We will write this tensor product as $Y \otimes_\theta Z$, where θ is the left action as above. However we will not focus much on rigged modules in this paper, since that theory is older and more developed.

We will use later the interior tensor product of weak* rigged modules [7, 8]. Here Y is a right w^*-rigged module over a dual operator algebra M, and Z is a right w^*-rigged module over a dual operator algebra N, and $\theta : M \to \mathbb{B}(Z)$ is a weak* continuous unital completely contractive homomorphism. Because Z is a left operator module $\mathbb{B}(Z)$-module (see [7, p. 349]), Z becomes an essential left dual operator module over M under the action $m \cdot z = \theta(m)z$. In this case we say Z is a right *M-N-correspondence* (an abusive notation because this concept is the weak* variant of the analogous notion studied earlier in this section under the same name). We form the normal module Haagerup tensor product $Y \otimes_M^{\sigma h} Z$ which we also write as $Y \otimes_\theta Z$ (again a somewhat abusive notation; the context will have to make it clear whether we are using the rigged or the w^*-rigged variant). By [7, 3.3] this is a right w^*-rigged module over N, called the *interior tensor product* of w^*-rigged modules.

3. Eilenberg–Watts type theorem. The norm on the matrix space $M_{m,n}(CB(Y,Z)_M)$ (and on its subspace $M_{m,n}(\mathbb{B}(Y,Z))$) is the operator space norm, namely giving $[f_{ij}]$ the 'completely bounded norm' in $CB(Y, M_{m,n}(Z))$ of the map $y \mapsto [f_{ij}(y)]$. We write this norm as $\|[f_{ij}]\|_{cb}$.

LEMMA 3.1. *Suppose that Y is a right w^*-rigged (resp. rigged) module, and Z is a right dual operator module (resp. right operator module) over a dual operator algebra (resp. operator algebra) M. For $m, n \in \mathbb{N}$ suppose that $[f_{ij}] \in M_{m,n}(CB(Y,Z)_M)$, with each f_{ij} weak* continuous in the w^*-rigged case. Then*

$$\|[f_{ij}]\|_{cb} = \sup_\alpha \|[f_{ij}(y_k^\alpha)]\|,$$

where $[f_{ij}(y_k^\alpha)]$ is indexed on rows by i and on columns by j and k, and where (y_k^α) are the 'coordinates' of the map ψ_α in Definition 1.1 (so that $\psi_\alpha([b_k]) = \sum_k y_k^\alpha b_k$). This norm also equals the 'completely bounded norm' in $CB(C_n(Y), C_m(Z))$ of the map $[y_j] \mapsto [\sum_j f_{ij}(y_j)]$ on $C_n(Z)$. In particular for w^*-rigged modules Y, Z over M we have

$$M_{m,n}(\mathbb{B}(Y,Z)) \cong \mathbb{B}(C_n(Y), C_m(Z))$$

completely isometrically.

Proof. The assertions for w^*-rigged modules follow by [7, Corollary 3.6], or by the weak* variant of the following. In the rigged module case the result follows by a trick which occurs very frequently in the theory (see e.g. [12]), so we will be brief. Write the map ϕ_α in Definition 1.1 as $\phi_\alpha(y) = [x_k^\alpha(y)]$, and set $y^\alpha = (y_k^\alpha) \in M_{1,n}(Y)$. Then for $[y_{pq}] \in M_r(Y)$ of norm 1 we have

$$[f_{ij}(y_{pq})] = \lim_\alpha \Big[f_{ij}\Big(\sum_k y_k^\alpha (x_k^\alpha, y_{pq}) \Big) \Big] = \lim_\alpha \Big[\sum_k f_{ij}(y_k^\alpha)(x_k^\alpha, y_{pq}) \Big].$$

The norm of this is dominated by $\sup_\alpha \|[f_{ij}(y_k^\alpha)]\|$, which in turn is dominated by $\|[f_{ij}]\|_{cb}$ since $\|y^\alpha\| \le 1$. This proves the displayed equation. A similar computation shows that $\|[\sum_j f_{ij}(y_j^{pq})]\| \le \sup_\alpha \|[f_{ij}(y_k^\alpha)]\|$ for a matrix $[y_i^{pq}]$ of norm 1 with entries y_i^{pq} in Y indexed on rows by i, p and on columns by q. In turn $\|[f_{ij}(y_k^\alpha)]\|$ is dominated by the completely bounded norm in $CB(C_n(Y), C_m(Z))$, as may be seen by viewing f as 'acting by left multiplication' on the $n \times (n \cdot n(\alpha))$ matrix $y^\alpha \otimes I_n$. ■

For a dual operator algebra M let $\mathcal{W}_M$ denote the category of right w^*-rigged modules over M. The morphisms are the weak* continuous (or equivalently, adjointable) completely bounded M-module maps. For an approximately unital operator algebra M let $\mathcal{R}_M$ be the category of right rigged modules over M, with morphisms the adjointable completely bounded M-module maps.

We will say that a functor F is *completely contractive* (resp. *linear*, *normal*, *strongly continuous*) if $T \mapsto F(T)$ is completely contractive (resp. linear, weak* continuous, takes bounded strongly convergent (that is, 'point-norm' convergent) nets to strongly convergent nets) on the space of morphisms.

PROPOSITION 3.2. *For approximately unital operator algebras* (*resp. dual operator algebras*) M *and* N *let* Z *be a right* M-N-*correspondence. Then the interior tensor product with* Z *is a strongly continuous normal* (*resp. normal*) *completely contractive linear functor from* $\mathcal{W}_M$ *to* $\mathcal{W}_M$ (*resp.* $\mathcal{R}_M$ *to* $\mathcal{R}_N$).

In particular, if M *and* N *are weak* Morita equivalent dual operator algebras in the sense of* [6], *then their categories of right* w^*-*rigged modules are isomorphic. Moreover this isomorphism is implemented by tensoring with the equivalence bimodule.*

Proof. Let $F(Y) = Y \otimes_\theta Z$ be the interior tensor product. That F is completely contractive follows from [8, Proposition 2.2 and the remark after it], and it is easily seen to be a linear functor. If a bounded net T_t converges to T in the strong (resp. weak*) topology in $\mathbb{B}(Y_1, Y_2)$ then $T_t \otimes I \to T \otimes I$ strongly (resp. weak*; see [9, Theorem 3.1]).

If (M, N, X, Y) is a weak* Morita context in the sense of [6] then by the above $\mathcal{F}(Z) = Z \otimes_M^{\sigma h} X$ is a completely contractive normal functor from $\mathcal{R}_M$ to $\mathcal{R}_N$, with 'inverse' the functor $\mathcal{G}$ from $\mathcal{R}_N$ to $\mathcal{R}_M$ defined by $\mathcal{G}(W) = W \otimes_N^{\sigma h} Y$. As in [6, Theorem 3.5], F and G are inverse functors via completely isometric isomorphisms, and so the categories $\mathcal{R}_M$ and $\mathcal{R}_N$ are isomorphic. ∎

THEOREM 3.3. *Let* M *and* N *be approximately unital operator algebras* (*resp. dual operator algebras*), *and suppose that* F *is a strongly continuous normal* (*resp. normal*) *completely contractive linear functor from* $\mathcal{W}_M$ *to* $\mathcal{W}_N$ (*resp.* $\mathcal{R}_M$ *to* $\mathcal{R}_N$). *Then there exists a right* M-N-*correspondence* Z *such that* F *is naturally unitarily isomorphic to the interior tensor product with* Z.

Proof. We are adapting the proof of the C^*-module variant in [2, Theorem 5.4]. Let $Z = F(M)$. We first prove that $C_n(F(M)) \cong F(C_n(M))$. The proof of the analogous statement in [2] does not work, instead we proceed as follows. If $i_k : M \to C_n(M)$ and $\pi_k : C_n(M) \to M$ are the canonical inclusions and projections, then these are clearly adjointable. We have $i = (i_k) \in M_{1,n}(\mathbb{B}(M, C_n(M)))$ as well as $\pi = [\pi_k] \in C_n(\mathbb{B}(C_n(M), M))$. Thus $F(i)$ is in $M_{1,n}(\mathbb{B}(Z, F(C_n(M)))$ and is a contraction, as also is $F(\pi) \in C_n(\mathbb{B}(F(C_n(M)), Z))$. By Lemma 3.1, we may view $F(i)$ as a contraction in $\mathbb{B}(C_n(Z), F(C_n(M)))$, and $F(\pi)$ as a contraction in $\mathbb{B}(F(C_n(M)), C_n(Z))$. The composition of these latter (complete) contractions in either order is easily seen to be the identity, so that indeed $F(C_n(M)) \cong C_n(Z)$ as desired. Because we shall need it shortly, we note that the unitary morphism $C_n(Z) \to F(C_n(M))$ here is $[z_k] \mapsto \sum_k F(i_k)(z_k)$.

As in [2, Theorem 5.4], Z is a right rigged (resp. w^*-rigged) module over N, and we make Z into an M-N-bimodule by defining $mz = F(L_m)(z)$ for $m \in M$, $z \in Z$. Here $L_m : M \to M$ is left multiplication by m, a completely bounded adjointable map. Since F is completely contractive, it is easy to argue that the associated homomorphism $\theta : M \to \mathbb{B}(Z)$ is completely contractive. Since F is strongly continuous (resp. normal), the left action of M on Z is nondegenerate (resp. separately weak* continuous), and Z is a right M-N-correspondence.

Define a bilinear map $\tau : Y \otimes_\theta Z \to F(Y)$ by $(y, z) \mapsto F(L_y)(z)$, where L_y is left multiplication by y on M. This map is an M-balanced right N-module map as in [2, Theorem 5.4], and in the w^*-rigged case it is clearly separately weak* continuous. It is completely contractive in the sense of Christensen and Sinclair, since if $y = [y_{ij}] \in \mathrm{Ball}(M_n(Y))$, $z = [z_{ij}] \in M_n(Z)$ then

$$[L_{y_{ik}}] \in \mathrm{Ball}(M_n(\mathbb{B}(M, Y))) = \mathrm{Ball}(\mathbb{B}(M, M_n(Y))),$$

so that $[F(L_{y_{ik}})] \in \mathrm{Ball}(M_n(\mathbb{B}(Z, F(Y))))$. Since $M_n(\mathbb{B}(Z, F(Y)))$ may be identified with $\mathbb{B}(C_n(Z), C_n(F(Y)))$ via Lemma 3.1, it is easy to see that

$$\Big\| \Big[\sum_k F(L_{y_{ik}})(z_{kj}) \Big] \Big\| \leq \|[z_{ij}]\|,$$

so that τ is completely contractive. By the universal property of the tensor product, we obtain a complete contractive N-module map $\tau_Y : Y \otimes_\theta Z \to F(Y)$ which is weak* continuous in the w^*-rigged case.

Showing that τ_Y is a complete isometry is similar to the (matrix normed version of the) computation in [2, Theorem 5.4]. However to take into account the w^*-rigged module case, the argument changes a bit. In either case, for $u \in Y \otimes_\theta Z$ we see that $\tau_Y(u)$ is the appropriate limit over α of $F(\psi_\alpha)F(\phi_\alpha)\tau_Y(u)$. Consequently,

$$\|(\tau_Y)_n(u)\|_{M_n(F(Y))} = \sup_\alpha \|[F(\phi_\alpha)\tau_Y(u_{ij})]\| \qquad \text{for } [u_{ij}] \in M_n(Y \otimes_\theta Z).$$

As in [2, bottom of p. 277], we have

$$F(\phi_\alpha)\tau_Y(u_{ij}) = \tau_{C_{n(\alpha)}(M)}((\varphi_\alpha \otimes I)(u_{ij})).$$

Since $\tau_{C_{n(\alpha)}(M)}$ is a complete isometry, we deduce that

$$\|(\tau_Y)_n(u)\|_{M_n(F(Y))} = \sup_\alpha \|[(\varphi_\alpha \otimes I)(u_{ij})]\| = \|u\|_{M_n(Y \otimes_\theta Z)},$$

with the last equality holding by the formula immediately after Definition 1.1, since $\varphi_\alpha \otimes I$ and $\psi_\alpha \otimes I$ are the asymptotic factorization maps for $Y \otimes_\theta Z$. Thus τ_Y is a complete isometry.

That τ_Y has dense range follows similarly to the argument for this in [2, Theorem 5.4], the key point being that the functions $\tau_Y \circ (\psi_\alpha \otimes I)$ and $F(\psi_\alpha) \circ \tau_{C_{n(\alpha)}(M)}$ agree on $C_{n(\alpha)}(M) \otimes_M Y$. So τ_Y is a completely isometric isomorphism, that is, a unitary isomorphism, and it is an easy exercise to see that it implements the natural equivalence in the desired sense. ∎

Remark. As in pure algebra, it is an easy exercise to see that this yields a bijection between (isomorphism classes of) right M-N-correspondences and (isomorphism classes of) such strongly continuous completely contractive functors. Composition of such functors corresponds to the interior tensor product of the bimodules.

4. The exterior tensor product of w^*-rigged modules. If Y is a right w^*-rigged module over M, and if Z is a right w^*-rigged module over N, we define the *weak* exterior tensor product* $Y \overline{\otimes} Z$ to be their *normal minimal* (or *spatial*) *tensor product* (see e.g. [10, 1.6.5]). We may view it as a module over $M \overline{\otimes} N$ as follows. Let $L(Y)$ and $L(Z)$ be the weak linking algebras for Y and Z respectively (as in [7, 3.2]). Viewing Y and Z as the 1-2-entries of $L(Y)$ and $L(Z)$ respectively, identify $Y \otimes Z$ with the obvious subspace of the dual operator algebra tensor product $L(Y) \overline{\otimes} L(Z)$. Write $Y \overline{\otimes} Z$ for its completion in the weak* topology of $L(Y) \overline{\otimes} L(Z)$. In this way, $Y \overline{\otimes} Z$ can be seen to be invariant under right multiplication by the 2-2-corner $M \overline{\otimes} N$ of $L(Y) \overline{\otimes} L(Z)$. Thus $Y \overline{\otimes} Z$ is a right dual operator $(M \overline{\otimes} N)$-module.

The normal minimal tensor product of any dual operator spaces Y and Z, and in particular hence the exterior tensor product of w^*-rigged modules, is completely isometrically and weak*-homeomorphically contained in $(Y_* \widehat{\otimes} Z_*)^*$, where $\widehat{\otimes}$ is the operator space projective tensor product. Thus it is contained completely isometrically and weak*-homeomorphically, via the canonical inclusions, in $CB(Y_*, Z)$ and $CB(Z_*, Y)$. Indeed, by basic operator space theory (see e.g. [13, 10]), we can identify $(Y_* \widehat{\otimes} Z_*)^* = CB(Y_*, Z) = CB(Z_*, Y)$ with the normal Fubini tensor product of Y and Z, and it is known that this contains a canonical copy of $Y \overline{\otimes} Z$ (see [13, Theorem 7.2.3]).

In what follows we will use the fact that the normal minimal tensor product is *functorial*. That is, if Y_k and Z_k are dual operator spaces, and if $T_k : Y_k \to Z_k$ are completely bounded weak* continuous maps, for $k = 1, 2$, then $T_1 \otimes T_2 : Y_1 \overline{\otimes} Z_1 \to Y_2 \overline{\otimes} Z_2$ defines a unique completely bounded weak* continuous map. Moreover, $\|T_1 \otimes T_2\|_{cb} \leq \|T_1\|_{cb}\|T_2\|_{cb}$. This also follows from some basic operator space theory (see e.g. [13, 10]). Tensoring the predual maps of T_k with respect to the operator space projective tensor product, and then dualizing, gives a weak* continuous map $u : ((Y_1)_* \widehat{\otimes} (Z_1)_*)^* \to ((Y_2)_* \widehat{\otimes} (Z_2)_*)^*$ with completely bounded norm $\leq \|T_1\|_{cb}\|T_2\|_{cb}$. As in the last paragraph, we can identify $((Y_k)_* \widehat{\otimes} (Z_k)_*)^*$ with the normal Fubini tensor product of Y_k and Z_k. Restricting u to the copy of $Y_1 \overline{\otimes} Z_1$, we get a completely bounded weak* continuous map from $Y_1 \overline{\otimes} Z_1 \to Y_2 \overline{\otimes} Z_2$.

Theorem 4.1. *The weak*-exterior tensor product of w^*-rigged modules Y and Z is a w^*-rigged module.*

Proof. Suppose that $\phi_\alpha : Y \to C_{n(\alpha)}(M)$ and $\psi_\alpha : C_{n(\alpha)}(M) \to Y$ are factorization maps for Y, and suppose that $\zeta_\beta : Z \to C_{m(\beta)}(N)$ and $\eta_\beta : C_{m(\beta)}(N) \to Z$ are factorization nets for Z, as in Definition 1.1. By operator space theory we know that $C_n(M) \overline{\otimes} C_m(N) \cong C_{nm}(M \overline{\otimes} N)$ completely isometrically and weak*-homeomorphically. By functoriality of $\overline{\otimes}$, we can define $\phi_\alpha \otimes \zeta_\beta$ and $\psi_\alpha \otimes \eta_\beta$ of $Y \overline{\otimes} Z$ through spaces $C_{n(\alpha)}(M) \overline{\otimes} C_{m(\beta)}(N) \cong C_{n(\alpha)m(\beta)}(M \overline{\otimes} N)$, and check that the conditions of Definition 1.1 are met. ∎

Corollary 4.2. *Suppose that Y_1 and Z_1 are right w^*-rigged modules over M, and that Y_2 and Z_2 are right w^*-rigged modules over N. Suppose that $T_k : Y_k \to Z_k$ are completely bounded and weak* continuous module maps over M and N respectively, for $k = 1, 2$. Then $T_1 \otimes T_2 : Y_1 \overline{\otimes} Z_1 \to Y_2 \overline{\otimes} Z_2$ defines a unique completely bounded weak* continuous $(M \overline{\otimes} N)$-module map. Moreover, $\|T_1 \otimes T_2\|_{cb} \leq \|T_1\|_{cb}\|T_2\|_{cb}$.*

Proof. Nearly all of this is just the functoriality discussed above Theorem 4.1. It is easy to argue by weak* density arguments that $T_1 \otimes T_2$ is an $(M \overline{\otimes} N)$-module map. ∎

One may check that the weak* exterior tensor product has other properties analogous to the interior tensor product. For example it is associative, 'injective', and is appropriately projective for w^*-orthogonally complemented submodules and commutes with direct sums (we will prove this at the end of the next section).

5. Complemented submodules. We say that a w^*-rigged module Z over a dual operator algebra M is the *w^*-orthogonal direct sum* of weak* closed submodules Y and W if $Y + W = Z$, $Y \cap W = (0)$, and W and Y are the ranges of two completely contractive idempotent maps P and Q. We say that Y is *w^*-orthogonally complemented* in Z if there exists such a W. It follows from algebra that the latter two maps P, Q are unique, and are M-module maps adding to I_Z with $PQ = QP = 0$. Also, they are weak* continuous. Indeed suppose that $x_t = y_t + w_t$ is a bounded net with weak* limit $x = y + w$, where $y_t, y \in Y$ and $w_t, w \in W$. Then (y_t) is bounded, and if $y_{t_\nu} \to z$ is a weak* convergent subnet, then $z \in Y$ and $w_{t_\nu} \to x - z \in W$. It follows that $z = y$ and $x - z = w$. By topology $y_t \to y$ weak*, so by the Krein–Šmulian theorem P is weak* continuous. It follows from e.g. [1, Theorem 7.2] that Z is the w^*-rigged module direct sum $Y \oplus^c W$ completely isometrically and unitarily. From [7, Section 3.5], we see that the w^*-orthogonally complemented submodules of a w^*-rigged module Z are precisely the ranges of completely contractive idempotents in $\mathbb{B}(Z)$.

Proposition 5.1. *The right M-summands in a w^*-rigged module Z in the sense of* [4] (*see also* [10, Sections 4.5 and 4.8]) *are precisely the w^*-orthogonally complemented submodules of Z. For any such submodule W of Z there is a unique contractive linear projection from Z onto W.*

Proof. This is similar to the proof of Corollary 2.2, but using the fact from [7, Theorem 2.3] that the left multiplier operator algebra of Z is $\mathbb{B}(Z)$, so that the orthogonal projections here are the completely contractive idempotents in $\mathbb{B}(Z)$. ∎

Example 5.2. Unlike the case when M is a von Neumann algebra (see e.g. [10, 8.5.16]), weak* closed submodules of w^*-rigged modules (or even of weak* Morita equivalence bimodules) need not be w^*-orthogonally complemented. For example, if f is a nontrivial inner function in $M = H^\infty(\mathbb{D})$ (such as the monomial z) then $Y = fH^\infty(\mathbb{D})$ is not complemented in the M-module $Z = H^\infty(\mathbb{D})$. We note that Y is a weak* Morita equivalence bimodule, with $\tilde{Y} = f^{-1}M$. The latter is not a subset of M, and indeed the adjoint $\tilde{i}$ of the inclusion map $i : Y \to Z$ is not a projection.

Lemma 5.3. *Let Z be a w^*-rigged module over M and let $P : Z \to Z$ be a w^*-continuous completely contractive idempotent module map. Then the range of P is a w^*-rigged module over M, which is w^*-orthogonally complemented in Z. Also P is adjointable both as a map into Z and into $P(Z)$. The dual module $\widetilde{P(Z)}$ of $P(Z)$ can be identified completely isometrically and w^*-homeomorphically with the weak* orthogonally complemented submodule $\tilde{P}(\tilde{Z})$ of $\tilde{Z}$, with the dual pairing being the restriction of the pairing $\tilde{Z} \times Z \to M$.*

Proof. It is easy to see from Remark after [7, Theorem 2.7], and considering the maps between Z and $Y = P(Z)$, that Y is a w^*-rigged module over M. By [7, Proposition 3.4], P is adjointable both as a map into Y and into Z. Since P is an orthogonal projection in $\mathbb{B}(Z)$, Y is w^*-orthogonally complemented in Z (cf. [7, Theorem 3.9]). We define $W = \operatorname{Ran}(\tilde{P}) = \{f \circ P : f \in \tilde{Z}\}$. This is easily seen to be a weak* closed submodule of $\tilde{Y}$. Note that $CB^\sigma(Y, M) = \{f_{|Y} : f \in CB^\sigma(Z, M)\}$. The map $f \mapsto f_{|Y} : W \to$

$CB^\sigma(Y, M)$ is a complete isometric M-module map, so that $\tilde{Y} \cong \tilde{P}(\tilde{Z})$. The remaining assertion is now easy to check. ∎

Proposition 5.4. *If Y is a weak* orthogonally complemented submodule in a w^*-rigged module Z, then $\mathbb{B}(Y)$ is completely isometrically isomorphic to a weak* closed completely contractively weak* complemented subalgebra of $\mathbb{B}(Z)$.*

Proof. Let $i : Y \to Z$ be the inclusion and $P : Z \to Y$ the projection. Then by functoriality of the tensor product,

$$i \otimes \tilde{P} : Y \otimes^{\sigma h}_M \tilde{Y} = \mathbb{B}(Y) \to Z \otimes^{\sigma h}_M \tilde{Z} = \mathbb{B}(Z)$$

is completely contractive and weak* continuous, and is easily checked to be a homomorphism. Similarly one obtains a completely contractive weak* continuous retraction $P \otimes \tilde{i} : Z \otimes^{\sigma h}_M \tilde{Z} = \mathbb{B}(Z) \to Y \otimes^{\sigma h}_M \tilde{Y} = \mathbb{B}(Y)$ with $(P \otimes \tilde{i}) \circ (i \otimes \tilde{P}) = I$. ∎

For the following result we recall that the W^*-*dilation* of a right w^*-rigged module Z over a dual operator algebra M is the canonical right W^*-module over a von Neumann algebra N generated by M given by $Y \otimes_\theta N$. Here $\theta : M \to N$ is the inclusion.

Corollary 5.5. *Let Z be a right w^*-rigged module over a dual operator algebra M, and suppose that Y is a subspace of Z, with $i : Y \to Z$ the inclusion map. The following are equivalent:*

1. *Y is weak* orthogonally complemented in Z.*
2. *Y is a w^*-rigged module over M and there exists a completely contractive weak* continuous M-module map $j : \tilde{Y} \to \tilde{Z}$ such that $\tilde{i} \circ j = I_{\tilde{Y}}$.*
3. *Y is a w^*-rigged module over M, and there is a von Neumann algebra N generated by M such that the induced map $i \otimes I_N$ between the W^*-dilations of Y and Z with respect to N is an isometry whose W^*-module adjoint $(i \otimes I_N)^*$ maps $Z \otimes 1_N$ into $Y \otimes 1_N$.*
4. *Same as (3), but for every von Neumann algebra N generated by M.*

Proof. (1)⇒(4). If $P : Z \to Y$ is the projection then we have adjointable contractions $f = (i \otimes I_N) : Y \otimes^{\sigma h}_M N \to Z \otimes^{\sigma h}_M N$ and $g = (P \otimes I_N) : Z \otimes^{\sigma h}_M N \to Y \otimes^{\sigma h}_M N$ with $g \circ f = I$. It follows that f is an isometry, $g = f^*$, and $f^* = g$ maps $Z \otimes 1_N$ into $Y \otimes 1_N$.

(3)⇒(1). Let j be the canonical isometry from Y into its W^*-dilation, which is a complete isometry by [7, 3.4]. It follows that $W = (i \otimes I_N)(Y \otimes^{\sigma h}_M N)$ is a weak* closed submodule of $Z \otimes^{\sigma h}_M N$, and the latter is a W^*-module. By e.g. [10, 8.5.16], the C^*-module adjoint of $i \otimes I_N$ is a contractive weak* continuous projection P from $Z \otimes^{\sigma h}_M N$ onto W. Thus $P \circ (i \otimes I_N) = I$ on $Y \otimes^{\sigma h}_M N$. Define $Q(z) = j^{-1}(i \otimes I_N)^{-1} P(z \otimes 1)$; this is a weak* continuous completely contractive M-module projection onto Y. Indeed,

$$Q(i(y)) = j^{-1}(i \otimes I_N)^{-1}P(i(y) \otimes 1) = j^{-1}(y \otimes 1) = y, \quad y \in Y.$$

Clearly (4) implies (3), and (1) implies all the others.

(2)⇒(1). $P = j \circ \tilde{i}$ is a weak* continuous completely contractive projection onto $j(\tilde{Y})$. So the latter is weak* orthogonally complemented in $\tilde{Z}$. Hence by Lemma 5.3 its dual module may be identified with $\tilde{P}(Z) = i(\tilde{j}(Z)) = Y$ (note that $\tilde{j} \circ i = I_Y$, so $\tilde{j}$ maps onto Y), and this is weak* orthogonally complemented in Z. ∎

Remark. It seems possible that the equivalences in the last result still hold with some of the words 'weak* continuous' or 'M-module' removed from (2). However this seems quite difficult at present, although the last assertion of Proposition 5.1 seems pertinent here. Things are better if Z is a module of the kind considered in Theorem 2.3. If we are in that case, suppose that there exists a completely contractive M-module map $j : \tilde{Y} \to \tilde{Z}$ such that $\tilde{i} \circ j = I_{\tilde{Y}}$ as in (2). Then $P = j \circ \tilde{i}$ is a completely contractive M-module projection on $\tilde{Z} = CB^{\sigma}(Z, M)_M = CB(Y, M)_M$. Hence it is weak* continuous by Theorem 2.3, and we can continue as in the proof of (2)⇒(1) above.

At the end of [7, Section 3] we mentioned with a sketchy proof the fact that direct sums commute with the interior tensor product; indeed, we have left and right distributivity of $\otimes_M^{\sigma h}$ over column direct sums of w^*-rigged modules. It is also true that direct sums commute with the exterior tensor product. The proof we give of the latter fact will cover the interior tensor product cases too, or is easily adaptable to those.

Proposition 5.6. *Suppose that M, N are dual operator algebras. If $(Y_k)_{k \in I}$ is a family of right w^*-rigged modules over M, and Z is a right w^*-rigged module over N, then*

$$\Big(\bigoplus_k^c Y_k\Big) \overline{\otimes}\, Z \cong \bigoplus_k^c (Y_k \,\overline{\otimes}\, Z),$$

unitarily as right w^*-rigged modules.

Proof. We shall prove the more general statement that

$$\Big(\bigoplus_k^c Y_k\Big) \otimes_\beta Z \cong \bigoplus_k^c (Y_k \otimes_\beta Z),$$

unitarily as right w^*-rigged modules, where $\otimes_\beta$ is any functorial tensor product (that is, the tensor product of weak* continuous completely contractive right module maps is also a weak* continuous completely contractive right module map) that produces a right w^*-rigged module from right w^*-rigged modules, and for which the canonical map $Y \times Z \to Y \otimes_\beta Z$ is separately weak* continuous and has range whose span is weak* dense. These assertions are true for the interior and exterior tensor product (see [6], particularly Section 2 there, and e.g. [10, 1.6.5]).

We will use the functoriality of $\otimes_\beta$ and [7, Theorem 3.9]: If $Y = \bigoplus_k^c Y_k$ and i_k, π_k are as in that result, then $\pi_k \otimes I$ and $i_k \otimes I$ are weak* continuous completely contractive right module maps that compose to the identity on $Y_k \otimes_\beta Z$ (since their composition is weak* continuous and equals I on the weak* dense subset $Y \otimes Z$). Similarly, they also satisfy $(\pi_k \otimes I)(i_j \otimes I) = 0$ if $j \neq k$. Thus we will be done by [7, Theorem 3.9] if $\sum_k (i_k \otimes I)(\pi_k \otimes I) = I$ in the weak* topology of $\mathbb{B}(Y \otimes_\beta Z)$. To see this, let $T_\Delta = \sum_{k \in \Delta} i_k \pi_k$ for finite $\Delta \subset I$. We will be done if $T_\Delta \otimes I \to I$ weak* in $\mathbb{B}(Y \otimes_\beta Z)$, since $T_\Delta \otimes I = \sum_{k\in\Delta} (i_k \otimes I)(\pi_k \otimes I)$.

Indeed, we shall prove a more general fact: if a bounded net S_t converges to S weak* in $\mathbb{B}(Y)$ then $S_t \otimes I \to S \otimes I$ weak* in $\mathbb{B}(Y \otimes_\beta Z)$. Suppose that we have a weak* convergent subnet $S_{t_\nu} \otimes I \to R$. By [7, Theorem 3.5] we see that $R \in \mathbb{B}(Y \otimes_\beta Z)$, and hence R is weak* continuous. For $y \in Y$, $z \in Z$ we have $S_t(y) \to S(y)$ weak* (this follows since the latter describes the weak* convergence of bounded nets in $CB(Y)$ by e.g. [10, 1.6.1], and since by [7, Theorem 2.3], $\mathbb{B}(Y)$ is a weak* closed subalgebra of $CB(Y)$). Hence

$$(S_t \otimes I)(y \otimes z) = S_t(y) \otimes z \to S(y) \otimes z = (S \otimes I)(y \otimes z).$$

Thus $R(y \otimes z) = (S \otimes I)(y \otimes z)$. Hence $R = S \otimes I$ since they are weak* continuous and agree on a dense subset. By topology it follows that $S_t \otimes I \to S \otimes I$ weak* as desired. ∎

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