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Supnorm of modular forms of half-integral weight in the weight aspect

by

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1. Introduction. The supremum norms of holomorphic and Maass Hecke eigenforms are connected to L-functions attached to them. In the case of holomorphic half-integral weight Hecke eigenforms they are directly related to the critical values of quadratic twists of the L-functions associated to their Shimura lift. Therefore supnorms have been studied by many in various ways: Iwaniec–Sarnak [6] in the eigenvalue aspect, Harcos–Templier [4], [5] and Saha [18] in the level aspect, as well as Kiral [7] in the case of half-integral weight, Templier [22] in the level as well as the eigenvalue aspect, unifying both best known results. In the weight aspect, supnorms have been studied by Xia [24], Das–Sengupta [2], Rudnick [17], Friedman–Jorgenson–Kramer [3] and the author himself [21], where in the last three articles the condition of being a Hecke eigenform is not necessary.

In this paper we are concerned with the supremum norm in the weight aspect of holomorphic half-integral weight Hecke eigenforms in the Kohnen plus space of level 4. Assuming the Lindelöf hypothesis we are able to prove an analogue of Xia's result [24] which states that for a holomorphic Hecke eigenform f of integral weight for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ we have $\sup_{z\in\mathbb{H}} y^{k/2}|f(z)| \ll_\epsilon k^{1/4+\epsilon}$. Our theorem reads as follows.

Theorem 1. *Let $k \in 1/2+\mathbb{Z}$ with $k \geq 5/2$ and let $f \in S_k^+(\Gamma_0(4)^\star)$ be an L^2-normalised Hecke eigenform ($\langle f, f\rangle_{\Gamma_0(4)} = 1$) of half-integral weight k contained in the Kohnen plus space. Assume the Lindelöf hypothesis for the family of L-functions $L(F,\chi,s)$, where F is any modular form of weight $2k-1$ on $\mathrm{SL}_2(\mathbb{Z})$, and χ any primitive quadratic character. Then*

$$\sup_{z\in\mathbb{H}} y^{k/2}|f(z)| \ll_\epsilon k^{1/4+\epsilon}.$$

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Unconditionally we are able to prove the following.

THEOREM 2. *Let $k \in 1/2 + \mathbb{Z}$ with $k \geq 5/2$ and let $f \in S_k^+(\Gamma_0(4)^\star)$ be an L^2-normalised Hecke eigenform ($\langle f, f\rangle_{\Gamma_0(4)} = 1$) of half-integral weight k contained in the Kohnen plus space. Then*

$$\sup_{z\in\mathbb{H}} y^{k/2}|f(z)| \ll_\epsilon k^{3/7+\epsilon}.$$

We now give a brief overview of the significance of the various exponents and the methods which go into them. If f is not assumed to be a Hecke eigenform, then the best exponent one can prove in general is $3/4$. Indeed, this has been shown for arbitrary real weight k by the author [21], and relies on estimates for the Fourier coefficients of Poincaré series. However, when f is an eigenform of half-integral weight as in the current paper, it follows from a result of Kohnen and Zagier [11] (or more generally Waldspurger [23]) that the squares of its Fourier coefficients are essentially central L-values. Using the convexity bound on the said L-functions one achieves a bound for the Fourier expansion, which is especially good near the cusps. Combing this estimate with a Bergman kernel for the case away from the cusps gives the bound $k^{1/2+\epsilon}$ for the supnorm. Any subconvexity result on those central L-values easily allows the removal of the ϵ to achieve the bound $k^{1/2}$; this was shown by the author in his master's thesis [20]. To decrease the exponent further, one can either use deeper techniques or assume unproven bounds, e.g. the Lindelöf hypothesis as in Theorem 1. The bound $k^{1/4+\epsilon}$ is essentially best possible, as the next theorem shows that the best uniform bound one can hope for is $k^{1/4}$, if one takes the dimension of the space into consideration. The bound $k^{3/7+\epsilon}$ comes from the best known bound for these central L-values given by Petrow [14] and Young [25] combined with the amplification method using the Bergman kernel.

THEOREM 3. *Let $k \in 1/2 + \mathbb{Z}$ with $k \geq 5/2$ and $\{f_j\} \subseteq S_k^+(\Gamma_0(4)^\star)$ be an orthonormal basis of Hecke eigenforms of half-integral weight k contained in the Kohnen plus space. Let $\{F_j\} \subseteq S_{2k-1}(\mathrm{SL}_2(\mathbb{Z}))$ be the corresponding arithmetically normalised Hecke eigenforms ($\widehat{F_j}(1) = 1$) under the Shimura map. Then*

$$\sup_{z\in\mathbb{H}} y^{k/2}|f_j(z)| \gg_\epsilon \max\left\{1, k^{1/4-\epsilon} \sup_{\substack{D \text{ fund. disc.} \\ (-1)^{k-1/2}D>0}} L\left(F_j, \left(\frac{D}{\cdot}\right), \frac{1}{2}\right)^{1/2} |D|^{-1/2}\right\}$$

and

$$\sum_j \sup_{z\in\mathbb{H}} y^k|f_j(z)|^2 \geq \sup_{z\in\mathbb{H}} \sum_j y^k|f_j(z)|^2 \gg k^{3/2}.$$

Although we restrict ourselves to the Kohnen plus space of level 4, the methods certainly generalise to larger level, but slightly weaker results are to be expected. Nevertheless the author strongly believes that even the convexity bound on the critical value of the corresponding L-functions is sufficient to break the convexity bound of $k^{3/4}$, as this is indeed the case in the Kohnen plus space of level 4.

2. Notation and preliminaries. Throughout, let $k \in 1/2 + \mathbb{Z}$ be a half-integer with $k \geq 5/2$. For a complex number $z \in \mathbb{C}^\times$ we define $z^k = \exp(k \cdot \mathrm{Log}(z))$, where $\mathrm{Log}(z) = \log|z| + i\arg(z)$ with $-\pi < \arg(z) \leq \pi$. The notation $f(x) \ll_{A,B} g(x)$ means that $|f(x)| \leq Kg(x)$, where K is some function depending at most on A and B. Further, let $e(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}$.

In what follows we give a brief account of necessary facts on modular forms of half-integral weight. For a more detailed account we refer the reader for example to [8]. As usual we define the Möbius action of $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$, the set of all 2×2 matrices with rational coefficients and positive determinant, on $\mathbb{H}$, the upper half-plane, as

$$\gamma \cdot z = \gamma z = \frac{az+b}{cz+d}, \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}),\ \forall z \in \mathbb{H}.$$

The action is extended to the set of cusps $\overline{\mathbb{Q}} = \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \sqcup \{\infty\}$. We further define

$$j(\gamma, z) = cz + d, \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}),\ \forall z \in \mathbb{H},$$

$$j_\Theta(\gamma, z) = \frac{\Theta(\gamma z)}{\Theta(z)}, \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),\ \forall z \in \mathbb{H},$$

where $\Theta(z) = \sum_{n\in\mathbb{Z}} e(n^2 z)$. By $\mathfrak{G}_k$ we denote the group whose elements are of the form (γ, φ), where $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\varphi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $|\varphi(z)| = (\det\gamma)^{-k/2}|j(\gamma, z)|^k$, and whose composition is given by

$$(\gamma, \varphi) \circ (\gamma', \varphi') = (\gamma\gamma', (\varphi \circ \gamma') \cdot \varphi').$$

For each k we have a group homomorphism

$$\star : \Gamma_0(4) \to \mathfrak{G}_k, \quad \gamma \mapsto \gamma^\star = (\gamma, j_\Theta(\gamma, \cdot)^{2k}).$$

We further have an inclusion of sets $\mathrm{GL}_2^+(\mathbb{Q}) \hookrightarrow \mathfrak{G}_k$, where we identify $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ with $(\gamma, (\det\gamma)^{-k/2} j(\gamma, \cdot)^k)$. Among all elements in $\mathfrak{G}_k$ we would like to distinguish two special elements W_4 and V_4, which we are going to use to translate the cusps $0, 1/2$ to ∞:

$$W_4 = \left(\begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}, (-2iz)^k\right), \quad V_4 = \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, (-i(2z+1))^k\right).$$

Definition 1. For $\tau \in \mathrm{SL}_2(\mathbb{Z})$ we define the *cusp width* n_τ and the *cusp parameter* $\kappa_\tau \in [0,1)$ of the cusp $\tau\infty$ in such a way that the stabiliser group at ∞ of $\tau^{-1}\Gamma_0(4)^\star\tau$ is generated by

$$\pm\left(\begin{pmatrix}1 & n_\tau\\ 0 & 1\end{pmatrix}, e(\kappa_\tau)\right).$$

Remark 1. The cusp width as well as the cusp parameter depend only on the equivalence class of $\tau\infty$ modulo $\Gamma_0(4)$. For $\Gamma_0(4)^\star$, the cusps $0, 1/2, \infty$ have cusp width $4, 1, 1$ and cusp parameter $0, 1/2-(-1)^{k-1/2}1/4, 0$ respectively.

The group $\mathfrak{G}_k$ acts on the set of meromorphic functions on $\mathbb{H}$ as follows:

$$(f|_k(\gamma,\varphi))(z) = \varphi(z)^{-1}f(\gamma z).$$

Definition 2. A holomorphic function f on the upper half-plane satisfying

$$f|_k\xi = f, \qquad \forall \xi \in \Gamma_0(4)^\star,$$

and having a Fourier expansion of the form

$$(f|_k\tau)(z) = \sum_{m+\kappa_\tau>0} \widehat{(f|_k\tau)}(m)e\left(\frac{m+\kappa_\tau}{n_\tau}z\right)$$

for every $\tau \in \mathrm{SL}_2(\mathbb{Z})$ is called a *cusp form* of weight k with respect to $\Gamma_0(4)^\star$. The set of such functions is denoted by $S_k(\Gamma_0(4)^\star)$. The Fourier coefficients at infinity $\widehat{(f|_kI)}(m)$ are further abbreviated to $\widehat{f}(m)$.

The space $S_k(\Gamma_0(4)^\star)$ is finite-dimensional and can be made into a Hilbert space by defining the Petersson inner product

$$\langle f, g\rangle_{\Gamma_0(4)} = \frac{1}{6}\int_{\mathbb{F}_{\Gamma_0(4)}} f(z)\overline{g(z)}y^k\,\frac{dx\,dy}{y^2},$$

where $\mathbb{F}_{\Gamma_0(4)}$ is a fundamental domain for $\Gamma_0(4)$ and $z = x+iy$. Furthermore, a theory of Hecke operators can be established on $S_k(\Gamma_0(4)^\star)$. For l an odd square, one defines

$$f|_kT(l) = l^{k/2-1}\sum_{\xi\in\Gamma_0(4)^\star\backslash\Gamma_0(4)^\star\xi_{1,l}\Gamma_0(4)^\star} f|_k\xi \quad \text{where} \quad \xi_{1,l} = \left(\begin{pmatrix}1 & 0\\ 0 & l\end{pmatrix}, l^{k/2}\right).$$

These operators are self-adjoint and commute. Thus one gets an orthonormal basis of Hecke eigenforms. Shimura [19] has shown that given such a Hecke eigenform f, one can use its Fourier coeficients to construct a classical Hecke eigenform $F \in S_{2k-1}(\Gamma_0(M))$ of weight $2k-1$ for some level M with the same Hecke eigenvalues. Later Niwa [13] has shown that one can always take $M = 2$, moreover Kohnen [9] has shown one can take $M = 1$ if the eigenform is coming from a certain subspace, the *Kohnen plus space*, which is defined as follows:

$$S_k^+(\Gamma_0(4)^\star) = \{f \in S_k(\Gamma_0(4)^\star) \mid \widehat{f}(m) = 0 \text{ for all } m \\ \text{such that } (-1)^{k-1/2}m \equiv 2, 3 \pmod 4\}.$$

The plus space has some nice properties, one of which is that one can define a Hecke operator T_4^+ which commutes with the odd Hecke operators as well as with the Shimura map, and another one is that the plus space comes with a projection $S_k(\Gamma_0(4)^\star) \to S_k^+(\Gamma_0(4)^\star)$. For this reason the subspace has its own Poincaré series, which have been computed by Kohnen [10].

Proposition 1 (see [10, Proposition 4]). *Let $k \in 1/2 + \mathbb{Z}$ with $k \geq 5/2$ and $m \in \mathbb{N}$, $(-1)^{k-1/2}m \equiv 0, 1 \pmod 4$. Define the Poincaré series $G_I^+(\Gamma_0(4)^\star, k, z, m)$ by the Fourier expansion*

$$G_I^+(\Gamma_0(4)^\star, k, z, m) = \sum_{\substack{n \geq 1 \\ (-1)^{k-1/2}n \equiv 0,1 \pmod 4}} g_{k,m}(n) e(nz)$$

with

$$g_{k,m}(n) = \frac{2}{3}\left[\delta_{m,n} + (-1)^{\left\lfloor \frac{k+1/2}{2} \right\rfloor} \pi\sqrt{2}\left(\frac{n}{m}\right)^{(k-1)/2} \sum_{c \geq 1} H_c(n, m) J_{k-1}\left(\frac{\pi}{c}\sqrt{nm}\right)\right],$$

where

$$H_c(n, m) = (1 - (-1)^{k-1/2} i)\left(1 + \left(\frac{4}{c}\right)\right)\frac{1}{4c} \sum_{\substack{\delta \pmod{4c} \\ (\delta, 4c) = 1}} \left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^k e\left(\frac{n\delta + m\delta^{-1}}{4c}\right).$$

Then $G_I^+(\Gamma_0(4)^\star, k, \cdot, m) \in S_k^+(\Gamma_0(4)^\star)$ and

$$\langle f, G_I^+(\Gamma_0(4)^\star, k, \cdot, m)\rangle_{\Gamma_0(4)} = \frac{\Gamma(k-1)}{6 \cdot (4\pi m)^{k-1}} \widehat{f}(m) \qquad \forall f \in S_k^+(\Gamma_0(4)^\star).$$

The following corollary is immediate.

Corollary 1. *Let $k \in 1/2 + \mathbb{Z}$ with $k \geq 5/2$ and let $\{f_j\}$ be an orthonormal basis of $S_k^+(\Gamma_0(4)^\star)$. Then*

$$\sum_j |\widehat{f}_j(m)|^2 = \frac{6 \cdot (4\pi m)^{k-1}}{\Gamma(k-1)} \cdot \frac{2}{3}\left[1 + (-1)^{\left\lfloor \frac{k+1/2}{2} \right\rfloor} \pi\sqrt{2} \sum_{c \geq 1} H_c(m, m) J_{k-1}\left(\frac{\pi m}{c}\right)\right].$$

Furthermore cusp forms in the Kohnen plus space have special relations among their Fourier coefficients at different cusps, as the next lemma shows.

LEMMA 1. *Let $k \in 1/2 + \mathbb{Z}$, $f \in S_k^+(\Gamma_0(4)^\star)$. Then the Fourier coefficients of f at the cusps $0, 1/2$ can be given in terms of the Fourier coefficients at ∞:*

$$(f|_k W_4)(z) = \left(\frac{2}{2k}\right) 2^{1/2-k} \sum_{m\geq 1} \widehat{f}(4m) e(mz),$$

$$(f|_k V_4)(z) = \left(\frac{2}{2k}\right) 2^{1/2-k} \sum_{\substack{m\geq 1\\ (-1)^{k-1/2}m\equiv 1 \,(\mathrm{mod}\, 4)}} i^{m/2} \widehat{f}(m) e\left(\frac{m}{4}z\right).$$

Here $\left(\frac{2}{2k}\right)$ denotes the Jacobi symbol.

Proof. In [9, Proposition 2] Kohnen showed that $(f|U_4|_k W_4)(z) = \left(\frac{2}{2k}\right) 2^{k-1/2} f(z)$, where $(f|U_4)(z) = \sum_{m\geq 1} \widehat{f}(4m)e(mz)$. Applying $|_k W_4$ to both sides gives the desired result, by noting that $|_k W_4^2$ is the identity map. The second identity follows from

$$\begin{aligned}
(f|_k V_4)(z) &= (-i(2z+1))^{-k} \sum_{\substack{m\geq 1\\ (-1)^{k-1/2}m\equiv 0,1 \,(\mathrm{mod}\, 4)}} \widehat{f}(m) e\left(\frac{m}{2} - \frac{m}{4z+2}\right)\\
&= (-i(2z+1))^{-k} \Bigg[2 \sum_{\substack{m\geq 1\\ m\equiv 0 \,(\mathrm{mod}\, 4)}} - \sum_{\substack{m\geq 1\\ (-1)^{k-1/2}m\equiv 0,1 \,(\mathrm{mod}\, 4)}}\Bigg] \widehat{f}(m) e\left(\frac{-m}{4z+2}\right)\\
&= (-i(2z+1))^{-k} 2 (f|U_4)\left(\frac{-1}{z+1/2}\right) - (f|_k W_4)(z+1/2)\\
&= 4^{1/2-k} (f|U_4|_k W_4)\left(\frac{z+1/2}{4}\right) - (f|_k W_4)(z+1/2)\\
&= \left(\frac{2}{2k}\right) 2^{1/2-k} \left[f\left(\frac{z+1/2}{4}\right) - (f|U_4)(z+1/2)\right]\\
&= \left(\frac{2}{2k}\right) 2^{1/2-k} \sum_{\substack{m\geq 1\\ (-1)^{k-1/2}m\equiv 1 \,(\mathrm{mod}\, 4)}} i^{m/2} \widehat{f}(m) e\left(\frac{m}{4}z\right). \quad \blacksquare
\end{aligned}$$

If we now assume that $f \in S_k^+(\Gamma_0(4)^\star)$ is a Hecke eigenform, we can say more about its Fourier coefficients. In this case Waldspurger has shown that the squares of the Fourier coefficients are proportional to the central value of a certain twist of the L-function associated to its Shimura lift. We only need a special case, which has been made explicit by Kohnen–Zagier.

PROPOSITION 2 ([11]). *Let $k \in 1/2+\mathbb{Z}$ with $k \geq 5/2$, let $f \in S_k^+(\Gamma_0(4)^\star)$ be a Hecke eigenform and let $F \in S_{2k-1}(\mathrm{SL}_2(\mathbb{Z}))$ be the corresponding arith-*

metically normalised Hecke eigenform ($\widehat{F}(1) = 1$) *of* f *under the Shimura map. Further let* D *be a fundamental discriminant with* $(-1)^{k-1/2}D > 0$ *and* $L\big(F, \big(\frac{D}{\cdot}\big), s\big)$ *the analytic continuation of the Dirichlet L-series*

$$\sum_{n=1}^{\infty} \left(\frac{D}{n}\right) \frac{\widehat{F}(n)}{n^{k-1}} n^{-s}.$$

Then

$$\frac{|\widehat{f}(|D|)|^2}{\langle f, f\rangle_{\Gamma_0(4)}} = \frac{\Gamma(k-1/2)}{\pi^{k-1/2}} |D|^{k-1} \frac{L\big(F, \big(\frac{D}{\cdot}\big), \frac{1}{2}\big)}{\langle F, F\rangle_{\mathrm{SL}_2(\mathbb{Z})}},$$

where

$$\langle F, F\rangle_{\mathrm{SL}_2(\mathbb{Z})} = \int_{\mathbb{F}_{\mathrm{SL}_2(\mathbb{Z})}} |F(z)|^2 y^{2k-1} \frac{dx\, dy}{y^2}$$

and $\mathbb{F}_{\mathrm{SL}_2(\mathbb{Z})}$ *is a fundamental domain of* $\mathrm{SL}_2(\mathbb{Z})$.

Concerning the size of $\langle F, F\rangle_{\mathrm{SL}_2(\mathbb{Z})}$ we have the following two results.

Proposition 3 ([15]). *Let* $F \in S_{2k-1}(\mathrm{SL}_2(\mathbb{Z}))$ *be an arithmetically normalised Hecke eigenform. Then*

$$\langle F, F\rangle_{\mathrm{SL}_2(\mathbb{Z})} = \frac{\Gamma(2k-1)}{2^{4k-3}\pi^{2k}} L(\mathrm{sym}^2 F, 1),$$

where $L(\mathrm{sym}^2 F, s)$ *is the analytic continuation of*

$$\prod_p (1 - \alpha_p^2 p^{2-2k-s})^{-1} (1 - \alpha_p \overline{\alpha_p} p^{2-2k-s})^{-1} (1 - \overline{\alpha_p}^2 p^{2-2k-s})^{-1} = \frac{\zeta(2s)}{\zeta(s)} \sum_{n=1}^{\infty} \frac{\widehat{F}(n)^2}{n^{s+2k-2}}$$

and $\alpha_p, \overline{\alpha_p}$ *are the solutions to* $\alpha_p + \overline{\alpha_p} = \widehat{F}(p)$, $\alpha_p \overline{\alpha_p} = p^{2k-2}$.

Proposition 4 ([12, p. 221, (2.16)]). *Let* $F \in S_{2k-1}(\mathrm{SL}_2(\mathbb{Z}))$ *be an arithmetically normalised Hecke eigenform. Then*

$$k^{-\epsilon} \ll_\epsilon L(\mathrm{sym}^2 F, 1) \ll_\epsilon k^{\epsilon}.$$

If we adopt the notation of Proposition 2, all the remaining Fourier coefficients of our Hecke eigenform $f \in S_k^+(\Gamma_0(4)^\star)$ satisfy

$$\widehat{f}(n^2|D|) = \widehat{f}(|D|) \sum_{d|n} \mu(d) \left(\frac{D}{d}\right) d^{k-3/2} \widehat{F}\left(\frac{n}{d}\right). \tag{2.1}$$

3. Proof of theorems. Let $f = f_1 \in S_k^+(\Gamma_0(4)^\star)$ be a Hecke eigenform of norm $\langle f, f\rangle_{\Gamma_0(4)} = 1$. Then $y^{k/2}|f(z)|$ is $\Gamma_0(4)$ invariant. Moreover, $(\mathrm{Im}\, \tau z)^{k/2} |f(\tau z)| = y^{k/2} |(f|_k \tau)(z)|$ for all $\tau \in \mathrm{GL}_2^+(\mathbb{Q})$. Since the set

$$\{z \mid \mathrm{Im}\, z \geq \sqrt{3}/8\} \cup \{W_4 z \mid \mathrm{Im}\, z \geq \sqrt{3}/8\} \cup \{V_4 z \mid \mathrm{Im}\, z \geq \sqrt{3}/8\}$$

covers a fundamental domain of $\Gamma_0(4)$, this implies

$$\sup_{z\in\mathbb{H}} y^{k/2}|f(z)| = \max_{\xi\in\{I,W_4,V_4\}} \sup_{y\geq\sqrt{3}/8} y^{k/2}|(f|_k\xi)(z)|. \tag{3.1}$$

The proof of Theorems 1 and 2 is split up into two parts. In the first part we use the Fourier expansion and bounds on the Fourier coefficients to bound the supnorm near a cusp. If we are far away from the cusp we can use the Bergman kernel in combination with an amplifier to get superior results, which is described in the second part. In a third part we give the proof of Theorem 3.

3.1. Bounding the Fourier expansion. On a first thought it is tempting to use classical estimates such as

$$\sum_{n\leq N} |\widehat{f}(n)|^2 \ll_f N^k$$

to bound the Fourier expansion, but it turns out that the implied constant is heavily dependent on f, in fact the supnorm of f itself appears as a factor. Thus one might try to use deeper techniques or use the currently best known result towards the Ramanujan–Petersson conjecture. We follow the latter path.

Throughout we assume we have a uniform bound of the shape

$$L(F,\chi,1/2) \ll k^\alpha q^\beta \tag{3.2}$$

for all arithmetically normalised Hecke eigenforms $F \in S_{2k-1}(\mathrm{SL}_2(\mathbb{Z}))$ and quadratic characters χ of conductor q. Through the work of Petrow [14] and Young [25] we now know that the pair $(\alpha,\beta) = (1/3+\epsilon, 1/3+\epsilon)$ is permissible for all $\epsilon > 0$. The Lindelöf hypothesis corresponds of course to $(\alpha,\beta) = (\epsilon,\epsilon)$.

Using Deligne's bound for the Fourier coefficients of $F \in S_{2k-1}(\mathrm{SL}_2(\mathbb{Z}))$ in (2.1) we find that

$$\begin{aligned}|\widehat{f}(n^2|D|)| &\ll_\epsilon |\widehat{f}(|D|)| \sum_{d|n} d^{k-3/2}\left(\frac{n}{d}\right)^{k-1+\epsilon} \\ &\ll_\epsilon |\widehat{f}(|D|)|(n^2)^{(k-1)/2+\epsilon}.\end{aligned} \tag{3.3}$$

Combining Propositions 2–4 with the bound (3.2) we get

$$|\widehat{f}(|D|)| \ll_\epsilon \frac{(4\pi)^{k/2}}{\Gamma(k)^{1/2}} \cdot |D|^{(k-1+\beta)/2} k^{\alpha/2+\epsilon}. \tag{3.4}$$

This yields the following proposition.

Proposition 5. *Let $k \in 1/2+\mathbb{Z}$ with $k \geq 5/2$ and let $f \in S_k^+(\Gamma_0(4)^\star)$ be an L^2-normalised Hecke eigenform. Further assume a uniform bound* (3.2) *with $\beta > 0$. Then*

$$|\widehat{f}(m)| \ll_\epsilon \frac{(4\pi)^{k/2} k^{\alpha/2+\epsilon}}{\Gamma(k)^{1/2}} \cdot m^{(k-1+\beta)/2}.$$

For convenience set

$$S(\alpha,\beta,\kappa) = \sum_{m+\kappa>0} (m+\kappa)^\alpha e^{-\beta(m+\kappa)}, \qquad \alpha,\beta,\kappa > 0. \tag{3.5}$$

We will need two lemmata for this sum.

LEMMA 2 ([21, Lemma 1]). *We have*

$$S(\alpha,\beta,\kappa) \le \beta^{-\alpha-1}\Gamma(\alpha+1) + \beta^{-\alpha}\alpha^\alpha e^{-\alpha},$$

and for $\alpha \le \beta\kappa$,

$$S(\alpha,\beta,\kappa) \le \beta^{-\alpha-1}\Gamma(\alpha+1) + \kappa^\alpha e^{-\beta\kappa}.$$

LEMMA 3 ([21, Lemma 2]). *For $\kappa \ge 6\alpha/\beta$, $\alpha,\beta > 0$,*

$$\kappa^\alpha e^{-\beta\kappa} \le \alpha^\alpha \beta^{-\alpha} e^{-\alpha} e^{-\beta\kappa/2}.$$

Using Proposition 5 in Lemma 1 we find:

$$\begin{aligned} y^{k/2}|f(z)| &\ll_\epsilon \frac{y^{k/2}(4\pi)^{k/2}k^{\alpha/2+\epsilon}}{\Gamma(k)^{1/2}} S\left(\frac{k-1+\beta}{2}, 2\pi y, 1\right),\\ y^{k/2}|(f|_k W_4)(z)| &\ll_\epsilon \frac{y^{k/2}(4\pi)^{k/2}k^{\alpha/2+\epsilon}}{\Gamma(k)^{1/2}} S\left(\frac{k-1+\beta}{2}, 2\pi y, 1\right),\\ y^{k/2}|(f|_k V_4)(z)| &\ll_\epsilon \frac{(\frac{y}{4})^{k/2}(4\pi)^{k/2}k^{\alpha/2+\epsilon}}{\Gamma(k)^{1/2}} S\left(\frac{k-1+\beta}{2}, \frac{2\pi y}{4}, 1\right). \end{aligned} \tag{3.6}$$

Using Lemma 2 we find in turn

$$S\left(\frac{k-1+\beta}{2}, 2\pi y, 1\right) \ll (4\pi)^{-k/2} y^{-k/2-1/2-\beta/2} k^{k/2+\beta/2} e^{-k/2}(1+yk^{-1/2}).$$

Thus we get the following proposition.

PROPOSITION 6. *Let $k \in 1/2 + \mathbb{Z}$ with $k \ge 5/2$ and let $f \in S_k^+(\Gamma_0(4)^\star)$ be an L^2-normalised Hecke eigenform. Assuming* (3.2) *with $\beta > 0$ we have, for $y \ge \sqrt{3}/8$:*

$$\begin{aligned} y^{k/2}|f(z)| &\ll_\epsilon \frac{k^{1/4+\alpha/2+\beta/2+\epsilon}}{y^{1/2+\beta/2}}(1+yk^{-1/2}),\\ y^{k/2}|(f|_k W_4)(z)| &\ll_\epsilon \frac{k^{1/4+\alpha/2+\beta/2+\epsilon}}{y^{1/2+\beta/2}}(1+yk^{-1/2}),\\ y^{k/2}|(f|_k V_4)(z)| &\ll_\epsilon \frac{k^{1/4+\alpha/2+\beta/2+\epsilon}}{y^{1/2+\beta/2}}(1+yk^{-1/2}). \end{aligned}$$

If $y \geq 3k/\pi$ (and $\beta \leq k$), we can use the second part of Lemma 2 with Lemma 3 to get

$$S\left(\frac{k-1+\beta}{2}, 2\pi y, 1\right) \ll (4\pi)^{-k/2} y^{-k/2-1/2-\beta/2} k^{k/2+\beta/2} e^{-k/2} (1+k^{1/2} e^{-\pi y}).$$

This yields the following proposition.

Proposition 7. *Let $k \in 1/2 + \mathbb{Z}$ with $k \geq 5/2$ and $f \in S^+(\Gamma_0(4)^\star)$ be an L^2-normalised Hecke eigenform. Assuming* (3.2) *with $k \geq \beta > 0$ we have, for $y \geq 12k/\pi$:*

$$y^{k/2}|f(z)| \ll_\epsilon \frac{k^{1/4+\alpha/2+\beta/2+\epsilon}}{y^{1/2+\beta/2}}(1+k^{1/2}e^{-\pi y}),$$

$$y^{k/2}|(f|_k W_4)(z)| \ll_\epsilon \frac{k^{1/4+\alpha/2+\beta/2+\epsilon}}{y^{1/2+\beta/2}}(1+k^{1/2}e^{-\pi y}),$$

$$y^{k/2}|(f|_k V_4)(z)| \ll_\epsilon \frac{k^{1/4+\alpha/2+\beta/2+\epsilon}}{y^{1/2+\beta/2}}(1+k^{1/2}e^{-\pi y/4}).$$

If we assume the Lindelöf hypothesis then the conjunction of Propositions 6 and 7 with the observation (3.1) gives Theorem 1. If we use the unconditional result $(\alpha, \beta) = (1/3+\epsilon, 1/3+\epsilon)$ instead, we find that

$$\max_{\xi \in \{I, W_4, V_4\}} \sup_{y \geq k^{1/4}} y^{k/2}|(f|_k\xi)(z)| \ll_\epsilon k^{5/12+\epsilon}. \tag{3.7}$$

The remaining region will be dealt with in the next section.

3.2. Amplification. We start by using the Bergman kernel as given in [21, Theorem 4] to deduce the identity

$$\sum_j \overline{f_j(w)} f_j(z) = \frac{3(k-1)}{4\pi} \sum_{\xi \in \Gamma_0(4)^\star} \frac{1}{\left(\frac{z-\overline{w}}{2i}\right)^k}\Bigg|_k \xi, \tag{3.8}$$

where $|_k\xi$ is taken with respect to the variable z and $\{f_j\}$ is an orthonormal basis of the whole space $S_k(\Gamma_0(4)^\star)$. If we apply the Hecke operator $|_kT(m)$ to both sides with respect to the variable z we get

$$\sum_j \lambda_j(m)\overline{f_j(w)} f_j(z) = \frac{3(k-1)}{4\pi} m^{k/2-1} \sum_{\xi \in \Gamma_0(4)^\star \xi_{1,m} \Gamma_0(4)^\star} \frac{1}{\left(\frac{z-\overline{w}}{2i}\right)^k}\Bigg|_k \xi, \tag{3.9}$$

with

$$\xi_{1,m} = \left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, m^{1/4}\right).$$

Let us denote by $A_j(m) = \lambda_j(m) m^{-(k-1)/2}$ the normalised Hecke eigenvalues. Further let $\mathcal{M}$ be a finite set of squares of odd integers and x_m arbitrary real numbers for $m \in \mathcal{M}$. For m, n odd we have the identity

$$\lambda_j(m^2)\lambda_j(n^2) = \sum_{d|(m,n)} d^{k-1}\lambda_j\left(\frac{m^2n^2}{d^4}\right),$$

which follows from the fact that the Shimura map from $S_k(\Gamma_0(4)^\star)$ to $S_{2k-1}(\Gamma_0(2))$ commutes with odd Hecke operators, and from the relation of Hecke operators in integral weight (see for example [16, Theorem 9.2.1]). Using this identity we get

$$\begin{aligned}
(3.10)\qquad &\sum_j \Big|\sum_{m\in\mathcal{M}} x_m A_j(m)\Big|^2 \overline{f_j(w)} f_j(z) \\
&= \sum_{m_1,m_2\in\mathcal{M}} x_{m_1}x_{m_2}(m_1m_2)^{-(k-1)/2} \sum_j \lambda_j(m_1)\lambda_j(m_2)\overline{f_j(w)} f_j(z) \\
&= \sum_l y_l l^{-(k-1)/2} \sum_j \lambda_j(l)\overline{f_j(w)} f_j(z) \\
&= \frac{3(k-1)}{4\pi} \sum_l y_l l^{-1/2} \sum_{\xi\in\Gamma_0(4)^\star\xi_{1,l}\Gamma_0(4)^\star} \frac{1}{\left(\frac{z-\overline{w}}{2i}\right)^k}\Bigg|_k \xi,
\end{aligned}$$

where

$$y_l = \sum_{\substack{m_1,m_2\in\mathcal{M}\\ d^2|(m_1,m_2)\\ l=m_1m_2/d^4}} x_{m_1}x_{m_2}.$$

Specialising to $w = z$ we get the inequality we are interested in:

$$(3.11)\qquad \Big|\sum_{m\in\mathcal{M}} x_m A_1(m)\Big|^2 y^k |f_1(z)|^2 \le \frac{3(k-1)}{4\pi}\sum_l |y_l| l^{-1/2} \sum_{\gamma\in G_l(4)} d_\gamma(z)^{-k},$$

where

$$d_\gamma(z) = \frac{|\gamma z - \overline{z}|\cdot|j(\gamma,z)|}{2yl^{1/2}}$$

and

$$G_l(4) = \left\{\gamma\in \mathrm{GL}_2(\mathbb{Z}) \,\middle|\, \det\gamma = l \text{ and } \gamma \equiv \begin{pmatrix}\star & \star\\ 0 & \star\end{pmatrix} \pmod 4\right\}.$$

Note that

$$\operatorname{Im}\gamma z = \frac{l\operatorname{Im} z}{|j(\gamma,z)|^2},$$

and thus

$$d_\gamma(z)^2 = \frac{|\gamma z-\overline{z}|^2}{4\operatorname{Im}\gamma z\operatorname{Im} z} = \frac{(\operatorname{Re}\gamma z - \operatorname{Re} z)^2 + (\operatorname{Im}\gamma z + \operatorname{Im} z)^2}{4\operatorname{Im}\gamma z\operatorname{Im} z} = u(\gamma z, z) + 1,$$

where

$$u(z,w)=\frac{|z-w|^2}{4\operatorname{Im}z\operatorname{Im}w}.$$

Now we want the same inequality with f_1 replaced with $f_1|_kW_4$ and $f_1|_kV_4$. For this we replace (3.8) with

$$\sum_j \overline{(f_j|_kB)(w)}(f_j|_kB)(z)=\frac{3(k-1)}{4\pi}\sum_{\xi\in B^{-1}\Gamma_0(4)^\star B}\frac{1}{\left(\frac{z-\overline{w}}{2i}\right)^k}\Bigg|_k\xi, \tag{3.12}$$

where $B\in\{W_4,V_4\}$. We apply $|_kB^{-1}T(m)B$ to both sides and proceed as before to obtain

$$\Big|\sum_{m\in\mathcal{M}}x_mA_1(m)\Big|^2 y^k|(f_1|_kB)(z)|^2 \le \frac{3(k-1)}{4\pi}\sum_l|y_l|l^{-1/2}\sum_{\gamma\in B^{-1}G_l(4)B}d_\gamma(z)^{-k}. \tag{3.13}$$

Now we just have to note that both W_4 and V_4 stabilise $G_l(4)$ for odd l.

We next consider two separate sets $\mathcal{M}_1,\mathcal{M}_2$ as in [1] given by

$$\mathcal{M}_1=\{p^2\,|\,p\text{ is a prime, }\Lambda\le p<2\Lambda\text{ and }p\neq 2\},$$
$$\mathcal{M}_2=\{p^4\,|\,p\text{ is a prime, }\Lambda\le p<2\Lambda\text{ and }p\neq 2\}.$$

This has the advantage that it reduces the amount of computations to bound the right hand side of (3.11) as there are fewer mixed terms to consider. Along with these choices for $\mathcal{M}$ we choose

$$x_m=\operatorname{sign}(A_1(m))\qquad\forall m\in\mathcal{M}_1\text{ (respectively }\mathcal{M}_2),$$

for which we have

$$\begin{aligned}|y_l|&\ll\begin{cases}\Lambda & \text{if } l=1,\\ 1 & \text{if } l=p^2q^2 \text{ with } p^2,q^2\in\mathcal{M}_1,\\ 0 & \text{otherwise,}\end{cases}\\ |y_l|&\ll\begin{cases}\Lambda & \text{if } l=1,\\ 1 & \text{if } l=p^4,p^4q^4 \text{ with } p^4,q^4\in\mathcal{M}_2,\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{3.14}$$

respectively. We now add the two equations (3.11) for $\mathcal{M}=\mathcal{M}_1,\mathcal{M}_2$, and by Cauchy–Schwarz we see that

$$\Big(\sum_{m\in\mathcal{M}_1\cup\mathcal{M}_2}|A_1(m)|\Big)^2y^k|f_1(z)|^2\gg_\epsilon \Lambda^{2-\epsilon}y^k|f_1(z)|^2$$

as $\max(|A_1(p^2)|,|A_1(p^4)|)\gg 1$. We get the same inequality also for the other cusps and conclude

$$\Lambda^{2-\epsilon} \max_{B\in\{I,W_4,V_4\}} y^k|(f_1|_k B)(z)|^2 \ll_\epsilon \sum_{\mathcal{M}=\mathcal{M}_1,\mathcal{M}_2} k \sum_l |y_l| l^{-1/2} \sum_{\gamma\in G_l(4)} (u(\gamma z,z)+1)^{-k/2}. \tag{3.15}$$

Thus we are left to bound the right hand side. For this, we define

$$\begin{aligned} M(z,l,\delta) &= |\{\gamma\in G_l(4) \mid u(\gamma z,z)\le\delta\}|,\\ M_\star(z,l,\delta) &= |\{\gamma\in G_l(4) \mid c\ne 0, (a+d)^2\ne 4l \text{ and } u(\gamma z,z)\le\delta\}|,\\ M_u(z,l,\delta) &= |\{\gamma\in G_l(4) \mid c=0, a\ne d \text{ and } u(\gamma z,z)\le\delta\}|,\\ M_p(z,l,\delta) &= |\{\gamma\in G_l(4) \mid (a+d)^2=4l \text{ and } u(\gamma z,z)\le\delta\}|. \end{aligned} \tag{3.16}$$

Lemma 4. *For $z=x+iy\in\mathbb{H}$ with $|x|\ll 1$ and $y\gg 1$ we have*

$$\sum_{\substack{1\le l\le L\\ l \text{ is a square}}} M_\star(z,l,\delta) \ll_\epsilon (L^{1/2}/y + L\delta^{1/2} + L^{3/2}\delta)L^\epsilon. \tag{3.17}$$

Proof. This is basically [22, Lemma 4.1]. The same proof carries through with ease as we do not need to take the level aspect into account. ∎

Lemma 5. *For $z=x+iy\in\mathbb{H}$ with $|x|\ll 1$, $y\gg 1$ and $l\in\mathbb{N}$ with $d(l)\ll 1$, where $d(l)$ denotes the number of divisors of l, we have*

$$M_u(z,l,\delta) \ll 1 + l^{1/2}\delta^{1/2}y. \tag{3.18}$$

Proof. This is part of the variant of [6, Lemma 1.3] given in [6, Appendix] ∎

Lemma 6. *For $z=x+iy\in\mathbb{H}$ with $|x|\ll 1$ and $y\gg 1$ we have*

$$M_p(z,l,\delta) \ll_\epsilon (1+l^{1/2}\delta^{1/2}y)^{1+\epsilon}. \tag{3.19}$$

Proof. This is essentially [22, Lemma 4.4]. Since the reference lacks details, we provide a proof that suffices for our case. Let γ be a matrix of the shape that we are counting. Such a matrix is parabolic and fixes a cusp $a/c\in\mathbb{P}^1(\mathbb{Q})$. We may assume $(a,c)=1$ and find a matrix

$$\sigma_{a/c} = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

It follows that $\gamma' = \sigma_{a/c}^{-1}\gamma\sigma_{a/c}$ is parabolic with determinant l and stabilises ∞. Hence γ' must be of the shape

$$\gamma' = \pm\begin{pmatrix} l^{1/2} & t\\ 0 & l^{1/2}\end{pmatrix}.$$

This forces l to be a square and leads to

$$\gamma = \pm\begin{pmatrix} l^{1/2} - act & a^2t \\ -c^2t & l^{1/2} + act \end{pmatrix}. \tag{3.20}$$

For $t = 0$ these are always the same two matrices independent of a and c, thus we account for this case only once. Assume from now on that $t \neq 0$. We find

$$u(\gamma z, z) = \frac{t^2|cz-a|^4}{4y^2l}.$$

Fixing t we find that we are counting lattice points in a ball of radius $O(l^{1/4}\delta^{1/4}y^{1/2}|t|^{-1/2})$ of the lattice generated by 1 and z. Its minimal distance is $\gg 1$ and its co-volume is y. By invoking [5, Lemma 1] we find the number of pairs of (a, c) is bounded by

$$O\left(1 + \frac{l^{1/4}\delta^{1/4}y^{1/2}}{|t|^{1/2}} + \frac{l^{1/2}\delta^{1/2}y}{y|t|}\right).$$

Summing over $|t| \ll l^{1/2}\delta^{1/2}y$ gives the desired result. ∎

Remark 2. The above argument also leads to a rigorous proof of [22, Lemma 4.4] up to epsilons in the exponents by making the following alterations. From (3.20) one deduces $N \mid ct$ since N is square-free. Then one splits up $N = n_1n_2$ with $n_1 \mid t$ and $n_2 \mid c$. Write $t = n_1t'$ and $c = n_2c'$. One now has $|cz - a|^2 \gg (c, N)/N \geq n_2/N$ for $z \in \mathcal{F}(N)$, and thus the minimal distance in the lattice generated by n_2z and 1 is $\gg n_2^{1/2}/N^{1/2}$, and its co-volume is n_2y. Furthermore $|t'| \ll l^{1/2}\delta^{1/2}y$. Estimating as before and summing over the factorisations of N yields the bound

$$N^{\epsilon}(1 + l^{1/2}\delta^{1/2}y)^{1+\epsilon}$$

when l is a square and 0 otherwise, which is only slightly worse than what Templier claims.

It is now not hard to bound the expression

$$\sum_{\gamma \in G_l(4)} (1 + u(\gamma z, z))^{-k/2}$$

polynomially in l, k, y for $k \geq 5/2$, so we omit the details. Instead we give the following insight. If $u(\gamma z, z) \geq k^{-1+\eta}$ for some positive real η, then the expression

$$(1 + u(\gamma z, z))^{-k/2}$$

has super-polynomial decay in k, thus if l, y only depend on k polynomially, we can completely neglect that part as follows:

$$(3.21)\qquad \sum_{\gamma\in G_l(4)} (1+u(\gamma z,z))^{-k/2} \leq \sum_{\substack{\gamma\in G_l(4)\\ u(\gamma z,z)\leq k^{-1+\eta}}} 1 + (1+k^{-1+\eta})^{-k/2+5/4} \sum_{\gamma\in G_l(4)} (1+u(\gamma z,z))^{-5/4}.$$

From now on we will assume that Λ and y depend polynomially on k, so that (3.21) becomes

$$(3.22)\qquad \sum_{\gamma\in G_l(4)} (1+u(\gamma z,z))^{-k/2} \ll_\eta M(z,l,k^{-1+\eta}).$$

We use this inequality to estimate the right hand side of (3.15). We first consider the case $\mathcal{M}=\mathcal{M}_1$. The contribution of $l=1$ is

$$k\Lambda^{1+\epsilon}(1+yk^{(-1+\eta)/2})$$

by Lemma 4 with $L=1$ and Lemmata 5 and 6. The contribution of $l>1$ is

$$k\Lambda^{\epsilon}(1/y+\Lambda^2 k^{(-1+\eta)/2}+\Lambda^4 k^{-1+\eta})$$

for the generic matrices by Lemma 4 with $L=2^4\Lambda^4$, and by Lemmata 5 and 6 the contribution of the upper triangular and the parabolic matrices is

$$k^{1+\epsilon}\sum_{l=p^2q^2} l^{-1/2}(1+l^{1/2}yk^{(-1+\eta)/2}) \ll_{\epsilon,\eta} k^{1+\epsilon}(1+\Lambda^2 yk^{(-1+\eta)/2}).$$

Thus the sum over $\mathcal{M}_1$ is bounded by

$$(3.23)\qquad k^{1+\epsilon}\Lambda^{\epsilon}(\Lambda+\Lambda^2 yk^{(-1+\eta)/2}+\Lambda^4 k^{-1+\eta}).$$

For $\mathcal{M}=\mathcal{M}_2$ the contribution of $l=1$ is again

$$k\Lambda^{1+\epsilon}(1+yk^{(-1+\eta)/2}).$$

For $l>1$ the contribution of the generic matrices is

$$k\Lambda^{\epsilon}(1/y+\Lambda^4 k^{(-1+\eta)/2}+\Lambda^8 k^{-1+\eta})$$

by Lemma 4 with $L=2^8\Lambda^8$, and by Lemmata 5 and 6 the contribution of the upper triangular and parabolic matrices is

$$k^{1+\epsilon}\Big(\sum_{l=p^4}+\sum_{l=p^4q^4}\Big) l^{-1/2}(1+l^{1/2}yk^{(-1+\eta)/2}) \ll_{\epsilon,\eta} k^{1+\epsilon}\left(\frac{1}{\Lambda}+\Lambda yk^{(-1+\eta)/2}+\frac{1}{\Lambda^2}+\Lambda^2 yk^{(-1+\eta)/2}\right).$$

Thus the sum over $\mathcal{M}_2$ is bounded by

$$(3.24)\qquad k^{1+\epsilon}\Lambda^{\epsilon}(\Lambda+\Lambda^2 yk^{(-1+\eta)/2}+\Lambda^4 k^{(-1+\eta)/2}+\Lambda^8 k^{-1+\eta}).$$

Combining (3.23) and (3.24) with (3.15) and letting ϵ, η be suitably small we get

$$\max_{B\in\{I,W_4,V_4\}} y^k|(f_1|_k B)(z)|^2 \ll_\epsilon k^{1+\epsilon}\Lambda^\epsilon\left(\frac{1}{\Lambda}+yk^{-1/2}+\Lambda^2k^{-1/2}+\Lambda^6k^{-1}\right). \tag{3.25}$$

Since we may assume $y \le k^{1/4}$ by (3.7), we can choose $\Lambda = k^{1/7}$ and we achieve

$$\sup_{z\in\mathbb{H}} y^{k/2}|f_1(z)| \ll_\epsilon k^{3/7+\epsilon}, \tag{3.26}$$

completing the proof of Theorem 2.

3.3. Lower bounds. As in Theorem 3, let $\{f_j\} \subseteq S_k^+(\Gamma_0(4)^\star)$ be an orthonormal basis of Hecke eigenforms of half-integral weight k contained in the Kohnen plus space and let $\{F_j\} \subseteq S_{2k-1}(\mathrm{SL}_2(\mathbb{Z}))$ be the corresponding arithmetically normalised Hecke eigenforms under the Shimura map.

The first part of the first lower bound is trivial as

$$\sup_{z\in\mathbb{H}} y^k|f_j(z)|^2 \gg \langle f_j, f_j\rangle_{\Gamma_0(4)} = 1.$$

The second part follows from the inequality

$$y^{k/2}|\widehat{f_j}(|D|)| = \Big|\int_0^1 y^{k/2}f(z)e(-|D|z)\,dx\Big| \le e^{2\pi|D|y}\int_0^1 y^{k/2}|f_j(z)|\,dx \le e^{2\pi|D|y}\sup_{z'\in\mathbb{H}} y'^{k/2}|f_j(z')|. \tag{3.27}$$

This inequality in conjunction with Propositions 2–4 gives

$$\sup_{z'\in\mathbb{H}} y'^{k/2}|f_j(z')| \gg_\epsilon \frac{(4\pi y)^{k/2}}{\Gamma(k)^{1/2}}\cdot|D|^{(k-1)/2}k^{-\epsilon}e^{-2\pi|D|y}L\left(F_j,\left(\frac{D}{\cdot}\right),\frac{1}{2}\right).$$

The choice $y = \frac{k}{4\pi|D|}$ gives the desired inequality. Similarly

$$\sup_{z'\in\mathbb{H}}\sum_j y'^k|f_j(z')|^2 \ge \sum_j\Big(\int_0^1 dx\Big)\Big(\int_0^1 y^k|f_j(z)|^2\,dx\Big) \ge \sum_j\Big(\int_0^1 y^{k/2}|f_j(z)|\,dx\Big)^2 \ge y^ke^{-4\pi y}\sum_j|\widehat{f_j}(1)|^2. \tag{3.28}$$

By Corollary 1 we have

(3.29)

$$\sum_j |\widehat{f}_j(1)|^2 = \frac{6\cdot(4\pi)^{k-1}}{\Gamma(k-1)}\cdot\frac{2}{3}\Big[1+(-1)^{\lfloor (k+1/2)/2\rfloor}\pi\sqrt{2}\sum_{c\ge 1} H_c(1,1)J_{k-1}\Big(\frac{\pi}{c}\Big)\Big]$$
$$\gg \frac{(4\pi)^{k-1}}{\Gamma(k-1)}\cdot\Big[1-2\pi\sum_{c\ge 1}\Big|J_{k-1}\Big(\frac{\pi}{c}\Big)\Big|\Big].$$

Now we use the following result.

PROPOSITION 8 ([21, Proposition 8]). *For $\rho\ge 2x^2$ one has*

$$|J_\rho(x)| \ll \frac{(x/2)^\rho}{\Gamma(\rho+1)}.$$

If $k\ge 21$, we are able to apply it in order to get the estimate

$$\sum_{c\ge 1}\Big|J_{k-1}\Big(\frac{\pi}{c}\Big)\Big| \ll \frac{(\pi/2)^{k-1}}{\Gamma(k)}\sum_{c\ge 1}\frac{1}{c^{k-1}} = o(1).$$

Combining this with (3.28) and (3.29) and making the choice $y=k/(4\pi)$ gives the last lower bound in Theorem 3.

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