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Spectral conditions for Jordan *-isomorphisms on operator algebras

by

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Abstract. We study non-linear transformations between the unitary groups of von Neumann algebras and the twisted subgroups of positive invertible elements in unital C^*-algebras with various preserver properties concerning the spectrum, spectral radius, and generalized distance measures. We present several results which show that those transformations are closely related to Jordan *-isomorphisms between the underlying full algebras. In fact, our results can easily be used to characterize such isomorphisms.

1. Introduction. In this paper all algebras are assumed to be complex and unital, the unit usually being denoted by 1. Let A_1, A_2 be algebras and let $\sigma(\cdot)$ stand for the spectrum. A map (no linearity is assumed) $\phi : A_1 \to A_2$ is called *spectrally multiplicative* if

$$\sigma(\phi(a)\phi(b)) = \sigma(ab)$$

for all $a, b \in A_1$. There has recently been considerable interest in such transformations since in many cases they turn out to be closely related to isomorphisms, hence the spectral condition above may faithfully compress the linearity and multiplicativity properties of maps into one two-variable equality between sets of scalars. As a typical result, we recall that any spectrally multiplicative bijection between the algebras of all continuous complex valued functions over compact Hausdorff spaces is an algebra isomorphism followed by multiplication by a fixed real valued continuous function of modulus 1. In fact, for first countable spaces this was proved in [7] (which was the starting point of that line of investigations), while in [11] the authors removed the first countability assumption. Concerning operator algebras, we proved in [7] that for an infinite-dimensional Hilbert space H, any spectrally multiplicative bijection on the algebra of all bounded linear operators on H is

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either an algebra isomorphism or the negative of an algebra isomorphism. Hence in those cases any spectrally multiplicative bijective map is a transformation which can be written as an algebra isomorphism multiplied by a central symmetry (which is a self-adjoint unitary in the center of the algebra in question). For further reference we mention the survey paper [4] exhibiting a collection of recent results (mainly concerning function algebras) as well as the interesting paper [2] where a variant of spectrally multiplicative maps (involving three variables, and not only two) has been investigated on general algebras.

In this paper we continue that line of research and present results which can be viewed as characterizations of Jordan *-isomorphisms between operator algebras via their spectral multiplicativity properties of different kinds and their other characteristic invariance properties involving the spectral radius. However, there is a significant difference between the previous investigations and the one reported here: the properties we consider in this paper are assumed to be satisfied not on the whole algebra but only on certain subsets which are substructures of the general linear group. This means, and again it is the main novelty here, that the spectral multiplicativity and other related conditions are required only on some smaller sets (the so-called twisted subgroup of positive invertible elements, or the unitary group); as we shall see, they are still strong enough to imply that the transformations under consideration are closely related to Jordan *-isomorphisms between the underlying full algebras. In addition to our conditions concerning the spectral radius we investigate transformations which preserve certain distance measures of very general kinds. Furthermore, we study spectral-multiplicativity-like and other conditions for pairs of maps defined on arbitrary sets with values in the above mentioned substructures of operator algebras.

Before presenting our results we collect the following facts concerning Jordan *-isomorphisms between C^*-algebras. We first recall that by [12, Proposition 1.3] every surjective Jordan homomorphism J between arbitrary algebras A_1, A_2 preserves invertibility and satisfies $J(a^{-1}) = J(a)^{-1}$ for any invertible $a \in A_1$. This implies that a Jordan isomorphism maps the general linear group onto the general linear group and preserves the spectra of elements. We recall the important correspondence between the spectra of the elements ab and ba, where a, b belong to an algebra A: we always have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$, and hence if a, b are invertible, then $\sigma(ab) = \sigma(ba)$.

Let now A_1, A_2 be arbitrary C^*-algebras and $J : A_1 \to A_2$ a Jordan isomorphism. By [1, Theorem 6.3.4] there exists a central projection q (by a *projection* we always mean a self-adjoint idempotent) in the so-called bounded central closure of A_2 (a C^*-algebra that contains A_2 as a C^*-subalgebra) such that

$$J(ab) = qJ(a)J(b) + (1-q)J(b)J(a), \qquad a, b \in A_1.$$

Let $a, b \in A_1$ be invertible. Set $x_1 = qJ(a)$, $y_1 = qJ(b)$ and $x_2 = (1-q)J(a)$, $y_2 = (1-q)J(b)$. We compute

$$\begin{aligned}\sigma(ab) \cup \{0\} = \sigma(J(ab)) \cup \{0\} &= \sigma(x_1 y_1) \cup \sigma(y_2 x_2)\\ &= \sigma(x_1 y_1) \cup \sigma(x_2 y_2) = \sigma(J(a)J(b)) \cup \{0\}.\end{aligned}$$

Since J preserves invertibility, $J(a)J(b)$ is invertible and by the above equality we have

$$\sigma(ab) = \sigma(J(a)J(b)),$$

which proves that J is spectrally multiplicative on the general linear group.

For a C^*-algebra A_j, we denote by A_{js} the real linear subspace of all self-adjoint elements in A_j. The set of all positive elements (i.e., self-adjoint elements with non-negative spectrum) in A_j is denoted by A_{j+}. The set A_{j+}^{-1} of all invertibles in A_{j+} is a so-called *twisted subgroup* of the general linear group, meaning that it is closed under taking the inverted Jordan triple product $ab^{-1}a$. For obvious reasons, it is also called the positive definite cone (or *positive cone* for short). Note that $A_{j+}^{-1} = \exp A_{js}$. The unitary group of A_j is denoted by U_j. Recall that $U_j = \exp iA_{js}$ if A_j is a von Neumann algebra. By a *symmetry* we mean a self-adjoint unitary element (or, equivalently a unitary whose square is the identity).

Recall that the spectral radius r satisfies $r(a) \leq \|a\|$ for every a in the C^*-algebra A_j, and $r(a) = \|a\|$ for any normal $a \in A_j$.

2. The case of the spaces of positive invertible elements. Besides a characterization via the spectral multiplicativity property, the first main result of the paper, Theorem 5, contains a sort of characterization of Jordan *-isomorphisms in terms of a preserver property involving so-called generalized distance measures. For this we need a recently obtained very general Mazur–Ulam type result that we cite below as Theorem 3. To formulate it we need some preparation. From [8] we recall the following.

Definition 1. Let X be a set equipped with a binary operation $\diamond$ which satisfies the following conditions:

(a1) $a \diamond a = a$ for every $a \in X$;
(a2) $a \diamond (a \diamond b) = b$ for any $a, b \in X$;
(a3) the equation $x \diamond a = b$ has a unique solution $x \in X$ for any given $a, b \in X$.

In this case the pair $(X, \diamond)$ (or X itself) is called a *point-reflection geometry*.

For a C^*-algebra A and elements $a, b \in A_+^{-1}$ define $a \diamond b = ab^{-1}a$. Then A_+^{-1} becomes a point-reflection geometry. Indeed, conditions (a1), (a2) are trivial to check. Concerning (a3) we recall that for any given $a, b \in A_+^{-1}$, the Ricatti equation $xa^{-1}x = b$ has a unique solution $x = a \# b$ which is just the geometric mean of a and b defined by

$$a \# b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}.$$

This result is usually termed the *Anderson–Trapp theorem*.

We need another concept, the one of generalized distance measures.

Definition 2. Given an arbitrary non-empty set X, a function $d : X \times X \to [0, \infty[$ is called a *generalized distance measure* if for any $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$.

Hence, in this definition we require only the definiteness property of a metric, but neither symmetry nor the triangle inequality. Our general Mazur–Ulam type theorem of [8] reads as follows.

Theorem 3. *Let X, Y be non-empty sets equipped with binary operations $\diamond, \star$, respectively, with which they form point-reflection geometries. Let $d : X \times X \to [0, \infty[$ and $\rho : Y \times Y \to [0, \infty[$ be generalized distance measures. Pick $a, b \in X$, set*

$$L_{a,b} = \{x \in X : d(a, x) = d(x, b \diamond a) = d(a, b)\}$$

and assume the following:

(b1) *$d(b \diamond x, b \diamond x') = d(x', x)$ for all $x, x' \in X$;*
(b2) *$\sup\{d(x, b) : x \in L_{a,b}\} < \infty$;*
(b3) *there exists $K > 1$ such that $d(x, b \diamond x) \geq K d(x, b)$ for every $x \in L_{a,b}$.*

Let $\phi : X \to Y$ be a surjective map such that

$$\rho(\phi(x), \phi(x')) = d(x, x'), \qquad x, x' \in X,$$

and assume that

(b4) *for the unique element $c \in Y$ satisfying $c \star \phi(a) = \phi(b \diamond a)$ we have $\rho(c \star y, c \star y') = \rho(y', y)$ for all $y, y' \in Y$.*

Then

$$\phi(b \diamond a) = \phi(b) \star \phi(a).$$

We shall also need the following properties defined for a continuous function $h :]0, \infty[\to \mathbb{R}$:

(c1) $h(t) = 0$ if and only if $t = 1$;
(c2) for some $\theta > 0$, we have $|h(t)| \geq \theta$ for all $t \in]0, \infty[$ outside a neighborhood of 1;
(c3) h is differentiable at $t = 1$ and $h'(1) \neq 0$;
(c4) $|h(t_0)| \neq |h(t_0^{-1})|$ for some $t_0 \in]0, \infty[$.

We begin the exposition of our new results with the next proposition.

Proposition 4. *Let A_j be a C^*-algebra for $j = 1, 2$. Suppose that ϕ is a surjection from A_{1+}^{-1} onto A_{2+}^{-1}. Suppose that there exist continuous functions $h_1, h_2\colon]0,\infty[\to \mathbb{R}$ which satisfy* (c1)–(c3) *and*

$$\|h_1(b^{-1/2}ab^{-1/2})\| = \|h_2(\phi(b)^{-1/2}\phi(a)\phi(b)^{-1/2})\|, \qquad a, b \in A_{1+}^{-1}.$$

*Then there exists a Jordan *-isomorphism J from A_1 onto A_2, an element $b_0 \in A_{2+}^{-1}$, a central projection $p \in A_2$ and a positive number c such that*

$$\phi(a) = b_0\big(pJ(a)^c + (1-p)J(a)^{-c}\big)b_0, \qquad a \in A_{1+}^{-1}.$$

Before proving Proposition 4 we make some remarks. First of all, whenever a normal element of a C^*-algebra is plugged into a continuous real function (with domain containing the spectrum of that element), it is understood the well-known continuous functional calculus is applied.

Let A be a C^*-algebra.

(R1) If a continuous function $h :]0,\infty[\to \mathbb{R}$ satisfies (c1), then for any $a \in A_+^{-1}$, the equality $h(a) = 0$ implies $a = 1$. Indeed, by the spectral mapping theorem we have $h(\sigma(a)) = \sigma(h(a)) = 0$, which by (c1) implies $\sigma(a) = \{1\}$, i.e., $\sigma(a-1) = \{0\}$, which yields $a - 1 = 0$. It follows that the formula

$$d(a,b) = \|h(b^{-1/2}ab^{-1/2})\|, \qquad a, b \in A_+^{-1},$$

defines a generalized distance measure on A_+^{-1}.

(R2) If a continuous function $h :]0,\infty[\to \mathbb{R}$ satisfies (c1) and (c2), then for any sequence $t_n \in]0,\infty[$ with $h(t_n) \to 0$ we have $t_n \to 1$. This easily implies that, similarly, for any sequence $a_n \in A_+^{-1}$ with $h(a_n) \to 0$ in the norm topology, we have $a_n \to 1$.

(R3) Let $h :]0,\infty[\to \mathbb{R}$ be a continuous function which is differentiable at $t = 1$. Then the transformation $x \mapsto h(x)$, $x \in A_+^{-1}$, is Fréchet-differentiable at $x = 1$ with derivative $(Dh)(1)y = h'(1) \cdot y$, $y \in A_s$. Indeed, by the differentiability of h we have a continuous function $\omega\colon]0,\infty[\to \mathbb{R}$ with $\omega(1) = 0$ such that $h(t) - h(1) - h'(1)(t-1) = \omega(t)(t-1)$ for all $t \in]0,\infty[$. Hence $h(x) - h(1)1 - h'(1)(x-1) = \omega(x)(x-1)$ for all $x \in A_+^{-1}$, so $\|h(x) - h(1)1 - h'(1)(x-1)\| \le \|\omega(x)\|\,\|x-1\|$. This implies

$$\frac{\|h(x) - h(1)1 - h'(1)(x-1)\|}{\|x-1\|} \to 0$$

as $x \to 1$, which proves the assertion.

(R4) If $h :]0,\infty[\to \mathbb{R}$ is a continuous function which satisfies (c1) and (c3), then there exists $K > 1$ such that $|h(t^2)| \ge K|h(t)|$ for all t in the ϵ-neighborhood of 1 for some $0 < \epsilon < 1$. Indeed,

$$\frac{h(t^2)}{h(t)} = (t+1)\frac{h(t^2)/(t^2-1)}{h(t)/(t-1)} \to 2$$

as $t \to 1$, proving our claim.

(R5) For any invertible $x \in A$ and $a, b \in A_+^{-1}$, the element $a^{1/2}b^{-1}a^{1/2}$ is unitarily equivalent to $b^{-1/2}ab^{-1/2}$, and $(xbx^*)^{-1/2}xax^*(xbx^*)^{-1/2}$ is unitarily equivalent to $b^{-1/2}ab^{-1/2}$. Indeed, for the former statement,

$$a^{1/2}b^{-1}a^{1/2} = u^*(b^{-1/2}ab^{-1/2})u,$$

where u is the unitary element in the polar decomposition of $b^{-1/2}a^{1/2} \in A$. For the latter statement,

$$\begin{aligned}(xbx^*)^{-1/2}&xax^*(xbx^*)^{-1/2}\\ &= |b^{1/2}x^*|^{-1}(b^{1/2}x^*)^*(b^{-1/2}ab^{-1/2})(b^{1/2}x^*)|b^{1/2}x^*|^{-1}\\ &= v^*(b^{-1/2}ab^{-1/2})v,\end{aligned}$$

where v is the unitary element in the polar decomposition of $b^{1/2}x^*$.

(R6) For any scalar valued continuous function h on $]0,\infty[$, unitary $u \in A$ and $a \in A_+^{-1}$ we have $h(uau^{-1}) = uh(a)u^{-1}$. Indeed, this follows easily from the fact that h can be uniformly approximated by polynomials on any compact subinterval of $]0,\infty[$ and from the isometric property of the continuous functional calculus.

Proof of Proposition 4. Suppose that there exist $h_1, h_2 :]0,\infty[\to \mathbb{R}$ as in the statement. Define

$$\begin{aligned} d(a,b) &= \|h_1(b^{-1/2}ab^{-1/2})\|, \quad a,b \in A_{1+}^{-1},\\ \rho(a,b) &= \|h_2(b^{-1/2}ab^{-1/2})\|, \quad a,b \in A_{2+}^{-1}.\end{aligned}$$

By (R1), d, ρ are generalized distance measures and

$$\rho(\phi(a),\phi(b)) = d(a,b), \qquad a,b \in A_{1+}^{-1}. \tag{2.1}$$

Applying (R5) and (R6) we obtain $d(zaz^*, zbz^*) = d(a,b)$ and

$$d(bx^{-1}b, bx'^{-1}b) = d(x^{-1}, x'^{-1}) = d(x', x) \tag{2.2}$$

for all $a, b, x \in A_{1+}^{-1}$ and invertible $z \in A_1$. Clearly, similar properties hold for the generalized distance measure ρ.

Now define $\phi_0 : A_{1+}^{-1} \to A_{2+}^{-1}$ by $\phi_0(a) = \phi(1)^{-1/2}\phi(a)\phi(1)^{-1/2}$ for $a \in A_{1+}^{-1}$. Plainly, ϕ_0 is a well defined map from A_{1+}^{-1} onto A_{2+}^{-1}, it is unital in the sense that $\phi_0(1) = 1$, and **(2.1)** also holds for ϕ_0, i.e., $\rho(\phi_0(a),\phi_0(b)) = d(a,b)$ for all $a,b \in A_{1+}^{-1}$.

We are going to apply Theorem 3. To check that the conditions of that theorem are satisfied, we first define the point-reflection geometry structures on A_{j+}^{-1} in the standard way, i.e., just as after Definition 1. Condition (b1) is fulfilled by **(2.2)**.

To proceed, we claim the following. Let H be a subset of A_{1+}^{-1} with the property that there are $\alpha, \beta > 0$ such that $\alpha 1 \le y \le \beta 1$ for all $y \in H$. (This means that H is bounded away from zero and also from above with respect to the usual order $\leq$ defined on the set of all self-adjoint elements coming from the notion of positivity. Recall that positive elements are the self-adjoint ones with spectrum in the set of non-negative reals.) Then we assert that there exists $\delta > 0$ with the property that whenever $a, b \in H$ are such that $\|a - b\| < \delta$, we necessarily have $\|b^{-1/2} x b^{-1/2} - 1\| < \epsilon$ (i.e., the spectrum of $b^{-1/2} x b^{-1/2}$ is in $]1-\epsilon, 1+\epsilon[$) for all $x \in L_{a,b}$, where ϵ appears in (R4) in relation to h_1.

Assume for a moment that this assertion is already proven. We can check easily that for $a, b \in A_{1+}^{-1}$ with $\|a - b\| < \delta$ properties (b2) and (b3) are satisfied. Indeed, by the isometric property of the continuous functional calculus, (b2) is clear since h_1 is bounded in $[1-\epsilon, 1+\epsilon]$. As for (b3), applying the second equality of **(2.2)**, the first part of (R5), and (R6), we can compute

$$\begin{aligned} d(x, bx^{-1}b) &= d(b^{-1}xb^{-1}, x^{-1}) = \|h_1(x^{1/2}(b^{-1}xb^{-1})x^{1/2})\| \\ &= \|h_1((x^{1/2}b^{-1}x^{1/2})^2)\| = \|h_1((b^{-1/2}xb^{-1/2})^2)\| \\ &\geq K\|h_1(b^{-1/2}xb^{-1/2})\| = Kd(x, b) \end{aligned}$$

for all $x \in L_{a,b}$, meaning that (b3) is also satisfied.

Now, to verify the starting assertion, assume on the contrary that there are sequences $a_n, b_n \in H$ and $x_n \in L_{a_n,b_n}$ such that $\|a_n - b_n\| < 1/n$ but $\|b_n^{-1/2} x_n b_n^{-1/2} - 1\| \geq \epsilon$. We compute

$$\|b_n^{-1/2} a_n b_n^{-1/2} - 1\| = \|b_n^{-1/2}(a_n - b_n)b_n^{-1/2}\| \leq \|b_n^{-1}\| \, \|a_n - b_n\|,$$

and this last term converges to 0 since $\|b_n^{-1}\| \leq 1/\alpha$. Therefore, $b_n^{-1/2} a_n b_n^{-1/2} \to 1$, and by (c1) it follows that $d(a_n, b_n) = \|h_1(b_n^{-1/2} a_n b_n^{-1/2})\| \to 0$. Since $x_n \in L_{a_n,b_n}$, we conclude that $d(a_n, x_n) = d(a_n, b_n) \to 0$, meaning that $h_1(x_n^{-1/2} a_n x_n^{-1/2}) \to 0$. Applying (R2) one can check easily that this implies $x_n^{-1/2} a_n x_n^{-1/2} \to 1$. Therefore, for every $0 < \gamma < 1$ there is an index n_0 such that for all $n \geq n_0$ we have $1 - \gamma 1 \leq x_n^{-1/2} a_n x_n^{-1/2} \leq 1 + \gamma 1$, which yields $(1/(1+\gamma))a_n \leq x_n \leq (1/(1-\gamma))a_n$ for all $n \geq n_0$. Since also $b_n^{-1/2} a_n b_n^{-1/2} \to 1$, in a similar manner, we may also assume that $(1/(1+\gamma))a_n \leq b_n \leq (1/(1-\gamma))a_n$ for all $n \geq n_0$. These imply that

$$(1/(1+\gamma) - 1/(1-\gamma))a_n \leq x_n - b_n \leq (1/(1-\gamma) - 1/(1+\gamma))a_n$$

for all $n \geq n_0$. Since $\gamma > 0$ is arbitrary and $a_n \leq \beta 1$ for all n, we infer that $x_n - b_n \to 0$, which immediately yields $b_n^{-1/2} x_n b_n^{-1/2} \to 1$, contrary to $\|b_n^{-1/2} x_n b_n^{-1/2} - 1\| \geq \epsilon$. This proves the above assertion, and hence (b2) and (b3) in Theorem 3 are satisfied. Observe that (b4) holds too, which can be checked just as (b1) above. By Theorem 3 there is $\delta > 0$ such that whenever $a, b \in A_{1+}^{-1}$ and $\|a - b\| < \delta$, we necessarily have

$$\phi_0(ba^{-1}b) = \phi_0(b)\phi_0(a)^{-1}\phi_0(b). \tag{2.3}$$

Now pick $a, b \in A_{1+}^{-1}$. We prove that **(2.3)** holds for a and b. To verify this, consider the curve

$$\Gamma(t) = a^{1/2}\big(\exp(t\log(a^{-1/2}ba^{-1/2}))\big)a^{1/2}, \qquad t \in [0,2],$$

connecting a and $ba^{-1}b$ and passing through b. The range of this curve is a norm-compact subset of A_{1+}^{-1}, and hence it satisfies the condition we imposed on the subset H of A_{1+}^{-1} in the previous part of the proof. Therefore, there is $\delta > 0$ such that for any $a', b' \in \Gamma([0,2])$ we have $\phi_0(b'a'^{-1}b') = \phi_0(b')\phi_0(a')^{-1}\phi_0(b')$. By the uniform continuity of Γ, for close enough $t, s \in [0,2]$ we have $\|\Gamma(t) - \Gamma(s)\| < \delta$. Now, we can select a large enough n such that the elements $a_k = \Gamma(k/2^n)$, $k = 0, 1, \dots, 2^{n+1}$, satisfy $\|a_k - a_{k+1}\| < \delta$. Clearly, $a_0 = a$, $a_{2^n} = b$, $a_{2^{n+1}} = ba^{-1}b$, and $a_{k+1}a_k^{-1}a_{k+1} = a_{k+2}$ for every $0 \le k \le 2^{n+1} - 2$. Moreover, by the closeness of a_k and a_{k+1} we have

$$\phi_0(a_{k+1}a_k^{-1}a_{k+1}) = \phi_0(a_{k+1})\phi_0(a_k)^{-1}\phi_0(a_{k+1})$$

for every $0 \le k \le 2^{n+1} - 1$. Purely algebraic computations yield

$$\phi_0(a_{2^n}a_0^{-1}a_{2^n}) = \phi_0(a_{2^n})\phi_0(a_0)^{-1}\phi_0(a_{2^n}).$$

In fact, this is just the content of [3, Lemma 4.2]. As $a_0 = a$ and $a_{2^n} = b$, we see that indeed **(2.3)** holds for all $a, b \in A_{1+}^{-1}$.

Setting $b = 1$ we deduce that $\phi_0(a^{-1}) = \phi_0(a)^{-1}$ for all $a \in A_{1+}^{-1}$, and so

$$\textbf{(2.4)} \qquad \phi_0(bab) = \phi_0(b)\phi_0(a)\phi_0(b), \qquad a, b \in A_{1+}^{-1}.$$

Using **(2.3)** and **(2.4)** one can trivially deduce that for any $a \in A_{1+}^{-1}$ we have $\phi_0(a^m) = \phi_0(a)^m$, first for any integer m and then for any rational number.

Pick $x \in A_{1s}$ and define $S : \mathbb{R} \to A_{2+}^{-1}$ by

$$S(t) = \phi_0(\exp(tx)), \qquad t \in \mathbb{R}.$$

We assert that S is continuous in the norm-topology. To see this, first observe that applying (R2) for a sequence $x_n \in A_{1+}^{-1}$ we have

$$\begin{aligned} &\|x_n - 1\| \to 0 \;\Rightarrow\; \|h_1(x_n)\| \to 0 \;\Rightarrow\; d(x_n, 1) \to 0 \;\Rightarrow \\ &\qquad \rho(\phi_0(x_n), \phi_0(1)) = \rho(\phi_0(x_n), 1) \to 0 \;\Rightarrow\; \|h_2(\phi_0(x_n))\| \to 0 \\ &\qquad\qquad \Rightarrow\; \|\phi_0(x_n) - 1\| \to 0. \end{aligned}$$

Now, picking $t, t_0 \in \mathbb{R}$ we compute

$$\begin{aligned} \|S(t+t_0) - S(t_0)\| &= \|\phi_0(\exp((t+t_0)x)) - \phi_0(\exp(t_0x))\| \\ &\le \|\phi_0(\exp(t_0x))^{1/2}\|^2 \\ &\quad \times \|\phi_0(\exp(t_0x))^{-1/2}\phi_0(\exp((t+t_0)x))\phi_0(\exp(t_0x))^{-1/2} - 1\| \\ &= \|\phi_0(\exp(t_0x))\| \left\| \phi_0\left(\exp\left(\frac{-t_0}{2}x\right)\exp((t+t_0)x)\exp\left(\frac{-t_0}{2}x\right)\right) - 1 \right\| \\ &= \|\phi_0(\exp(t_0x))\|\,\|\phi_0(\exp(tx)) - 1\|. \end{aligned}$$

It follows that as $t \to 0$ we have $S(t + t_0) \to S(t_0)$ in the norm topology, implying the norm-continuity of S.

We next deduce that S is a one-parameter group in A_{2+}^{-1}. Indeed, let m, n, m', n' be integers with $m, m' \neq 0$. We calculate

$$\begin{aligned} S\left(\frac{n}{m} + \frac{n'}{m'}\right) &= \phi_0\left(\exp\left(\left(\frac{n}{m} + \frac{n'}{m'}\right)x\right)\right) = \phi_0\left(\exp\frac{1}{mm'}x\right)^{m'n+mn'} \\ &= \phi_0\left(\exp\frac{1}{mm'}x\right)^{m'n}\phi_0\left(\exp\frac{1}{mm'}x\right)^{mn'} = S\left(\frac{n}{m}\right)S\left(\frac{n'}{m'}\right). \end{aligned}$$

Since S is continuous, it follows that

$$S(t + t') = S(t)S(t'), \qquad t, t' \in \mathbb{R}.$$

Therefore, S is a continuous one-parameter group in A_2.

By [10, Proposition 6.4.6(a)] there exists $y \in A_2$ with

$$S(t) = \exp(ty), \qquad t \in \mathbb{R}.$$

Since $S(t)$ is self-adjoint, so is y (use, e.g., [10, Proposition 6.4.6(c)]).

Defining $f(x) = y$ we obtain a map $f : A_{1s} \to A_{2s}$ for which

$$\phi_0(\exp(tx)) = S(t) = \exp(tf(x)), \qquad t \in \mathbb{R},\ x \in A_{1s}.$$

As ϕ_0 preserves or more precisely respects the pair d, ρ of generalized distance measures, it is clearly injective. This implies that so is f. Considering ϕ_0^{-1} in place of ϕ_0, we clearly have an injective map $g : A_{2s} \to A_{1s}$ such that $\phi_0^{-1}(\exp(ty)) = \exp(tg(y))$ for all $y \in A_{2s}$ and $t \in \mathbb{R}$. This easily implies that $y = f(g(y))$ for all $y \in A_{2s}$. Hence f is surjective and therefore it is a bijection from A_{1s} onto A_{2s}. Note that $f(0) = 0$ by the definition of f.

Our next claim is that f is a scalar multiple of a norm-isometry. To verify this, we assert that as $t \to 0$,

$$\textbf{(2.5)} \qquad \frac{d(\exp(tx), \exp(ty))}{|t|} \to |h_1'(1)|\, \|x - y\|$$

for all $x, y \in A_{1s}$. Clearly,

$$\begin{aligned} &\frac{\exp\left(-\frac{t}{2}y\right)\exp(tx)\exp\left(-\frac{t}{2}y\right) - 1}{t} \\ &\quad = \exp\left(-\frac{t}{2}y\right)\frac{(\exp(tx) - 1) - (\exp(ty) - 1)}{t}\exp\left(-\frac{t}{2}y\right) \to x - y. \end{aligned}$$

Since

$$\frac{d(\exp(tx), \exp(ty))}{|t|} = \frac{\left\|h_1\left(\exp\left(-\frac{t}{2}y\right)\exp(tx)\exp\left(-\frac{t}{2}y\right)\right) - h_1(1)\right\|}{|t|},$$

(2.5) follows from (R3) and the chain rule. Similarly,

$$\frac{\rho(\exp(tx), \exp(ty))}{|t|} \to |h_2'(1)| \, \|x - y\|$$

for all $x, y \in A_{2s}$. Since ϕ_0 respects the pair d, ρ of generalized distance measures, it follows that $|h_1'(1)| \, \|x - y\| = |h_2'(1)| \, \|f(x) - f(y)\|$ for all x, y in A_{1s}. This implies that there is $c > 0$ such that $(1/c)f$ is an isometry from A_{1s} onto A_{2s}. Since $f(0) = 0$, by the classical Mazur–Ulam theorem we infer that f is linear. The structure of linear isometries between the self-adjoint parts of C^*-algebras is well-known: According to a theorem of Kadison [6, Theorem 2], $(1/c)f(1)$ is a central symmetry in A_2 and there is a Jordan *-isomorphism J from A_1 onto A_2 such that

$$f(x) = f(1)J(x), \qquad x \in A_{1s}.$$

Set $p = (1 + (1/c)f(1))/2$. Then p is a central projection in A_2 and

$$f(x) = c(pJ(x) - (1 - p)J(x)), \qquad x \in A_{1s}.$$

We now calculate

$$\begin{aligned}
\phi_0(\exp x) &= \exp(c(pJ(x) - (1 - p)J(x))) \\
&= \sum_{n=0}^{\infty} \frac{(c(pJ(x) - (1 - p)J(x)))^n}{n!} = \sum_{n=0}^{\infty} \frac{pJ((cx)^n) + (1 - p)J((-cx)^n)}{n!} \\
&= pJ(\exp(cx)) + (1 - p)J(\exp(-cx)) = pJ(\exp x)^c + (1 - p)J(\exp x)^{-c}
\end{aligned}$$

for every $x \in A_{1s}$. Thus

$$\textbf{(2.6)} \qquad \phi_0(a) = pJ(a)^c + (1 - p)J(a)^{-c}, \qquad a \in A_{1+}^{-1},$$

and we arrive at the desired conclusion. ∎

Now, the first main result of the paper, which yields several characterizations of Jordan *-isomorphisms, reads as follows.

THEOREM 5. *Let A_j be a C^*-algebra for $j = 1, 2$. Suppose that ϕ is a surjection from A_{1+}^{-1} onto A_{2+}^{-1}. Consider the following statements:*

(5.1) $\sigma(ab^{-1}) = \sigma(\phi(a)\phi(b)^{-1})$ *for all* $a, b \in A_{1+}^{-1}$*;*
(5.2) $r(ab^{-1} - 1) = r(\phi(a)\phi(b)^{-1} - 1)$ *for all* $a, b \in A_{1+}^{-1}$*;*
(5.3) *there exist continuous functions* $h_1, h_2 :]0, \infty[\to \mathbb{R}$ *which satisfy* (c1)–(c3) *and*

$$\|h_1(b^{-1/2}ab^{-1/2})\| = \|h_2(\phi(b)^{-1/2}\phi(a)\phi(b)^{-1/2})\|, \qquad a, b \in A_{1+}^{-1};$$

(5.4) *there exists a Jordan *-isomorphism J from A_1 onto A_2, an element $b_0 \in A_{2+}^{-1}$, a central projection $p \in A_2$ and $c > 0$ such that*

$$\phi(a) = b_0(pJ(a)^c + (1 - p)J(a)^{-c})b_0, \qquad a \in A_{1+}^{-1};$$

(5.5) *there exists a Jordan *-isomorphism J from A_1 onto A_2, an element $b_0 \in A_{2+}^{-1}$ and a central projection $p \in \mathcal{A}_2$ such that*

$$\phi(a) = b_0(pJ(a) + (1-p)J(a)^{-1})b_0, \qquad a \in A_{1+}^{-1};$$

(5.6) *there exists a Jordan *-isomorphism J from A_1 onto A_2 and an element $b_0 \in A_{2+}^{-1}$ such that*

$$\phi(a) = b_0 J(a) b_0, \qquad a \in A_{1+}^{-1}.$$

Then (5.1)⇒(5.2)⇒(5.3)⇒(5.4). *If $h_1 = h_2$, then* (5.3)⇒(5.5). *If $h_1 = h_2$ and they satisfy* (c4), *then* (5.1)⇔(5.2)⇔(5.3)⇔(5.6).

Proof. It is obvious that (5.1) implies (5.2). To verify (5.2)⇒(5.3) observe that for any $a, b \in A_{1+}^{-1}$ we have

$$\sigma(ab^{-1} - 1) = \sigma(ab^{-1}) - 1 = \sigma(b^{-1/2}ab^{-1/2}) - 1 = \sigma(b^{-1/2}ab^{-1/2} - 1),$$

which implies that

$$r(ab^{-1} - 1) = r(b^{-1/2}ab^{-1/2} - 1) = \|b^{-1/2}ab^{-1/2} - 1\|.$$

Therefore, assuming (5.2) and defining $h_1(t) = h_2(t) = t - 1$, $t \in \,]0, \infty[$, we plainly obtain (5.3).

Proposition 4 ensures that (5.3)⇒(5.4). Observe further that if we assume $h_1 = h_2$ and the central projection p above is non-trivial, then inserting $a = t1$, $t \in \,]0, \infty[$ and $b = 1$ into **(2.6)**, and using the generalized distance measure preserving property of ϕ_0, we easily obtain

$$|h_1(t)| = \max\{|h_1(t^c)|, |h_1(t^{-c})|\}$$

for all $t > 0$. Hence $|h_1(t)| = |h_1(t^{-1})|$ and $|h_1(t)| = |h_1(t^c)|$, for all $t \in \,]0, \infty[$. Differentiating h_1 at $t = 1$ we easily see that $c = 1$. Therefore, if p is non-trivial, we have $c = 1$. A similar argument applies when p is trivial, i.e., when $|h_1(t)| = |h_1(t^c)|$ or $|h_1(t)| = |h_1(t^{-c})|$, for all $t \in \,]0, \infty[$. This gives (5.3)⇒(5.5) under the assumption that $h_1 = h_2$.

If $h_1 = h_2$ and $|h_1(t_0)| \neq |h_1(t_0^{-1})|$ for some $t_0 \in \,]0, \infty[$, then going through the last part of the argument above, we see that p is necessarily trivial, in fact $p = 1$, and $c = 1$, proving (5.3)⇒(5.6).

To complete the proof, suppose now that (5.6) holds. For any $a, b \in A_{1+}^{-1}$ we infer that

$$\begin{aligned}\sigma(\phi(a)\phi(b)^{-1}) &= \sigma(b_0 J(a) J(b)^{-1} b_0^{-1}) = \sigma(b_0 J(a) J(b^{-1}) b_0^{-1}) \\ &= \sigma(J(a)J(b^{-1})) = \sigma(ab^{-1}),\end{aligned}$$

and hence we obtain (5.1). ■

Observe that the implication (5.3)⇒(5.5) gives a substantial generalization of our former result [5, Theorem 9] about the structure of Thompson isometries between the positive cones of C^*-algebras. Indeed, one has only to choose $h_1 = h_2 = \log$ to obtain that result from Theorem 5.

We also remark that in [8] we have presented structural results for surjective maps between the positive cones of factor von Neumann algebras which respect a pair of generalized distance measures of the form similar to what appears in (5.3) above, with the difference that in [8] we have considered arbitrary unitarily invariant norms in place of the unique C^*-algebra norm $\|\cdot\|$ (operator norm). So in a sense those results concern more general distance measures, but in a more restricted context. Indeed, due to the (mainly algebraic) tools we have applied there, the results [8] have been obtained only for factor von Neumann algebras and not, like here, for general C^*-algebras. Related results in the context of matrix algebras appeared in [9].

Now we present several sorts of extensions of our theorem.

Corollary 6. *Let A_j be a C^*-algebra for $j = 1, 2$ and suppose that ϕ and ψ are surjections from A_{1+}^{-1} onto A_{2+}^{-1}. Then the following assertions are equivalent:*

(6.1) $\sigma(ab) = \sigma(\phi(a)\psi(b))$ *for all* $a, b \in A_{1+}^{-1}$;

(6.2) $r(ab - 1) = r(\phi(a)\psi(b) - 1)$ *for all* $a, b \in A_{1+}^{-1}$;

(6.3) *there is a continuous function* $h :]0, \infty[\to \mathbb{R}$ *which satisfies conditions* (c1)–(c4) *and*

$$\|h(b^{1/2}ab^{1/2})\| = \|h(\psi(b)^{1/2}\phi(a)\psi(b)^{1/2})\|, \qquad a, b \in A_{1+}^{-1};$$

(6.4) *there exists a Jordan* **-isomorphism J from A_1 onto A_2 and an element $b_0 \in A_{2+}^{-1}$ such that*

$$\phi(a) = b_0 J(a) b_0, \qquad \psi(a) = b_0^{-1} J(a) b_0^{-1}, \qquad a \in A_{1+}^{-1}.$$

Proof. The implication (6.1)⇒(6.2) is obvious, and to see (6.2)⇒(6.3) set $h(t) = t - 1$ for $t \in]0, \infty[$.

Suppose that (6.3) holds. For any $a \in A_{1+}^{-1}$ we have

$$0 = \|h(a^{-1/2}aa^{-1/2})\| = \|h(\psi(a^{-1})^{1/2}\phi(a)\psi(a^{-1})^{1/2})\|,$$

which implies $\psi(a^{-1})^{1/2}\phi(a)\psi(a^{-1})^{1/2} = 1$, i.e., $\phi(a) = \psi(a^{-1})^{-1}$. Then

$$\begin{aligned}\|h(b^{-1/2}ab^{-1/2})\| &= \|h(\psi(b^{-1})^{1/2}\phi(a)\psi(b^{-1})^{1/2})\| \\ &= \|h(\phi(b)^{-1/2}\phi(a)\phi(b)^{-1/2})\|.\end{aligned}$$

Theorem 5 shows that there is a Jordan *-isomorphism J from A_1 onto A_2 and $b_0 \in A_{2+}^{-1}$ such that

$$\phi(a) = b_0 J(a) b_0, \qquad a \in A_{1+}^{-1}.$$

Moreover,

$$\psi(a) = \phi(a^{-1})^{-1} = b_0^{-1} J(a) b_0^{-1}, \qquad a \in A_{1+}^{-1},$$

and we obtain (6.4).

Suppose now that (6.4) holds. For any $a, b \in A_{1+}^{-1}$ we calculate

$$\sigma(\phi(a)\psi(b)) = \sigma(b_0 J(a)J(b)b_0^{-1}) = \sigma(J(a)J(b)) = \sigma(ab).$$

Thus (6.1) holds, and the proof is complete. ■

From the above statement we immediately obtain the following corollary which gives a complete description of spectrally multiplicative maps between the positive cones of C^*-algebras.

Corollary 7. *Let A_j be a C^*-algebra for $j = 1, 2$. Suppose that ϕ is a surjection from A_{1+}^{-1} onto A_{2+}^{-1}. Then the following statements are equivalent:*

(7.1) $\sigma(ab) = \sigma(\phi(a)\phi(b))$ *for all* $a, b \in A_{1+}^{-1}$*;*
(7.2) $r(ab - 1) = r(\phi(a)\phi(b) - 1)$ *for all* $a, b \in A_{1+}^{-1}$*;*
(7.3) *there is a continuous function* $h :]0, \infty[\to \mathbb{R}$ *which satisfies conditions* (c1)–(c4) *and*

$$\|h(b^{1/2}ab^{1/2})\| = \|h(\phi(b)^{1/2}\phi(a)\phi(b)^{1/2})\|, \qquad a, b \in A_{1+}^{-1};$$

(7.4) *there exists a Jordan *-isomorphism J from A_1 onto A_2 such that*

$$\phi(a) = J(a), \qquad a \in A_{1+}^{-1}.$$

Proof. In the light of the previous proofs, we only need to verify (7.3)⇒(7.4). Assuming (7.3), by Corollary 6 there exists a Jordan *-isomorphism J from A_1 onto A_2 and $b_0 \in A_{2+}^{-1}$ such that

$$\phi(a) = b_0 J(a) b_0, \qquad \phi(a) = b_0^{-1} J(a) b_0^{-1}, \qquad a \in A_{1+}^{-1}.$$

Choosing $a = 1$ yields $b_0^2 = b_0^{-2}$, which implies $b_0 = 1$ and we are done. ■

With some extra effort, from Corollary 6 we can deduce the following formally more general result on the structure of maps defined on arbitrary sets with values in positive cones of C^*-algebras with a specific property closely related to spectral multiplicativity.

Theorem 8. *Let A_j be a C^*-algebra for $j = 1, 2$ and F a non-empty set. Suppose that Φ_1 and Ψ_1 are surjections from F onto A_{1+}^{-1} and that Φ_2 and Ψ_2 are surjections from F onto A_{2+}^{-1}. The following statements are equivalent:*

(8.1) $\sigma(\Phi_1(x)\Psi_1(y)) = \sigma(\Phi_2(x)\Psi_2(y))$ *for all* $x, y \in F$*;*
(8.2) $r(\Phi_1(x)\Psi_1(y) - 1) = r(\Phi_2(x)\Psi_2(y) - 1)$ *for all* $x, y \in F$*;*
(8.3) *there is a continuous function* $h :]0, \infty[\to \mathbb{R}$ *which satisfies conditions* (c1)–(c4) *and*

$$\|h(\Psi_1(y)^{1/2}\Phi_1(x)\Psi_1(y)^{1/2})\| = \|h(\Psi_2(y)^{1/2}\Phi_2(x)\Psi_2(y)^{1/2})\|, \quad x, y \in F;$$

(8.4) *there exists a Jordan *-isomorphism J from A_1 onto A_2 and an element $b_0 \in A_{2+}^{-1}$ such that*

$$\Phi_2(x) = b_0 J(\Phi_1(x)) b_0, \qquad \Psi_2(x) = b_0^{-1} J(\Psi_1(x)) b_0^{-1}, \qquad x, y \in F.$$

Proof. Again, by the previous proofs the implications (8.1)⇒(8.2)⇒(8.3) are clear.

Suppose that (8.3) holds. To prove (8.4), we first observe that $\Phi_1(x) = \Phi_1(x')$ implies $\Phi_2(x)=\Phi_2(x')$. Indeed, let $x, x' \in F$ and assume $\Phi_1(x)=\Phi_1(x')$. Since $\Psi_1(F) = A_{1+}^{-1}$, there exists $y \in F$ with $\Psi_1(y) = \Phi_1(x)^{-1}$. Then

$$0 = \|h(\Psi_1(y)^{1/2}\Phi_1(x)\Psi_1(y)^{1/2})\| = \|h(\Psi_2(y)^{1/2}\Phi_2(x)\Psi_2(y)^{1/2})\|,$$

implying $\Psi_2(y)^{1/2}\Phi_2(x)\Psi_2(y)^{1/2} = 1$. Thus $\Psi_2(y)^{-1} = \Phi_2(x)$. In a similar way we obtain $\Psi_2(y)^{-1} = \Phi_2(x')$. Hence $\Phi_2(x) = \Phi_2(x')$. In the same way one can deduce that $\Psi_1(x) = \Psi_1(x')$ implies $\Psi_2(x) = \Psi_2(x')$. Now we define $\phi, \psi : A_{1+}^{-1} \to A_{2+}^{-1}$ by $\phi(\Phi_1(x)) = \Phi_2(x)$ and $\psi(\Psi_1(x)) = \Psi_2(x)$ for $x \in F$. Clearly, ϕ, ψ are well defined and surjective. Rewriting the equality in (8.3) we have

$$\|h(b^{1/2}ab^{1/2})\| = \|h(\psi(b)^{1/2}\phi(a)\psi(b)^{1/2})\|, \quad a, b \in A_{1+}^{-1}.$$

By Corollary 6 there exists a Jordan *-isomorphism J from A_1 onto A_2 and an element $b_0 \in A_{2+}^{-1}$ such that

$$\phi(a) = b_0 J(a) b_0, \quad \psi(a) = b_0^{-1} J(a) b_0^{-1}, \quad a \in A_{1+}^{-1}.$$

In other words,

$$\Phi_2(x) = b_0 J(\Phi_1(x)) b_0, \quad \Psi_2(x) = b_0^{-1} J(\Psi_1(x)) b_0^{-1}, \quad x \in F,$$

and this proves (8.4).

Finally, in a way similar to the proof of Corollary 6 one can check that (8.4) implies (8.1), which finishes the proof of the theorem. ∎

We conclude this section with a few other corollaries which provide characterizations of Jordan *-isomorphisms of the self-adjoint parts of C^*-algebras by means of spectral-multiplicativity-type properties.

COROLLARY 9. *Let A_j be a C^*-algebra for $j = 1, 2$. Suppose that f and g are surjections from A_{1s} onto A_{2s}. Then the following assertions are equivalent:*

(9.1) $\sigma(\exp x \exp y) = \sigma(\exp f(x) \exp g(y))$ *for all* $x, y \in A_{1s}$*;*
(9.2) $r(\exp x \exp y - 1) = r(\exp f(x) \exp g(y) - 1)$ *for all* $x, y \in A_{1s}$*;*
(9.3) *there is a continuous function* $h :]0, \infty[\to \mathbb{R}$ *which satisfies conditions* (c1)–(c4) *and*

$$\|h(\exp(y/2)\exp(x)\exp(y/2))\| = \|h(\exp(g(y)/2)\exp(f(x))\exp(g(y)/2))\|, \quad x, y \in A_{1s};$$

(9.4) *there exists a Jordan *-isomorphism J from A_{1s} onto A_{2s} and an element $b_0 \in A_{2+}^{-1}$ such that*

$$\exp f(x) = b_0(\exp J(x)) b_0, \quad \exp g(x) = b_0^{-1}(\exp J(x)) b_0^{-1}, \quad x, y \in A_{1s}.$$

Moreover, in any of the above cases, if $f(0) = 0$, then $f = g = J$ on A_{1s}.

Proof. Define $\phi(a) = \exp(f(\log a))$ for $a \in A_{1+}^{-1}$ and $\psi(b) = \exp(g(\log b))$ for $b \in A_{1+}^{-1}$. Apply Corollary 6 to see the equivalence of (9.1)–(9.4). If $f(0) = 0$, we easily obtain $b_0 = 1$, which implies $f = g = J$ on A_{1s}. ∎

If $f = g$ in the previous corollary, we trivially obtain the following statement.

COROLLARY 10. *Let A_j be a C^*-algebra for $j = 1, 2$. Suppose that f is a surjection from A_{1s} onto A_{2s}. Then the following assertions are equivalent:*

(10.1) $\sigma(\exp x \exp y) = \sigma(\exp f(x) \exp f(y))$ *for all* $x, y \in A_{1s}$;
(10.2) $r(\exp x \exp y - 1) = r(\exp f(x) \exp f(y) - 1)$ *for all* $x, y \in A_{1s}$;
(10.3) *there is a continuous function* $h :]0, \infty[\to \mathbb{R}$ *which satisfies conditions* (c1)–(c4) *and*

$$\|h(\exp(y/2)\exp(x)\exp(y/2))\| = \|h(\exp(f(y)/2)\exp(f(x))\exp(f(y)/2))\|, \quad x, y \in A_{1s};$$

(10.4) *there exists a Jordan *-isomorphism J from A_{1s} onto A_{2s} such that $f = J$ on A_{1s}.*

3. The case of unitary groups. In the last part of our paper we present spectral conditions for Jordan *-isomorphisms between the unitary groups of von Neumann algebras. In the proof of our second main result, Theorem 12 below, we apply a general Mazur–Ulam type result concerning groups. It appeared in [8, Proposition 20] (cf. [3, Corollary 3.9]).

THEOREM 11. *Suppose that G and H are groups equipped with generalized distance measures d and ρ, respectively. Pick $a, b \in G$, set*

$$L_{a,b} = \{x \in G : d(a, x) = d(x, ba^{-1}b) = d(a, b)\},$$

and assume the following:

(d1) $d(bx^{-1}b, bx'^{-1}b) = d(x', x)$ *for all* $x, x' \in G$;
(d2) $\sup\{d(x, b) : x \in L_{a,b}\} < \infty$;
(d3) *there exists $K > 1$ such that*

$$d(x, bx^{-1}b) \geq K d(x, b), \quad x \in L_{a,b};$$

(d4) $\rho(cy^{-1}c', cy'^{-1}c') = \rho(y', y)$ *for all* $c, c', y, y' \in H$.

Then for any surjective map $\phi : G \to H$ which satisfies

$$\rho(\phi(x), \phi(x')) = d(x, x'), \quad x, x' \in G,$$

we have

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

One may ask if the above statement can be deduced from Theorem 3. The easy answer is "no", since the natural operation $ab^{-1}a$, called the inverted

Jordan triple product in a group does not generally satisfy the uniqueness part of the condition in (a3).

Analogously to Section 2, below we shall consider generalized distance measures on unitary groups obtained from continuous functions defined on the unit circle $\mathbb{T}$.

For any continuous function $h : \mathbb{T} \to \mathbb{C}$ we shall consider the following properties:

(e1) $h(z) = 0$ if and only if $z = 1$;
(e2) h is differentiable at $z = 1$, meaning that the limit $\lim_{z\to 1} \frac{h(z)-h(1)}{z-1}$ exists, and we assume that it is non-zero.

Observe that just as in (R4) one can prove that conditions (e1)–(e2) imply that for any $0 \le K < 2$ we have $|h(z^2)| \ge K|h(z)|$ for all $z \in \mathbb{T}$ close enough to 1.

The second main result of the paper reads as follows.

Theorem 12. *Let M_j be a von Neumann algebra with unitary group U_j for $j = 1, 2$. Suppose that ϕ is a surjection from U_1 onto U_2. The following conditions are equivalent:*

(12.1) $\sigma(ab^{-1}) = \sigma(\phi(a)\phi(b)^{-1})$ *for all* $a, b \in U_1$;
(12.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and an element $u_0 \in U_2$ such that*

$$\phi(a) = u_0 J(a), \qquad a \in U_1.$$

Moreover, the following conditions are also equivalent:

(12.3) $r(ab^{-1} - 1) = r(\phi(a)\phi(b)^{-1} - 1)$ *for all* $a, b \in U_1$;
(12.4) *there exist continuous functions $h_1, h_2 : \mathbb{T} \to \mathbb{C}$ which satisfy conditions* (e1)–(e2) *and*

$$\|h_1(ab^{-1})\| = \|h_2(\phi(a)\phi(b)^{-1})\|, \qquad a, b \in U_1;$$

(12.5) *there exists a Jordan *-isomorphism J from M_1 onto M_2, an element $u_0 \in U_2$ and a central projection $p \in A_2$ such that*

$$\phi(a) = u_0(pJ(a) + (1-p)J(a)^{-1}), \qquad a \in U_1.$$

Proof. We begin with the second part of the theorem. To see that (12.3)⇒(12.4) set $h_1(z) = h_2(z) = z - 1$ for $z \in \mathbb{T}$.

In the next part of the proof we follow the proof of Proposition 4 rather closely. Assume that (12.4) holds with continuous functions $h_1, h_2 : \mathbb{T} \to \mathbb{C}$ satisfying (e1)–(e2). Define

$$d(a, b) = \|h_1(ab^{-1})\|, \quad a, b \in U_1,$$
$$\rho(a, b) = \|h_2(ab^{-1})\|, \quad a, b \in U_2.$$

By (e1), d, ρ are generalized distance measures and

$$\rho(\phi(a), \phi(b)) = d(a, b), \qquad a, b \in U_1. \tag{3.1}$$

One can easily check that $d(zaw, zbw) = d(a, b)$ and

$$d(bx^{-1}b, bx'^{-1}b) = d(x^{-1}, x'^{-1}) = d(x', x) \tag{3.2}$$

for all $a, b, x, z, w \in U_1$. Clearly, similar properties hold for ρ.

Define $\phi_0 : U_1 \to U_2$ by $\phi_0(a) = \phi(1)^{-1}\phi(a)$ for $a \in U_1$. Plainly, ϕ_0 is a well defined map from U_1 onto U_2, it is unital in the sense that $\phi_0(1) = 1$, and **(3.1)** also holds for ϕ_0, i.e.,

$$\rho(\phi_0(a), \phi_0(b)) = d(a, b), \qquad a, b \in U_1. \tag{3.3}$$

We are going to apply Theorem 11 for $G = U_1, H = U_2$, for the above defined distance measures d, ρ and for the surjective map ϕ_0. We have seen in **(3.2)** that conditions (d1), (d4) of Theorem 11 are satisfied. Condition (d2) also holds by the boundedness of h_1. Now we show that (d3) is satisfied for $a, b \in U_1$ close enough to each other in norm. To see this, we shall need the following simple observation: for any sequences a_n, b_n in U_1 we have

$$\|a_n - b_n\| = \|a_n b_n^{-1} - 1\| \to 0 \;\Leftrightarrow\; \|h_1(a_n b_n^{-1})\| = d(a_n, b_n) \to 0$$

and a similar observation holds for ρ as well. In fact, this follows easily from the continuity of h_1 and property (e1). In particular, "convergence" in any of the generalized distance measures d, ρ is equivalent to convergence in the norm $\|\cdot\|$.

In order to show that condition (d3) holds for $a, b \in U_1$ close enough, assume on the contrary that we have sequences $a_n, b_n \in U_1$ and $x_n \in L_{a_n, b_n}$ such that $\|a_n - b_n\| \to 0$ and

$$d(x_n, b_n x_n^{-1} b_n) < \tfrac{3}{2} d(x_n, b_n)$$

for all n. This last inequality means that

$$\|h_1((x_n b_n^{-1})^2)\| < \tfrac{3}{2}\|h_1(x_n b_n^{-1})\|$$

for all n. Since $d(a_n, x_n) = d(a_n, b_n) \to 0$, we have $a_n x_n^{-1}, a_n b_n^{-1} \to 1$ in norm, which apparently implies that $x_n b_n^{-1} \to 1$ in norm. On the other hand, $|h_1(z^2)| \geq \frac{3}{2}|h_1(z)|$ for all $z \in \mathbb{T}$ close enough to 1. Therefore, $\|h_1((x_n b_n^{-1})^2)\| \geq \frac{3}{2}\|h_1(x_n b_n^{-1})\|$ for large enough n, a contradiction. This shows that (d3) is satisfied for $a, b \in U_1$ close enough. Applying Theorem 11 it follows that there is $\delta > 0$ such that for any $a, b \in U_1$ with $\|a - b\| < \delta$, we have

$$\phi_0(ba^{-1}b) = \phi_0(b)\phi_0(a)^{-1}\phi_0(b).$$

Just as in the first part of the proof of [5, Theorem 1] we then deduce that

$$\phi_0(ba^{-1}b) = \phi_0(b)\phi_0(a)^{-1}\phi_0(b)$$

holds not only locally, but also globally, i.e., for any $a, b \in U_1$. Setting $b = 1$ we get $\phi_0(a^{-1}) = \phi_0(a)^{-1}$ for every $a \in U_1$, and so

$$\phi_0(bab) = \phi_0(b)\phi_0(a)\phi_0(b), \qquad a, b \in U_1. \tag{3.4}$$

By the equivalence of convergence in d, ρ and in norm we deduce that ϕ_0 is norm-continuous. Therefore, following [5, proof of Theorem 1, pp. 160–161] employing one-parameter unitary groups, we infer that there is a bijective map $f : M_{1s} \to M_{2s}$ with $f(0) = 0$ for which

$$\phi_0(\exp(itx)) = \exp(itf(x)), \qquad t \in \mathbb{R},\ x \in M_{1s}.$$

Just as in the proof of Theorem 5 we claim that f is a scalar multiple of a norm-isometry. To verify this, one can prove similarly to **(2.5)** that

$$\frac{d(\exp(itx), \exp(ity))}{|t|} \to |h_1'(1)|\, \|x - y\|$$

for all $x, y \in M_{1s}$ as $t \to 0$. We omit the details. Similarly,

$$\frac{\rho(\exp(itx), \exp(ity))}{|t|} \to |h_2'(1)|\, \|x - y\|$$

for all $x, y \in M_{2s}$ as $t \to 0$. Since ϕ_0 respects the pair d, ρ of generalized distance measures, i.e., satisfies **(3.3)**, it follows that $|h_1'(1)|\, \|x - y\| = |h_2'(1)|\, \|f(x) - f(y)\|$ for all $x, y \in M_{1s}$. This implies that there is a positive scalar c such that $(1/c)f$ is an isometry from M_{1s} onto M_{2s}. Just as in the proof of Theorem 5, since $f(0) = 0$, by the Mazur–Ulam theorem we infer that f is linear and next apply Kadison's theorem [6, Theorem 2] to conclude that $(1/c)f(1)$ is a central symmetry in M_2 and there is a Jordan *-isomorphism J from M_1 onto M_2 such that

$$f(x) = f(1)J(x), \qquad x \in M_{1s}.$$

Set $p = (1 + (1/c)f(1))/2$. Then p is a central projection in M_2 and

$$f(x) = c\big(pJ(x) - (1 - p)J(x)\big), \qquad x \in M_{1s}.$$

Next an easy calculation yields

$$\begin{aligned} \phi_0(\exp ix) &= \exp\big(c(pJ(ix) - (1 - p)J(ix))\big) \\ &= pJ(\exp(icx)) + (1 - p)J(\exp(-icx)) \end{aligned} \tag{3.5}$$

for every $x \in M_{1s}$. We assert that c is necessarily an integer. Indeed, since ϕ_0 is unital and satisfies **(3.4)**, it follows that ϕ_0 sends symmetries to symmetries. Therefore, for any non-zero projection q in M_1, the spectrum of

$$\phi_0(\exp i\pi q) = pJ(\exp(ic\pi q)) + (1 - p)J(\exp(-ic\pi q))$$

is contained in $\{-1, 1\}$. Since J preserves the spectrum and p is central, it follows that at least one of the sets $\{1, \exp(ic\pi)\}$, $\{1, \exp(-ic\pi)\}$ (p might be trivial) is contained in $\{-1, 1\}$. This implies that c is an integer; recall that it is also assumed to be positive. Therefore, by **(3.5)**,

$$\phi_0(a) = pJ(a^c) + (1-p)J(a^{-c}), \qquad a \in U_1. \tag{3.6}$$

Now we prove that $c = 1$. Indeed, assuming that the central projection p above is non-trivial, inserting the scalars $a = z1$, $z \in \mathbb{T}$ and $a = 1$ into **(3.6)** respectively, and using the generalized distance measure preserving property of ϕ_0, we easily obtain

$$|h_1(z)| = \max\{|h_2(z^c)|, |h_2(z^{-c})|\}$$

for all $z \in \mathbb{T}$. Since h_1, h_2 each have a unique root at $z = 1$, we infer that c must be 1. A similar argument works when p is trivial. This completes the proof of (12.4)⇒(12.5).

Assume now that (12.5) holds. We compute

$$\begin{aligned} r(\phi(a)\phi(b)^{-1} - 1) &= \|\phi(a)\phi(b)^{-1} - 1\| \\ &= \|u_0(pJ(a)J(b)^{-1} + (1-p)J(a)^{-1}J(b))u_0^{-1} - 1\| \\ &= \|pJ(a)J(b)^{-1} + (1-p)J(a)^{-1}J(b) - 1\| \\ &= \max\{\|p(J(a)J(b)^{-1} - 1)\|, \|(1-p)(J(a)^{-1}J(b) - 1)\|\}. \end{aligned} \tag{3.7}$$

Furthermore, by taking adjoints we get

$$\begin{aligned} \|(1-p)(J(a)^{-1}J(b) - 1)\| &= \|(1-p)(J(b)^{-1}J(a) - 1)\| \\ = \|J(b)^{-1}(1-p)(J(a) - J(b))\| &= \|(1-p)(J(a) - J(b))J(b)^{-1}\| \\ &= \|(1-p)(J(a)J(b)^{-1} - 1)\| \end{aligned} \tag{3.8}$$

since $1 - p$ commutes with every element in M_2. It follows from **(3.7)** and **(3.8)** that

$$\begin{aligned} r(\phi(a)\phi(b)^{-1} - 1) &= \max\{\|p(J(a)J(b)^{-1} - 1)\|, \|(1-p)(J(a)J(b)^{-1} - 1)\|\} \\ &= \|p(J(a)J(b)^{-1} - 1) + (1-p)(J(a)J(b)^{-1} - 1)\| = \|J(a)J(b)^{-1} - 1\| \\ &= r(J(a)J(b^{-1}) - 1) = r(ab^{-1} - 1). \end{aligned}$$

The last equality follows from the spectral multiplicativity of J. Thus we obtain (12.3).

Let us now consider the first part of the theorem. Assume (12.1) holds. It trivially implies (12.3), which implies (12.5). Consequently, there exists a Jordan *-isomorphism J from M_1 onto M_2, an element $u_0 \in U_2$ and a central projection $p \in A_2$ such that

$$\phi(a) = u_0\big(pJ(a) + (1-p)J(a)^{-1}\big), \qquad a \in U_1.$$

It is not hard to verify that p must be the identity, which yields (12.2). Since (12.2)⇒(12.1) is trivial, the proof is complete. ∎

COROLLARY 13. *Let M_j be a von Neumann algebra with unitary group U_j for $j = 1, 2$. Suppose that ϕ and ψ are surjections from U_1 onto U_2. Then the following conditions are equivalent:*

(13.1) *$\sigma(ab) = \sigma(\phi(a)\psi(b))$ for all $a, b \in U_1$;*
(13.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and an element $u_0 \in U_2$ such that*

$$\phi(a) = u_0 J(a), \psi(a) = J(a)u_0^{-1}, \quad a \in U_1.$$

Moreover the following conditions are also equivalent:

(13.3) *$r(ab - 1) = r(\phi(a)\psi(b) - 1)$ for all $a, b \in U_1$;*
(13.4) *there exist continuous functions $h_1, h_2 : \mathbb{T} \to \mathbb{C}$ which satisfy conditions* (e1)–(e2) *and*

$$\|h_1(ab)\| = \|h_2(\phi(a)\psi(b))\|, \quad a, b \in U_1;$$

(13.5) *there exists a Jordan *-isomorphism J from M_1 onto M_2, a central projection $p \in M_2$, and $u_0 \in U_2$ such that*

$$\phi(a) = u_0(pJ(a) + (1-p)J(a)^{-1}),\ \psi(a) = (pJ(a) + (1-p)J(a)^{-1})u_0^{-1},\ a \in U_1.$$

Proof. Setting $b = a^{-1}$, from both (13.1) and (13.4) we obtain $\psi(a^{-1}) = \phi(a)^{-1}$. An easy application of Theorem 12 gives (13.1)⇒(13.2) and (13.4)⇒(13.5). The rest of the proof is similar to previous arguments. For example, (13.5)⇒(13.3) can be proved by a reasoning similar to the one for (12.5)⇒(12.3). We omit the details. ■

COROLLARY 14. *Let M_j be a von Neumann algebra with unitary group U_j for $j = 1, 2$. Suppose that ϕ is a surjection from U_1 onto U_2. Then the following conditions are equivalent:*

(14.1) *$\sigma(ab) = \sigma(\phi(a)\phi(b))$ for all $a, b \in U_1$;*
(14.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and a central symmetry $u_0 \in U_2$ such that*

$$\phi(a) = u_0 J(a), \quad a \in U_1.$$

Moreover the following conditions are also equivalent:

(14.3) *$r(ab - 1) = r(\phi(a)\phi(b) - 1)$ for all $a, b \in U_1$;*
(14.4) *there exist continuous functions $h_1, h_2 : \mathbb{T} \to \mathbb{C}$ which satisfy conditions* (e1)–(e2) *and*

$$\|h_1(ab)\| = \|h_2(\phi(a)\phi(b))\|, \quad a, b \in U_1;$$

(14.5) *there exists a Jordan *-isomorphism J from M_1 onto M_2, a central projection $p \in M_2$, and a central symmetry $u_0 \in U_2$ such that*

$$\phi(a) = u_0(pJ(a) + (1-p)J(a)^{-1}), \quad a \in U_1.$$

Proof. We apply Theorem 13 for $\psi = \phi$. The only implications that need a closer look are (14.1)⇒(14.2) and (14.4)⇒(14.5). Assuming (14.1) we have a Jordan *-isomorphism $J : M_1 \to M_2$ and $u_0 \in U_2$ such that $\phi(a) = u_0 J(a) = J(a)u_0^{-1}$ for all $a \in U_1$. Since the unitary group linearly generates the whole algebra, it follows that $u_0 x = x u_0^{-1}$ for all $x \in M_2$, which readily implies that $u_0 = u_0^{-1}$, and so u_0 is central. A similar argument applies for (14.4)⇒(14.5). The rest can be proved by already employed arguments. ∎

To simplify the formulations of the remaining results, in what follows we shall omit conditions regarding the invariance properties of the transformations under consideration with respect to generalized distance measures. We are convinced that after having read the paper carefully up to this point, the reader will be able to easily complete the results with such additional equivalent conditions.

THEOREM 15. *Let M_j be a von Neumann algebra with unitary group U_j for $j = 1, 2$, and F a non-empty set. Suppose that Φ_j and Ψ_j are surjections from F onto U_j for $j = 1, 2$. Then the following conditions are equivalent:*

(15.1) $\sigma(\Phi_1(x)\Psi_1(y)) = \sigma(\Phi_2(x)\Psi_2(y))$ *for all* $x, y \in F$;

(15.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and an element $u_0 \in U_2$ such that*

$$\Phi_2(x) = u_0 J(\Phi_1(x)), \quad \Psi_2(x) = J(\Psi_1(x))u_0^{-1}, \quad x \in F.$$

Moreover, the following conditions are also equivalent:

(15.3) $r(\Phi_1(x)\Psi_1(y) - 1) = r(\Phi_2(x)\Psi_2(y) - 1)$ *for all* $x, y \in F$;

(15.4) *there exists a Jordan *-isomorphism J from M_1 onto M_2, a central projection $p \in M_2$, and $u_0 \in U_2$ such that*

$$\Phi_2(x) = u_0(pJ(\Phi_1(x)) + (1-p)J(\Phi_1(x))^{-1}), \quad x \in F,$$
$$\Psi_2(x) = (pJ(\Psi_1(x)) + (1-p)J(\Psi_1(x))^{-1})u_0^{-1}, \quad x \in F.$$

Proof. Suppose that (15.2) holds. We easily infer that

$$\sigma(\Phi_2(x)\Psi_2(y)) = \sigma(J(\Phi_1(x))J(\Psi_1(y))) = \sigma(\Phi_1(x)\Psi_1(y)), \quad x, y \in F.$$

Conversely, suppose that (15.1) holds. We first observe that $\Phi_1(x) = \Phi_1(x')$ implies $\Phi_2(x) = \Phi_2(x')$. Indeed, assume $\Phi_1(x) = \Phi_1(x')$. Then

$$\begin{aligned}\sigma(\Phi_2(x)\Psi_2(y)) &= \sigma(\Phi_1(x)\Psi_1(y)) = \sigma(\Phi_1(x')\Psi_1(y)) \\ &= \sigma(\Phi_2(x')\Psi_2(y)), \quad y \in F.\end{aligned}$$

Pick $y \in F$ with $\Psi_2(y) = \Phi_2(x)^{-1}$. Such a y exists since $\Psi_2(F) = U_2$. Then

$$\{1\} = \sigma(\Phi_2(x)\Psi_2(y)) = \sigma(\Phi_2(x')\Psi_2(y)).$$

Hence $1 = \Phi_2(x')\Psi_2(y)$, and thus

$$\Phi_2(x') = \Psi_2(y)^{-1} = \Phi_2(x).$$

In the same way we see that $\Psi_1(x) = \Psi_1(x')$ implies $\Psi_2(x) = \Psi_2(x')$. Define $\phi, \psi : U_1 \to U_2$ by $\phi(\Phi_1(x)) = \Phi_2(x)$ and $\psi(\Psi_1(x)) = \Psi_2(x)$ for $x \in F$. Clearly, ϕ, ψ are well defined surjections from U_1 onto U_2. Moreover,

$$\sigma(ab) = \sigma(\phi(a)\psi(b)), \quad a, b \in U_1.$$

By Theorem 13 there exists a Jordan *-isomorphism from M_1 onto M_2 and $u_0 \in U_2$ such that

$$\phi(a) = u_0 J(a), \quad \psi(a) = J(a)u_0^{-1}, \quad a \in U_1,$$

and we easily conclude that (15.2) holds.

The implication (15.4)$\Rightarrow$(15.3) can be proved by a reasoning similar to that for (12.5)$\Rightarrow$(12.3).

Suppose now that (15.3) holds. We first observe that $\Phi_1(x) = \Phi_1(x')$ implies $\Phi_2(x) = \Phi_2(x')$ for any $x, x' \in F$. Indeed, assume $\Phi_1(x) = \Phi_1(x')$. Then

$$\begin{aligned} r(\Phi_2(x)\Psi_2(y) - 1) &= r(\Phi_1(x)\Psi_1(y) - 1) = r(\Phi_1(x')\Psi_1(y) - 1) \\ &= r(\Phi_2(x')\Psi_2(y) - 1), \quad y \in F. \end{aligned}$$

As $\Psi_2(F) = U_2$, there exists $y \in F$ with $\Psi_2(y) = \Phi_2(x)^{-1}$. Then

$$0 = r(\Phi_2(x)\Psi_2(y) - 1) = r(\Phi_2(x')\Psi_2(y) - 1).$$

As $\Phi_2(x')\Psi_2(y)$ is unitary, we have

$$\|\Phi_2(x')\Psi_2(y) - 1\| = r(\Phi_2(x')\Psi_2(y) - 1) = 0,$$

implying

$$\Phi_2(x') = \Psi_2(y)^{-1} = \Phi_2(x).$$

In a similar way we see that $\Psi_1(x) = \Psi_1(x')$ implies $\Psi_2(x) = \Psi_2(x')$. Once again, define maps $\phi, \psi : U_1 \to U_2$ by $\phi(\Phi_1(x)) = \Phi_2(x)$ and $\psi(\Psi_1(x)) = \Psi_2(x)$ for $x \in F$, which turn out to be well defined and surjective. Moreover,

$$r(ab - 1) = r(\phi(a)\psi(b) - 1), \quad a, b \in U_1.$$

Then by Theorem 13 there exists a Jordan *-isomorphism, a central projection $p \in M_2$ and $u_0 \in U_2$ such that

$$\phi(a) = u_0(pJ(a)+(1-p)J(a)^{-1}),\ \psi(a) = (pJ(a)+(1-p)J(a)^{-1})u_0^{-1},\ a\in U_1.$$

This shows that (15.4) holds. ∎

Finally, we present corollaries of the former results from which non-linear spectral-multiplicativity-type conditions can be deduced for maps between the self-adjoint parts of von Neumann algebras to be Jordan *-isomorphisms.

Corollary 16. *Let M_j be a von Neumann algebra for $j = 1, 2$. Suppose that f and g are bijections from M_{1s} onto M_{2s}. Then the following conditions are equivalent:*

(16.1) $\sigma(\exp(ix)\exp(iy)) = \sigma(\exp(if(x))\exp(ig(y)))$ *for all* $x, y \in M_{1s}$;

(16.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and an element $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0 \exp(iJ(x)), \quad \exp(ig(x)) = (\exp(iJ(x)))u_0^{-1}, \quad x \in M_{1s}.$$

In particular, if f and g are homogeneous, then $f = g = J$ and $u_0 = 1$.

Moreover, the following conditions are also equivalent:

(16.3) $r(\exp(ix)\exp(iy)-1) = r(\exp(if(x))\exp(ig(y))-1)$ *for* $x, y \in M_{1s}$*;*
(16.4) *there exists a Jordan *-isomorphism J from M_1 onto M_2, a central projection $p \in M_2$ and $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0\big(p\exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1}\big),$$
$$\exp(ig(x)) = \big(p\exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1}\big)u_0^{-1},$$

for every $x \in M_{1s}$.

In particular, if f and g are homogeneous, then $f = g = (2p-1)J$ and $u_0 = 1$.

Proof. Suppose that (16.2) holds. We infer that

$$\begin{aligned}\sigma\big(\exp(if(x))\exp(ig(y))\big) &= \sigma\big(u_0\exp(iJ(x))\exp(iJ(y))u_0^{-1}\big)\\ &= \sigma(J(\exp(ix))J(\exp(iy))) = \sigma(\exp(ix)\exp(iy)), \quad x, y \in M_{1s}.\end{aligned}$$

In particular, if f is homogeneous, then $f(0) = 0$. It follows that $u_0 = 1$ and

$$\exp(itf(x)) = \exp(if(tx)) = \exp(itJ(x)), \quad t \in \mathbb{R},\ x \in M_{1s}.$$

Letting $t \to 0$, from

$$(\exp(itf(x)) - 1)/t = (\exp(itJ(x)) - 1)/t$$

we obtain $f(x) = J(x)$ for all $x \in M_{1s}$. In the same way we deduce that $g(x) = J(x)$ for all $x \in M_{1s}$ if g is homogeneous.

Suppose that (16.1) holds. Set $F = M_{1s}$ and define $\Phi_1, \Psi_1 : M_{1s} \to U_1$ by $\Phi_1(x) = \Psi_1(x) = \exp(ix)$ for $x \in M_{1s}$. Also define $\Phi_2, \Psi_2 : M_{1s} \to U_2$ by $\Phi_2(x) = \exp(if(x))$ and $\Psi_2(x) = \exp(ig(x))$ for $x \in M_{1s}$. As $\exp iM_{js} = U_j$, the maps Φ_j and Ψ_j are surjective for $j = 1, 2$. Obviously,

$$\sigma(\Phi_1(x)\Psi_1(y)) = \sigma(\Phi_2(x)\Psi_2(y)), \quad x, y \in F.$$

Then by Theorem 15 there exists a Jordan *-isomorphism $J : M_1 \to M_2$ and $u_0 \in U_2$ such that

$$\exp(if(x)) = \Phi_2(x) = u_0 J(\Phi_1(x)) = u_0 J(\exp(ix)) = u_0\exp(iJ(x)),$$
$$\exp(ig(x)) = \Psi_2(x) = J(\Psi_1(x))u_0^{-1} = J(\exp(ix))u_0^{-1} = (\exp(iJ(x)))u_0^{-1},$$

for every $x \in M_{1s}$, and hence we obtain (16.2).

Now suppose that (16.4) holds. Then by a simple calculation we have

$$\begin{aligned}&\exp(if(x))\exp(ig(y))\\&\quad= u_0\big(pJ(\exp(ix))J(\exp(iy)) + (1-p)J(\exp(ix))^{-1}J(\exp(iy))^{-1}\big)u_0^{-1}.\end{aligned}$$

Using a calculation similar to the one we have applied in the proof of (12.5)⇒(12.3) we find that

$$r(\exp(ix)\exp(iy)-1) = r(\exp(if(x))\exp(ig(x))-1), \qquad x,y\in M_{1s},$$

and hence we obtain (16.3). In particular, if f is homogeneous, then $f(0)=0$. Thus

$$1 = \exp(if(0)) = u_0\big(pJ(\exp(i0)) + (1-p)J(\exp(i0))^{-1}\big) = u_0.$$

It follows that

$$\begin{aligned}\exp(itf(x)) = \exp(if(tx)) &= p\exp(iJ(tx)) + (1-p)(\exp(iJ(tx)))^{-1}\\&= p\exp(itJ(x)) + (1-p)\exp(-itJ(x)), \qquad x\in M_{1s}.\end{aligned}$$

Letting $t\to 0$, from

$$\big(\exp(itf(x))-1\big)/it = p\big(\exp(itJ(x))-1\big)/it + (1-p)\big(\exp(-itJ(x))-1\big)/it,$$

we deduce

$$f(x) = (2p-1)J(x), \qquad x\in M_{1s}.$$

In a similar manner we obtain $g(x)=(2p-1)J(x)$ for $x\in M_{1s}$.

Suppose that (16.3) holds. Set $F=M_{1s}$ and once again define $\Phi_1,\Psi_1 : M_{1s}\to U_1$ by $\Phi_1(x)=\Psi_1(x)=\exp(ix)$ and $\Phi_2,\Psi_2 : M_{1s}\to U_2$ by $\Phi_2(x)=\exp(if(x))$, $\Psi_2(x)=\exp(ig(x))$ for $x\in M_{1s}$. Then Φ_j and Ψ_j are both surjective maps for $j=1,2$. Clearly,

$$r(\Phi_1(x)\Psi_1(y)-1) = r(\Phi_2(x)\Psi_2(y)-1), \qquad x,y\in M_{1s}.$$

By Theorem 15 there exists a Jordan *-isomorphism J from M_1 onto M_2, a central projection $p\in M_2$ and a unitary $u_0\in U_2$ such that

$$\begin{aligned}\Phi_2(x) &= u_0\big(pJ(\Phi_1(x)) + (1-p)J(\Phi_1(x))^{-1}\big),\\ \Psi_2(x) &= \big(pJ(\Psi_1(x)) + (1-p)J(\Psi_1(x))^{-1}\big)u_0^{-1},\end{aligned}$$

for every $x\in M_{1s}$. Then

$$\begin{aligned}\exp(if(x)) &= u_0\big(J(\exp(ix)) + (1-p)J(\exp(ix))^{-1}\big)\\&= u_0\big(p\exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1}\big)\end{aligned}$$

and

$$\begin{aligned}\exp(ig(x)) &= \big(pJ(\exp(ix)) + (1-p)J(\exp(ix))^{-1}\big)u_0^{-1}\\&= \big(p\exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1}\big)u_0^{-1}\end{aligned}$$

for every $x\in M_{1s}$. This completes the proof. ∎

The following statement is an easy consequence of Corollary 16—one just needs to take $g = f$ (and have a short look at the argument in the proof of Corollary 14 concerning centrality).

Corollary 17. *Let M_j be a von Neumann algebra for $j = 1, 2$. Suppose that f is a bijection from M_{1s} onto M_{2s}. Then the following conditions are equivalent:*

(17.1) $\sigma(\exp(ix)\exp(iy)) = \sigma(\exp(if(x))\exp(if(y)))$ *for all* $x, y \in M_{1s}$*;*
(17.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and a central symmetry $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0 \exp(iJ(x)), \qquad x \in M_{1s}.$$

In particular, if f is homogeneous, then $f = J$ and $u_0 = 1$.

The following conditions are also equivalent:

(17.3) $r(\exp(ix)\exp(iy)-1) = r(\exp(if(x))\exp(if(y))-1)$ *for* $x, y \in M_{1s}$*;*
(17.4) *there exists a Jordan *-isomorphism $J : M_1 \to M_2$, a central projection $p \in M_2$ and a central symmetry $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0\big(p\exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1}\big), \qquad x \in M_{1s}.$$

In particular, if f is homogeneous, then $f = (2p-1)J$ and $u_0 = 1$.

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