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On (conditional) positive semidefiniteness in a matrix-valued context

by

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Abstract. In a nutshell, we intend to extend Schoenberg's classical theorem connecting conditionally positive semidefinite functions $F : \mathbb{R}^n \to \mathbb{C}$, $n \in \mathbb{N}$, and their positive semidefinite exponentials $\exp(tF)$, $t > 0$, to the case of matrix-valued functions $F: \mathbb{R}^n \to \mathbb{C}^{m\times m}$, $m \in \mathbb{N}$. Moreover, we study the closely associated property that $\exp(tF(-i\nabla))$, $t > 0$, is positivity preserving and its failure to extend directly in the matrix-valued context.

Contents

1. Introduction. To set the stage and hence describe the matrix-valued extensions of some of the classical results on (conditional) positive semidefiniteness we are interested in, we first briefly recall the basic definitions of positive semidefinite and conditionally positive semidefinite matrices $A \in \mathbb{C}^{m\times m}$ and positive semidefinite and conditionally positive semidefinite functions $F: \mathbb{R}^n \to \mathbb{C}$, and then state three classical results in this context:

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Definition 1.1. Let $m \in \mathbb{N}$ and $A \in \mathbb{C}^{m\times m}$, and suppose $F\colon \mathbb{R}^n \to \mathbb{C}$, $n \in \mathbb{N}$.

(i) A is called *positive semidefinite*, denoted by $A \geq 0$, if

$$(c, Ac)_{\mathbb{C}^m} = \sum_{j,k=1}^{m} \overline{c_j} A_{j,k} c_k \geq 0 \quad \text{for all } c = (c_1, \dots, c_m)^\top \in \mathbb{C}^m. \tag{1.1}$$

(ii) $A = \{A_{j,k}\}_{1\leq j,k\leq m} = A^* \in \mathbb{C}^{m\times m}$ is said to be *conditionally positive semidefinite* if

$$(c, Ac)_{\mathbb{C}^m} \geq 0 \quad \text{for all } c = (c_1, \dots, c_m)^\top \in \mathbb{C}^m \text{ with } \sum_{j=1}^{m} c_j = 0. \tag{1.2}$$

(iii) F is called *positive semidefinite* if for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, the matrix $\{F(x_p - x_q)\}_{1\leq p,q\leq N} \in \mathbb{C}^{N\times N}$ is positive semidefinite.

(iv) F is called *conditionally positive semidefinite* if for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, the matrix $\{F(x_p - x_q)\}_{1\leq p,q\leq N} \in \mathbb{C}^{N\times N}$ is conditionally positive semidefinite.

(v) F is called *positive semidefinite in the sense of Schoenberg* if $F(-x) = \overline{F(x)}$, $x \in \mathbb{R}^n$, and for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, the matrix $\{F(x_p - x_q) - F(x_p) - \overline{F(x_q)}\}_{1\leq p,q\leq N} \in \mathbb{C}^{N\times N}$ is positive semidefinite.

(vi) Let $T \in \mathcal{B}(L^2(\mathbb{R}^n))$. Then T is called *positivity preserving* (in $L^2(\mathbb{R}^n)$) if for any $0 \leq f \in L^2(\mathbb{R}^n)$ also $Tf \geq 0$.

In connection with Definition 1.1(iv) one can show that if F is conditionally positive semidefinite, then $F(-x) = \overline{F(x)}$ for all $x \in \mathbb{R}^n$. In addition, one observes that for T to be positivity preserving it suffices to take $0 \leq f \in C_0^\infty(\mathbb{R}^n)$ in Definition 1.1(vi).

Given the notions just introduced in Definition 1.1, we now recall three classical results. We start with Schoenberg's Theorem [35], who studied isometric imbeddability of separable spaces with appropriate distance functions into a Hilbert space.

Theorem 1.2 (cf., e.g., [4], [24, Sect. 3.6], [34, Proposition 4.4]). *Assume that $F\colon \mathbb{R}^n \to \mathbb{C}$. Then the following conditions are equivalent:*

(i) $F(0) \leq 0$ *and F is conditionally positive semidefinite.*

(ii) $F(0) \leq 0$ *and for all $t > 0$,* $\exp(tF)$ *is positive semidefinite.*

(iii) *F is positive semidefinite in the sense of Schoenberg.*

If, in addition, F is locally bounded and one of conditions (i)–(iii) *holds, then there exists $C > 0$ such that*

$$|F(x)| \leq C[1 + |x|^2], \qquad x \in \mathbb{R}^n. \tag{1.3}$$

In this context see also [5, Sects. 4.3, 4.4] and [6, Sect. II.7].

Given $F \in C(\mathbb{R}^n)$ and F polynomially bounded, one can define

$$F(-i\nabla)\colon \begin{cases} C_0^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \\ f \mapsto F(-i\nabla)f = (f^\wedge F)^\vee. \end{cases} \tag{1.4}$$

More generally, if $F \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, one introduces the maximally defined operator of multiplication by F in $L^2(\mathbb{R}^n)$, denoted by M_F, by

$$\begin{aligned} &(M_F f)(x) = F(x)f(x), \\ &f \in \operatorname{dom}(M_F) = \{g \in L^2(\mathbb{R}^n) \mid Fg \in L^2(\mathbb{R}^n)\}, \end{aligned} \tag{1.5}$$

and then defines $F(-i\nabla)$ as a normal operator in $L^2(\mathbb{R}^n)$ via

$$F(-i\nabla) = \mathcal{F}^{-1} M_F \mathcal{F} \tag{1.6}$$

(cf. (1.16), (1.17) and their unitary extensions to $L^2(\mathbb{R}^n)$).

Theorem 1.3 (cf., e.g., [21], [25], [33, Theorems XIII.52 and XIII.53]). *Assume that $F \in C(\mathbb{R}^n)$ and there exists $c \in \mathbb{R}$ such that* $\operatorname{Re}(F(x)) \le c$. *Then the following conditions are equivalent:*

(i) *For all $t > 0$,* $\exp(tF(-i\nabla))$ *is positivity preserving.*
(ii) *For each $t > 0$, e^{tF} is a positive semidefinite function.*
(iii) *$F(-x) = \overline{F(x)}$ for all $x \in \mathbb{R}^n$, and F is conditionally positive semidefinite.*
(iv) (The Lévy–Khintchine formula) *There exist $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^n$, $0 \le A \in \mathbb{C}^{n\times n}$, and a nonnegative finite measure ν on $\mathbb{R}^n$ with $\nu(\{0\}) = 0$ such that*

$$\begin{aligned} F(x) = \alpha + i(\beta\cdot x) - (x\cdot(Ax)) \\ + \int_{\mathbb{R}^n} \bigg[\exp(i(x\cdot y)) - 1 - \frac{i(x\cdot y)}{1+|y|^2}\bigg] \frac{1+|y|^2}{|y|^2}\, d\nu(y), \quad x \in \mathbb{R}^n. \end{aligned} \tag{1.7}$$

The principal aim of this paper is to investigate to which degree Theorems 1.2 and 1.3(i)–(iii) extend to the matrix-valued context, where $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$, $m \in \mathbb{N}$, $m \ge 2$, and if direct generalizations are impossible, what modified forms of extensions exist. We also note that a matrix-valued extension of the Lévy–Khintchine formula, Theorem 1.3(iv), while not the subject of this paper, is part of ongoing investigations. For a historical survey on infinitely divisible distributions and their connection to the Lévy–Khintchine formula we refer to [28] (and the extensive list of references cited therein).

For completeness we also recall Bochner's theorem [9] as it naturally fits in with Theorems 1.2 and 1.3:

Theorem 1.4 (Bochner's Theorem, cf., e.g., [2, Sect. 5.4], [32, p. 13], [34, p. 46]). *Assume that $F \in C(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(i) *F is positive semidefinite.*

(ii) *There exists a nonnegative finite measure μ on $\mathbb{R}^n$ such that*

$$F(x) = \mu^\wedge(x), \quad x \in \mathbb{R}^n. \tag{1.8}$$

In addition, if (i) *or* (ii) *holds, then*

$$F(-x) = \overline{F(x)}, \quad |F(x)| \leq |F(0)|, \quad x \in \mathbb{R}^n, \tag{1.9}$$

in particular, F is bounded on $\mathbb{R}^n$.

In this context we emphasize that the extension of Bochner's Theorem 1.4 has been obtained by Berberian [3] not only in the matrix context (cf. Theorem 4.3), but in the infinite-dimensional case in connection with Abelian groups. As a result, in the following we exclusively focus on extensions of Theorems 1.2 and 1.3(i)–(iii).

Turning to the matrix-valued case, $F \colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$, $m \in \mathbb{N}$, and taking the notions of positive semidefinite and conditionally positive semidefinite matrix-valued functions F in Definition 2.4 (and the obvious matrix-valued extension of Definition 1.1(v)) for granted, we can now briefly describe the form in which Theorems 1.2 and 1.3(i)–(iii) extend to the matrix-valued context. First and foremost,

- the exponential $\exp(tF)$ must consistently be replaced by the Hadamard exponential $\exp_{\mathrm{H}}(tF)$ in the matrix context.

Here the *Hadamard exponential* $\exp_{\mathrm{H}}(G(x))$ of $G \colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$, $m \in \mathbb{N}$, is defined by

$$\exp_{\mathrm{H}}(G(x)) = \{\exp_{\mathrm{H}}(G(x))_{j,k} := \exp(G(x)_{j,k})\}_{1\leq j,k\leq m}, \quad x \in \mathbb{R}^n. \tag{1.10}$$

It is understood in the following that $\exp(tF)$ is always replaced by the Hadamard exponential $\exp_{\mathrm{H}}(tF)$ in the matrix context of $m \in \mathbb{N}$, $m \geq 2$.

In connection with the matrix-valued extension of Schoenberg's Theorem 1.2 (for $m \in \mathbb{N}$, $m \geq 2$) we prove the following facts in Theorem 4.9 and Remark 4.10:

- Items (i) and (ii) of Theorem 1.2 remain equivalent (disregarding the condition $F(0) \leq 0$).
- If $F(0) \leq 0$ and (i) or (ii) of Theorem 1.2 holds, then (iii) of Theorem 1.2 follows, but the converse is false in the matrix-valued context.

In connection with the matrix-valued extension of Theorem 1.3 (for $m \in \mathbb{N}$, $m \geq 2$) we prove the following facts in Theorems 4.11 and 4.15:

- Conditions (ii) and (iii) of Theorem 1.3 remain equivalent in the matrix-valued context; however, (i) does not extend at all (employing $\exp_{\mathrm{H}}(tF)$ as agreed upon). We do, however, find a proper extension of condition (i) (cf. Theorem 4.11(i)).

These comments illustrate that much of Theorems 1.2 and 1.3 extends to the matrix-valued context, but some items require specific modifications. In particular, the positivity preserving condition (i) of Theorem 1.3 has to be altered considerably.

Next, we briefly turn to the contents of each section. Section 2 is of preparatory nature and recalls the basic facts on positive semidefinite and conditionally positive semidefinite matrices and matrix-valued functions on $\mathbb{R}^n$, $n \in \mathbb{N}$, introduces the notion of the Hadamard exponential, and derives the equivalence of items (i) and (ii) of Schoenberg's Theorem 1.2 in the matrix-valued context. Introductory remarks on convolution operators involving matrix-valued measures are the content of Section 3. We recall the spaces $L^p(\mathbb{R}^n, \mathbb{C}^{m\times m})$, $p \in [1,\infty) \cup \{\infty\}$, discuss the operator $F(-i\nabla)$, $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$, via Fourier transform, discuss various consequences of positivity preserving of $F(-i\nabla)$, and conclude with two approximation results (Lemmas 3.11 and 3.13). Our principal results are formulated in Section 4. The classical L^1 and L^2 Fourier multiplier results are discussed in the matrix-valued context in Theorems 4.4 and 4.6. The matrix-valued extension of Schoenberg's Theorem 1.2 is formulated in Theorem 4.9; the fact that no complete extension of Theorem 1.2 is possible (in the sense that either of conditions (i) and (ii) of Theorem 1.2 implies (iii), but the converse is false) is demonstrated in Remark 4.10. The extent to which Theorem 1.3 extends to the matrix-valued case is dealt with in detail in Theorems 4.11 and 4.15, as well as in Remark 4.12. The analog of the bound (1.3) in the matrix-valued context is derived in Theorem 4.18. In Appendix A we construct a counterexample verifying the claim made in Remark 4.2, and Appendix B provides a proof of (4.41).

Finally, we briefly summarize the basic notation employed. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\cdot,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$.

The Banach spaces of bounded and compact linear operators on a separable complex Hilbert space $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively; the corresponding ℓ^p-based Schatten–von Neumann trace ideal (cf. [16, Ch. III], [36, Ch. 1]) is denoted by $\mathcal{B}_p(\mathcal{H})$, with norm $\|\cdot\|_{\mathcal{B}_p(\mathcal{H})}$, $p \geq 1$ (defined in terms of the ℓ^p-norm of the singular values of the operator in question). Moreover, $\operatorname{tr}_{\mathcal{H}}(A)$ denotes the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$. We also employ the analogous notation $\mathcal{B}(X_1, X_2)$ for bounded linear operators mapping the Banach space X_1 into the Banach space X_2.

For X a set, $X^{m\times n}$, $m, n \in \mathbb{N}$, represents the set of $m \times n$ matrices with entries in X.

Unless explicitly stated otherwise, $\mathbb{C}^m$ is always equipped with the Euclidean scalar product $(\cdot,\cdot)_{\mathbb{C}^m}$ and associated norm $\|\cdot\|_{\mathbb{C}^m}$.

For $A \in \mathbb{C}^{m\times m}$, $m \in \mathbb{N}$, we denote by $A^\top$ the transpose of A, and by $\|A\|_{\mathcal{B}(\mathbb{C}^m)}$ the operator norm of A, when we consider A as a linear operator on $\mathbb{C}^m$ (equipped with $\|\cdot\|_{\mathbb{C}^m}$). In this context we note that

$$(1.11)\qquad (\mathbb{C}^{m\times m}, \|\cdot\|_{\mathcal{B}(\mathbb{C}^m)})^* = (\mathbb{C}^{m\times m}, \|\cdot\|_{\mathcal{B}_1(\mathbb{C}^m)}).$$

We also introduce

$$(1.12)\qquad \|A\|_{\max} = \max_{1\le j,k\le m} |A_{j,k}|.$$

The symbol $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ denotes the space of all $\mathbb{C}^{m\times m}$-valued rapidly decreasing functions on $\mathbb{R}^n$ with each entry in the usual Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In addition, we introduce the spaces

$$(1.13)\qquad C_0(\mathbb{R}^n, \mathbb{C}^{m\times m}) = \{f \in C(\mathbb{R}^n, \mathbb{C}^{m\times m}) \mid \operatorname{supp}(f) \text{ compact}\},$$

$$(1.14)\qquad C_b(\mathbb{R}^n, \mathbb{C}^{m\times m}) = \{f \in C(\mathbb{R}^n, \mathbb{C}^{m\times m}) \mid \|f\|_\infty < \infty\},$$

$$(1.15)\qquad C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}) = \Big\{f = \{f_{j,k}\}_{1\le j,k\le m} \colon \mathbb{R}^n \to \mathbb{C}^{m\times m} \mid f_{j,k} \in C(\mathbb{R}^n), \\ \lim_{|x|\to\infty} f_{j,k}(x) = 0,\ 1 \le j,k \le m\Big\}.$$

Unless explicitly stated otherwise, the spaces (1.13)–(1.15) will always be equipped with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|f(x)\|_{\mathcal{B}(\mathbb{C}^m)}$.

For the sake of brevity, we omit displaying the Lebesgue measure $d^n x$ in $L^p(\mathbb{R}^n, \mathbb{C}^{m\times m})$, $p \in [1,\infty]$, whenever the latter is understood.

The Fourier and inverse Fourier transforms on $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ are given by the pair of formulas

$$(1.16)\qquad (\mathcal{F}f)(y) = f^\wedge(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(y\cdot x)} f(x)\, d^n x,$$

$$(1.17)\qquad (\mathcal{F}^{-1}g)(x) = g^\vee(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x\cdot y)} g(y)\, d^n y,$$

for $f, g \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$, and we use the same notation for the appropriate extensions to $L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$ or $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$.

The open ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $r_0 > 0$ is denoted by $B_n(x_0, r_0)$, the norm of $x \in \mathbb{R}^n$ is denoted by $|x|$, the scalar product of $x, y \in \mathbb{R}^n$ is abbreviated by $x \cdot y$.

We denote by $\mathfrak{B}_n$ the σ-algebra of all Borel subsets of $\mathbb{R}^n$, and for $E \in \mathfrak{B}_n$, $|E|$ is the n-dimensional Lebesgue measure of E.

2. Matrix-valued (conditional) positive semidefinite functions: a variant of Schoenberg's theorem. In this preparatory section we recall the basic facts on positive semidefinite and conditionally positive semidefinite matrices and matrix-valued functions on $\mathbb{R}^n$, $n \in \mathbb{N}$, introduce the notion of the Hadamard exponential, and derive the equivalence of items (i) and (ii) in Schoenberg's Theorem 1.2 (see, e.g., [4], [24, Sect. 3.6], and [34, Proposition 4.4]) in the matrix-valued context.

We start with the following definition (cf., e.g., [8, p. 180], [23, p. 451]).

DEFINITION 2.1. Let $m \in \mathbb{N}$ and $A = \{A_{j,k}\}_{1 \leq j,k \leq m} \in \mathbb{C}^{m \times m}$.

(i) A is called *positive semidefinite*, denoted by $A \geq 0$, if

$$(c, Ac)_{\mathbb{C}^m} = \sum_{j,k=1}^{m} \overline{c_j} A_{j,k} c_k \geq 0 \quad \text{for all } c = (c_1, \dots, c_m)^\top \in \mathbb{C}^m. \tag{2.1}$$

(ii) $A = \{A_{j,k}\}_{1 \leq j,k \leq m} = A^* \in \mathbb{C}^{m \times m}$ is said to be *conditionally positive semidefinite* if

$$(c, Ac)_{\mathbb{C}^m} \geq 0 \quad \text{for all } c = (c_1, \dots, c_m)^\top \in \mathbb{C}^m \text{ with } \sum_{j=1}^{m} c_j = 0. \tag{2.2}$$

Given $S \in \mathbb{C}^{m \times m}$, $m \in \mathbb{N}$, its *Hadamard exponential*, denoted by $\exp_{\mathrm{H}}(S)$, is defined by

$$\exp_{\mathrm{H}}(S) = \{\exp_{\mathrm{H}}(S)_{j,k} := \exp(S_{j,k})\}_{1 \leq j,k \leq m}. \tag{2.3}$$

LEMMA 2.2 (see, e.g., [23, Theorem 6.3.6]). *Let $A \in \mathbb{C}^{m \times m}$, $m \in \mathbb{N}$, be conditionally positive semidefinite. Then $\exp_{\mathrm{H}}(A) \geq 0$, that is, the Hadamard exponential of A is positive semidefinite.*

The following result can be viewed as a complexified version of [8, Exercise 5.6.15], [23, Theorem 6.3.13]:

LEMMA 2.3. *Let $\varepsilon > 0$, assume $A = A^* \in \mathbb{C}^{m \times m}$, $m \in \mathbb{N}$, and suppose $\exp_{\mathrm{H}}(tA)$ is positive semidefinite for all $t \in (0, \varepsilon)$. Then A is conditionally positive semidefinite.*

Proof. Let $c = (c_1, \dots, c_m)^\top \in \mathbb{C}^m$ with $\sum_{j=1}^{m} c_j = 0$. Then for all $t \in (0, \varepsilon)$,

$$\begin{aligned} 0 \leq t^{-1}(c, \exp_{\mathrm{H}}(tA)c)_{\mathbb{C}^m} &= \sum_{j,k=1}^{m} \overline{c_j} t^{-1} [\exp_{\mathrm{H}}(tA_{j,k}) - 1] c_k \\ &\underset{t \downarrow 0}{\longrightarrow} \sum_{j,k=1}^{m} \overline{c_j} A_{j,k} c_k = (c, Ac)_{\mathbb{C}^m}. \ \blacksquare \end{aligned} \tag{2.4}$$

Combining Lemmas 2.2 and 2.3 shows that for $A = A^* \in \mathbb{C}^{m \times m}$, $m \in \mathbb{N}$,

$$\begin{aligned} &\exp_{\mathrm{H}}(tA) \geq 0 \text{ for all } t \in (0, \varepsilon_0) \text{ for some fixed } \varepsilon_0 > 0 \ \Leftrightarrow \\ &\exp_{\mathrm{H}}(tA) \geq 0 \text{ for all } t \geq 0. \end{aligned} \tag{2.5}$$

DEFINITION 2.4. Let $F \colon \mathbb{R}^n \to \mathbb{C}^{m \times m}$, $m, n \in \mathbb{N}$.

(i) F is called *positive semidefinite* if for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, the block matrix $\{F(x_p - x_q)\}_{1 \leq p,q \leq N} \in \mathbb{C}^{mN \times mN}$ is positive semidefinite.

(ii) F is *conditionally positive semidefinite* if for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, the block matrix $\{F(x_p - x_q)\}_{1 \leq p,q \leq N} \in \mathbb{C}^{mN \times mN}$ is conditionally positive semidefinite.

LEMMA 2.5. *Let $F\colon \mathbb{R}^n \to \mathbb{C}^{m \times m}$, $m, n \in \mathbb{N}$.*

(i) *F is positive semidefinite if and only if for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $c_p \in \mathbb{C}^m$, $1 \leq p \leq N$,*

$$\sum_{p,q=1}^{N} (c_p, F(x_p - x_q) c_q)_{\mathbb{C}^m} \geq 0. \tag{2.6}$$

(ii) ([3, p. 178]) *F is positive semidefinite if and only if for all $N \in \mathbb{N}$, $x_p \in \mathbb{R}^n$, $c_p \in \mathbb{C}$, $1 \leq p \leq N$, and $f = (f_1, \ldots, f_m)^\top \in \mathbb{C}^m$,*

$$\sum_{p,q=1}^{N} \overline{c_p} (f, F(x_p - x_q) f)_{\mathbb{C}^m} c_q = \sum_{p,q=1}^{N} \sum_{j,k=1}^{m} \overline{c_p}\, \overline{f_j} F(x_p - x_q)_{j,k} f_k c_q \geq 0. \tag{2.7}$$

(iii) *F is conditionally positive semidefinite if and only if the following conditions (α) and (β) hold:*

(α) *$F(-x) = F(x)^*$ for all $x \in \mathbb{R}^n$.*

(β) *For all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $c_p = (c_{p,1}, \ldots, c_{p,m}) \in \mathbb{C}^m$, $1 \leq p \leq N$, satisfying*

$$\sum_{p=1}^{N} \sum_{j=1}^{m} c_{p,j} = 0, \tag{2.8}$$

one has

$$\sum_{p,q=1}^{N} (c_p, F(x_p - x_q) c_q)_{\mathbb{C}^m} \geq 0. \tag{2.9}$$

In addition, $F\colon \mathbb{R}^n \to \mathbb{C}^{m \times m}$ satisfies (α) if and only if it satisfies

(α') *for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, the block matrix $\{F(x_p - x_q)\}_{1 \leq p,q \leq N} \in \mathbb{C}^{mN \times mN}$ is self-adjoint in $\mathbb{C}^{mN}$.*

Given $S\colon \mathbb{R}^n \to \mathbb{C}^{M \times M}$, $M, n \in \mathbb{N}$, its *Hadamard exponential*, denoted by $\exp_{\mathrm{H}}(S)$, is defined by

$$\exp_{\mathrm{H}}(S(x)) = \{\exp_{\mathrm{H}}(S(x))_{j,k} := \exp(S(x)_{j,k})\}_{1 \leq j,k \leq M}, \quad x \in \mathbb{R}^n. \tag{2.10}$$

The next two theorems represent a matrix generalization of a variant of Schoenberg's theorem (cf., e.g., [34, Proposition 4.4]), namely, the equivalence of items (i) and (ii) in Theorem 1.2, the principal result of this section:

THEOREM 2.6. *Let $F\colon \mathbb{R}^n \to \mathbb{C}^{m \times m}$, $m, n \in \mathbb{N}$, be conditionally positive semidefinite. Then $\exp_{\mathrm{H}}(F)$ is positive semidefinite.*

Proof. For all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \le p \le N$, the block matrix $\{F(x_p - x_q)\}_{1 \le p,q \le N} \in \mathbb{C}^{mN \times mN}$ is conditionally positive semidefinite. Thus, Lemma 2.2 implies the block matrix $\exp_{\mathrm{H}}(\{F(x_p - x_q)\}_{1 \le p,q \le N}) \in \mathbb{C}^{mN \times mN}$ is positive semidefinite. Since

$$\exp_{\mathrm{H}}(\{F(x_p - x_q)\}_{1 \le p,q \le N}) = \{\exp_{\mathrm{H}}(F(x_p - x_q))\}_{1 \le p,q \le N}, \tag{2.11}$$

this completes the proof. ∎

THEOREM 2.7. *Suppose that $\varepsilon > 0$, $F\colon \mathbb{R}^n \to \mathbb{C}^{m \times m}$, and $\exp_{\mathrm{H}}(tF) : \mathbb{R}^n \to \mathbb{C}^{m \times m}$ is positive semidefinite for all $t \in (0, \varepsilon)$. Then F is conditionally positive semidefinite.*

Proof. Suppose that $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \le p \le N$, and assume that $c_p = (c_{p,1}, \dots, c_{p,m}) \in \mathbb{C}^m$, $1 \le p \le N$, satisfy

$$\sum_{p=1}^{N} \sum_{j=1}^{m} c_{p,j} = 0. \tag{2.12}$$

Then for all $t \in (0, \varepsilon)$, Lemma 2.5(i) yields

$$\begin{aligned} 0 &\le t^{-1} \sum_{p,q=1}^{N} (c_p, \exp_{\mathrm{H}}(tF(x_p - x_q))c_q)_{\mathbb{C}^m} \\ &= \sum_{p,q=1}^{N} \sum_{j,k=1}^{m} \overline{c_{p,j}} t^{-1} [\exp(tF(x_p - x_q)_{j,k}) - 1] c_{q,k} \\ &\underset{t \downarrow 0}{\longrightarrow} \sum_{p,q=1}^{N} \sum_{j,k=1}^{m} \overline{c_{p,j}} F(x_p - x_q)_{j,k} c_{q,k} = \sum_{p,q=1}^{N} (c_p, F(x_p - x_q)c_q)_{\mathbb{C}^m}. \end{aligned} \tag{2.13}$$

By Lemma 2.5(iii), it remains to show that

$$F(-x) = F(x)^*, \qquad x \in \mathbb{R}^n. \tag{2.14}$$

To this end one observes that the block matrix

$$\begin{pmatrix} \exp_{\mathrm{H}}(tF(0)) & \exp_{\mathrm{H}}(tF(x)) \\ \exp_{\mathrm{H}}(tF(-x)) & \exp_{\mathrm{H}}(tF(0)) \end{pmatrix} \in \mathbb{C}^{2m \times 2m} \tag{2.15}$$

is positive semidefinite and hence self-adjoint. Thus,

$$\exp_{\mathrm{H}}(tF(-x)) = [\exp_{\mathrm{H}}(tF(x))]^*, \qquad x \in \mathbb{R}^n,\ t \in (0, \varepsilon). \tag{2.16}$$

Next, let $E_{2m} \in \mathbb{C}^{2m \times 2m}$ be the matrix all of whose entries equal 1. Then

$$t^{-1}[\exp_{\mathrm{H}}(tF(-x)) - E_{2m}] = t^{-1}\{[\exp_{\mathrm{H}}(tF(x))]^* - E_{2m}\}, \qquad x \in \mathbb{R}^n,\ t \in (0, \varepsilon), \tag{2.17}$$

and letting $t \downarrow 0$ in (2.17), one obtains

$$F(-x)_{j,k} = \overline{F(x)_{k,j}}, \qquad x \in \mathbb{R}^n,\ 1 \le j, k \le m, \tag{2.18}$$

proving (2.14). ∎

Combining Theorems 2.6 and 2.7 shows that for $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$,

$$\begin{aligned} &\exp_{\mathrm{H}}(tF) \geq 0 \text{ for all } t \in (0,\varepsilon_0) \text{ for some fixed } \varepsilon_0 > 0 \iff \\ &\exp_{\mathrm{H}}(tF) \geq 0 \text{ for all } t \geq 0. \end{aligned} \tag{2.19}$$

Next, we intend to show that Definitions 2.1 and 2.4 are compatible.

Corollary 2.8. *Let $0 \leq A \in \mathbb{C}^{m\times m}$ (i.e., A is positive semidefinite) and introduce $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ by*

$$F(x) = A, \quad x \in \mathbb{R}^n. \tag{2.20}$$

Then $F \geq 0$, that is, F is positive semidefinite in the sense of Definition 2.4(i).

Proof. For any $c = (c_1,\dots,c_N)^\top \in \mathbb{C}^N$,

$$\sum_{p,q=1}^N \overline{c_p}(f,Af)_{\mathbb{C}^m} c_q = (f,Af)_{\mathbb{C}^m} \sum_{p,q=1}^N \overline{c_p} c_q = (f,Af)_{\mathbb{C}^m}(c,H_N c)_{\mathbb{C}^N}, \tag{2.21}$$

where H_N denotes the $N\times N$-matrix with all entries equal to 1. Since it is well-known that H_N is positive semidefinite,

$$\sum_{p,q=1}^N \overline{c_p}(f,Af)_{\mathbb{C}^m} c_q \geq 0. \tag{2.22}$$

Thus, Lemma 2.5(ii) implies the conclusion. ∎

Corollary 2.9. *Let $A \in \mathbb{C}^{m\times m}$ be conditionally positive semidefinite and introduce $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ by*

$$F(x) = A, \quad x \in \mathbb{R}^n. \tag{2.23}$$

Then F is conditionally positive semidefinite in the sense of Definition 2.4(ii).

Proof. By Lemma 2.2, for all $t > 0$, $\exp_{\mathrm{H}}(tA) \geq 0$ is positive semidefinite. Thus, by Corollary 2.8, for all $t > 0$, $\exp_{\mathrm{H}}(tF)(x) = \exp_{\mathrm{H}}(tA)$, $x \in \mathbb{R}^n$, is positive semidefinite. Hence, by Theorem 2.7, F is conditionally positive semidefinite. ∎

Corollaries 2.8 and 2.9 indeed verify compatibility of Definitions 2.1 and 2.4. For other elementary examples of conditionally positive semidefinite matrix-valued functions we refer to Example 4.19.

The classical (i.e., scalar-valued, $m = 1$) version of Schoenberg's theorem, at first sight, suggests an alternative "weak" definition of conditionally positive semidefinite functions (cf. also [27]) as follows:

Definition 2.10. Let $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$. Then F is called *weakly conditionally positive semidefinite* if for all $N \in \mathbb{N}$, $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, and all $f = (f_1,\dots,f_m)^\top \in \mathbb{C}^m$, the matrix $\{(f,F(x_p - x_q)f)_{\mathbb{C}^m}\}_{1\leq p,q\leq N} \in \mathbb{C}^{N\times N}$

is conditionally positive semidefinite, that is, for all $c_p \in \mathbb{C}$, $1 \le p \le N$, with $\sum_{p=1}^{N} c_p = 0$, one has

$$\sum_{p,q=1}^{N} \sum_{j,k=1}^{m} \overline{c_p}\,\overline{f_j} F(x_p - x_q)_{j,k} f_k c_q \ge 0. \tag{2.24}$$

We conclude this section with a simple example showing that Definitions 2.4(ii) and 2.10 are inequivalent.

EXAMPLE 2.11. Consider

$$A = \begin{pmatrix} \ln(1/2) & 0 \\ 0 & \ln(1/2) \end{pmatrix} \tag{2.25}$$

and introduce $F\colon \mathbb{R}^n \to \mathbb{C}^{2\times 2}$ by

$$F(x) = A, \quad x \in \mathbb{R}^n. \tag{2.26}$$

Then, for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $c_p \in \mathbb{C}$, $1 \le p \le N$, with $\sum_{p=1}^{N} c_p = 0$, and all $f = (f_1, f_2)^\top \in \mathbb{C}^2$,

$$\sum_{p,q=1}^{N} \sum_{j,k=1}^{2} \overline{c_p}\,\overline{f_j} F(x_p - x_q)_{j,k} f_k c_q = (f, Af)_{\mathbb{C}^2} \sum_{p,q=1}^{N} \overline{c_p} c_q = 0, \tag{2.27}$$

and hence F is weakly conditionally positive semidefinite. On the other hand,

$$\exp_{\mathrm{H}}(F) = \exp_{\mathrm{H}}(A) = \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}, \quad x \in \mathbb{R}^n. \tag{2.28}$$

However, $\exp_{\mathrm{H}}(A)$ has a simple negative eigenvalue $\lambda_1 = -1/2$; denoting by $v_1 \in \mathbb{C}^2$ an associated normalized eigenvector, for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $c_p \in \mathbb{C}$, $1 \le p \le N$, one computes

$$\begin{aligned} \sum_{p,q=1}^{N} \overline{c_p} (v_1, \exp_{\mathrm{H}}(F)(x_p - x_q) v_1)_{\mathbb{C}^2} c_q &= \sum_{p,q=1}^{N} \overline{c_p} (v_1, \exp_{\mathrm{H}}(A) v_1)_{\mathbb{C}^2} c_q \\ &= -\frac{1}{2} \bigg| \sum_{p=1}^{N} c_p \bigg|^2 \le 0. \end{aligned} \tag{2.29}$$

In particular, as long as $\sum_{p=1}^{N} c_p \ne 0$, we have

$$\sum_{p,q=1}^{N} \overline{c_p} (v_1, \exp_{\mathrm{H}}(F)(x_p - x_q) v_1)_{\mathbb{C}^2} c_q < 0, \tag{2.30}$$

and hence $\exp_{\mathrm{H}}(F)$ is *not* positive semidefinite by Lemma 2.5(ii). Consequently, F is *not* conditionally positive semidefinite by Theorem 2.6, and Definitions 2.4(ii) and 2.10 are indeed inequivalent.

Remark 2.12. There are other inequivalent extensions of scalar conditionally positive semidefinite functions to the matrix context in the literature. One of the principal goals of this paper is to extend the classical results of Theorems 1.2 and 1.3(i)–(iii) to the matrix context. So we chose to use the more restrictive definition of matrix-valued conditionally positive semidefinite functions in Definition 2.4. For treatments of other inequivalent extensions of scalar conditionally positive semidefinite functions to the matrix case, see, for instance, [15, Ch. II], [44, Chs. 3, 4]. For detailed surveys of the theory of scalar positive semidefinite functions we refer, for example, to [17], [39].

3. Preliminaries on operators associated to matrix-valued positive semidefinite functions. In this section we develop the basic material on convolutions involving matrix-valued measures and matrix-valued convolution operators needed in our principal Section 4. We rely on [10, Sect. 2] and [11, Sects. 2.1, 3.1] (see also [20]). For readers who are interested in convolution involving operator-valued measures in the infinite-dimensional Hilbert space context, we refer to [14].

Throughout the remainder of this paper we fix $m \in \mathbb{N}$.

A $\mathbb{C}^{m\times m}$-*valued measure* on $\mathbb{R}^n$ is a countably additive function $\mu\colon \mathfrak{B}_n \to \mathbb{C}^{m\times m}$. Equivalently, $\mu = \{\mu_{j,k}\}_{1\le j,k\le m}$ is a $\mathbb{C}^{m\times m}$-valued measure on $\mathbb{R}^n$ if and only if each entry $\mu_{j,k}\colon \mathfrak{B}_n \to \mathbb{C}$, $1 \le j,k \le m$, is a complex measure on $\mathbb{R}^n$. The variation $|\mu|$ of μ is defined as the finite nonnegative measure on $\mathbb{R}^n$ given by

$$|\mu|(E) = \sup_{\mathcal{P}} \Big\{ \sum_{E_\ell \in \mathcal{P}} \|\mu(E_\ell)\|_{\mathcal{B}(\mathbb{C}^m)} \Big\}, \qquad E \in \mathfrak{B}_n, \tag{3.1}$$

where the supremum is taken over all partitions $\mathcal{P}$ of E into a finite number of pairwise disjoint subsets $E_\ell \in \mathfrak{B}_n$. The norm $\|\mu\|$ of μ is defined by

$$\|\mu\| = |\mu|(\mathbb{R}^n), \tag{3.2}$$

and we also introduce the notation

$$N(\mu) = \max_{1\le j,k\le m} |\mu_{j,k}|(\mathbb{R}^n) = \max_{1\le j,k\le m} \|\mu_{j,k}\|. \tag{3.3}$$

A function $f = \{f_{j,k}\}_{1\le j,k\le m}\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ is called μ-*integrable* if the integrals

$$\int_{\mathbb{R}^n} f(x)_{j,k}\, d\mu_{r,s}(x), \qquad 1 \le j,k,r,s \le m, \tag{3.4}$$

exist, in which case one defines, for all $E \in \mathfrak{B}_n$,

$$(3.5)\qquad \int_E f(x)\,d\mu(x) = \bigg\{\Big(\int_E f(x)\,d\mu(x)\Big)_{j,k}\bigg\}_{1\le j,k\le m},$$

$$(3.6)\qquad \Big(\int_E f(x)\,d\mu(x)\Big)_{j,k} = \sum_{\ell=1}^{m}\int_E f(x)_{j,\ell}\,d\mu_{\ell,k}(x), \quad 1\le j,k\le m.$$

Then, for all μ-integrable functions f,

$$(3.7)\qquad \Big\|\int_E f(x)\,d\mu(x)\Big\|_{\mathcal{B}(\mathbb{C}^m)} \le \int_E \|f(x)\|_{\mathcal{B}(\mathbb{C}^m)}\,d|\mu|(x), \quad E\in\mathfrak{B}_n.$$

Next, we introduce $\mathcal{M}(\mathbb{R}^n,\mathbb{C}^{m\times m})$ as the space of all (finite) measures on $\mathbb{R}^n$ of the form $\mu\colon \mathfrak{B}_n \to (\mathbb{C}^{m\times m}, \|\cdot\|_{\mathcal{B}(\mathbb{C}^m)})$. As shown in [10, Lemma 5], there exists a linear, isometric order isomorphism between $\mathcal{M}(\mathbb{R}^n,\mathbb{C}^{m\times m})$ and the dual space of $C_\infty(\mathbb{R}^n,\mathbb{C}^{m\times m})$ such that the duality pairing $\langle\cdot,\cdot\rangle$: $C_\infty(\mathbb{R}^n,\mathbb{C}^{m\times m}) \times \mathcal{M}(\mathbb{R}^n,\mathbb{C}^{m\times m})$ is given by

$$(3.8)\qquad \langle f,\mu\rangle = \operatorname{tr}_{\mathbb{C}^m}\Big(\int_{\mathbb{R}^n} f(x)\,d\mu(x)\Big) = \sum_{j,k=1}^{m}\int_{\mathbb{R}^n} f(x)_{j,k}\,d\mu_{k,j}(x).$$

Given $\mu\in\mathcal{M}(\mathbb{R}^n,\mathbb{C}^{m\times m})$ and a μ-integrable $f\colon\mathbb{R}^n\to\mathbb{C}^{m\times m}$, we define their convolution by

$$(3.9)\qquad f*\mu\colon \begin{cases} \mathbb{R}^n \to \mathbb{C}^{m\times m}, \\ x\mapsto (f*\mu)(x) = \int_{\mathbb{R}^n} f(x-y)\,d\mu(y), \end{cases} \quad x\in\mathbb{R}^n.$$

Moreover, for $p\in[1,\infty)$ we introduce

$$(3.10)\qquad L^p(\mathbb{R}^n,\mathbb{C}^{m\times m}) = \bigg\{f\colon\mathbb{R}^n\to\mathbb{C}^{m\times m} \text{ measurable} \,\bigg|\, \|f\|_{p,m} = \Big(\int_{\mathbb{R}^n}\|f(x)\|^p_{\mathcal{B}(\mathbb{C}^m)}\,d^nx\Big)^{1/p} < \infty\bigg\},$$

and similarly, for $p=\infty$,

$$(3.11)\qquad L^\infty(\mathbb{R}^n,\mathbb{C}^{m\times m}) = \bigg\{f\colon\mathbb{R}^n\to\mathbb{C}^{m\times m} \text{ measurable} \,\bigg|\, \|f\|_{\infty,m} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\|f(x)\|_{\mathcal{B}(\mathbb{C}^m)} < \infty\bigg\}.$$

Then one estimates

$$\begin{aligned}(3.12)\qquad \|(f*\mu)(x)\|_{\mathcal{B}(\mathbb{C}^m)} &= \Big\|\int_{\mathbb{R}^m} f(x-y)\,d\mu(y)\Big\|_{\mathcal{B}(\mathbb{C}^m)} \\ &\le \int_{\mathbb{R}^n}\|f(x-y)\|_{\mathcal{B}(\mathbb{C}^m)}\,d|\mu|(y) \\ &\le \Big(\int_{\mathbb{R}^n}\|f(x-y)\|^p_{\mathcal{B}(\mathbb{C}^m)}\,d|\mu|(y)\Big)^{1/p}[|\mu|(\mathbb{R}^n)]^{1/p'},\end{aligned}$$

with $p^{-1} + (p')^{-1} = 1$, and hence

$$\begin{aligned}(3.13)\qquad \|f * \mu\|_{p,m} &= \Big(\int_{\mathbb{R}^n} \|(f * \mu)(x)\|_{\mathcal{B}(\mathbb{C}^m)}^p \, d^n x \Big)^{1/p} \\ &\leq \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|f(x-y)\|_{\mathcal{B}(\mathbb{C}^m)}^p \, d|\mu|(y) \, d^n x \Big)^{1/p} [|\mu|(\mathbb{R}^n)]^{1/p'} \\ &= \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|f(x-y)\|_{\mathcal{B}(\mathbb{C}^m)}^p \, d^n x \, d|\mu|(y) \Big)^{1/p} [|\mu|(\mathbb{R}^n)]^{1/p'} \\ &= |\mu|(\mathbb{R}^n) \|f\|_{p,m}, \quad p \in [1, \infty).\end{aligned}$$

Thus, for $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ one can introduce the associated convolution operator $T_\mu \in \mathcal{B}(L^p(\mathbb{R}^n, \mathbb{C}^{m\times m}))$, $p \in [1, \infty)$, by

$$(3.14)\qquad T_\mu f = f * \mu, \quad f \in L^p(\mathbb{R}^n, \mathbb{C}^{m\times m}),$$

satisfying (cf. (3.13))

$$(3.15)\qquad \|T_\mu\|_{\mathcal{B}(L^p(\mathbb{R}^n, \mathbb{C}^{m\times m}))} \leq |\mu|(\mathbb{R}^n), \quad p \in [1, \infty).$$

Next, we introduce the following equivalent norm in $L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$:

$$(3.16)\qquad |||f|||_{1,m} := \sum_{j,k=1}^m \|f_{j,k}\|_1, \quad f \in L^1(\mathbb{R}^n, \mathbb{C}^{m\times m}),$$

such that

$$(3.17)\qquad (c_m')^{-1} |||f|||_{1,m} \leq \|f\|_{1,m} \leq c_m' |||f|||_{1,m}, \quad f \in L^1(\mathbb{R}^n, \mathbb{C}^{m\times m}),$$

where $c_m' \geq 1$ is chosen such that

$$(3.18)\qquad (c_m')^{-1} \sum_{j,k=1}^m |A_{j,k}| \leq \|A\|_{\mathcal{B}(\mathbb{C}^m)} \leq c_m' \sum_{j,k=1}^m |A_{j,k}|, \quad A \in \mathbb{C}^{m\times m}.$$

Similarly, if we introduce the following equivalent norm in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$:

$$(3.19)\qquad |||f|||_{2,m} := \sum_{j,k=1}^m \|f_{j,k}\|_2, \quad f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}),$$

then there exists $c_m'' \geq 1$ such that

$$(3.20)\qquad (c_m'')^{-1} |||f|||_{2,m} \leq \|f\|_{2,m} \leq c_m'' |||f|||_{2,m}, \quad f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}).$$

Next, we also introduce the following equivalent norm in $L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$:

$$(3.21)\qquad |||f|||_{\infty,m} := \max_{1\leq j,k\leq m} \|f_{j,k}\|_\infty, \quad f \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m});$$

then there exists $d_m \geq 1$ such that

$$(3.22)\qquad (d_m)^{-1} |||f|||_{\infty,m} \leq \|f\|_{\infty,m} \leq d_m |||f|||_{\infty,m}, \quad f \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}).$$

In the special case $m = 1$ we will omit the extra subscript 1 in (3.16), (3.19), and (3.21).

For future use in Section 4, we also introduce $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}})$, where $\mathbb{C}^{m\times m}_{\mathrm{HS}}$ denotes the space $(\mathbb{C}^{m\times m}, \|\cdot\|_{\mathcal{B}_2(\mathbb{C}^m)})$ (i.e., the operator norm $\|\cdot\|_{\mathcal{B}(\mathbb{C}^m)}$ is now replaced by the Hilbert–Schmidt norm $\|\cdot\|_{\mathcal{B}_2(\mathbb{C}^m)}$), as follows: First, $\mathbb{C}^{m\times m}$ can be identified with $\mathbb{C}^{m^2}$, and then the standard Euclidean norm on $\mathbb{C}^{m^2}$ becomes the Hilbert–Schmidt norm $\|\cdot\|_{\mathcal{B}_2(\mathbb{C}^m)}$ (cf., e.g., [7, p. 93]), and hence the space $\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})$ can be identified with $\mathbb{C}^{m^2\times m^2}$. Summarizing,

$$\mathbb{C}^{m\times m}_{\mathrm{HS}} \simeq \mathcal{B}_2(\mathbb{C}^m) \simeq \mathbb{C}^{m^2}, \quad \mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}}) \simeq \mathcal{B}(\mathbb{C}^{m^2}) \simeq \mathbb{C}^{m^2\times m^2}. \tag{3.23}$$

Then we introduce

$$\begin{aligned} L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}}) = \Big\{ f\colon \mathbb{R}^n \to \mathbb{C}^{m\times m} \text{ measurable} \,\Big|\, & \\ \|f\|_{L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}})} &= \Big(\int_{\mathbb{R}^n} \|f(x)\|^2_{\mathcal{B}_2(\mathbb{C}^m)}\, d^n x \Big)^{1/2} \\ &= \Big(\int_{\mathbb{R}^n} \sum_{j,k=1}^m |f_{j,k}(x)|^2\, d^n x \Big)^{1/2} < \infty \Big\}, \end{aligned} \tag{3.24}$$

so that as sets, $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}})$ coincide, but their norms (and scalar products) differ. The classical Plancherel theorem then yields

$$\|f\|_{L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}})} = \|f^\wedge\|_{L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}})}, \quad f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}}). \tag{3.25}$$

Next, we define left translations L_x, $x \in \mathbb{R}^n$, acting on $f\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ via

$$(L_x f)(y) = f(y - x), \quad y \in \mathbb{R}^n. \tag{3.26}$$

Definition 3.1. Let $T \in \mathcal{B}(L^p(\mathbb{R}^n, \mathbb{C}^{m\times m}))$, $p \in [1, \infty) \cup \{\infty\}$. Then T is called $\mathbb{C}^{m\times m}$-*linear* if

$$T(Af) = A(Tf), \quad A \in \mathbb{C}^{m\times m},\ f \in L^p(\mathbb{R}^n, \mathbb{C}^{m\times m}). \tag{3.27}$$

Proposition 3.2 ([11, p. 27]). *Let $p \in [1, \infty)$ and $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then $T_\mu \in \mathcal{B}(L^p(\mathbb{R}^n, \mathbb{C}^{m\times m}))$ is $\mathbb{C}^{m\times m}$-linear and*

$$L_x T_\mu f = T_\mu L_x f, \quad x \in \mathbb{R}^n,\ f \in L^p(\mathbb{R}^n, \mathbb{C}^{m\times m}). \tag{3.28}$$

Proposition 3.3 ([11, Proposition 3.1.10, Corollary 3.1.11]).

(i) *Let $p \in [1, \infty)$ and assume that $T \in \mathcal{B}(L^p(\mathbb{R}^n, \mathbb{C}^{m\times m}))$ is $\mathbb{C}^{m\times m}$-linear. Then the following assertions are equivalent:*

(α) *$T = T_\mu$ for some $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$.*

(β) *$L_x T = T L_x$ for all $x \in \mathbb{R}^n$, and*

$$T \in \mathcal{B}(C_0(\mathbb{R}^n, \mathbb{C}^{m\times m}), C_b(\mathbb{R}^n, \mathbb{C}^{m\times m})).$$

(ii) *Assume that* $T \in \mathcal{B}(L^1(\mathbb{R}^n, \mathbb{C}^{m\times m}))$ *is* $\mathbb{C}^{m\times m}$*-linear. Then the following assertions are equivalent:*

(γ) $T = T_\mu$ *for some* $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$.
(δ) $L_x T = T L_x$ *for all* $x \in \mathbb{R}^n$.

Next, given $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$, we also define the associated operator $F(-i\nabla) \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}))$ by

$$F(-i\nabla)f = (f^\wedge F)^\vee, \quad f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}). \tag{3.29}$$

More generally, if $F \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^{m\times m})$, one introduces the maximally defined operator of right multiplication by F in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$, denoted by M_F, by

$$\begin{aligned} &(M_F f)(x) = f(x)F(x), \\ &f \in \mathrm{dom}(M_F) = \{g \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}) \mid gF \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})\}, \end{aligned} \tag{3.30}$$

and then defines $F(-i\nabla)$ as the closed operator in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ via

$$F(-i\nabla)f = \mathcal{F}^{-1}(M_F(\mathcal{F}f)) \tag{3.31}$$

(cf. (1.16), (1.17) and their unitary extensions to $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}})$ as indicated in (3.25)).

Lemma 3.4. *If* $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$, *then*

$$L_x F(-i\nabla) = F(-i\nabla) L_x, \quad x \in \mathbb{R}^n. \tag{3.32}$$

Since $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ is dense in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$, and all operators in (3.32) are bounded, it suffices to prove (3.32) for $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$. The latter follows from a straightforward calculation.

For future reference we also recall the following results: Introducing

$$j_a(x) = e^{-a|x|}, \quad a > 0,\ x \in \mathbb{R}, \tag{3.33}$$

one verifies

$$j_a^\wedge(y) = \frac{1}{(2\pi)^{1/2}} \frac{2a}{y^2 + a^2}, \quad y \in \mathbb{R}. \tag{3.34}$$

Similarly, introducing

$$k_a(x) = \prod_{\ell=1}^n j_a(x_\ell), \quad x \in \mathbb{R}^n, \tag{3.35}$$

one obtains

$$k_a^\wedge(y) = \frac{1}{(2\pi)^{n/2}} \prod_{\ell=1}^n \frac{2a}{y_\ell^2 + a^2}, \quad y \in \mathbb{R}^n, \tag{3.36}$$

and hence

$$\|k_a^\wedge\|_1 = \int_{\mathbb{R}^n} |k_a^\wedge(y)|\, d^n y = (2\pi)^{n/2}. \tag{3.37}$$

LEMMA 3.5. *Let $a > 0$ and introduce the following diagonal matrix:*

$$M_a(x) = k_a(x) I_{\mathbb{C}^m}, \qquad x \in \mathbb{R}^n. \tag{3.38}$$

Then there exists $c_m \geq 1$ such that

$$\|(M_a^{\wedge} F)^{\vee}\|_{\infty,m} \leq c_m^2 \|F\|_{\infty,m}, \qquad F \in L^{\infty}(\mathbb{R}^n, \mathbb{C}^{m\times m}). \tag{3.39}$$

Proof. Recalling the definition of $\|\cdot\|_{\max}$ in (1.12), there exists $c_m \geq 1$ such that

$$c_m^{-1} \|A\|_{\mathcal{B}(\mathbb{C}^m)} \leq \|A\|_{\max} \leq c_m \|A\|_{\mathcal{B}(\mathbb{C}^m)}, \qquad A \in \mathbb{C}^{m\times m}. \tag{3.40}$$

Next, let $x \in \mathbb{R}^n$, $1 \leq j,k \leq m$. Then

$$\begin{aligned} |(M_a^{\wedge} F)^{\vee}(x)_{j,k}| &= (2\pi)^{-n/2} \Big| \int_{\mathbb{R}^n} e^{i(x\cdot y)} (M_a^{\wedge} F)(y)_{j,k}\, d^n y \Big| \\ &= (2\pi)^{-n/2} \Big| \int_{\mathbb{R}^n} e^{i(x\cdot y)} k_a^{\wedge}(y) F(y)_{j,k}\, d^n y \Big| \\ &\leq (2\pi)^{-n/2} \Big[\operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \|F(y)\|_{\max} \Big] \int_{\mathbb{R}^n} |k_a^{\wedge}(y)|\, d^n y \\ &= \operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \|F(y)\|_{\max}, \end{aligned} \tag{3.41}$$

employing (3.37). Thus,

$$\|(M_a^{\wedge} F)^{\vee}(x)\|_{\max} \leq \operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \|F(y)\|_{\max},\ x \in \mathbb{R}^n,\ F \in L^{\infty}(\mathbb{R}^n, \mathbb{C}^{m\times m}), \tag{3.42}$$

and hence

$$\begin{aligned} \|(M_a^{\wedge} F)^{\vee}\|_{\infty,m} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(M_a^{\wedge} F)^{\vee}(x)\|_{\mathcal{B}(\mathbb{C}^m)} \\ &\leq c_m \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(M_a^{\wedge} F)^{\vee}(x)\|_{\max} \\ &\leq c_m \operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \|F(y)\|_{\max} \\ &\leq c_m^2 \operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \|F(y)\|_{\mathcal{B}(\mathbb{C}^m)} = c_m^2 \|F\|_{\infty,m}. \ \blacksquare \end{aligned} \tag{3.43}$$

In the following we use the notation $0 \leq g \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ if $g \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $g(x) \geq 0$ (i.e., $g(x) \in \mathbb{C}^{m\times m}$ is positive semidefinite) for (Lebesgue) a.e. $x \in \mathbb{R}^n$.

DEFINITION 3.6. Let $T \in \mathcal{B}\big(L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})\big)$. Then T is called *positivity preserving* (in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$) if for any $0 \leq f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ also $Tf \geq 0$.

As will be shown in Lemma 3.13, for T to be positivity preserving it suffices to take $0 \leq f \in C_0^{\infty}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ in Definition 3.6.

Lemma 3.7. *Suppose that $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $F(-i\nabla)$ is positivity preserving in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then, with $c_m \geq 1$ as in* (3.40),

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)f)(x)\|_{\max} \leq 2c_m^4 \|F\|_{\infty,m} \tag{3.44}$$

for all $f \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ satisfying

$$\begin{aligned} &\text{(i) } \operatorname{supp}(f) \text{ is compact,} \\ &\text{(ii) } \sup\nolimits_{x\in\mathbb{R}^n} \|f(x)\|_{\max} \leq 1, \\ &\text{(iii) } f(x) \geq 0 \text{ for a.e. } x \in \mathbb{R}^n. \end{aligned} \tag{3.45}$$

Proof. By the spectral theorem one obtains, for a.e. $x \in \mathbb{R}^n$,

$$0 \leq f(x) \leq \|f(x)\|_{\mathcal{B}(\mathbb{C}^m)} I_{\mathbb{C}^m} \leq c_m \|f(x)\|_{\max} I_{\mathbb{C}^m} \leq c_m I_{\mathbb{C}^m}, \tag{3.46}$$

employing $c_m \geq 1$ in (3.40). Since $\operatorname{supp}(f)$ is compact, there exists a sufficiently small $a > 0$ such that for a.e. $x \in \mathbb{R}^n$,

$$0 \leq f(x) \leq 2c_m k_a(x) I_{\mathbb{C}^m}, \tag{3.47}$$

with k_a introduced in (3.35) ([1]). Since $F(-i\nabla)$ is positivity preserving by hypothesis,

$$0 \leq F(-i\nabla)f \leq 2c_m F(-i\nabla)(k_a I_{\mathbb{C}^m}), \tag{3.48}$$

implying

$$\|(F(-i\nabla)f)(x)\|_{\mathcal{B}(\mathbb{C}^m)} \leq 2c_m \|(F(-i\nabla)(k_a I_{\mathbb{C}^m}))(x)\|_{\mathcal{B}(\mathbb{C}^m)} \tag{3.49}$$

for a.e. $x \in \mathbb{R}^n$. Thus,

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)f)(x)\|_{\max} &\leq c_m \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)f)(x)\|_{\mathcal{B}(\mathbb{C}^m)} \\ &\leq 2c_m^2 \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)(k_a I_{\mathbb{C}^m}))(x)\|_{\mathcal{B}(\mathbb{C}^m)} \\ &= 2c_m^2 \|F(-i\nabla)(k_a I_{\mathbb{C}^m})\|_{\infty,m} = 2c_m^2 \|(M_a^\wedge F)^\vee\|_{\infty,m} \leq 2c_m^4 \|F\|_{\infty,m}, \end{aligned} \tag{3.50}$$

applying Lemma 3.5. ∎

Next, let $A \in \mathcal{B}(\mathcal{H})$ and denote, as usual,

$$\operatorname{Re}(A) = 2^{-1}(A + A^*), \quad \operatorname{Im}(A) = (2i)^{-1}(A - A^*). \tag{3.51}$$

Since $\operatorname{Re}(A)$ and $\operatorname{Im}(A)$ are self-adjoint in $\mathcal{H}$, we define their positive and negative parts, denoted by $\operatorname{Re}(A)_\pm$ and $\operatorname{Im}(A)_\pm$, as well as $|\operatorname{Re}(A)|$ and $|\operatorname{Im}(A)|$, with the help of the spectral theorem (with $|T| = (T^*T)^{1/2}$, $T \in \mathcal{B}(\mathcal{H})$), and hence obtain

$$\operatorname{Re}(A)_\pm = 2^{-1}[|\operatorname{Re}(A)| \pm \operatorname{Re}(A)], \quad \operatorname{Im}(A)_\pm = 2^{-1}[|\operatorname{Im}(A)| \pm \operatorname{Im}(A)]. \tag{3.52}$$

([1]) Actually, the factor 2 in (3.47) can be replaced by $1+\varepsilon$ for $\varepsilon > 0$ sufficiently small, provided that we choose $a = a(\varepsilon) > 0$ sufficiently small, but since this plays no role in the following, we ignore this improvement.

Moreover, since $\|T\|_{\mathcal{B}(\mathcal{H})} = \big\||T|\big\|_{\mathcal{B}(\mathcal{H})}$, one obtains (with $T = \mathrm{Re}(A)$)

$$\|\mathrm{Re}(A)_{\pm}\|_{\mathcal{B}(\mathcal{H})} \le \|A\|_{\mathcal{B}(\mathcal{H})}, \quad \|\mathrm{Im}(A)_{\pm}\|_{\mathcal{B}(\mathcal{H})} \le \|A\|_{\mathcal{B}(\mathcal{H})}. \tag{3.53}$$

Next, we drop the nonnegativity hypothesis (iii) in Lemma 3.7 and hence obtain the following result.

Lemma 3.8. *Suppose that $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $F(-i\nabla)$ is positivity preserving in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then, with c_m as in* (3.40),

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)f)(x)\|_{\max} \le 8c_m^6 \|F\|_{\infty,m} \tag{3.54}$$

for all $f \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ satisfying

$$\begin{aligned} &\text{(i) } \operatorname{supp}(f) \text{ is compact,} \\ &\text{(ii) } \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|f(x)\|_{\max} \le 1. \end{aligned} \tag{3.55}$$

Proof. With c_m as in (3.40), one concludes from the latter and from (3.53) that for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned} \|\mathrm{Re}(f(x))_{\pm}\|_{\max} &\le c_m \|\mathrm{Re}(f(x))_{\pm}\|_{\mathcal{B}(\mathbb{C}^m)} \\ &\le c_m \|f(x)\|_{\mathcal{B}(\mathbb{C}^m)} \le c_m^2 \|f(x)\|_{\max}. \end{aligned} \tag{3.56}$$

Thus, $\mathrm{Re}(f)_{\pm} \colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ satisfies

(α) $\operatorname{supp}(\mathrm{Re}(f)_{\pm})$ is compact,
(β) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|\mathrm{Re}(f(x))_{\pm}\|_{\max} \le c_m^2$,
(γ) $\mathrm{Re}(f(x))_{\pm} \ge 0$ for a.e. $x \in \mathbb{R}^n$.

By Lemma 3.7,

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)\mathrm{Re}(f)_{\pm})(x)\|_{\max} \le 2c_m^6 \|F\|_{\infty,m}, \tag{3.57}$$

and similarly

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)\mathrm{Im}(f)_{\pm})(x)\|_{\max} \le 2c_m^6 \|F\|_{\infty,m}, \tag{3.58}$$

implying

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)f)(x)\|_{\max} \le 8c_m^6 \|F\|_{\infty,m}. \ \blacksquare \tag{3.59}$$

In order to prove a consequence of Lemma 3.8, we need the following auxiliary result.

Lemma 3.9 (cf., e.g., [1, Theorem 2.29 and p. 250]).

(i) *If $f \in L^1(\mathbb{R}^n)$, then $f^\wedge \in C_\infty(\mathbb{R}^n)$ and $\|f^\wedge\|_\infty \le (2\pi)^{-n/2}\|f\|_1$.*
(ii) *Let $f \in C_0(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subseteq \overline{B_n(0,r)}$ for some $r > 0$. Then there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(f_j) \subseteq \overline{B_n(0,2r)}$, $j \in \mathbb{N}$, and $\lim_{j\to\infty} \|f_j - f\|_\infty = 0$.*

Remark 3.10. Let $\sigma\colon \mathfrak{B}_n \to [0,\infty)$ be a finite nonnegative measure on $\mathbb{R}^n$ and let $\mu\colon \mathfrak{B}_n \to \mathbb{C}^{m\times m}$ be the nonnegative matrix-valued measure defined by

$$\mu(E) = \sigma(E) I_m, \quad E \in \mathfrak{B}_n. \tag{3.60}$$

Then $T_\mu \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}))$ is positivity preserving. Indeed, let $f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then

$$\int_{\mathbb{R}^n} f(y)\, d\mu(y) = \Big\{ \int_{\mathbb{R}^n} f_{j,k}(y)\, d\sigma(y) \Big\}_{1\le j,k\le m}. \tag{3.61}$$

Hence, if $0 \le f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$, then for all $v = (v_1, \dots, v_m)^\top \in \mathbb{C}^m$,

$$(v, (T_\mu f)(x) v)_{\mathbb{C}^m} = \int_{\mathbb{R}^n} \sum_{j,k=1}^m \overline{v_j} f_{j,k}(x-y) v_k \, d\sigma(y) \ge 0. \tag{3.62}$$

Lemma 3.11. *Assume that $0 \le f \in C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that $f_j(x) \ge 0$, $j \in \mathbb{N}$, and $\lim_{j\to\infty} f_j = f$ in $(C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}), \|\cdot\|_{\infty,m})$.*

Proof. Clearly one can find a sequence $\{g_j\}_{j\in\mathbb{N}} \subset C_0(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that

$$g_j \ge 0, \quad j \in \mathbb{N}, \quad \lim_{j\to\infty} g_j = f \text{ in } (C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}), \|\cdot\|_{\infty,m}). \tag{3.63}$$

Indeed, let

$$k_n \in C_0(\mathbb{R}^n), \quad 0 \le k_n \le 1, \quad k_n(x) = \begin{cases} 1, & 0 \le |x| \le n, \\ 0, & |x| \ge n+1, \end{cases} \tag{3.64}$$

k_n decreasing from 1 to 0 as $|x|$ increases from n to $n+1$, and set $g_n = k_n f$, $n \in \mathbb{N}$. Then $g_n \ge 0$ on $\mathbb{R}^n$ and $f(x) - g_n(x) = 0$ for $0 \le |x| \le n$. Since $\|g_n(x)\|_{\max} \le \|f(x)\|_{\max}$ and $\lim_{|x|\to\infty} \|f(x)\|_{\max} = 0$, one obtains (3.63). Thus, without loss of generality we may assume that $f \in C_0(\mathbb{R}^n, \mathbb{C}^{m\times m})$.

Next, we recall the definition of standard Friedrichs mollifiers $\{\phi_\varepsilon\}_{\varepsilon>0}$ (cf., e.g., [1, pp. 36, 37]) and introduce

$$\Phi_\varepsilon(x) = \phi_\varepsilon(x) I_m, \quad x \in \mathbb{R}^n,\ \varepsilon > 0. \tag{3.65}$$

In addition, we define the measure $\sigma_\varepsilon \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ by

$$\sigma_\varepsilon(E) = \Big(\int_E \phi_\varepsilon(x)\, d^n x \Big) I_m, \quad E \in \mathfrak{B}_n. \tag{3.66}$$

Then, using the fact that T_{σ_ε} is positivity preserving in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$, one introduces $f_j = T_{\sigma_{1/j}} f$, $j \in \mathbb{N}$, and concludes $f_j \ge 0$, $j \in \mathbb{N}$. Moreover,

$$f_j(x)_{k,\ell} = (f_{k,\ell} * \phi_\varepsilon)(x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N},\ 1 \le k, \ell \le m. \tag{3.67}$$

By standard properties of mollifiers, $(f_j)_{k,\ell} \in C_0^\infty(\mathbb{R}^n)$ and

$$\lim_{j\to\infty} (f_j)_{k,\ell} = f_{k,\ell} \quad \text{in } (C_0(\mathbb{R}^n), \|\cdot\|_\infty),\ 1 \le k,\ell \le m. \tag{3.68}$$

Thus, $0 \le f_j \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $\lim_{j\to\infty} f_j = f$ in $(C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}), \|\cdot\|_{\infty,m})$. ■

COROLLARY 3.12. *Suppose that $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $F(-i\nabla)$ is positivity preserving in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then*

$$F(-i\nabla)\colon (C_0(\mathbb{R}^n, \mathbb{C}^{m\times m}), \|\cdot\|_{\infty,m}) \to (C_b(\mathbb{R}^n, \mathbb{C}^{m\times m}), \|\cdot\|_{\infty,m}) \quad \textit{continuously.} \tag{3.69}$$

In addition, there exists a nonnegative measure $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that $F(-i\nabla) = T_\mu$.

Proof. Suppose $f \in C_0(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $\operatorname{supp}(f) \subseteq \overline{B_n(0,r)}$. Then an application of Lemma 3.9(ii) implies the existence of a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that $\operatorname{supp}(f_j) \subseteq \overline{B_n(0,2r)}$, $j \in \mathbb{N}$, and $\lim_{j\to\infty} \|(f_j)_{k,\ell} - f_{k,\ell}\|_\infty = 0$, $1 \le k,\ell \le m$.

Without loss of generality we may assume that for each $j \in \mathbb{N}$, $f_j - f$ satisfies the hypotheses of Lemma 3.8. Thus, since

$$\lim_{j\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|f_j(x) - f(x)\|_{\max} = 0, \tag{3.70}$$

Lemma 3.8 yields

$$\lim_{j\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|(F(-i\nabla)f_j)(x) - (F(-i\nabla)f)(x)\|_{\max} = 0. \tag{3.71}$$

Since $f_j \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$, $f_j^\wedge \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$, one concludes that $f_j^\wedge F \in L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and hence Lemma 3.9(i) implies $F(-i\nabla)f_j = \big(f_j^\wedge F\big)^\vee \in C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Hence, $F(-i\nabla)f$ is the uniform limit of a bounded sequence $\{F(-i\nabla)f_j\}_{j\in\mathbb{N}} \subset C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and thus $F(-i\nabla)f \in C_b(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Lemma 3.8 implies that $F(-i\nabla)$ maps $(C_0(\mathbb{R}^n, \mathbb{C}^{m\times m}), \|\cdot\|_{\infty,m})$ to the space $(C_b(\mathbb{R}^n, \mathbb{C}^{m\times m}), \|\cdot\|_{\infty,m})$ continuously. That there exists a measure $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that $F(-i\nabla) = T_\mu$ follows from Proposition 3.3(i) (upon choosing $T = F(-i\nabla)$ in Proposition 3.3(i)(β)) and Lemma 3.4. Identifying $\mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ with $C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})^*$, it remains to show that

$$\operatorname{tr}_{\mathbb{C}^m}\Big(\int_{\mathbb{R}^n} f(x)\, d\mu(x)\Big) \ge 0, \qquad 0 \le f \in C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}). \tag{3.72}$$

By Lemma 3.11 it suffices to prove this for all $0 \le f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Thus, let $0 \le f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then $f^\wedge \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and hence by Lemma 3.9(i),

$$F(-i\nabla)f = \big(f^\wedge F\big)^\vee \in C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}). \tag{3.73}$$

In addition, since $F(-i\nabla)$ is positivity preserving,

$$0 \leq (F(-i\nabla)f)(0) = (T_\mu f)(0) = \int_{\mathbb{R}^n} f(-y)\, d\mu(y). \tag{3.74}$$

Thus,

$$\operatorname{tr}_{\mathbb{C}^m}\Big(\int_{\mathbb{R}^n} f(-y)\, d\mu(y)\Big) \geq 0, \tag{3.75}$$

and hence μ is nonnegative. ∎

We also add the following auxiliary result.

Lemma 3.13. *Let $f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and suppose $f(x) \geq 0$ for a.e. $x \in \mathbb{R}^n$. Then there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that for all $j \in \mathbb{N}$, $f_j(x) \geq 0$ for a.e. $x \in \mathbb{R}^n$, and $\lim_{j\to\infty} \|f_j - f\|_{2,m} = 0$.*

Proof. Let ϕ_ε, Φ_ε, and σ_ε, $\varepsilon > 0$, be as in the proof of Lemma 3.11, and recall that T_{σ_ε} is positivity preserving in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Next, let $0 \leq f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and introduce

$$g_j = (\chi_{[-j,j]^n} I_m) f, \quad j \in \mathbb{N}, \tag{3.76}$$

where χ_A denotes the characteristic function of $A \subset \mathbb{R}^n$. Clearly, $0 \leq g_j \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$, $\operatorname{supp}(g_j)$ is compact, $j \in \mathbb{N}$, and $\lim_{j\to\infty} \|g_j - f\|_{2,m} = 0$. Hence, it suffices to show that if $0 \leq g \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $\operatorname{supp}(g)$ is compact, then there exists a sequence $\{h_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that $h_j \geq 0$, $j \in \mathbb{N}$, and $\lim_{j\to\infty} \|h_j - g\|_{2,m} = 0$. Thus, let

$$h_j = T_{\sigma_{1/n}} g, \quad j \in \mathbb{N}. \tag{3.77}$$

Then $h_j \geq 0$ since $T_{\sigma_{1/n}}$ is positivity preserving and

$$h_j(x)_{k,\ell} = (g_{k,\ell} * \phi_{1/n})(x), \quad x \in \mathbb{R}^n,\ 1 \leq k, \ell \leq m. \tag{3.78}$$

By standard properties of Friedrichs mollifiers (cf., e.g., [1, pp. 36, 37]), $(h_j)_{k,\ell} \in C_0^\infty(\mathbb{R}^n)$ and

$$\lim_{j\to\infty} \|(h_j)_{k,\ell} - g_{k,\ell}\|_2 = 0, \quad 1 \leq k, \ell \leq m, \tag{3.79}$$

implying $\{h_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $\lim_{j\to\infty} \|h_j - g\|_{2,m} = 0$. ∎

Introducing the *Hadamard product* $A \circ_{\mathrm{H}} B$ of two matrices $A, B \in \mathbb{C}^{m\times m}$, by

$$(A \circ_{\mathrm{H}} B)_{j,k} = A_{j,k} B_{j,k}, \quad 1 \leq j, k \leq m, \tag{3.80}$$

we conclude this section with the following remark, addressing the lack of the semigroup property of $\exp_{\mathrm{H}}(tF)(-i\nabla)$.

Remark 3.14. Suppose that $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ is conditionally positive semidefinite such that for some $c \in \mathbb{R}$,

$$\operatorname{Re}(F(x)_{j,k}) \leq c \quad \text{for a.e. } x \in \mathbb{R}^n,\ 1 \leq j, k \leq m. \tag{3.81}$$

In addition, introduce

$$f(t) = (\exp_{\mathrm{H}}(tF)(-i\nabla))f, \quad f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}),\ t \geq 0. \tag{3.82}$$

Then

$$\frac{d}{dt}(f(t)) = \big(f^\wedge((\exp_{\mathrm{H}}(tF)) \circ_{\mathrm{H}} F)\big)^\vee, \quad t > 0. \tag{3.83}$$

4. Operators associated with matrix-valued positive semidefinite functions. In this section we prove our principal results. In particular, we give analogs of the classical Theorems 1.2 and 1.3(i)–(iii) in the matrix-valued context to the extent possible and along the way introduce the necessary modifications needed to obtain such extensions. We also recall Fourier multiplier theorems in the L^1 and L^2 context extending classical results in the scalar case to the matrix-valued situation.

We start with the following fact.

THEOREM 4.1. *Suppose that $F \in C(\mathbb{R}^n, \mathbb{C}^{m\times m}) \cap L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and $F(-i\nabla)$ is positivity preserving in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then there exists a nonnegative measure $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that*

$$F(x) = \mu^\wedge(x), \quad x \in \mathbb{R}^n, \tag{4.1}$$

equivalently,

$$F(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x\cdot\xi)}\, d\mu(\xi), \quad x \in \mathbb{R}^n. \tag{4.2}$$

Proof. Define ϕ_ε and Φ_ε as in the proof of Lemma 3.13 and introduce

$$\Phi_{\varepsilon,x}(y) = \Phi_\varepsilon(x-y), \quad x, y \in \mathbb{R}^n,\ \varepsilon > 0. \tag{4.3}$$

Suppose $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then

$$\begin{aligned}(F(-i\nabla)f)(x) &= (f^\wedge F)^\vee(x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi\cdot(x-\eta))} f(\eta)F(\xi)\, d^n\eta\, d^n\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi\cdot\omega)} f(x-\omega)F(\xi)\, d^n\omega\, d^n\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (f(x-\cdot))^\vee(\xi)F(\xi)\, d^n\xi.\end{aligned} \tag{4.4}$$

Introducing $f_{\varepsilon,x} \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ by

$$f_{\varepsilon,x}(y) = (\Phi_{\varepsilon,x})^\wedge(x-y), \quad x, y \in \mathbb{R}^n,\ \varepsilon > 0, \tag{4.5}$$

one obtains, for $\varepsilon > 0$,

$$
\begin{aligned}
(4.6) \qquad (F(-i\nabla) f_{\varepsilon,x})(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (f_{\varepsilon,x}(x-\cdot))^{\vee}(\xi) F(\xi)\, d^n\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (\Phi_{\varepsilon,x}^{\wedge})^{\vee}(\xi) F(\xi)\, d^n\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi_{\varepsilon,x}(\xi) F(\xi)\, d^n\xi \underset{\varepsilon\downarrow 0}{\longrightarrow} (2\pi)^{-n/2} F(x), \quad x \in \mathbb{R}^n.
\end{aligned}
$$

By Corollary 3.12, there exists a nonnegative measure $\mu_0 \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that $F(-i\nabla) = T_{\mu_0}$. Hence,

$$
\begin{aligned}
(4.7) \qquad (F(-i\nabla) f_{\varepsilon,x})(x) &= (T_{\mu_0} f_{\varepsilon,x})(x) = (f_{\varepsilon,x} * \mu_0)(x) \\
&= \int_{\mathbb{R}^n} f_{\varepsilon,x}(x-\eta)\, d\mu_0(\eta) = \int_{\mathbb{R}^n} \Phi_{\varepsilon,x}^{\wedge}(\eta)\, d\mu_0(\eta) \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(\eta\cdot\xi)} \Phi_{\varepsilon,x}(\xi)\, d^n\xi\, d\mu_0(\eta), \quad x \in \mathbb{R}^n,\ \varepsilon > 0.
\end{aligned}
$$

Since $\Phi_{\varepsilon,x}$ has compact support and $\mu_{k,\ell}$, $1 \le k,\ell \le m$, are finite complex measures on $\mathbb{R}^n$, one can interchange the order of integration in the last double integral in (4.7) to arrive at

$$
\begin{aligned}
(4.8) \qquad (F(-i\nabla) f_{\varepsilon,x})(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi_{\varepsilon,x}(\xi) \Big(\int_{\mathbb{R}^n} e^{-i(\xi\cdot\eta)}\, d\mu_0(\eta) \Big)\, d^n\xi \\
&= \int_{\mathbb{R}^n} \Phi_{\varepsilon}(x-\xi) \mu_0^{\wedge}(\xi)\, d^n\xi \underset{\varepsilon\downarrow 0}{\longrightarrow} \mu_0^{\wedge}(x), \quad x \in \mathbb{R}.
\end{aligned}
$$

Thus, (4.1) follows with $\mu = (2\pi)^{n/2}\mu_0$. ∎

Remark 4.2. In Appendix A we will prove that the converse to Theorem 4.1, that is, if $F = \mu^{\wedge}$ for some nonnegative $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ then $F(-i\nabla)$ is positivity preserving in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$, does **not** hold (unless, of course, μ is of the type $\mu_\sigma = \sigma I_{\mathbb{C}^m}$ with $\sigma\colon \mathfrak{B}_n \to [0,\infty)$ a finite measure).

Next, we recall the finite-dimensional special case of an infinite-dimensional version of Bochner's theorem (cf. Theorem 1.4) in connection with locally compact Abelian groups due to Berberian [3] (see also [12], [13], [29], [43]):

Theorem 4.3 ([3, p 178, Theorem 3 and Corollary on p. 177]). *Assume that $F \in C(\mathbb{R}^n, \mathbb{C}^{m\times m}) \cap L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then the following conditions are equivalent:*

(i) *F is positive semidefinite.*
(ii) *There exists a nonnegative measure $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that*

$$
(4.9) \qquad F(x) = \mu^{\wedge}(x), \quad x \in \mathbb{R}^n.
$$

In addition, if (i) *or* (ii) *holds, then*

$$F(-x) = F(x)^*, \quad \|F(x)\|_{\mathcal{B}(\mathbb{C}^m)} \leq \|F(0)\|_{\mathcal{B}(\mathbb{C}^m)}, \quad x \in \mathbb{R}^n. \tag{4.10}$$

We note that Berberian [3, p. 178, Theorem 3] discusses a seemingly more general result in which boundedness of F is not assumed—it is, however, a consequence of his results.

Next, we extend the classical L^1-multiplier theorem due to Bochner (cf., e.g., [18, Theorem 2.5.8 and pp. 143, 144], [37, p. 28], [38, pp. 29, 30]) to the matrix-valued context. An infinite-dimensional version of this result appeared in Gaudry, Jefferies, and Ricker [14, Proposition 3.15 and Corollary 3.20]. For completeness, we present an elementary proof in the matrix-valued case and add the estimates (4.12) which appear to be new in this context.

We recall the definition (3.3) of $N(\mu)$ and the definition of $\||\cdot\||_{1,m}$ in (3.16).

THEOREM 4.4. *Assume that* $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$. *Then the following conditions are equivalent:*

(i) $F(-i\nabla)|_{C_0^\infty(\mathbb{R}^n,\mathbb{C}^{m\times m})}$ *can be extended to a bounded operator* (*denoted by the same symbol, for simplicity*) $F(-i\nabla) \in \mathcal{B}(L^1(\mathbb{R}^n, \mathbb{C}^{m\times m}))$.

(ii) *There exists a measure* $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ *such that*

$$F(x) = \mu^\wedge(x), \quad x \in \mathbb{R}^n. \tag{4.11}$$

In addition, if (i) *or* (ii) *holds, then*

$$(2\pi)^{-n/2} N(\mu) \leq \|F(-i\nabla)\|_{\mathcal{B}((L^1(\mathbb{R}^n,\mathbb{C}^{m\times m}),\||\cdot\||_{1,m}))} \leq m(2\pi)^{-n/2} N(\mu). \tag{4.12}$$

Both estimates in (4.12) *are sharp.*

Proof. First, suppose that (ii) holds. Let $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$. Then

$$(F(-i\nabla)f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi\cdot(x-\eta))} f^\wedge(\xi)\, d\mu(\eta)\, d^n\xi, \quad x \in \mathbb{R}^n. \tag{4.13}$$

Since $f^\wedge \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m}) \subset L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$, one can interchange the order of integration in (4.13) to obtain

$$\begin{aligned}(F(-i\nabla)f)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi\cdot(x-\eta))} f^\wedge(\xi)\, d^n\xi\, d\mu(\eta) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (f^\wedge)^\vee(x-\eta)\, d\mu(\eta) = (2\pi)^{-n/2}(T_\mu f)(x), \quad x \in \mathbb{R}^n.\end{aligned} \tag{4.14}$$

Thus, applying (3.9)–(3.15) one gets

$$\|F(-i\nabla)\|_{\mathcal{B}(L^1(\mathbb{R}^n,\mathbb{C}^{m\times m}))} \leq (2\pi)^{-n/2}\|\mu\|, \tag{4.15}$$

implying condition (i).

To prove the converse, suppose that (i) holds. We introduce $I(j,k) \in \mathbb{C}^{m\times m}$ by

$$I(j,k)_{p,q} = \begin{cases} 1 & \text{if } p=j \text{ and } q=k, \\ 0 & \text{if } p\neq j \text{ or } q\neq k, \end{cases} \qquad 1\leq j,k,p,q\leq m. \tag{4.16}$$

In addition, let

$$U(j,k)\colon \begin{cases} L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n,\mathbb{C}^{m\times m}), \\ g\mapsto U(j,k)g = gI(j,k), \end{cases} \qquad 1\leq j,k\leq m, \tag{4.17}$$

and

$$D(j,k)\colon \begin{cases} L^1(\mathbb{R}^n,\mathbb{C}^{m\times m}) \to L^1(\mathbb{R}^n), \\ f\mapsto D(j,k)f = f_{j,k}, \end{cases} \qquad 1\leq j,k\leq m. \tag{4.18}$$

One verifies that $U(j,k)$ and $D(j,k)$ are bounded for each $1\leq j,k\leq m$, and hence also

$$P(p,q,j,k) = D(p,q)F(-i\nabla)U(j,k)\colon L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n), \qquad 1\leq j,k,p,q\leq m, \tag{4.19}$$

are bounded. Employing the fact that

$$P(1,k,1,j)g = (g^{\wedge}F_{j,k})^{\vee}, \quad g\in L^1(\mathbb{R}^n), \tag{4.20}$$

one infers that the linear operator $L^1(\mathbb{R}^n)\ni g\mapsto (g^{\wedge}F_{j,k})^{\vee}\in L^1(\mathbb{R}^n)$ is bounded, that is, $F_{j,k}$ is an $L^1(\mathbb{R}^n)$-multiplier. By the classical Bochner theorem, there exists a (finite) complex measure $\mu_{k,j}$ on $\mathbb{R}^n$ such that $F_{j,k} = \mu_{j,k}^{\wedge}$. Introducing $\mu = \{\mu_{j,k}\}_{1\leq j,k\leq m}\in\mathcal{M}(\mathbb{R}^n,\mathbb{C}^{m\times m})$, we have $F=\mu^{\wedge}$, and hence (ii) holds.

Next we turn to the lower bound in (4.12). Choose $p,q\in\{1,\dots,m\}$ such that

$$N(\mu) = |\mu_{p,q}|(\mathbb{R}^n). \tag{4.21}$$

Since $F_{p,q} = \mu_{p,q}^{\wedge}$, the classical (i.e., scalar-valued) L^1-multiplier theorem applies, and hence $F_{p,q}(-i\nabla)|_{C_0^\infty(\mathbb{R}^n)}$ can be extended to a bounded operator $F_{p,q}(-i\nabla)\in\mathcal{B}(L^1(\mathbb{R}^n))$ with norm

$$\|F_{p,q}(-i\nabla)\|_{\mathcal{B}(L^1(\mathbb{R}^n))} = (2\pi)^{-n/2}\|\mu_{p,q}\| = (2\pi)^{-n/2}|\mu_{p,q}|(\mathbb{R}^n). \tag{4.22}$$

Thus, there exists a sequence $\{f_\ell\}_{\ell\in\mathbb{N}}$ in $L^1(\mathbb{R}^n)$ with $\|f_\ell\|_1 = 1$, $\ell\in\mathbb{N}$, such that

$$\lim_{\ell\to\infty}\|F_{p,q}(-i\nabla)f_\ell\|_1 = (2\pi)^{-n/2}|\mu_{p,q}|(\mathbb{R}^n). \tag{4.23}$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, we can assume that $f_\ell\in C_0^\infty(\mathbb{R}^n)$, $\ell\in\mathbb{N}$. Introduce (cf. (4.17))

$$g_\ell = U(1,p)f_\ell, \quad \ell\in\mathbb{N}. \tag{4.24}$$

Then

$$(F(-i\nabla)g_\ell)_{r,s} = \begin{cases} 0, & 2 \le r \le m, \\ (f_\ell^\wedge F_{p,s})^\vee, & r = 1, \end{cases} \tag{4.25}$$

and hence

$$|||\, g_\ell |||_{1,m} = \|f_\ell\|_1, \quad \ell \in \mathbb{N}, \tag{4.26}$$

and

$$\begin{aligned} |||F(-i\nabla)g_\ell|||_{1,m} &= \sum_{s=1}^m \|(f_\ell^\wedge F_{p,s})^\vee\|_1 \\ &\ge \|(f_\ell^\wedge F_{p,q})^\vee\|_1 = \|F_{p,q}(-i\nabla)f_\ell\|_1 \xrightarrow[\ell\to\infty]{} (2\pi)^{-n/2}|\mu_{p,q}|(\mathbb{R}^n), \end{aligned} \tag{4.27}$$

implying the lower bound in (4.12).

To show that this lower bound is best possible it suffices to look at the following example. With $\gamma_n \colon \mathfrak{B}_n \to [0,1]$ the standard Gaussian measure on $\mathbb{R}^n$,

$$\gamma_n(E) = (2\pi)^{-n/2} \int_E \exp(-|x|^2/2)\, d^n x, \quad E \in \mathfrak{B}_n, \tag{4.28}$$

introduce the measure $\mu_0 \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ via

$$\mu_{0,j,k}(E) = \gamma_n(E)\delta_{j,1}\delta_{k,1}, \quad 1 \le j,k \le m,\, E \in \mathfrak{B}_n, \tag{4.29}$$

and let $F_0 = \mu_0^\wedge$. For $f \in L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$ with $|||f|||_{1,m} = 1$ one obtains

$$\begin{aligned} |||F_0(-i\nabla)f|||_{1,m} &= \sum_{j=1}^m \|(f_{j,1}^\wedge \gamma_n^\wedge)^\vee\|_1 \\ &\le \sum_{j=1}^m \|\gamma_n^\wedge(-i\nabla)\|_{\mathcal{B}(L^1(\mathbb{R}^n))} \sum_{j=1}^m \|f_{j,1}\|_1 \\ &\le (2\pi)^{-n/2}\gamma_n(\mathbb{R}^n)\|f\|_{1,m} = (2\pi)^{-n/2}\gamma_n(\mathbb{R}^n) \\ &= (2\pi)^{-n/2}N(\mu_0), \end{aligned} \tag{4.30}$$

implying $\|F_0(-i\nabla)\|_{\mathcal{B}((L^1(\mathbb{R}^n,\mathbb{C}^{m\times m}),\, |||\cdot|||_{1,m}))} \le (2\pi)^{-n/2}N(\mu_0)$.

Turning to the upper bound in (4.12), let $\varphi \in L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$ with $|||\varphi|||_{1,m} = 1$. Then

$$(F(-i\nabla)\varphi)_{j,k} = \sum_{r=1}^m (\varphi_{j,r}^\wedge F_{r,k})^\vee, \quad 1 \le j,k \le m. \tag{4.31}$$

Applying the classical (i.e., scalar-valued) L^1-multiplier theorem once more, one estimates

$$
\begin{aligned}
(4.32)\qquad |||F(-i\nabla)\varphi|||_{1,m} &= \sum_{j,k=1}^{m} \|(F(-i\nabla)\varphi)_{j,k}\|_1 \\
&\leq \sum_{j,k,r=1}^{m} \|(\varphi_{j,r}^{\wedge} F_{r,k})^{\vee}\|_1 = \sum_{j,k,r=1}^{m} \|F_{r,k}(-i\nabla)\varphi_{j,r}\|_1 \\
&\leq \sum_{j,k,r=1}^{m} \|F_{r,k}(-i\nabla)\|_{\mathcal{B}(L^1(\mathbb{R}^n))}\|\varphi_{j,r}\|_1 = (2\pi)^{-n/2} \sum_{j,k,r=1}^{m} |\mu_{r,k}|(\mathbb{R}^n)\|\varphi_{j,r}\|_1 \\
&\leq (2\pi)^{-n/2} N(\mu) \sum_{k=1}^{m} \sum_{j,r=1}^{m} \|\varphi_{j,r}\|_1 = (2\pi)^{-n/2} N(\mu) m |||\varphi|||_{1,m} \\
&= (2\pi)^{-n/2} N(\mu) m.
\end{aligned}
$$

To demonstrate that this upper bound is best possible, we once more employ the Gaussian measure (4.28) on $\mathbb{R}^n$ and hence introduce the measure $\mu_1 \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ via

$$
(4.33)\qquad \mu_{1,j,k}(E) = \gamma_n(E), \quad 1 \leq j, k \leq m,\ E \in \mathfrak{B}_n,
$$

and let $F_1 = \mu_1^{\wedge}$, such that $F_{1,j,k} = \gamma_n^{\wedge}$, $1 \leq j, k \leq m$. Applying the classical multiplier theorem again, one obtains

$$
(4.34)\qquad \|F_{1,j,k}(-i\nabla)\|_{\mathcal{B}(L^1(\mathbb{R}^n))} = \gamma_n(\mathbb{R}^n) = |\gamma_n|(\mathbb{R}^n) = 1.
$$

Thus, there exists a sequence $\{f_\ell\}_{\ell\in\mathbb{N}}$ in $L^1(\mathbb{R}^n)$ with $\|f_\ell\|_1 = 1$, $\ell \in \mathbb{N}$, such that for all $r, s \in \{1, \ldots, m\}$,

$$
(4.35)\qquad \lim_{\ell\to\infty} \|\gamma_n^{\wedge}(-i\nabla)f_\ell\|_1 = \lim_{\ell\to\infty} \|F_{1,r,s}(-i\nabla)f_\ell\|_1 = 1.
$$

Let $\varphi_\ell \in L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$, $\ell \in \mathbb{N}$, be defined via

$$
(4.36)\qquad \varphi_{\ell,j,k} = m^{-2} f_\ell, \quad \ell \in \mathbb{N},\ 1 \leq j, k \leq m.
$$

Then

$$
(4.37)\qquad |||\varphi_\ell|||_{1,m} = \sum_{j,k=1}^{m} \|\varphi_{\ell,j,k}\|_1 = \sum_{j,k=1}^{m} m^{-2}\|f_\ell\|_1 = 1, \quad \ell \in \mathbb{N}.
$$

Consequently,

$$
(4.38)\qquad (F_1(-i\nabla)\varphi_\ell)_{j,k} = \sum_{r=1}^{m} (\varphi_{\ell,j,r}^{\wedge} F_{1,r,k})^{\vee} = \sum_{r=1}^{m} m^{-2} F_{1,r,k}(-i\nabla) f_\ell, \qquad \ell \in \mathbb{N},\ 1 \leq j, k \leq m,
$$

and thus

$$
\begin{aligned}
(4.39)\qquad |||F_1(-i\nabla)\varphi_\ell|||_{1,m} &= \sum_{j,k=1}^{m} \|(F_1(-i\nabla)\varphi_\ell)_{j,k}\|_1 \\
&= \sum_{j,k=1}^{m} \Big\| \sum_{r=1}^{m} m^{-2} F_{1,r,k}(-i\nabla) f_\ell \Big\|_1 \\
&= \sum_{j,k=1}^{m} m^{-1} \|\gamma_n^\wedge(-i\nabla) f_\ell\|_1 \\
&\xrightarrow[\ell\to\infty]{} (2\pi)^{-n/2} m = (2\pi)^{-n/2} m N(\mu_1). \ \blacksquare
\end{aligned}
$$

Alternatively, one can prove the equivalence (i)⇔(ii) in Theorem 4.4 using (3.15), Proposition 3.3(ii), and Lemma 3.4.

REMARK 4.5. (i) We stress once more that the equivalence of (i)⇔(ii) in Theorem 4.4 was proved by Gaudry, Jefferies, and Ricker [14, Proposition 3.15 and Corollary 3.20] in the infinite-dimensional context. For completeness we decided to present a rather elementary and straightforward proof. The bounds (4.12) appear to be new.

(ii) In the special case $m = 1$, the upper and lower bounds in (4.12) coincide and hence reduce to the classical result

$$\|F(-i\nabla)\|_{\mathcal{B}(L^1(\mathbb{R}^n))} = (2\pi)^{-n/2}\|\mu\|.$$

Next, we also present the L^2-analog of the multiplier Theorem 4.4 (see, e.g., [18, Theorem 2.5.10], [37, p. 28], [38, pp. 28, 29] for the classical version where $m = 1$). An infinite-dimensional version of this result appeared in Gaudry, Jefferies, and Ricker [14, Lemma 2.5 and Proposition 2.8]. For completeness, we present an elementary proof in the matrix-valued case (deferring the proof of (4.41) to Appendix B) and add the estimates (4.42) which appear to be new in this context.

We recall the definition of $|||\cdot|||_{2,m}$ in (3.19) and $|||\cdot|||_{\infty,m}$ in (3.21).

THEOREM 4.6. *Assume that* $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ *is measurable such that* $f^\wedge F \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$, $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ *and define*

$$(4.40)\qquad F(-i\nabla)\colon \begin{cases} C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}) \to L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}), \\ f \mapsto F(-i\nabla)f = (f^\wedge F)^\vee. \end{cases}$$

Then the following conditions are equivalent:

(i) $F(-i\nabla)|_{C_0^\infty(\mathbb{R}^n,\mathbb{C}^{m\times m})}$ *can be extended to a bounded operator* (*denoted by the same symbol*) $F(-i\nabla) \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}))$.

(ii) $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$.

In addition, if (i) *or* (ii) *holds, then*

$$\|F(-i\nabla)\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|F(x)\|_{\mathcal{B}(\mathbb{C}^m)} = \|F\|_{\infty,m}, \tag{4.41}$$

and moreover,

$$|||F|||_{\infty,m} \leq \|F(-i\nabla)\|_{\mathcal{B}((L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}),\,|||\cdot|||_{2,m}))} \leq m|||F|||_{\infty,m}. \tag{4.42}$$

Both estimates in (4.42) *are sharp.*

Proof. Assume that (i) holds. We recall the definitions of $I(j,k)$, $U(j,k)$, $D(j,k)$, and $P(p,q,j,k)$ in (4.16)–(4.19), with L^1 replaced by L^2. Then as in (4.20), $P(1,k,1,j)f = (f^\wedge F_{j,k})^\vee$, $f \in L^2(\mathbb{R}^n)$, and hence the linear operator $L^2(\mathbb{R}^n) \ni g \mapsto (g^\wedge F_{j,k})^\vee \in L^2(\mathbb{R}^n)$ is bounded, that is, $F_{j,k}$ is an $L^2(\mathbb{R}^n)$-multiplier. By the classical L^2-multiplier theorem, $F_{j,k} \in L^\infty(\mathbb{R}^n)$, $1 \leq j,k \leq m$, that is, $F \in L^\infty(\mathbb{R}^n,\mathbb{C}^{m\times m})$, and hence (ii) holds.

The bound (4.41) has been proved in [14, Lemma 2.5] in the infinite-dimensional context; for completeness we rederive it in the present matrix-valued case in Appendix B. Clearly, the bound (4.41) also shows that (ii) implies (i).

Next we turn to the lower bound in (4.42). Choose $p,q \in \{1,\dots,m\}$ such that

$$|||F|||_{\infty,m} = \|F_{p,q}\|_\infty. \tag{4.43}$$

Then the classical L^2-multiplier theorem (for $m=1$) implies that

$$\|F_{p,q}(-i\nabla)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|F_{p,q}\|_\infty. \tag{4.44}$$

Thus, there exists a sequence $\{f_\ell\}_{\ell\in\mathbb{N}}$ in $L^2(\mathbb{R}^n)$ with $\|f_\ell\|_2 = 1$, $\ell \in \mathbb{N}$, such that

$$\lim_{\ell\to\infty} \|F_{p,q}(-i\nabla)f_\ell\|_2 = \|F_{p,q}\|_\infty. \tag{4.45}$$

Introducing (cf. (4.17))

$$g_\ell = U(1,p)f_\ell, \quad \ell \in \mathbb{N}, \tag{4.46}$$

we have

$$(F(-i\nabla)g_\ell)_{r,s} = \begin{cases} 0, & 2 \leq r \leq m, \\ (f_\ell^\wedge F_{p,s})^\vee, & r = 1, \end{cases} \tag{4.47}$$

and hence

$$|||g_\ell|||_{2,m} = \|f_\ell\|_2 = 1, \quad \ell \in \mathbb{N}, \tag{4.48}$$

and

$$\begin{aligned} |||F(-i\nabla)g_\ell|||_{2,m} &= \sum_{s=1}^m \|(f_\ell^\wedge F_{p,s})^\vee\|_2 \\ &\geq \|(f_\ell^\wedge F_{p,q})^\vee\|_2 = \|F_{p,q}(-i\nabla)f_\ell\|_2 \xrightarrow[\ell\to\infty]{} \|F_{p,q}\|_\infty, \end{aligned} \tag{4.49}$$

implying the lower bound in (4.42).

To show that this lower bound is best possible it suffices to look at the following example. Let

$$F_{0,j,k} = \delta_{j,1}\delta_{k,1}, \quad 1 \le j, k, \le m. \tag{4.50}$$

For $f \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ with $|||f|||_{2,m} = 1$ one obtains

$$\begin{aligned} |||F_0(-i\nabla)f|||_{2,m} &= \sum_{j=1}^m \|(f_{j,1}^\wedge)^\vee\|_2 = \sum_{j=1}^m \|f_{j,1}\|_2 \\ &\le \|f\|_{2,m} = 1 = |||F_0|||_{\infty,m}, \end{aligned} \tag{4.51}$$

implying $\|F_0(-i\nabla)\|_{\mathcal{B}((L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}),\,|||\cdot|||_{2,m}))} \le |||F_0|||_{\infty,m}$.

Turning to the upper bound in (4.42), let $\varphi \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ with $|||\varphi|||_{2,m} = 1$. Then

$$(F(-i\nabla)\varphi)_{j,k} = \sum_{r=1}^m (\varphi_{j,r}^\wedge F_{r,k})^\vee, \quad 1 \le j, k \le m. \tag{4.52}$$

Applying the classical L^2-multiplier theorem once more, one estimates

$$\begin{aligned} |||F(-i\nabla)\varphi|||_{2,m} &= \sum_{j,k=1}^m \|(F(-i\nabla)\varphi)_{j,k}\|_2 = \sum_{j,k=1}^m \Big\|\sum_{r=1}^m (\varphi_{j,r}^\wedge F_{r,k})^\vee\Big\|_2 \\ &\le \sum_{j,k,r=1}^m \|F_{r,k}(-i\nabla)\varphi_{j,r}\|_2 \\ &\le \sum_{j,k,r=1}^m \|F_{r,k}(-i\nabla)\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\|\varphi_{j,r}\|_2 \\ &= \sum_{j,k,r=1}^m \|F_{r,k}\|_\infty \|\varphi_{j,r}\|_2 \le |||F|||_{\infty,m} \sum_{k=1}^m \sum_{j,r=1}^m \|\varphi_{j,r}\|_2 \\ &= m|||F|||_{\infty,m}\, |||\varphi|||_{2,m} = m|||F|||_{\infty,m}\,. \end{aligned} \tag{4.53}$$

To demonstrate that this upper bound is best possible, we introduce $F_1 \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ by

$$F_{1,j,k} = 1, \quad 1 \le j, k \le m. \tag{4.54}$$

Let $f \in L^2(\mathbb{R}^n)$ with $\|f\|_2 = 1$, and define $\varphi \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$ via

$$\varphi_{j,k} = m^{-2} f, \quad 1 \le j, k \le m. \tag{4.55}$$

Then

$$|||\varphi|||_{2,m} = \sum_{j,k=1}^m \|\varphi_{j,k}\|_2 = \sum_{j,k=1}^m m^{-2}\|f\|_2 = 1. \tag{4.56}$$

Consequently,

$$(F_1(-i\nabla)\varphi)_{j,k} = \sum_{r=1}^{m} (\varphi_{j,r}^{\wedge} F_{1,r,k})^{\vee} = \sum_{r=1}^{m} m^{-2} f = m^{-1} f, \quad 1 \leq r, s \leq m, \tag{4.57}$$

and thus

$$|||F_1(-i\nabla)\varphi|||_{2,m} = \sum_{j,k=1}^{m} \|(F_1(-i\nabla)\varphi)_{j,k}\|_2 = \sum_{j,k=1}^{m} m^{-1}\|f\|_2 = m = m|||F_1|||_{\infty,m} \,. \ \blacksquare \tag{4.58}$$

Remark 4.7. (i) We stress once more that the equivalence (i)⇔(ii) in Theorem 4.6 (as well as the fact (4.41)) was proved by Gaudry, Jefferies, and Ricker [14, Lemma 2.5 and Proposition 2.8] in the infinite-dimensional context (we also refer to [31] for related results). For completeness we again decided to present a rather elementary and straightforward proof. The bounds (4.42) appear to be new.

(ii) In the special case $m = 1$, the upper and lower bounds in (4.42) coincide and hence reduce to the classical result $\|F(-i\nabla)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|F\|_\infty$.

Next, we provide a matrix-valued extension of a part of Schoenberg's Theorem [34, Proposition 4.4] (cf. Theorem 1.2). To be precise, we will show that (i) implies (iii) in Schoenberg's Theorem 1.2 in the matrix-valued context:

Theorem 4.8. *Let $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ and suppose that F is conditionally positive semidefinite and $F(0) \leq 0$. Then for all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, the block matrix $\{F(x_p - x_q) - F(x_p) - F(x_q)^*\}_{1\leq p,q\leq N} \in \mathbb{C}^{mN\times mN}$ is positive semidefinite.*

Proof. Let $x_p \in \mathbb{R}^n$, $c_p \in \mathbb{C}^m$, $1 \leq p \leq N$. Writing $c_0 := -\sum_{p=1}^{N} c_p$ and $c_p = (c_{p,1}, \ldots, c_{p,m})^\top$, $0 \leq p \leq N$, one has

$$\sum_{p=0}^{N}\sum_{j=1}^{m} c_{p,j} = 0. \tag{4.59}$$

In addition, set $x_0 = 0 \in \mathbb{R}^n$. Then by Lemma 2.5(iii) one obtains

$$\begin{aligned} 0 &\leq \sum_{p,q=0}^{N} (c_p, F(x_p - x_q)c_q)_{\mathbb{C}^m} \\ &= (c_0, F(0)c_0)_{\mathbb{C}^m} + \sum_{p=1}^{N} (c_p, F(x_p)c_0)_{\mathbb{C}^m} + \sum_{q=1}^{N} (c_0, F(-x_q)c_q)_{\mathbb{C}^m} \\ &\quad + \sum_{p,q=1}^{N} (c_p, F(x_p - x_q)c_q)_{\mathbb{C}^m} \end{aligned} \tag{4.60}$$

$$
\begin{aligned}
&= (c_0, F(0)c_0)_{\mathbb{C}^m} + \sum_{p=1}^{N} (c_p, F(-x_p)^* c_0)_{\mathbb{C}^m} + \sum_{q=1}^{N} (c_0, F(-x_q)c_q)_{\mathbb{C}^m} \\
&\quad + \sum_{p,q=1}^{N} (c_p, F(x_p - x_q)c_q)_{\mathbb{C}^m} \\
&= (c_0, F(0)c_0)_{\mathbb{C}^m} - \sum_{p,q=1}^{N} (c_p, F(-x_p)^* c_q)_{\mathbb{C}^m} - \sum_{p,q=1}^{N} (c_p, F(-x_q)c_q)_{\mathbb{C}^m} \\
&\quad + \sum_{p,q=1}^{N} (c_p, F(x_p - x_q)c_q)_{\mathbb{C}^m}.
\end{aligned}
$$

Since $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, were arbitrary, replacing x_p by $-x_p$, $1 \leq p \leq N$, implies

$$
0 \leq -(c_0, F(0)c_0)_{\mathbb{C}^m} \leq \sum_{p.q=1}^{N} (c_p, [F(x_q - x_p) - F(x_q) - F(x_p)^*]c_q)_{\mathbb{C}^m}, \tag{4.61}
$$

completing the proof. ∎

Combining Theorems 2.6, 2.7, and 4.8, one obtains the following matrix variant of Schoenberg's Theorem 1.2:

THEOREM 4.9. *Let $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$. Then the following conditions are equivalent:*

(i) *F is conditionally positive semidefinite.*
(ii) *For all $t > 0$, $\exp_{\mathrm{H}}(tF)$ is positive semidefinite.*

If (i) *or* (ii) *holds, and if $F(0) \leq 0$, then the following holds:*

(iii) *For all $N \in \mathbb{N}$ and $x_p \in \mathbb{R}^n$, $1 \leq p \leq N$, the block matrix $\{F(x_p - x_q) - F(x_p) - F(x_q)^*\}_{1\leq p,q\leq N} \in \mathbb{C}^{mN\times mN}$ is positive semidefinite.*

REMARK 4.10. It should be noted that the converse of Theorem 4.8, and hence the complete analog of Schoenberg's Theorem 1.2, cannot hold in the matrix-valued context, as the following example for $m = 2$ shows: Choose $n = 1$, $m = 2$ and

$$
F_0(x) = ixS, \quad S = S^* \in \mathbb{C}^{2\times 2},\ x \in \mathbb{R}, \tag{4.62}
$$

with

$$
S_{j,j} \in \mathbb{R}, \quad j = 1,2, \qquad S_{1,2} = \overline{S_{2,1}} = is, \quad s > 0. \tag{4.63}
$$

Then

$$
F_0(x_p - x_q) - F_0(x_p) - F_0(x_q)^* = 0, \quad x_p, x_q \in \mathbb{R}, \tag{4.64}
$$

and hence condition (iii) in Theorem 4.9 holds for F_0 in the special case $n = 1$, $m = 2$.

Next, pick $x_1, x_2 \in \mathbb{R}$, $x_1 > x_2$. Then

$$\{F_0(x_p - x_q)\}_{1\le p,q\le 2} = \begin{pmatrix} F_0(0) & F_0(x_1 - x_2) \\ F_0(x_2 - x_1) & F_0(0) \end{pmatrix} = (x_1 - x_2)\begin{pmatrix} 0 & iS \\ -iS & 0 \end{pmatrix}. \tag{4.65}$$

Thus, choosing $c \in \mathbb{R}^4$ with $c_1 = c_4 = 0$, $c_3 = -c_2 \neq 0$ one obtains

$$\sum_{k=1}^{4} c_k = 0, \quad (c, \{F_0(x_p - x_q)\}_{1\le p,q\le 2}\, c)_{\mathbb{C}^4} = -(x_1 - x_2)2sc_2^2 < 0, \tag{4.66}$$

and hence F_0 is not conditionally positive semidefinite.

Now we turn to a matrix-valued extension of [33, Theorem XIII.52] (cf. Theorem 1.3 and the subsequent Remark 4.12).

THEOREM 4.11. *Let $F \in C(\mathbb{R}^n, \mathbb{C}^{m\times m})$ and suppose there exists $c \in \mathbb{R}$ such that*

$$\mathrm{Re}(F(x)_{j,k}) \le c, \quad x \in \mathbb{R}^n,\ 1 \le j,k \le m. \tag{4.67}$$

Then the following conditions are equivalent:

(i) *For all $t > 0$, $(\exp_{\mathrm{H}}(tF))(-i\nabla)|_{C_0^\infty(\mathbb{R}^n,\mathbb{C}^{m\times m})}$ extends to a bounded operator* (*denoted by the same symbol*) *in $\mathcal{B}(L^1(\mathbb{R}^n, \mathbb{C}^{m\times m}))$ and* ([2])

$$\mathrm{tr}_{\mathbb{C}^m}\big(((\exp_{\mathrm{H}}(tF))(-i\nabla)f)(0)\big) \ge 0,\ 0 \le f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}),\ t > 0. \tag{4.68}$$

(ii) *For all $t > 0$, $\exp_{\mathrm{H}}(tF)\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ is positive semidefinite.*

(iii) *F is conditionally positive semidefinite.*

In addition, if one of the conditions (i)–(iii) *holds, then inequality* (4.68) *can be replaced by*

$$\mathrm{tr}_{\mathbb{C}^m}\big(((\exp_{\mathrm{H}}(tF))(-i\nabla)f)(x)\big) \ge 0, \quad 0 \le f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}),\ x \in \mathbb{R}^n,\ t > 0. \tag{4.69}$$

Proof. Fix $t > 0$ and suppose condition (i) holds. Then $\exp_{\mathrm{H}}(tF)$ is an $L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$ multiplier and hence Theorem 4.4 guarantees the existence

([2]) By Lemma 3.9(i), $(\exp_{\mathrm{H}}(tF))(-i\nabla)f \in C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ for $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$, hence the pointwise evaluation $((\exp_{\mathrm{H}}(tF))(-i\nabla)f)(x_0)$, $x_0 \in \mathbb{R}$, is well-defined. Indeed, if $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$, then $f^\wedge \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m}) \subset L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$. In addition, since $\mathrm{Re}(F(\cdot)_{j,k}) \le c$, $\exp_{\mathrm{H}}(tF) \in L^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$, and so each entry of $f^\wedge \exp_{\mathrm{H}}(tF)$ lies in $L^1(\mathbb{R}^n)$, Lemma 3.9(i) yields $(\exp_{\mathrm{H}}(tF))(-i\nabla)f = (f^\wedge \exp_{\mathrm{H}}(tF))^\vee \in C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$.

of $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$ such that $\exp_{\mathrm{H}}(tF) = \mu^\wedge$. In addition,

$$\begin{aligned}(4.70)\quad \big((\exp_{\mathrm{H}}(tF))(-i\nabla)f\big)(x) &= (f^\wedge \exp_{\mathrm{H}}(tF))^\vee(x)\\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i((x-\eta)\cdot\xi)} f^\wedge(\xi)\, d\mu(\eta)\, d^n\xi,\\ &\qquad f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}),\ x \in \mathbb{R}^n.\end{aligned}$$

Since $f^\wedge \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^{m\times m}) \subset L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})$, one can interchange the order of integration in (4.70) to obtain

$$\begin{aligned}(4.71)\quad \big((\exp_{\mathrm{H}}(tF))(-i\nabla)f\big)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i((x-\eta)\cdot\xi)} f^\wedge(\xi)\, d^n\xi\, d\mu(\eta)\\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (f^\wedge)^\vee(x-\eta)\, d\mu(\eta) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x-\eta)\, d\mu(\eta)\\ &= (2\pi)^{-n/2}(T_\mu f)(x), \quad f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}).\end{aligned}$$

Thus, by (i),

(4.72)

$$0 \le \operatorname{tr}_{\mathbb{C}^m}\big(((\exp_{\mathrm{H}}(tF))(-i\nabla)f)(0)\big) = (2\pi)^{-n/2}\operatorname{tr}_{\mathbb{C}^m}\Big(\int_{\mathbb{R}^n} f(-\eta)\, d\mu(\eta)\Big),$$
$$0 \le f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}),$$

and hence

$$(4.73)\qquad \operatorname{tr}_{\mathbb{C}^m}\Big(\int_{\mathbb{R}^n} f(x)\, d\mu(x)\Big) \ge 0, \quad 0 \le f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}).$$

By Lemma 3.11, (4.73) extends to

$$(4.74)\qquad \operatorname{tr}_{\mathbb{C}^m}\Big(\int_{\mathbb{R}^n} f(x)\, d\mu(x)\Big) \ge 0, \quad 0 \le f \in C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}).$$

By the duality result preceding (3.8), this implies $\mu \ge 0$. In view of Theorems 4.3 and 4.4, $\exp_{\mathrm{H}}(tF) = \mu^\wedge$ is positive semidefinite and hence (ii) holds.

Conversely, suppose that (ii) holds. Then Theorem 4.4 implies that $(\exp_{\mathrm{H}}(tF))(-i\nabla)|_{C_0^\infty(\mathbb{R}^n,\mathbb{C}^{m\times m})}$ extends to an operator $(\exp_{\mathrm{H}}(tF))(-i\nabla) \in \mathcal{B}\big(L^1(\mathbb{R}^n, \mathbb{C}^{m\times m})\big)$. As in the first part of this proof (cf. (4.71)), one infers

$$(4.75)\qquad (\exp_{\mathrm{H}}(tF))(-i\nabla)f = (2\pi)^{-n/2}T_\mu f, \quad f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}).$$

Thus,

$$\begin{aligned}(4.76)\quad \operatorname{tr}_{\mathbb{C}^m}\big(((\exp(tF))(-i\nabla)f)(0)\big) &= (2\pi)^{-n/2}\operatorname{tr}_{\mathbb{C}^m}((T_\mu f)(0))\\ &= (2\pi)^{-n/2}\operatorname{tr}_{\mathbb{C}^m}\Big(\int_{\mathbb{R}^n} f(-y)\, d\mu(y)\Big) \ge 0, \quad 0 \le f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}),\end{aligned}$$

by the duality result preceding (3.8). Thus, (i) holds.

The equivalence (ii)$\Leftrightarrow$(iii) is a consequence of Theorems 2.6 and 2.7.

Finally, if one of (i)–(iii) holds, then (4.69) follows from (4.68) since by Lemma 3.4, $(\exp_{\mathrm{H}}(tF))(-i\nabla)$ commutes with translations. ■

Remark 4.12. In the classical case where $m = 1$, condition (i) in Theorem 4.11 is equivalent to

(i′) for all $t > 0$, $(\exp(tF))(-i\nabla) \in \mathcal{B}(L^2(\mathbb{R}^n))$ is positivity preserving in $L^2(\mathbb{R}^n)$.

Thus Theorem 4.11 resembles Theorem 1.3 for $m = 1$. In this context we note that $(\exp(tF))(-i\nabla) = \exp(tF(-i\nabla))$ for all $t \geq 0$, for $m = 1$.

Proof of (i′)⇒(i). If (i′) holds, then

$$\operatorname{tr}_{\mathbb{C}}\big(((\exp(tF))(-i\nabla)f)(x)\big) = \big((\exp(tF))(-i\nabla)f\big)(x) \geq 0, \quad 0 \leq f \in C_0^\infty(\mathbb{R}^n),\ x \in \mathbb{R}^n,\ t > 0 \tag{4.77}$$

(see also the footnote accompanying Theorem 4.11). In particular,

$$\operatorname{tr}_{\mathbb{C}}\big(((\exp(tF))(-i\nabla)f)(0)\big) = \big((\exp(tF))(-i\nabla)f\big)(0) \geq 0, \quad t > 0, \tag{4.78}$$

under the assumptions in (4.77). Since $(\exp(tF))(-i\nabla)$ is positivity preserving, Corollary 3.12 guarantees the existence of a scalar-valued, nonnegative, finite measure μ on $\mathbb{R}$ such that

$$(\exp(tF))(-i\nabla) = T_\mu, \quad t > 0. \tag{4.79}$$

Thus, the estimate (3.15) for $p = 1$ shows that $(\exp(tF))(-i\nabla)|_{C_0^\infty(\mathbb{R}^n)}$ extends to a bounded operator $(\exp(tF))(-i\nabla) \in \mathcal{B}(L^1(\mathbb{R}^n))$, implying (i).

Proof of (i)⇒(i′). If (i) holds, then

$$\big((\exp(tF))(-i\nabla)f\big)(0) = \operatorname{tr}_{\mathbb{C}}\big(((\exp(tF))(-i\nabla)f)(0)\big) \geq 0, \quad 0 \leq f \in C_0^\infty(\mathbb{R}^n),\ t > 0. \tag{4.80}$$

By Lemma 3.4, this yields

$$\big((\exp(tF))(-i\nabla)f\big)(x) = \operatorname{tr}_{\mathbb{C}}\big(((\exp(tF))(-i\nabla)f)(x)\big) \geq 0, \quad 0 \leq f \in C_0^\infty(\mathbb{R}^n),\ x \in \mathbb{R}^n,\ t > 0. \tag{4.81}$$

Since $\{f \in C_0(\mathbb{R}^n) \mid f \geq 0\}$ is dense in $\{f \in L^2(\mathbb{R}^n) \mid f \geq 0\}$, one concludes that $(\exp(tF))(-i\nabla) \in \mathcal{B}(L^2(\mathbb{R}^n))$ is positivity preserving, that is, (i′) holds.

Next, we will show that the analog of (i′) for $m = 1$, with $\exp(\cdot)$ replaced by $\exp_{\mathrm{H}}(\cdot)$, *cannot* hold for $m \geq 2$. We start with two preliminaries:

Lemma 4.13. *Let $F \in C(\mathbb{R}^n, \mathbb{C}^{m\times m})$ be conditionally positive semidefinite and suppose there exists $c \in \mathbb{R}$ such that*

$$\operatorname{Re}(F(x)_{j,k}) \leq c, \quad x \in \mathbb{R}^n,\ 1 \leq j, k \leq m. \tag{4.82}$$

By Theorem 4.11, for all $t > 0$, $\exp_{\mathrm{H}}(tF)\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ is positive semidefinite, and hence by Theorem 4.3, there exists a nonnegative finite measure

$\mu_t \in \mathcal{M}(\mathbb{R}^n, \mathbb{C}^{m\times m})$, $t > 0$, *such that*

$$\exp_{\mathrm{H}}(tF)(x) = \mu_t^{\wedge}(x), \quad x \in \mathbb{R}^n,\ t > 0. \tag{4.83}$$

Then

$$\mu_{t,j,k}(\mathbb{R}^n) \neq 0, \quad 1 \leq j,k \leq m,\ t > 0. \tag{4.84}$$

Thus, for all $t > 0$, there exists $R_t > 0$ such that

$$\mu_{t,j,k}(\overline{B(0,R_t)}) \neq 0, \quad 1 \leq j,k \leq m. \tag{4.85}$$

Proof. Since $\exp_{\mathrm{H}}(tF)(x) = \mu_t^{\wedge}(x)$ for all $x \in \mathbb{R}^n$ and $t > 0$, one concludes that

$$\begin{aligned} 0 \neq \exp(tF(0)_{j,k}) = \exp_{\mathrm{H}}(tF)_{j,k}(0) &= (2\pi)^{-n/2}\Big(\int_{\mathbb{R}^n} d\mu_t(x)\Big)_{j,k} \\ &= (2\pi)^{-n/2}\mu_{t,j,k}(\mathbb{R}^n), \quad 1 \leq j,k \leq m, \end{aligned} \tag{4.86}$$

and hence (4.84) holds. Since μ_t is nonnegative, $\mu_t(\overline{B(0,R)}) \uparrow \mu_t(\mathbb{R}^n)$ as $R \to \infty$, and thus

$$\mu_{t,j,k}(\overline{B(0,R)}) \xrightarrow[R\to\infty]{} \mu_{t,j,k}(\mathbb{R}^n), \quad 1 \leq j,k \leq m, \tag{4.87}$$

implying (4.85). ∎

Lemma 4.14. *Let $D \in \mathbb{C}^{m\times m}$ with $m \in \mathbb{N}$, $m \geq 2$, be a strictly positive diagonal matrix with*

$$D_{j,k} = d_j\delta_{j,k}, \quad d_j > 0, \quad 1 \leq j,k \leq m, \quad d_1 \neq d_2, \tag{4.88}$$

and let $S = S^* \in \mathbb{C}^{m\times m}$ be self-adjoint with $S_{1,2} \neq 0$. Then DS is not self-adjoint in $\mathbb{C}^{m\times m}$.

Proof. This is clear from $(DS)_{1,2} = d_1S_{1,2}$ and $\overline{(DS)_{2,1}} = d_2\overline{S_{2,1}} = d_2S_{1,2}$. ∎

Theorem 4.15. *Let $F \in C(\mathbb{R}^n, \mathbb{C}^{m\times m})$, $m \geq 2$, be conditionally positive semidefinite and suppose there exists $c \in \mathbb{R}$ such that*

$$\mathrm{Re}(F(x)_{j,k}) \leq c, \quad x \in \mathbb{R}^n,\ 1 \leq j,k \leq m. \tag{4.89}$$

Then for all $t > 0$,

$$(\exp_{\mathrm{H}}(tF))(-i\nabla) \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})) \text{ is } \textbf{not} \text{ positivity preserving.} \tag{4.90}$$

Proof. Fix $t > 0$ and let μ_t and R_t be as in Lemma 4.13, and $D \in \mathbb{C}^{m\times m}$ be the strictly positive diagonal matrix of Lemma 4.14. For sufficiently small $\varepsilon > 0$ we introduce

$$h_\varepsilon \in C^\infty([0,\infty)), \qquad h_\varepsilon(r) = \begin{cases} 1, & r \in [0,R_t], \\ 0, & r \in [R_t+\varepsilon,\infty), \end{cases} \tag{4.91}$$

and

$$0 \leq g_\varepsilon \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}), \quad g_\varepsilon(x) = h_\varepsilon(|x|)D,\ x \in \mathbb{R}^n. \tag{4.92}$$

Then

$$\text{(4.93)}\qquad \big(((\exp_{\mathrm{H}}(tF))(-i\nabla))f\big)(x) = (2\pi)^{-n/2}(T_{\mu_t}f)(x), \quad f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{m\times m}),\ x \in \mathbb{R}^n,$$

by the (ii)⇒(i) part of the proof of Theorem 4.4. Thus,

$$\text{(4.94)}\qquad \begin{aligned} &\big(((\exp_{\mathrm{H}}(tF))(-i\nabla))g_\varepsilon\big)(0) = (2\pi)^{-n/2}(T_{\mu_t}g_\varepsilon)(0)\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} g_\varepsilon(-y)\,d\mu_t(y) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} g_\varepsilon(y)\,d\mu_t(y)\\ &= (2\pi)^{-n/2}\Big[\int_{\overline{B_n(0,R_t)}} g_\varepsilon(y)\,d\mu_t(y) + \int_{B_n(0,R_t+\varepsilon)\setminus\overline{B_n(0,R_t)}} g_\varepsilon(y)\,d\mu_t(y)\Big]\\ &= (2\pi)^{-n/2}D\mu_t(\overline{B_n(0,R_t)}) + (2\pi)^{-n/2}\int_{B_n(0,R_t+\varepsilon)\setminus\overline{B_n(0,R_t)}} g_\varepsilon(y)\,d\mu_t(y). \end{aligned}$$

By estimate (3.7),

$$\text{(4.95)}\qquad \begin{aligned} \Big\|\int_{B_n(0,R_t+\varepsilon)\setminus\overline{B_n(0,R_t)}} g_\varepsilon(y)\,d\mu_t(y)\Big\| &\le \int_{B_n(0,R_t+\varepsilon)\setminus\overline{B_n(0,R_t)}} \|g_\varepsilon(y)\|_{\mathcal{B}(\mathbb{C}^m)}\,d|\mu_t|(y)\\ &\le \int_{B_n(0,R_t+\varepsilon)\setminus\overline{B_n(0,R_t)}} \|D\|_{\mathcal{B}(\mathbb{C}^m)}\,d|\mu_t|(y) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0. \end{aligned}$$

Using the fact that

$$\text{(4.96)}\qquad \mathcal{N}_m = \mathbb{C}^{m\times m}\setminus\{A^*A \in \mathbb{C}^{m\times m} \mid A \in \mathbb{C}^{m\times m}\} \text{ is open in } \mathbb{C}^{m\times m}$$

(since the nonnegative $m\times m$ matrices form a closed cone in $\mathbb{C}^{m\times m}$), employing

$$\text{(4.97)}\qquad D\mu_t(\overline{B_n(0,R_t)}) \in \mathcal{N}_m,$$

applying Lemma 4.14 with $S = \mu_t\big(\overline{B_n(0,R_t)}\big)$, and utilizing

$$\text{(4.98)}\qquad ((\exp_{\mathrm{H}}(tF))(-i\nabla))(g_\varepsilon) \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}) \cap C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$$

by Lemma 3.9(i), one concludes that for all sufficiently small $\varepsilon > 0$, $(\exp_{\mathrm{H}}(tF)(-i\nabla)g_\varepsilon)(0)$ is not nonnegative. Thus, for all sufficiently small $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that $(\exp_{\mathrm{H}}(tF)(-i\nabla)g_\varepsilon)(x)$, $x \in B_n(0,\eta(\varepsilon))$, is not nonnegative. Since $g_\varepsilon \ge 0$, this completes the proof. ∎

Thus, unlike the classical case $m = 1$ discussed in Remark 4.12, the straightforward extension of Theorem 1.3 replacing its condition (i) by

(i′) for all $t > 0$, $(\exp_{\mathrm{H}}(tF))(-i\nabla)$ is positivity preserving

cannot hold in the matrix-valued context, $m \ge 2$.

Finally, we derive the bound (1.3) in the matrix-valued context following [24, Lemma 3.6.22]. First, we recall the following fact:

Proposition 4.16 ([22, p. 112]). *Let* $0 \leq M_\ell \in \mathbb{C}^{m_\ell \times m_\ell}$, $m_\ell \in \mathbb{N}$, $\ell = 1, 2$ *(i.e.,* M_ℓ, $\ell = 1, 2$, *are positive semidefinite), and* $X \in \mathbb{C}^{m_1 \times m_2}$. *Introduce the block matrix*

$$A = \begin{pmatrix} M_1 & X \\ X^* & M_2 \end{pmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)}. \tag{4.99}$$

Then A *is positive semidefinite (i.e.,* $A \geq 0$*) if and only if there exists a contraction* $C \in \mathbb{C}^{m_1 \times m_2}$ *such that* $X = M_1^{1/2} C M_2^{1/2}$.

Here C is viewed as a linear map $C \colon \mathbb{C}^{m_2} \to \mathbb{C}^{m_1}$, and, according to our convention, we employ the standard Euclidean scalar product and norm on $\mathbb{C}^{m_\ell}$, $\ell = 1, 2$.

Next, we state a preparatory result:

Lemma 4.17. *Suppose that* $F \in C(\mathbb{R}^n, \mathbb{C}^{m \times m})$ *is conditionally positivie semidefinite with* $F(0) \leq 0$. *Then*

$$0 \leq F(0) - 2\mathrm{Re}(F(x)) \leq -2\mathrm{Re}(F(x)), \quad x \in \mathbb{R}^n, \tag{4.100}$$

$$\begin{aligned} \|F(0) - 2\mathrm{Re}(F(x))\|_{\mathcal{B}(\mathbb{C}^m)} &\leq 2\|\mathrm{Re}(F(x))\|_{\mathcal{B}(\mathbb{C}^m)} \\ &\leq 2\|F(x)\|_{\mathcal{B}(\mathbb{C}^m)}, \quad x \in \mathbb{R}^n, \end{aligned} \tag{4.101}$$

$$\begin{aligned} &\|F(x-y) - F(x) - F(y)^*\|_{\mathcal{B}(\mathbb{C}^m)} \\ &\qquad \leq 2\|F(x)\|_{\mathcal{B}(\mathbb{C}^m)}^{1/2}\|F(y)\|_{\mathcal{B}(\mathbb{C}^m)}^{1/2}, \quad x, y \in \mathbb{R}^n, \end{aligned} \tag{4.102}$$

$$\|F(x+y)\|_{\mathcal{B}(\mathbb{C}^m)}^{1/2} \leq \|F(x)\|_{\mathcal{B}(\mathbb{C}^m)}^{1/2} + \|F(y)\|_{\mathcal{B}(\mathbb{C}^m)}^{1/2}, \quad x, y \in \mathbb{R}^n. \tag{4.103}$$

Proof. Inequality (4.100) follows from Theorem 4.8 and from $F(0) \leq 0$, and (4.101) is a consequence of (3.51)–(3.53).

Next, denote $G(x) = F(0) - F(x) - F(x)^*$, $H(x,y) = F(x-y) - F(x) - F(y)^*$, and $K(y) = F(0) - F(y) - F(y)^*$. Applying once more Theorem 4.8 one infers that

$$0 \leq \begin{pmatrix} G(x) & H(x,y) \\ H(x,y)^* & K(y) \end{pmatrix} \in \mathbb{C}^{2m \times 2m}, \quad x, y \in \mathbb{R}^n. \tag{4.104}$$

By (4.100),

$$G(x) \geq 0, \quad K(y) \geq 0, \quad x, y \in \mathbb{R}^n, \tag{4.105}$$

and hence Proposition 4.16 guarantees the existence of a linear contraction $C(x,y) \in \mathbb{C}^{m \times m}$, $x, y \in \mathbb{R}^n$, such that

$$H(x,y) = G(x)^{1/2} C(x,y) K(y)^{1/2}, \quad x, y \in \mathbb{R}^n. \tag{4.106}$$

Thus, (4.101) yields

$$\begin{aligned} \|H(x,y)\|_{\mathcal{B}(\mathbb{C}^m)} &\leq \|G(x)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)}\|K(y)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)} \\ &\leq 2\|F(x)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)}\|F(y)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)}, \quad x,y\in\mathbb{R}^n, \end{aligned} \tag{4.107}$$

proving (4.102).

By (4.102) one obtains

$$\begin{aligned} \|F(x-y)\|_{\mathcal{B}(\mathbb{C}^m)} - \|F(x)\|_{\mathcal{B}(\mathbb{C}^m)} - \|F(y)^*\|_{\mathcal{B}(\mathbb{C}^m)} &\leq \|F(x-y)\|_{\mathcal{B}(\mathbb{C}^m)} - \|F(x)+F(y)^*\|_{\mathcal{B}(\mathbb{C}^m)} \\ &\leq \|F(x-y)-F(x)-F(y)^*\|_{\mathcal{B}(\mathbb{C}^m)} \\ &\leq 2\|F(x)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)}\|F(y)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)}, \quad x,y\in\mathbb{R}^n, \end{aligned} \tag{4.108}$$

implying

$$\|F(x-y)\|_{\mathcal{B}(\mathbb{C}^m)} \leq [\|F(x)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)} + \|F(y)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)}]^2, \quad x,y\in\mathbb{R}^n. \tag{4.109}$$

Replacing y by $-y$ and using $F(-y)=F(y)^*$ yields (4.103). ∎

Theorem 4.18. *Suppose that $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ is locally bounded and conditionally positive semidefinite with $F(0)\leq 0$. Then there exists $C>0$ such that*

$$\|F(x)\|_{\mathcal{B}(\mathbb{C}^m)} \leq C[1+|x|^2], \quad x\in\mathbb{R}^n. \tag{4.110}$$

Proof. By local boundedness of F it suffices to prove the existence of $C'>0$ such that $\|F(x)\|_{\mathcal{B}(\mathbb{C}^m)} \leq C'|x|^2$ for $|x|$ sufficiently large. Thus, for $x\in\mathbb{R}^n$ with $|x|\geq 2$, let $m(x)\in\mathbb{N}$ be the positive integer such that $|x|\in[m(x),m(x)+1)$. Then by (4.103),

$$\begin{aligned} \|F(x)\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)} &= \|F(m(x)(x/m(x)))\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)} \\ &\leq m(x)\|F((x/m(x)))\|^{1/2}_{\mathcal{B}(\mathbb{C}^m)} \\ &\leq m(x)\Big[\sup_{y\in\mathbb{R}^n}\{\|F(y)\|_{\mathcal{B}(\mathbb{C}^m)} \mid 0\leq|y|\leq 2\}\Big]^{1/2} \\ &\leq [C']^{1/2}|x|, \quad |x|\geq 2, \end{aligned} \tag{4.111}$$

where

$$C' = \sup_{y\in\mathbb{R}^n}\{\|F(y)\|_{\mathcal{B}(\mathbb{C}^m)} \mid 0\leq|y|\leq 2\}. \tag{4.112}$$ ∎

We conclude with some elementary examples of conditionally positive semidefinite matrix-valued functions on $\mathbb{R}^n$.

Example 4.19. (i) Fix $y_j\in\mathbb{R}^n$, $j=1,2$, with $y_1\neq y_2$. Then $F_2\colon \mathbb{R}^n\to\mathbb{C}^2$ defined via

$$F_2(x) = -i\begin{pmatrix} x\cdot(y_1+y_2) & x\cdot y_2 \\ x\cdot y_2 & x\cdot(y_1+y_2)\end{pmatrix}, \quad x\in\mathbb{R}^n, \tag{4.113}$$

is conditionally positive semidefinite.

(ii) Suppose that $G_0\colon \mathbb{R}^n \to \mathbb{C}$ is conditionally positive semidefinite and introduce the constant matrix $H = \{H_{j,k}\}_{1\le j,k\le m} \in \mathbb{C}^{m\times m}$ by

$$H_{j,k} = 1, \quad 1\le j,k\le m. \tag{4.114}$$

Then $F_0\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ defined by

$$F_0(x) = G_0(x)H, \quad x\in\mathbb{R}^n, \tag{4.115}$$

is conditionally positive semidefinite.

Proof. (i) Introduce the 2×2 matrix-valued measure $\mu_{2,t}$, $t>0$, via

$$\mu_{2,t} = (2\pi)^{n/2}\begin{pmatrix} \delta_{ty_1}+\delta_{ty_2} & \delta_{ty_2} \\ \delta_{ty_2} & \delta_{ty_1}+\delta_{ty_2}\end{pmatrix}, \quad t>0. \tag{4.116}$$

Here δ_{x_0} denotes the usual Dirac measure at $x_0\in\mathbb{R}^n$. One readily computes, for $x\in\mathbb{R}^n$,

$$(\mu_{2,t}^{\wedge}(x))_{j,j} = e^{-it(x\cdot y_1)} + e^{-it(x\cdot y_2)} = (\exp_{\mathrm{H}}(tF_2(x)))_{j,j}, \quad j=1,2, \tag{4.117}$$

$$\begin{aligned}(\mu_{2,t}^{\wedge}(x))_{1,2} &= e^{-it(x\cdot y_2)} = (\exp_{\mathrm{H}}(tF_2(x)))_{1,2} \\ &= (\exp_{\mathrm{H}}(tF_2(x)))_{2,1} = (\mu_{2,t}^{\wedge}(x))_{2,1},\end{aligned} \tag{4.118}$$

and hence

$$\exp_{\mathrm{H}}(tF_2(x)) = \mu_{2,t}^{\wedge}(x), \quad x\in\mathbb{R}^n,\ t>0. \tag{4.119}$$

By Theorem 4.3 and the equivalence (ii)⇔(iii) in Theorem 4.11, it suffices to prove that $\mu_{2,t}$ is nonnegative for all $t>0$. Since for all $E\in\mathfrak{B}_n$, $\mu_{2,t}(E)$ can only take on the values

$$\begin{aligned}&\begin{pmatrix}0&0\\0&0\end{pmatrix} \text{ if } ty_j\notin E,\ j=1,2, \quad \begin{pmatrix}1&0\\0&1\end{pmatrix} \text{ if } ty_1\in E,\ ty_2\notin E,\\ &\begin{pmatrix}2&1\\1&2\end{pmatrix} \text{ if } ty_j\in E,\ j=1,2, \quad \begin{pmatrix}1&1\\1&1\end{pmatrix} \text{ if } ty_1\notin E,\ ty_2\in E,\end{aligned} \tag{4.120}$$

and all matrices in (4.120) are nonnegative, so is $\mu_{2,t}$, $t>0$.

(ii) Since G_0 is conditionally positive semidefinite, $\exp(tG_0)\colon \mathbb{R}^n\to\mathbb{C}$ is positive semidefinite for all $t>0$. So by the classical Bochner theorem, for all $t>0$, there exists a nonnegative scalar-valued measure ν_t on $\mathbb{R}^n$ such that

$$e^{tG_0} = \nu_t^{\wedge}, \quad t>0. \tag{4.121}$$

Set

$$\mu_{0,t}(E) = \nu_t(E)H, \quad E\in\mathfrak{B}_n,\ t>0. \tag{4.122}$$

Then $\mu_{0,t}$, $t > 0$, is nonnegative and

$$\exp_{\mathrm{H}}(tF_0) = e^{tG_0} H = \nu_t^{\wedge} H = \mu_{0,t}^{\wedge}, \quad t > 0. \tag{4.123}$$

Thus F_0 is conditionally positive semidefinite by utilizing once more Theorem 4.3 and the equivalence (ii)⇔(iii) in Theorem 4.11. ∎

Appendix A. A counterexample. In this appendix we verify the claim made in Remark 4.2. For brevity, we construct the counterexample for $m = 2$, but the construction extends to general $m \in \mathbb{N}$, $m \geq 3$.

Let $\gamma_n : \mathfrak{B}_n \to [0,1]$ be the standard Gaussian measure on $\mathbb{R}^n$,

$$\gamma_n(E) = (2\pi)^{-n/2} \int_E \exp(-|x|^2/2)\, d^n x, \quad E \in \mathfrak{B}_n, \tag{A.1}$$

and introduce

$$\mu(E) = \gamma_n(E) A, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \geq 0,\ E \in \mathfrak{B}_n, \qquad F = \mu^{\wedge}, \tag{A.2}$$

and

$$M = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \geq 0 \quad \text{such that} \quad MA = \begin{pmatrix} 3 & 2 \\ 1 & 6 \end{pmatrix} \text{ is not self-adjoint,} \tag{A.3}$$

let alone positive semidefinite.

As in the proof of (ii)⇒(i) in Theorem 4.4, one obtains

$$(F(-i\nabla)f)(x) = (2\pi)^{-n/2}(T_\mu f)(x), \quad f \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{2\times 2}),\ x \in \mathbb{R}^n. \tag{A.4}$$

Next, for sufficiently small $\varepsilon > 0$, consider $h_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$0 \leq h_\varepsilon(x) \leq 1,\ x \in \mathbb{R}^n, \quad h_\varepsilon(x) = \begin{cases} 1, & x \in \overline{B_n(0,1)}, \\ 0, & x \in \mathbb{R}^n \setminus B_n(0, 1+\varepsilon), \end{cases} \tag{A.5}$$

and let

$$g_\varepsilon(x) = h_\varepsilon(x) M, \quad x \in \mathbb{R}^n. \tag{A.6}$$

Then

$$\begin{aligned} (F(-i\nabla)g_\varepsilon)(0) &= (2\pi)^{-n/2}(T_\mu g_\varepsilon)(0) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} g_\varepsilon(-y)\, d\mu(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g_\varepsilon(y)\, d\mu(y) \\ &= (2\pi)^{-n/2} \int_{B_n(0,1)} g_\varepsilon(y)\, d\mu(y) + (2\pi)^{-n/2} \int_{B_n(0,1+\varepsilon)\setminus B_n(0,1)} g_\varepsilon(y)\, d\mu(y) \\ &= (2\pi)^{-n/2} \gamma_n(B_n(0,1)) MA + (2\pi)^{-n/2} \int_{B_n(0,1+\varepsilon)\setminus B_n(0,1)} g_\varepsilon(y)\, d\mu(y). \end{aligned} \tag{A.7}$$

By (3.7),

$$\text{(A.8)}\quad \Big\| \int\limits_{B_n(0,1+\varepsilon)\setminus B_n(0,1)} g_\varepsilon(y)\, d\mu(y) \Big\|_{\mathcal{B}(\mathbb{C}^m)} \leq \int\limits_{B_n(0,1+\varepsilon)\setminus B_n(0,1)} \|g_\varepsilon(y)\|_{\mathcal{B}(\mathbb{C}^m)}\, d|\mu|(y) \leq \int\limits_{B_n(0,1+\varepsilon)\setminus B_n(0,1)} \|M\|_{\mathcal{B}(\mathbb{C}^m)}\, d|\mu|(y) = \|M\|_{\mathcal{B}(\mathbb{C}^m)} \|A\|_{\mathcal{B}(\mathbb{C}^m)} \gamma_n\big(B_n(0,1+\varepsilon)\setminus B_n(0,1)\big) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

Since the set

$$\text{(A.9)}\qquad \mathcal{N}_2 = \mathbb{C}^{2\times 2} \setminus \{A^*A \in \mathbb{C}^{2\times 2} \mid A \in \mathbb{C}^{2\times 2}\}$$

is open in $\mathbb{C}^{2\times 2}$ (cf. (4.96)), since

$$\text{(A.10)}\qquad \gamma_n(B_n(0,1))MA \in \mathcal{N}_2,$$

and since $F(-i\nabla)g_\varepsilon \in L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}) \cap C_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})$ by Lemma 3.9(i), for $\varepsilon > 0$ sufficiently small, $(F(-i\nabla)g_\varepsilon)(0)$ is *not* positive semidefinite, and thus there exists $\delta(\varepsilon) > 0$ such that $(F(-i\nabla)g_\varepsilon)(x)$ is not positive semidefinite for all $x \in B_n(0, \delta(\varepsilon))$, even though $g_\varepsilon \geq 0$, illustrating Remark 4.2.

In the special case where $\mu_\sigma(E) = \sigma(E) I_{\mathbb{C}^m}$, $E \in \mathfrak{B}_n$, with $\sigma\colon \mathfrak{B}_n \to [0,\infty)$ a finite meausure, and $F = \mu_\sigma^\wedge$, $F(-i\nabla) = (2\pi)^{-n/2} T_{\mu_\sigma}$ is of course positivity preserving in $L^2(\mathbb{R}^n, \mathbb{C}^{m\times m})$.

Appendix B. The multiplier norm equality (4.41). The purpose of this appendix is an elementary and straightforward proof of the multiplier norm equality (4.41).

We start with some preliminary observations. First, each matrix in $\mathbb{C}^{m\times m}$ will be identified with a column vector in $\mathbb{C}^{m^2}$ by listing the entries of the matrix from left to right, and from top to bottom. We also recall the identifications

$$\text{(B.1)}\qquad \mathbb{C}^{m\times m}_{\text{HS}} \simeq \mathcal{B}_2(\mathbb{C}^m) \simeq \mathbb{C}^{m^2}, \quad \mathcal{B}(\mathbb{C}^{m\times m}_{\text{HS}}) \simeq \mathcal{B}\big(\mathbb{C}^{m^2}\big) \simeq \mathbb{C}^{m^2\times m^2},$$

consistently employing the Euclidean norm on $\mathbb{C}^m$ and $\mathbb{C}^{m^2}$.

In addition, given $A \in \mathbb{C}^{m\times m}$ we introduce the linear operator M_A of right multiplication by A on $\mathbb{C}^{m\times m}$ via

$$\text{(B.2)}\qquad M_A(B) := BA, \quad B \in \mathbb{C}^{m\times m}.$$

Since M_A is a linear operator on $\mathbb{C}^{m^2}$, it is representable by a matrix $K_A \in \mathbb{C}^{m^2\times m^2}$, and the latter may be described upon inspection as follows:

Lemma B.1. *K_A is a block matrix with m^2 blocks, m blocks across horizontally and m blocks vertically. Each block is an $m\times m$ matrix, the diagonal* *blocks each equal* $A^\top$ *(the transpose of* A*), and all off-diagonal blocks equal the zero matrix in* $\mathbb{C}^{m\times m}$*.*

Then one obtains the following result for the operator norm of M_A.

PROPOSITION B.2. *Let* $A \in \mathbb{C}^{m\times m}$*. Then*

$$\|M_A\|_{\mathcal{B}(\mathbb{C}^{m^2})} = \|K_A\|_{\mathcal{B}(\mathbb{C}^{m^2})} = \|A\|_{\mathcal{B}(\mathbb{C}^m)}, \tag{B.3}$$

where, according to our conventions, $\mathbb{C}^m$ *and* $\mathbb{C}^{m^2}$ *are equipped with the Euclidean norm.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{C}^m$ such that $\|u_j\|_{\mathbb{C}^m} = 1$, $j \in \mathbb{N}$, and $\lim_{j\to\infty} \|A^\top u_j\|_{\mathbb{C}^m} = \|A^\top\|_{\mathcal{B}(\mathbb{C}^m)}$. For each $j \in \mathbb{N}$, let $v_j \in \mathbb{C}^{m^2}$ be the column vector obtained by repeating u_j m times down the column, and introduce

$$\omega_j = m^{-1/2} v_j \in \mathbb{C}^{m^2}, \quad \text{such that} \quad \|\omega_j\|_{\mathbb{C}^{m^2}} = 1, \quad j \in \mathbb{N}. \tag{B.4}$$

Then for all $j \in \mathbb{N}$, $K_A\omega_j \in \mathbb{C}^{m^2}$ is the column vector obtained upon repeating $m^{-1/2}A^\top u_j$ m times down the column such that

$$\|K_A\omega_j\|_{\mathbb{C}^{m^2}} = \|A^\top u_j\|_{\mathbb{C}^m} \xrightarrow[j\to\infty]{} \|A^\top\|_{\mathcal{B}(\mathbb{C}^m)}. \tag{B.5}$$

Thus,

$$\|K_A\|_{\mathcal{B}(\mathbb{C}^{m^2})} \geq \|A^\top\|_{\mathcal{B}(\mathbb{C}^m)} = \|A\|_{\mathcal{B}(\mathbb{C}^m)}, \quad A \in \mathbb{C}^{m\times m}. \tag{B.6}$$

To prove the opposite inequality we identify $\mathbb{C}^{m\times m}_{\mathrm{HS}} = (\mathbb{C}^{m\times m}, \|\cdot\|_{\mathrm{HS}})$ with $\mathbb{C}^{m^2}$ and observe that for all $B \in \mathbb{C}^{m\times m} \simeq \mathbb{C}^{m^2}$ one has

$$\begin{aligned}\|M_A(B)\|_{\mathbb{C}^{m^2}} &= \|BA\|_{(\mathbb{C}^{m\times m},\|\cdot\|_{\mathrm{HS}})} = \|BA\|_{\mathcal{B}_2(\mathbb{C}^m)} \\ &\leq \|B\|_{\mathcal{B}_2(\mathbb{C}^m)}\|A\|_{\mathcal{B}(\mathbb{C}^m)} = \|B\|_{\mathbb{C}^{m^2}}\|A\|_{\mathcal{B}(\mathbb{C}^m)},\end{aligned} \tag{B.7}$$

implying

$$\|M_A(B)\|_{\mathbb{C}^{m^2}} \leq \|A\|_{\mathcal{B}(\mathbb{C}^m)}. \ \blacksquare \tag{B.8}$$

At this point we can turn to the principal aim of this appendix:

Proof of (4.41). Suppose that

$$\begin{aligned}&\Phi : \mathbb{R}^n \to \mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}}) \simeq \mathbb{C}^{m^2\times m^2} \text{ is measurable,} \\ &\|\Phi\|_{\infty,m^2} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|\Phi(x)\|_{\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})} < \infty,\end{aligned} \tag{B.9}$$

and introduce

$$S_\Phi \colon \begin{cases} L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}}) \to L^2(\mathbb{R}^n, \mathbb{C}^{m\times m}_{\mathrm{HS}}), \\ (S_\Phi f)^\wedge(y) = \Phi(y) f^\wedge(y) \quad \text{for a.e. } y \in \mathbb{R}^n. \end{cases} \tag{B.10}$$

Lemma B.3. *Assume* (B.9). *Then*

$$\|S_{\Phi}\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} \le \|\Phi\|_{\infty,m^2}. \tag{B.11}$$

Proof. Let $f \in L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}})$ with $\|f\|_{L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}})} = 1$. Then

$$\begin{aligned}
\|S_{\Phi} f\|_{L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}})} &= \|(S_{\Phi} f)^\wedge\|_{L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}})} \\
&= \Big(\int_{\mathbb{R}^n} \|\Phi(y) f^\wedge(y)\|^2_{\mathbb{C}^{m^2}}\, d^n x\Big)^{1/2} \\
&\le \Big(\int_{\mathbb{R}^n} \|\Phi(y)\|^2_{\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})} \|f^\wedge(y)\|^2_{\mathbb{C}^{m^2}}\, d^n x\Big)^{1/2} \\
&\le \|\Phi\|_{\infty,m^2} \Big(\int_{\mathbb{R}^n} \|f^\wedge(y)\|^2_{\mathbb{C}^{m^2}}\, d^n x\Big)^{1/2} \\
&= \|\Phi\|_{\infty,m^2} \|f^\wedge\|_{L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}})} \\
&= \|\Phi\|_{\infty,m^2} \|f\|_{L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}})} = \|\Phi\|_{\infty,m^2}. \ \blacksquare
\end{aligned} \tag{B.12}$$

Lemma B.4. *Assume that $\widetilde{\Phi}$ is a simple function, that is, there exist $J \in \mathbb{N}$, $a_j \in \mathbb{C}$, $\Phi_j \in \mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})$, with $\|\Phi_j\|_{\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})} = 1$, and $E_j \in \mathfrak{B}_n$, $1 \le j \le J$, such that $\widetilde{\Phi}$ is of the type*

$$\widetilde{\Phi} = \sum_{j=1}^{J} a_j \Phi_j \chi_{E_j}. \tag{B.13}$$

Then

$$\|S_{\widetilde{\Phi}}\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} = \|\widetilde{\Phi}\|_{\infty,m^2}. \tag{B.14}$$

Proof. Without loss of generality we may assume in addition that the sets E_j are pairwise disjoint, and that $|E_j| > 0$, $1 \le j \le J$, $0 < |E_1| < \infty$, $|a_1| \ge |a_j| > 0$, $2 \le j \le J$, implying

$$\|\widetilde{\Phi}\|_{\infty,m^2} = |a_1|. \tag{B.15}$$

Since by assumption $\|\Phi_1\|_{\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})} = 1$, there exists a sequence $\{u_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}^{m\times m}_{\mathrm{HS}}$ with $\|u_\ell\|_{\mathcal{B}_2(\mathbb{C}^m)} = 1$, $\ell \in \mathbb{N}$, such that

$$\lim_{\ell\to\infty} \|\Phi_1 u_\ell\|_{\mathcal{B}_2(\mathbb{C}^m)} = 1. \tag{B.16}$$

Introducing $f_\ell \in L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}})$, $\ell \in \mathbb{N}$, via

$$f_\ell = (|E_1|^{-1/2} u_\ell \chi_{E_1})^\vee, \quad \ell \in \mathbb{N}, \tag{B.17}$$

one infers

$$\begin{aligned}\text{(B.18)}\quad \|f_\ell\|_{L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m})} &= \|f_\ell^\wedge\|_{L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m})} \\ &= \big\||E_1|^{-1/2}u_\ell\chi_{E_1}\big\|_{L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m})} \\ &= |E_1|^{-1/2}\Big(\int_{\mathbb{R}^n}\sum_{j,k=1}^m |(u_\ell\chi_{E_1}(x))_{j,k}|^2\,d^nx\Big)^{1/2} \\ &= |E_1|^{-1/2}\Big(\int_{E_1}\sum_{j,k=1}^m |(u_\ell)_{j,k}|^2\,d^nx\Big)^{1/2} = 1, \quad \ell\in\mathbb{N},\end{aligned}$$

and

$$\begin{aligned}\text{(B.19)}\quad \|S_\Phi f_\ell\|^2_{L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m})} &= \|(S_\Phi f_\ell)^\wedge\|^2_{L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m})} \\ &= \Big(\int_{\mathbb{R}^n}\sum_{j,k=1}^m \big|(\widetilde{\Phi}(x)|E_1|^{-1/2}u_\ell\chi_{E_1}(x))_{j,k}\big|^2\,d^nx\Big)^{1/2} \\ &= \Big(|E_1|^{-1}\int_{\mathbb{R}^n}\sum_{j,k=1}^m |(a_1\Phi_1u_\ell\chi_{E_1}(x))_{j,k}|^2\,d^nx\Big)^{1/2} \\ &= \Big(|E_1|^{-1}|a_1|^2\int_{E_1}\sum_{j,k=1}^m |(\Phi_1u_\ell)_{j,k}|^2\,d^nx\Big)^{1/2} \\ &= |a_1|\,\|\Phi_1u_\ell\|_{\mathcal{B}_2(\mathbb{C}^m)} \underset{\ell\to\infty}{\longrightarrow} |a_1|.\end{aligned}$$

Thus,

$$\text{(B.20)}\qquad \|S_{\widetilde{\Phi}}\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m}))} \ge |a_1| = \|\widetilde{\Phi}\|_{\infty,m^2},$$

and Lemma B.3 provides the converse inequality. ∎

Lemma B.5. *Assume* (B.9). *Then*

$$\text{(B.21)}\qquad \|S_\Phi\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m}))} = \|\Phi\|_{\infty,m^2}.$$

Proof. Let $c_m \ge 1$ be such that

$$\begin{aligned}\text{(B.22)}\qquad c_m^{-1}\max_{1\le j,k\le m^2}|A_{j,k}| &\le \|A\|_{\mathcal{B}(\mathbb{C}_{\mathrm{HS}}^{m\times m})} \\ &\le c_m\max_{1\le j,k\le m^2}|A_{j,k}|, \quad A\in\mathbb{C}^{m^2\times m^2}.\end{aligned}$$

Then, for (Lebesgue) a.e. $x\in\mathbb{R}^n$,

$$\text{(B.23)}\quad |\Phi(x)_{j,k}| \le \max_{1\le r,s\le m^2}|\Phi(x)_{r,s}| \le c_m\|\Phi(x)\|_{\mathcal{B}(\mathbb{C}_{\mathrm{HS}}^{m\times m})} \le c_m\|\Phi\|_{\infty,m^2}.$$

Thus, for each $j,k\in\{1,\dots,,m^2\}$, there exists a sequence of simple functions

$\Psi_{j,k,\ell}\colon \mathbb{R}^n \to \mathbb{C}$, $\ell \in \mathbb{N}$, such that for a.e. $x \in \mathbb{R}^n$,

$$|\Phi(x)_{j,k} - \Psi(x)_{j,k,\ell}| \le 2^{1/2-\ell}, \tag{B.24}$$

$$|\Psi(x)_{j,k,\ell}| \le |\Phi(x)_{j,k}|. \tag{B.25}$$

Next, introduce $\Psi_\ell\colon \mathbb{R}^n \to \mathbb{C}^{m^2\times m^2}$ via $(\Psi_\ell(x))_{j,k} = \Psi(x)_{j,k,\ell}$, $1 \le j,k \le m^2$, $x \in \mathbb{R}^n$. Then for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned}\|\Phi(x) - \Psi_\ell(x)\|_{\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})} &\le c_m \max_{1\le j,k\le m^2} |\Phi(x)_{j,k} - \Psi(x)_{j,k,\ell}| \\ &\le 2^{1/2-\ell} c_m.\end{aligned} \tag{B.26}$$

Combining Lemma B.3 and (B.26) results in

$$\begin{aligned}\|S_\Phi - S_{\Psi_\ell}\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} &= \|S_{\Phi-\Psi_\ell}\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} \\ &\le \|\Phi - \Psi_\ell\|_{\infty,m^2} \le 2^{1/2-\ell} c_m,\end{aligned} \tag{B.27}$$

implying

$$\big|\|S_\Phi\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} - \|S_{\Psi_\ell}\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^{m\times m}_{\mathrm{HS}}))}\big| \le 2^{1/2-\ell} c_m. \tag{B.28}$$

Since Ψ_ℓ is a simple function, Lemma B.4 implies

$$\|S_{\Psi_\ell}\|_{\mathcal{B}(L^2(\mathbb{R}^n,\,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} = \|\Psi_\ell\|_{\infty,m^2}. \tag{B.29}$$

Employing (B.24) one obtains for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned}\big|\|\Phi(x)\|_{\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})} - \|\Psi_\ell(x)\|_{\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})}\big| &\le \|\Phi(x) - \Psi_\ell(x)\|_{\mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})} \\ &\le c_m \max_{1\le j,k\le m^2} |\Phi(x)_{j,k} - \Psi_\ell(x)_{j,k}| \le 2^{1/2-\ell} c_m,\end{aligned} \tag{B.30}$$

implying

$$\|\Phi\|_{\infty,m^2} = \lim_{\ell\to\infty} \|\Psi_\ell\|_{\infty,m^2}. \tag{B.31}$$

Combining (B.28), (B.29), and (B.31) finally yields

$$\|S_\Phi\|_{\mathcal{B}(L^2(\mathbb{R}^n,\,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} = \|\Phi\|_{\infty,m^2}. \tag{B.32}$$ ∎

We emphasize that Lemma B.5 has been proven in [14] in the infinite-dimensional context.

Corollary B.6. *Assume that* $F\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ *is measurable and that* $\|F\|_{\infty,m} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|F(x)\|_{\mathcal{B}(\mathbb{C}^m)} < \infty$. *Then*

$$\|F(-i\nabla)\|_{\mathcal{B}(L^2(\mathbb{R}^n,\,\mathbb{C}^{m\times m}_{\mathrm{HS}}))} = \|F\|_{\infty,m}. \tag{B.33}$$

Proof. Given $A \in \mathbb{C}^{m\times m}$, let $M_A \in \mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})$ be defined as in (B.2),

$$M_A(B) = BA, \quad B \in \mathbb{C}^{m\times m}_{\mathrm{HS}}, \tag{B.34}$$

and introduce $\Phi\colon \mathbb{R}^n \to \mathcal{B}(\mathbb{C}^{m\times m}_{\mathrm{HS}})$ by

$$\Phi(x) = M_{F(x)}, \quad x \in \mathbb{R}^n. \tag{B.35}$$

By Proposition B.2,

$$\|\Phi(x)\|_{\mathcal{B}(\mathbb{C}_{\mathrm{HS}}^{m\times m})} = \|F(x)\|_{\mathcal{B}(\mathbb{C}^m)}, \quad x \in \mathbb{R}^n, \tag{B.36}$$

and hence by Lemma B.5,

$$\begin{aligned} \|F(-i\nabla)\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m}))} &= \|S_\Phi\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}_{\mathrm{HS}}^{m\times m}))} \\ &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|\Phi(x)\|_{\mathcal{B}(\mathbb{C}_{\mathrm{HS}}^{m\times m})} \\ &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|F(x)\|_{\mathcal{B}(\mathbb{C}^m)} = \|F\|_{\infty,m}. \ \blacksquare \end{aligned} \tag{B.37}$$

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