

## Rosenthal compacta that are premetric of finite degree

by

**Antonio Avilés** (Murcia), **Alejandro Poveda** (Barcelona) and  
**Stevo Todorčević** (Toronto and Paris)

**Abstract.** We show that if a separable Rosenthal compactum  $K$  is a continuous  $n$ -to-one preimage of a metric compactum, but it is not a continuous  $n - 1$ -to-one preimage, then  $K$  contains a closed subset homeomorphic to either the  $n$ -split interval  $S_n(I)$  or the Alexandroff  $n$ -plicate  $D_n(2^{\mathbb{N}})$ . This generalizes a result of the third author that corresponds to the case  $n = 2$ .

**1. Introduction.** A compact space  $K$  is a Rosenthal compactum if it is homeomorphic to a compact subset of  $\mathcal{B}_1(X)$ , the space of real-valued functions of the first Baire class on a Polish space  $X$  endowed with the topology of pointwise convergence. This is a well studied class originated in relation with the study of separable Banach spaces without copies of  $\ell^1$  [OR75], [BFT78], [God80], [HMO07].

In [Tod99], three critical Rosenthal compacta are identified: The split interval  $S(I)$ , the Alexandroff duplicate of the Cantor set, denoted by  $D(2^{\mathbb{N}})$ , and the one-point compactification of a discrete set of size continuum,  $A(D)$ . The definitions of  $S(I)$  and  $D(2^{\mathbb{N}})$  are recalled in Section 3. One key property of these two compact spaces is that they are *premetric compacta of degree at most two*. A compact space  $K$  is a premetric compactum of degree at most two if there exists a continuous surjection  $f : K \rightarrow M$  onto a metric compactum  $M$  such that  $|f^{-1}(x)| \leq 2$  for all  $x \in M$ . It is proven in [Tod99] that a separable Rosenthal compactum which does not contain discrete subspaces of size continuum must be premetric of degree at most two. Another result is the following:

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**THEOREM 1.1** (S. Todorćević). *If a separable Rosenthal compactum  $K$  is a premetric compactum of degree at most two, then at least one of the following alternatives holds:*

- $K$  is metric.
- $K$  contains a homeomorphic copy of  $S(I)$ .
- $K$  contains a homeomorphic copy of  $D(2^{\mathbb{N}})$ .

For any natural number  $n$ , we can say more generally that a compact space  $K$  is a premetric compactum of degree at most  $n$  if there exists a continuous surjection  $f : K \rightarrow M$  onto a metric compactum such that  $|f^{-1}(x)| \leq n$  for all  $x \in M$ . For  $n = 1$  we get the class of metric compacta. In this work, we introduce  $n$ -dimensional versions  $S_n(I)$  and  $D_n(2^{\mathbb{N}})$  of the split interval and the Alexandroff duplicate, and prove the following generalization of the previous theorem:

**THEOREM 1.2.** *Fix a natural number  $n \geq 2$ . If a separable Rosenthal compactum  $K$  is a premetric compactum of degree at most  $n$ , then one of the following alternatives holds:*

- $K$  is a premetric compactum of degree at most  $n - 1$ .
- $K$  contains a homeomorphic copy of  $S_n(I)$ .
- $K$  contains a homeomorphic copy of  $D_n(2^{\mathbb{N}})$ .

Section 2 contains some preliminary results, in Section 3 we introduce the spaces  $S_n(I)$  and  $D_n(2^{\mathbb{N}})$ , and Section 4 contains the proof of the main result, which mimics [Tod99, Section E] with some adaptations needed for the new multidimensional setting.

**2. Preliminaries.** In this section we recall some results on descriptive set theory, general topology and Ramsey theory that we need. For further details we refer to [Kec95], [Deb14], [Eng89] and [Tod10].

We denote by  $\mathbb{N}$  the set of natural numbers and identify each natural number  $n$  with the set  $\{0, \dots, n - 1\}$  of its predecessors. Given a non-empty set  $X$ , let  $X^{<\mathbb{N}}$  be the set of finite sequences in  $X$ . For every finite sequence  $s = (s_0, \dots, s_{n-1})$  we denote by  $\text{length}(s)$  the natural number  $n$ , the domain of  $s$ . If  $n \leq \text{length}(s)$  then  $s \upharpoonright n$  will denote the sequence  $(s_0, \dots, s_{n-1})$ , the restriction of  $s$  to its first  $n$  coordinates. Given  $s, t \in X^{<\mathbb{N}}$ , we say that  $s$  is an *initial segment* of  $t$  or  $t$  is an *extension* of  $s$  (written  $s \preceq t$ ) if there exists a natural number  $m \leq \text{length}(t)$  such that  $s = t \upharpoonright m$ . If  $\text{length}(s) < \text{length}(t)$ , we say that this extension is *proper*. If  $s \not\preceq t$  and  $t \not\preceq s$  then  $s$  and  $t$  are *incomparable*. Given two sequences  $s = (s_0, \dots, s_{n-1})$  and  $t = (t_0, \dots, t_{k-1})$ , we denote by  $s \frown t$  the sequence  $(s_0, \dots, s_{n-1}, t_0, \dots, t_{k-1})$ .

We say that a subset  $T$  of  $X^{<\mathbb{N}}$  is a *tree* on  $X$  if it is non-empty and closed under initial segments, that is, if  $s \in T$  and  $t \in X^{<\mathbb{N}}$  with  $t \preceq s$  then  $t \in T$ . We will refer to the elements of  $T$  as the *nodes* of the tree  $T$ .

We say that a sequence  $x \in X^{\mathbb{N}}$  is a *branch* of the tree  $T$  if  $x \upharpoonright n \in T$  for every  $n \in \mathbb{N}$ . When a node  $s \in T$  is an initial segment of a branch  $x$ , we write  $s \subset x$  or  $x \in [s]$ . Finally,  $[T]$  denotes the set of all branches of  $T$ .

Given two non-empty sets  $X, Y$ , there is a natural way to build a tree  $T$  on the product  $X \times Y$ . In such a case, we restrict ourselves to nodes of the form  $s = (t, u) \in T$  such that  $t \in X^{<\mathbb{N}}$ ,  $u \in Y^{<\mathbb{N}}$  and  $\text{length}(t) = \text{length}(u)$ . A couple  $(t, u)$  will be an extension of  $(t', u')$  if and only if  $t'$  and  $u'$  are initial segments of  $t$  and  $u$  respectively. It is easy to prove that

$$[T] = \{(u, t) \in X^{\mathbb{N}} \times Y^{\mathbb{N}} : (u \upharpoonright n, t \upharpoonright n) \in T, \forall n \in \mathbb{N}\}.$$

Recall that every product  $X^{\mathbb{N}}$  can be endowed in a natural way with the product topology by taking the discrete topology on  $X$ . An open basis of the topology of  $X^{\mathbb{N}}$  is given by the sets  $\{x \in X^{\mathbb{N}} : s \subset x\}$  where  $s \in X^{<\mathbb{N}}$ .

Recall that a topological space is said to be *Polish* if it is separable and completely metrizable, that is, there exists a complete metric compatible with the topology. If  $X$  is a countable set then  $X^{\mathbb{N}}$  is a Polish space. Two significant examples of that kind of Polish spaces are the Cantor space  $2^{\mathbb{N}}$  and the Baire space  $\mathbb{N}^{\mathbb{N}}$ . It is well known, for example, that every perfect Polish space contains copies of  $2^{\mathbb{N}}$ , and hence has cardinality continuum. The following notion is often used when building copies of  $2^{\mathbb{N}}$  inside another space:

**DEFINITION 2.1** (Cantor scheme). Given a set  $X$ , we say that a family  $\{\mathcal{A}_s\}_{s \in 2^{<\mathbb{N}}}$  of subsets of  $X$  is a *Cantor scheme* over  $X$  if it the following conditions are satisfied:

- $\mathcal{A}_{s \smallfrown 0} \cap \mathcal{A}_{s \smallfrown 1} = \emptyset$ .
- $\mathcal{A}_{s \smallfrown i} \subseteq \mathcal{A}_s$  for all  $s \in 2^{<\mathbb{N}}$ ,  $i \in \{0, 1\}$ .

Given any totally ordered set  $(X, \preceq)$ , we can endow  $X$  with the topology generated by its  $\prec$ -rays  $\{x \in X : x \prec x_0\}$  and  $\{x \in X : x \succ x_0\}$  where  $x_0 \in X$ . This topology is called the *order topology induced by  $\preceq$* . Recall that given two partially ordered sets  $(X, \leq)$  and  $(Y, \preceq)$  the lexicographical order on  $X \times Y$  is defined by the following condition

$$(x_1, y_1) \leq_{\text{lex}} (x_2, y_2) \Leftrightarrow (x_1 < x_2) \text{ or } (x_1 = x_2 \text{ and } y_1 \preceq y_2).$$

A special property shared by every space whose topology arises from an order is the following

**THEOREM 2.2.** *For a topological space  $X$  endowed with an order topology, it is equivalent to be separable and hereditarily separable.*

*Proof.* If  $D$  is a countable dense subset of  $X$ , and  $A \subset X$ , then for every interval  $I$  (of any kind,  $(a, b)$ ,  $[a, b)$ ,  $(a, b]$ ,  $[a, b]$ , or even  $[a, a] = a$ ) whose extremes belong to  $D$ , choose  $a(I) \in A \cap I$  whenever  $A \cap I \neq \emptyset$ . The elements  $a(I)$  form a countable dense subset of  $A$ . ■

A real-valued function  $f$  on a Polish space  $X$  is of the *first Baire class* if it is the pointwise limit of real-valued continuous functions on  $X$ . This class of functions is usually denoted by  $\mathcal{B}_1(X)$ .

DEFINITION 2.3 (Rosenthal compactum). We say that a topological space  $K$  is a *Rosenthal compactum* if it is homeomorphic to a pointwise compact subspace of  $\mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$ .

If  $X$  is an arbitrary Polish space, then every pointwise compact subset of  $\mathcal{B}_1(X)$  is a Rosenthal compactum, because  $X$  is a continuous image of  $\mathbb{N}^{\mathbb{N}}$ . The class of Rosenthal compact spaces constitutes a generalization of the class of compact metrizable spaces, and many of the properties of metrizable compacta hold true for Rosenthal compacta as well. One example is Bourgain–Fremlin–Talagrand’s result on the Fréchet–Urysohn property of Rosenthal compacta [BFT78], meaning that every point in the closure of a set is the limit of a convergent sequence from that set. Another remarkable result along the same lines states that every Rosenthal compactum contains a dense metrizable subspace [Tod99]. We are specially interested in separable Rosenthal compacta. If we have countably many Borel functions on a Polish space  $X$ , we can add countably many open sets to the topology of  $X$  to get a larger Polish topology where all those functions are continuous. From this observation one gets:

PROPOSITION 2.4. *Every separable Rosenthal compactum is homeomorphic to the pointwise closure in  $\mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$  of a sequence of continuous functions.*

A Rosenthal compactum  $K$  is *premetric of degree at most  $n$*  if there exists a continuous surjection  $f : K \rightarrow M$  onto a metric compactum  $M$  such that  $|f^{-1}(x)| \leq n$  for all  $x \in M$ . Some classical examples of premetric compacta of degree at most  $n$  are the compact metric spaces ( $n = 1$ ) as well as the split interval  $S(I)$  and the Alexandroff duplicate of the Cantor space,  $D(2^{\mathbb{N}})$  ( $n = 2$ ). In this paper we present the  $n$ -dimensional versions  $S_n(I)$  and  $D_n(2^{\mathbb{N}})$  of  $S(I)$  and  $D(2^{\mathbb{N}})$  respectively, and we prove they are premetric compacta of degree at most  $n$ . In Section 3 the definition of  $S_n(I)$  and  $D_n(2^{\mathbb{N}})$  will be presented along with their classical versions for  $n = 2$ . There are also classical examples of Rosenthal compacta which are not premetric compacta of any degree. One of these is the Alexandroff compactification of a discrete space  $D$  of size continuum,  $A(D) = D \cup \{\infty\}$ . For simplicity, we will refer to compacta that are premetric of degree at most  $n$  for some  $n$  as *compacta of degree  $n$* .

There is a nice topological characterization of those compact spaces (not necessarily Rosenthal) that are of certain degree depending on projections over countable products.

PROPOSITION 2.5. *Given a set  $X$  and a compact subspace  $K$  of  $\mathbb{R}^X$ , the subspace  $K$  is of degree  $n$  if and only if there exists a countable set  $D_0 \subset X$  such that*

$$\pi_{D_0} : K \rightarrow \{f \upharpoonright_{D_0} : f \in K\}$$

*is at most  $n$ -to-1.*

*Proof.* If there exists a  $D_0$  such that  $\pi_{D_0}$  is at most  $n$ -to-1, then since  $\pi_{D_0}$  is continuous,  $\{f \upharpoonright_{D_0} : f \in K\}$  is a metric compactum, and thus  $K$  is of degree  $n$ . Conversely, let  $M$  be a compact metric space and  $\Phi : K \rightarrow M$  continuous and at most  $n$ -to-1. Since  $\Phi$  is uniformly continuous, there exists a countable subset  $D_0$  of  $X$  such that if  $f \upharpoonright_{D_0} = g \upharpoonright_{D_0}$  then  $\Phi(f) = \Phi(g)$ . This implies that  $\pi_{D_0}$  is at most  $n$ -to-1. ■

**3. The compact spaces  $S_n(I)$  and  $D_n(2^{\mathbb{N}})$ .** This section is devoted to presenting the compact spaces  $S_n(I)$  and  $D_n(2^{\mathbb{N}})$ . They are respective generalizations of the critical compacta  $S(I)$  and  $D(2^{\mathbb{N}})$ , and as we shall see in the last section, they play the same role as the split interval and the Alexandroff duplicate but for the class of separable Rosenthal compacta of degree  $n$ . For completeness, first we give the definition of  $S(I)$  and  $D(K)$  as well as some of their fundamental properties.

### 3.1. The compact space $S_n(I)$

DEFINITION 3.1 (Split interval). The *split interval*  $S(I)$  is the space  $I \times \{0, 1\} \setminus \{(0, 0), (1, 1)\}$  endowed with the topology induced by the lexicographical order.

It is easy to see that neighbourhood bases at  $(x, 0)$ ,  $(x, 1)$  in  $S(I)$  are given respectively by the sets

$$\begin{aligned} & ](y, 0), (x, 1)[ \quad \text{with } y \in I, y < x, \\ & ](x, 0), (y, 1)[ \quad \text{with } y \in I, x < y. \end{aligned}$$

It is convenient to make some comments about convergence in  $S(I)$ . Note that a sequence  $\{(x_m, i_m)\}_{m \in \mathbb{N}}$  in  $S(I)$  converges to a point  $(x, 0)$  of the first level if and only if  $\{x_m\}_{m \in \mathbb{N}}$  converges to  $x$  from the left, that is, for every  $\varepsilon > 0$  there exists  $m_0$  such that  $x - \varepsilon < x_m \leq x$  for every  $m \geq m_0$ . The analogous fact holds for points of the second level and convergence from the right.

It is easy to see that  $S(I)$  is a Rosenthal compactum. Indeed, the map given by  $(x, 0) \mapsto \mathbf{1}_{[0, x)}$  and  $(x, 1) \mapsto \mathbf{1}_{[0, x]}$  is a homeomorphism between  $S(I)$  and a closed subspace of the compact space  $[0, 1]^{[0, 1]}$  formed by functions in  $\mathcal{B}_1([0, 1])$ . Moreover, the split interval is a non-metrizable and hereditarily separable space. The latter statement is immediate since  $S(I)$  is separable, and thus hereditarily separable by Theorem 2.2. The former is also straightforward, in fact the only metrizable subspaces of  $S(I)$  are the countable

ones. We shall see later that  $D(2^{\mathbb{N}})$  is not separable, so  $S(I)$  does not contain copies of  $D(2^{\mathbb{N}})$  precisely because  $S(I)$  is hereditarily separable. In the following theorem we summarize some of the most important properties of  $S(I)$  that will be necessary later:

**THEOREM 3.2** (Properties of  $S(I)$ ). *The split interval  $S(I)$  has the following properties:*

- *It is a Rosenthal compactum of degree 2.*
- *It is hereditarily separable.*
- *It is non-metrizable, so it is not of degree 1.*
- *It does not contain copies of  $D(2^{\mathbb{N}})$ .*

Now, we define a compact space  $S_n(I)$  which is a natural generalization of the split interval. In fact, we give the following slightly more general definition, for any perfect subset  $R \subset I$ :

**DEFINITION 3.3** ( $n$ -split interval of  $R$ ). Given any natural number  $n \geq 2$  and any perfect subspace  $R \subset I$ , the  $n$ -split interval of  $R$ , denoted by  $S_n(R)$ , is the space  $R \times \{0, \dots, n-1\}$  endowed with the topology for which the points of the form  $(x, i)$  with  $i \in \{2, \dots, n-1\}$  are isolated and the points of the form  $(x, 0)$  and  $(x, 1)$  have respective basic neighbourhoods of the form

$$\begin{aligned} &\{(x, 0)\} \cup \{(y, i) : z_0 < y < x, i \in \{0, \dots, n-1\}\}, \\ &\{(x, 1)\} \cup \{(y, i) : z_1 > y > x, i \in \{0, \dots, n-1\}\}, \end{aligned}$$

where  $z_0, z_1 \in R$  with  $z_0 < x, z_1 > x$ . If  $x$  is the minimum of  $R$  then  $(x, 0)$  is isolated, and if it is the maximum then  $(x, 1)$  is isolated. If  $R$  is the whole space  $I$  then we simply say that  $S_n(I)$  is the  $n$ -split interval.

With this definition, the 2-split interval  $S_2(I)$  is the split interval  $S(I)$  with two extra isolated points. This formal difference between  $S(I)$  and  $S_2(I)$  is completely irrelevant for our discussion. Now we prove some basic results about  $S_n(I)$ .

**PROPOSITION 3.4.** *For any natural  $n \geq 2$  and any perfect subspace  $R$  of  $I$ , the space  $S_n(R)$  contains copies of  $S_n(I)$ .*

*Proof.* It is well known that every perfect Polish space contains copies of the Cantor space  $2^{\mathbb{N}}$ . Now, let us view  $2^{\mathbb{N}}$  as a subset  $C$  of the unit interval in the classical way. By the preceding comment,  $S_n(R)$  contains copies of  $S_n(C)$ . Note that the subspace  $C \times \{0, 1\}$  of  $S_n(C)$  has countably many isolated points: namely, the set of its isolated points is  $(\mathcal{L} \times \{1\}) \cup (\mathcal{R} \times \{0\})$  where  $\mathcal{L}$  and  $\mathcal{R}$  are respectively the set of left and right end points deleted while building  $C$ . Now let  $\mathcal{X}$  be the subspace of  $S_n(C)$  obtained by removing the points of  $(\mathcal{L} \times \{1\}) \cup (\mathcal{R} \times \{0, 2, \dots, n-1\})$ . Since all those

points are isolated in  $S_n(C)$ , the space  $\mathcal{X}$  is compact. Then the map

$$\Psi : \mathcal{X} \rightarrow S_n(I), \quad (x, i) \mapsto (\Phi(x), i),$$

where  $\Phi$  is the classical surjection of  $C$  onto  $I$ , is a homeomorphism. Thus,  $S_n(R)$  contains copies of  $S_n(I)$ . ■

PROPOSITION 3.5.  $S_n(I)$  is a topological Hausdorff compact space.

*Proof.* It is straightforward to see that  $S_n(I)$  is Hausdorff, so we only prove compactness. Given any open covering  $\{\mathcal{U}_i\}_{i \in \mathcal{I}}$  of  $S_n(I)$  we can suppose without loss of generality that  $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$  and  $\mathcal{U}_i$  is a singleton if  $i \in \mathcal{I}_0$ , while if  $i \in \mathcal{I}_1$  then

$$\mathcal{U}_i = ](x_i, 0), (y_i, 1)[ \cup ]x_i, y_i[ \times \{2, \dots, n - 1\},$$

where  $]a, b[$  denotes an interval in the lexicographical order of  $I \times \{0, 1\}$  or in the usual order of  $I$ . Since the subspace  $I \times \{0, 1\}$  of  $S_n(I)$  is homeomorphic to  $S_2(I)$ , which is compact, we can find a finite  $F \subset \mathcal{I}_1$  such that  $\{\mathcal{U}_i\}_{i \in F}$  covers  $I \times \{0, 1\}$ . Let also  $\{\mathcal{U}_i\}_{i \in G}$  be a finite subfamily of  $\{\mathcal{U}_i\}_{i \in \mathcal{I}}$  which contains the points of the form  $(x_i, j)$  and  $(y_i, j)$  with  $i \in F$  and  $0 \leq j \leq n - 1$ . We claim that  $\{\mathcal{U}_i\}_{i \in F \cup G}$  is a finite subcover of  $\{\mathcal{U}_i\}_{i \in \mathcal{I}}$ . Indeed, given  $(x, j) \in S_n(I)$ , we have  $(x, 0) \in I \times \{0, 1\}$ , so there exists  $i_0 \in F$  such that  $(x, 0) \in \mathcal{U}_{i_0}$ . If  $x \neq y_{i_0}$ , then  $(x, j) \in \mathcal{U}_{i_0}$ , by the form of  $\mathcal{U}_i$ . If otherwise  $x = y_{i_0}$ , then  $(x, j)$  lies in some open set  $\mathcal{U}_k$  with  $k \in G$ . ■

THEOREM 3.6.  $S_n(I)$  is a Rosenthal compactum.

*Proof.* Recall that the class of Rosenthal compacta is closed under taking closed subspaces and countable products. Now, consider the map  $\Phi : S_n(I) \rightarrow S_2(I) \times A(I)^{n-2}$  given by

$$\begin{aligned} \Phi(x, 0) &= ((x, 0), \infty, \dots, \infty), \\ \Phi(x, 1) &= ((x, 1), \infty, \dots, \infty), \\ \Phi(x, i) &= ((x, 1), \infty, \dots, x_i, \dots, \infty) \quad \text{if } i \in \{2, \dots, n - 1\}. \end{aligned}$$

We will prove that  $\Phi$  is an embedding, and thus  $S_n(I)$ , a Rosenthal compactum. Since  $\Phi$  is injective, it is enough to check that it is continuous. It is obvious that the projection onto the first coordinate,  $\pi_1 \circ \Phi : S_n(I) \rightarrow S_2(I)$ , is a continuous map. For the others, it is enough to study the points of the form  $(x, 0)$  since the case  $(x, 1)$  is analogous and the others are isolated. Note that, when  $i > 1$ ,

$$\pi_i \circ \Phi : S_n(I) \rightarrow A(I), \quad (x, j) \mapsto \begin{cases} \infty & \text{if } j \neq i, \\ x & \text{if } j = i. \end{cases}$$

Since a neighbourhood of  $\pi_i \circ \Phi(x, 0) = \infty$  is of the form  $\mathcal{V} = I \setminus F \cup \{\infty\}$  where  $F$  is a finite subset of  $I$ , the neighbourhood

$$\mathcal{U} = S_n(I) \setminus \{(y, i) : y \in F, i \in \{2, \dots, n - 1\}\}$$

satisfies  $\Phi(\mathcal{U}) \subset \mathcal{V}$ . ■

If  $n \geq 2$  then  $S_n(I)$  is non-metrizable. Moreover, we now prove

**THEOREM 3.7.**  *$S_n(I)$  is a Rosenthal compactum of degree  $n$  but not of degree  $n - 1$ .*

*Proof.*  $S_n(I)$  is obviously a Rosenthal compactum of degree  $n$  since the projection onto the first coordinate is continuous and at most  $n$ -to-1. To see that  $S_n(I)$  is not of degree  $n - 1$  we need Proposition 2.5. First of all notice that the family of characteristic functions

$$\mathcal{F} = \{\mathbb{1}_{[x,y] \times \{0, \dots, n-1\} \setminus \{(x,0), (y,1)\}}\}_{x,y \in I, x < y} \cup \{\mathbb{1}_{\{(x,i)\}}\}_{x \in I, i \in \{2, \dots, n-1\}}$$

is formed by continuous functions and separates points, and thus the evaluation map  $e : (x, i) \mapsto e_{(x,i)}$  establishes an embedding of  $S_n(I)$  into  $\mathbb{R}^{\mathcal{F}}$ . If  $S_n(I)$  were of degree  $n - 1$ , by Proposition 2.5 there would exist a countable subset  $D_0$  of  $\mathcal{F}$  such that  $\pi_{D_0}$  is at most  $(n - 1)$ -to-1 on the image of the above embedding. We are going to see that in fact for any countable subset of  $\mathcal{F}$  the corresponding projection map is at least  $n$ -to-1. Given a countable subset

$$N = \{\mathbb{1}_{[x_k, y_k] \times \{0, \dots, n-1\} \setminus \{(x_k,0), (y_k,1)\}}\}_{k \in \mathbb{N}} \cup \{\mathbb{1}_{\{(z_k, i_k)\}}\}_{k \in \mathbb{N}}$$

of  $\mathcal{F}$ , we can pick a point  $x \in I$  such that  $x \notin \{x_k, y_k, z_k : k \in \mathbb{N}\}$ . Now, if we take any  $i \in \{0, \dots, n - 1\}$  and any  $k \in \mathbb{N}$ , then

$$\mathbb{1}_{[x_k, y_k] \times \{0, \dots, n-1\} \setminus \{(x_k,0), (y_k,1)\}}((x, i)) = \mathbb{1}_{[x_k, y_k]}(x), \quad \mathbb{1}_{(z_k, i_k)}((x, i)) = 0.$$

Thus, all evaluations  $\{e_{(x,i)} : i = 0, \dots, n - 1\}$  have the same image under  $\pi_N$ , so  $\pi_N$  is at least  $n$ -to-1. ■

If  $R$  is a perfect subspace of  $I$ , then  $S_n(R)$  is a closed subset of  $S_n(I)$ , and by Proposition 3.4 also  $S(I)$  is homeomorphic to a closed subset of  $S_n(R)$ . Therefore, the previous results for  $S_n(I)$  hold for  $S_n(R)$  as well.

**THEOREM 3.8** (Properties of  $S_n(R)$ ). *For every perfect subspace  $R$  of the unit interval  $I$ , the  $n$ -split interval of  $R$  has the following properties:*

- It is a Rosenthal compactum of degree  $n$  but not of degree  $n - 1$ .
- It is non-separable.
- It is non-metrizable.
- It contains copies of the  $n$ -split interval  $S_n(I)$ .

### 3.2. The compact space $D_n(2^{\mathbb{N}})$

**DEFINITION 3.9** (Alexandroff duplicate). Given any topological Hausdorff space  $(X, \mathcal{T})$ , we define its *Alexandroff duplicate*  $D(X)$  as the space  $X \times \{0, 1\}$  endowed with the topology for which the points  $(x, 1)$  are isolated and the points  $(x, 0)$  have neighbourhoods of the form

$$\mathcal{U} \times \{0, 1\} \setminus \{(x, 1)\}$$

where  $\mathcal{U}$  is a  $\mathcal{T}$ -neighbourhood of  $x$ .



It is well-known that if  $K$  is a compact Hausdorff space then so is  $D(K)$ . In fact,  $D(K)$  is compact if and only if  $K$  is. If  $K$  is a compact metric space, then  $D(K)$  is a Rosenthal compactum [Tod99]. In this paper we are interested in the compact space  $D(2^{\mathbb{N}})$ . It is a non-separable space (because it has uncountably many isolated points), and thus it is not metrizable. The results from [Tod99] show that the Alexandroff duplicate  $D(2^{\mathbb{N}})$  is a critical example of Rosenthal compactum. For example, it embeds into any separable Rosenthal compactum of degree 2 that is not hereditarily separable.

**PROPOSITION 3.10** (Properties of  $D(2^{\mathbb{N}})$ ). *The Alexandroff duplicate  $D(2^{\mathbb{N}})$  has the following properties:*

- *It is a Rosenthal compactum of degree 2.*
- *It is not separable.*
- *It is not metrizable, so it is not of degree 1.*
- *It is monolithic (every separable subspace is metrizable), so it does not contain copies of  $S(I)$ .*

We shall prove the above facts for the more general spaces  $D_n(2^{\mathbb{N}})$  that we define now:

**DEFINITION 3.11** (Alexandroff  $n$ -plicate). Given any natural number  $n \geq 2$  and a topological Hausdorff space  $(X, \mathcal{T})$ , the *Alexandroff  $n$ -plicate*  $D_n(X)$  is the space  $X \times \{0, \dots, n-1\}$  endowed with the topology for which the points  $(x, i)$  with  $i \in \{1, \dots, n-1\}$  are isolated and the points  $(x, 0)$  have basic neighbourhoods of the form

$$\mathcal{U} \times \{0, \dots, n-1\} \setminus \bigcup_{i=1}^{n-1} \{(x, i)\}$$

where  $\mathcal{U}$  is a  $\mathcal{T}$ -neighbourhood of  $x$ .

Note that the Alexandroff 2-plicate  $D_2(X)$  coincides with its classical version  $D(X)$ . Moreover, the Alexandroff  $n$ -plicate shares with the Alexandroff duplicate all the properties mentioned in Proposition 3.10. Since we are interested in the concrete Alexandroff  $n$ -plicate  $D_n(2^{\mathbb{N}})$ , we only prove the fundamental properties of that space, although some of these hold for more general  $D_n(X)$  spaces.

**PROPOSITION 3.12.**  *$D_n(2^{\mathbb{N}})$  is a Rosenthal compactum.*

*Proof.* The space is clearly Hausdorff. The space  $D_n(X)$  is compact whenever  $X$  is compact, because if  $\mathcal{W}$  is an open cover of  $D_n(X)$ , then after taking a finite subcover of  $X \times \{0\}$ , only finitely many points can remain, or otherwise they would accumulate to  $X \times \{0\}$ . To see that it is a Rosenthal compactum, consider the map  $\Phi : D_n(2^{\mathbb{N}}) \rightarrow 2^{\mathbb{N}} \times A(2^{\mathbb{N}})^{n-1}$  given

by

$$\Phi(x, 0) = (x, \infty, \dots, \infty), \quad \Phi(x, i) = (x, \infty, \dots, x_i, \dots, \infty).$$

Since  $2^{\mathbb{N}} \times A(2^{\mathbb{N}})^{n-1}$  is a Rosenthal compactum, it is enough to see that  $\Phi$  is an embedding, and in fact it is enough to see that  $\Phi$  is continuous at the points  $(x, 0)$ . A basic neighbourhood of  $\Phi(x, 0)$  is of the form

$$\mathcal{U} \times \prod_{i=1}^{n-1} (2^{\mathbb{N}} \setminus F_i \cup \{\infty\})$$

where  $\mathcal{U}$  is a neighbourhood of  $x$  in  $2^{\mathbb{N}}$  and  $F_i$  are finite subsets of  $2^{\mathbb{N}}$ . Let

$$F = \bigcup_{i=1}^{n-1} \{(x, i)\} \cup \{(y, j) : i, j \in \{1, \dots, n-1\}, y \in F_i\}$$

and consider  $\mathcal{V} = \mathcal{U} \times \{0, \dots, n-1\} \setminus F$ , which is a neighbourhood of  $(x, 0)$  that satisfies

$$\Phi(\mathcal{V}) \subseteq \mathcal{U} \times \prod_{i=1}^{n-1} (\{y : y \in K \setminus F_i\} \cup \{\infty\}),$$

and thus  $\Phi$  is a continuous map. ■

PROPOSITION 3.13.  $D_n(2^{\mathbb{N}})$  is a Rosenthal compactum of degree  $n$  but not of degree  $n - 1$ .

*Proof.* Obviously,  $D_n(2^{\mathbb{N}})$  is a Rosenthal compactum of degree  $n$  since the corresponding projection onto  $2^{\mathbb{N}}$  is an  $n$ -to-1 continuous map. To see that  $D_n(2^{\mathbb{N}})$  is not of degree  $n - 1$  we proceed as in Theorem 3.7. Since  $2^{\mathbb{N}}$  is zero-dimensional and second-countable, we can take a countable clopen basis  $\{\mathcal{A}_m\}_{m \in \mathbb{N}}$ , and thus if  $\mathcal{F} = \{\mathbb{1}_{\mathcal{A}_m \times \{0, \dots, n-1\}}\}_{m \in \mathbb{N}} \cup \{\mathbb{1}_{\{(x,i)\}}\}_{x \in 2^{\mathbb{N}}, i \in \{1, \dots, n-1\}}$  then the evaluation map  $e : (x, i) \mapsto e_{(x,i)}$  establishes an embedding of  $D_n(2^{\mathbb{N}})$  into  $\mathbb{R}^{\mathcal{F}}$ .

Suppose that  $D_n(2^{\mathbb{N}})$  is of degree  $n - 1$ . Then by Proposition 2.5 there is a countable subset  $N$  of  $\mathcal{F}$  for which the corresponding projection  $\pi_N$  is at most  $(n - 1)$ -to-1. We can suppose that  $N$  is of the form  $\{\mathbb{1}_{\mathcal{A}_{m_k} \times \{0, \dots, n-1\}}\}_{k \in \mathbb{N}} \cup \{\mathbb{1}_{\{(x_k, i_k)\}}\}_{k \in \mathbb{N}}$ . Pick  $x \in 2^{\mathbb{N}}$  for which  $x \neq x_k$  for every  $k \in \mathbb{N}$ . Then, for all  $k \in \mathbb{N}$  and all  $i \in \{0, \dots, n - 1\}$ ,

$$e_{(x,i)}(\mathbb{1}_{\{(x_k, i_k)\}}) = 0, \quad e_{(x,i)}(\mathbb{1}_{\mathcal{A}_{m_k} \times \{0, \dots, n-1\}}) = \mathbb{1}_{\mathcal{A}_{m_k}}(x).$$

Therefore  $\pi_N$  is at least  $n$ -to-1 because all points  $(x, i)$  have the same image; a contradiction. ■

We finish this section by proving that the Alexandroff  $n$ -plicate is a monolithic space. As we shall see later, this will imply that none of the compact spaces  $S_n(I)$  and  $D_n(2^{\mathbb{N}})$  contains copies of the other.

PROPOSITION 3.14.  $D_n(2^{\mathbb{N}})$  is a monolithic space.

*Proof.* Let  $Z$  be a countable subset of  $D_n(2^{\mathbb{N}})$ , and let  $\mathcal{B}$  be a countable basis of open subsets of  $2^{\mathbb{N}}$ . Then

$$\{B \times \{0, \dots, n-1\} : B \in \mathcal{B}\} \cup \{\{(x, i)\} : (x, i) \in Z, i \neq 0\}$$

is a countable family of open sets that separates the points of the compact space  $\bar{Z}$  (notice that all points of  $\bar{Z} \setminus Z$  live in  $2^{\mathbb{N}} \times \{0\}$ ). Hence  $\bar{Z}$  is metrizable. ■

COROLLARY 3.15.  $S_n(I)$  does not contain copies of  $D_n(2^{\mathbb{N}})$ , and vice versa.

*Proof.* If  $S_n(I) \hookrightarrow D_n(2^{\mathbb{N}})$ , then since the Alexandroff  $n$ -plicate is a monolithic space and  $S(I)$  is a separable subspace of  $S_n(I)$  (remember that  $S(I)$  is the result of removing two isolated points from  $S_2(I)$ ),  $S(I)$  would be metrizable, contradicting Proposition 3.2. On the other hand, if  $D_n(2^{\mathbb{N}}) \hookrightarrow S_n(I)$ , then the non-isolated points of  $D_n(2^{\mathbb{N}})$  embed into the non-isolated points of  $S_n(I)$ . That would give a copy of  $2^{\mathbb{N}}$  inside  $S(I)$ , in contradiction with Proposition 3.2 again. ■

**4. The proof of the main result.** First, it is convenient to point out that the assumption of separability is essential. Indeed, consider the Rosenthal compact space  $K = D_n(2^{\mathbb{N}}) \setminus \mathcal{B}$  where  $\mathcal{B} = B \times \{1, \dots, n-1\}$  and  $B$  is a *Bernstein set* (that is,  $C \cap B$  and  $C \setminus B$  are uncountable for all uncountable Borel sets  $C$ ). The same proof as in Proposition 3.13 shows that  $K$  is of degree  $n$  but not  $n-1$ . It is clear that  $K$  does not contain copies of  $S_n(I)$ , nor even  $S_2(I)$ , since neither does  $D_n(2^{\mathbb{N}})$ .

On the other hand,  $K$  contains no copies of  $D_n(2^{\mathbb{N}})$ , or even  $D_2(2^{\mathbb{N}})$ . One way to see this is to use the fact that for every continuous function  $f : D_2(2^{\mathbb{N}}) \rightarrow 2^{\mathbb{N}}$  there exists a perfect subset  $P \subset 2^{\mathbb{N}}$  such that  $|f^{-1}(p)| = 2$  for all  $p \in P$ . The restriction of the first coordinate  $f : X \rightarrow 2^{\mathbb{N}}$ ,  $f(x, i) = x$ , has points with only one preimage on any perfect set, so  $D_2(2^{\mathbb{N}})$  does not embed into  $X$ . The proof of the above fact is an elementary exercise: For every clopen set  $A \subset 2^{\mathbb{N}}$ , write its clopen preimage in the form

$$f^{-1}(A) = C_A \times \{0, 1\} \Delta F_A \times \{1\},$$

where  $C_A \subset 2^{\mathbb{N}}$  is clopen,  $F_A$  is finite, and  $\Delta$  represents symmetric difference. It is enough to take for  $P$  a perfect set disjoint from all the  $F_A$ 's.

As in the proof of the classical result for separable Rosenthal compacta of degree 2, we will require two Ramsey-like results used in [Tod99]. For completeness, we state them below. The first is [Tod99, Theorem 11]. The second was proved by Galvin [Gal68] for open  $A$ , but by results of Mycielski [Myc67], it also holds for any measurable  $A$  (see [Bla82]).

**THEOREM 4.1.** *Let  $\{f_s : s \in 2^{<\mathbb{N}}\}$  be a relatively compact subset of Baire class 1 functions defined on a Polish space  $X$ . Then there is a perfect Polish space  $P \subset 2^{\mathbb{N}}$  and an infinite strictly increasing sequence  $\{m_k\}_k$  of natural numbers such that  $\{f_{a \upharpoonright m_k}\}_k$  is pointwise convergent for every  $a \in P$ .*

**THEOREM 4.2.** *For every perfect Polish space  $X$  and every symmetric analytic relation  $A \subset X^2$  there is a perfect set  $P \subset X$  such that  $P^{[2]} = \{(x, y) \in P^2 : x \neq y\}$  is either disjoint from or included in  $A$ .*

**4.1. The proof.** In what follows,  $n \geq 2$  will be a natural number,  $K \subset \mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$  will be a separable Rosenthal compactum of degree  $n$  but not  $n - 1$ , and  $K_0$  will be a countable dense subspace of  $K$  consisting of continuous functions (remember Proposition 2.4).

Since  $K$  is of degree  $n$ , by Proposition 2.5 there exists a countable subset  $D_0 \subset \mathbb{N}^{\mathbb{N}}$  such that  $\pi_{D_0}$  is at most  $n$ -to-1. With this in mind we will build, as in [Tod99], an  $\omega_1$ -sequence  $\{(f_\alpha^0, \dots, f_\alpha^{n-1})\}_{\alpha < \omega_1}$  of functions in  $K^n$  and an  $\omega_1$ -sequence  $\{(x_\alpha^{ij})_{ij} : 0 \leq i \neq j \leq n - 1, \alpha < \omega_1\}$  of points in  $(\mathbb{N}^{\mathbb{N}})^{n^2 - n}$  with the following properties:

- (1)  $f_\alpha^i \upharpoonright_{D_0} = f_\alpha^j \upharpoonright_{D_0}$  for all  $i, j \in \{0, \dots, n - 1\}$ .
- (2)  $f_\alpha^i(x_\alpha^{ij}) \neq f_\alpha^j(x_\alpha^{ij})$  and  $x_\alpha^{ij} = x_\alpha^{ji}$  for all  $\alpha < \omega_1$  and  $i, j \in \{0, \dots, n - 1\}$  with  $i \neq j$ .
- (3)  $f_\alpha^k(x_\beta^{ij}) = f_\alpha^l(x_\beta^{ij})$  for all  $\beta < \alpha$  and  $i, j, k, l \in \{0, \dots, n - 1\}$  with  $i \neq j$ .
- (4)  $f_\alpha^i \upharpoonright_{D_0} \neq f_\beta^j \upharpoonright_{D_0}$  when  $\alpha \neq \beta$  and  $i, j \in \{0, \dots, n - 1\}$ .

**PROPOSITION 4.3.** *There exist  $\omega_1$ -sequences*

$$\{(f_\alpha^0, \dots, f_\alpha^{n-1})\}_{\alpha < \omega_1}, \quad \{(x_\alpha^{01}, \dots, x_\alpha^{(n-1)(n-2)})\}_{\alpha < \omega_1}$$

*as above which satisfy (1)–(4).*

*Proof.* We proceed by induction. For any  $\alpha < \omega_1$  set

$$D_\alpha = D_0 \cup \{(x_\gamma^{01}, \dots, x_\gamma^{(n-1)(n-2)})\}_{\gamma < \alpha},$$

and suppose that for every  $\gamma < \alpha$  the tuples  $(f_\gamma^0, \dots, f_\gamma^{n-1})$  and  $(x_\gamma^{01}, \dots, x_\gamma^{(n-1)(n-2)})$  satisfy (1)–(4). Since  $K$  is not of degree  $n - 1$ , by Proposition 2.5 the projection  $\pi_{D_\alpha}$  is at least  $n$ -to-1. Therefore there exist  $f_\alpha^0, \dots, f_\alpha^{n-1}$  in  $K$  satisfying

$$f_\alpha^i \upharpoonright_{D_\alpha} = f_\alpha^j \upharpoonright_{D_\alpha}, \quad i, j \in \{0, \dots, n - 1\}.$$

In particular (1) and (3) are satisfied.

Since all the functions are distinct, there exist  $n^2 - n$  points  $x_\alpha^{ij}$  ( $x_\alpha^{ij} = x_\alpha^{ji}$  for every  $i \neq j$ ) distinguishing  $f_\alpha^i$  and  $f_\alpha^j$ , in the sense that (2) is satisfied. It remains to see that (4) is true. To reach a contradiction, suppose that there exist  $\gamma < \alpha$  and  $i_0, j_0 \in \{0, \dots, n - 1\}$  such that  $f_\alpha^{i_0} \upharpoonright_{D_0} = f_\gamma^{j_0} \upharpoonright_{D_0}$ . Then by construction  $f_\alpha^i \upharpoonright_{D_0} = f_\gamma^j \upharpoonright_{D_0}$  for every  $i, j$ . Since  $\pi_{D_0}$  is at most  $n$ -to-1,

there are  $k_0$  and  $k_1$  such that  $f_\alpha^{k_0} = f_\gamma^0$  and  $f_\alpha^{k_1} = f_\gamma^1$ . But this is impossible because  $f_\alpha^{k_0}(x_\gamma^{01}) = f_\alpha^{k_1}(x_\gamma^{01})$  by **(3)**, whereas  $f_\gamma^0(x_\gamma^{01}) \neq f_\gamma^1(x_\gamma^{01})$  by **(2)**. ■

Due to property **(2)**, passing to an uncountable subset of  $\omega_1$ , we can assume that there exist open intervals  $I^{ij} \subset \mathbb{R}$  with rational endpoints such that

**(5)** for every  $\alpha < \omega_1$  and distinct  $i, j \in \{0, \dots, n - 1\}$ , we have

$$\overline{I^{ij}} \cap \overline{I^{ji}} = \emptyset, \quad f_\alpha^i(x_\alpha^{ij}) \in I^{ij}, \quad f_\alpha^j(x_\alpha^{ij}) \in I^{ji}.$$

Also we can suppose without loss of generality that either  $I^{ij} = I^{kl}$  or  $I^{ij} \cap I^{kl} = \emptyset$  for any choice of indices.

As in [Tod99], for every  $m \in \mathbb{N}$  fix an enumeration  $\{B_{mk}\}_{k \in \mathbb{N}}$  of all open rational intervals with diameter  $\leq 2^{-m}$ , and an enumeration  $\{d_m\}_{m \in \mathbb{N}}$  of  $D_0$  in which every element is repeated infinitely many times. Given a finite sequence  $t \in \mathbb{N}^{<\mathbb{N}}$ , define

$$B(t) = \{h \in \mathbb{R}^{D_0} : h(d_m) \in B_{mt(m)}, m \leq \text{length}(t)\}.$$

Note that  $\{B(t)\}_{t \in \mathbb{N}^{<\mathbb{N}}}$  is an open basis of  $\mathbb{R}^{D_0}$ . Note also that  $B(s) \subset B(t)$  if  $t \preceq s$ .

A key point in the proof is to find an embedding of the Cantor set  $2^{\mathbb{N}}$  in some perfect subspaces of  $(\mathbb{N}^{\mathbb{N}})^{(n^2-n)}$  and  $\mathbb{R}^{D_0}$  respectively.

Given an ordinal  $\alpha < \omega_1$ , we will denote by  $h_\alpha$  the common restriction of all the  $f_\alpha^i$  to  $D_0$ . We will construct a tree  $T$  on  $(\mathbb{N}^{<\mathbb{N}})^{n^2-n} \times \mathbb{N}^{<\mathbb{N}}$  formed by pairs  $(t^0, t^1)$  where  $t^0$  will be a tuple that we write in the form  $(t_{ij}^0 : i, j = 0, \dots, n - 1, i \neq j)$ , and will be a restriction of some (in fact many)  $(x_\alpha^{ij} : i, j = 0, \dots, n - 1, i \neq j)$ , while the second coordinate  $t^1$  codifies the open sets  $B(t^1)$  where the function  $h_\alpha$  lies.

**DEFINITION 4.4.** Given  $(t^0, t^1)$  in  $(\mathbb{N}^{<\mathbb{N}})^{(n^2-n)} \times \mathbb{N}^{<\mathbb{N}}$  and  $\alpha < \omega_1$ , we say that the pair  $(x_\alpha, h_\alpha)$  in  $(\mathbb{N}^{\mathbb{N}})^{(n^2-n)} \times \mathbb{R}^{D_0}$  extends  $(t^0, t^1)$  if  $h_\alpha \in B(t^1)$  and  $t_{ij}^0 \subset x_\alpha^{ij}$  for all  $i, j = 0, \dots, n - 1, i \neq j$ .

All the components of the pairs  $(t^0, t^1)$  of our tree will have the same length, so from now on, we assume that for all tuples  $(t^0, t^1) \in (\mathbb{N}^{<\mathbb{N}})^{(n^2-n)} \times \mathbb{N}^{<\mathbb{N}}$ , we have  $\text{length}(t_{ij}^0) = \text{length}(t^1)$  for all  $i, j$ . In the following, the symbol  $\exists^{\omega_1} \alpha$  means *there exist  $\omega_1$  many  $\alpha$ 's such that . . .*

**DEFINITION 4.5.** Given a node  $(t^0, t^1) \in (\mathbb{N}^{<\mathbb{N}})^{(n^2-n)} \times \mathbb{N}^{<\mathbb{N}}$ , we say that two extensions  $(s^0, s^1)$  and  $(u^0, u^1)$  of  $(t^0, t^1)$  properly split if

$$s^0 \neq u^0 \wedge \exists m \in \mathbb{N} (s^1(m) \neq t^1(m) \wedge \overline{B_{ms^1(m)}} \cap \overline{B_{mt^1(m)}} = \emptyset).$$

**PROPOSITION 4.6.** *There exists a tree  $T$  on the product  $(\mathbb{N}^{<\mathbb{N}})^{(n^2-n)} \times \mathbb{N}^{<\mathbb{N}}$  formed by pairs  $(t^0, t^1)$  for which*

$$\exists^{\omega_1} \alpha ((x_\alpha, h_\alpha) \text{ extends } (t^0, t^1)).$$

Moreover, for every  $(t^0, t^1)$  in  $T$  there exist extensions  $(s^0, s^1)$  and  $(u^0, u^1)$  of  $(t^0, t^1)$  in  $T$  that properly split.

*Proof.* First of all define

$$\mathcal{Y} = \{(t^0, t^1) \in (\mathbb{N}^{<\mathbb{N}})^{(n^2-n)} \times \mathbb{N}^{<\mathbb{N}} : \exists \omega_1 \alpha ((x_\alpha, h_\alpha) \text{ extends } (t^0, t^1))\}.$$

We will prove that  $\mathcal{Y}$  has a subtree  $T$  as desired. Let

$$(4.1) \quad \mathcal{A} = \{\alpha < \omega_1 : \text{if } (x_\alpha, h_\alpha) \text{ extends } (t^0, t^1) \text{ then } (t^0, t^1) \in \mathcal{Y}\}.$$

For any  $(t^0, t^1)$  denote by  $\mathcal{E}_{(t^0, t^1)}$  the set of all ordinals  $\alpha$  such that  $(x_\alpha, h_\alpha)$  extends  $(t^0, t^1)$ . Notice that  $\mathcal{A}$  is uncountable. In fact,  $\omega_1 \setminus \mathcal{A}$  is countable because

$$\omega_1 \setminus \mathcal{A} \subseteq \bigcup_{(t^0, t^1) \notin \mathcal{Y}} \mathcal{E}_{(t^0, t^1)}$$

and  $\mathcal{E}_{(t^0, t^1)}$  is countable whenever  $(t^0, t^1) \notin \mathcal{Y}$ . With this in mind define

$$T = \{(t^0, t^1) \in (\mathbb{N}^{<\mathbb{N}})^{(n^2-n)} \times \mathbb{N}^{<\mathbb{N}} : \exists \alpha \in \mathcal{A} ((x_\alpha, h_\alpha) \text{ extends } (t^0, t^1))\}.$$

Since  $\omega_1 \setminus \mathcal{A}$  is countable and  $T \subset \mathcal{Y}$ , it is clear that for any  $(t^0, t^1) \in T$  the set of all ordinals in  $\alpha \in \mathcal{A}$  such that  $(x_\alpha, h_\alpha)$  extends  $(t^0, t^1)$  is uncountable.

Let us show that  $T$  satisfies the statement of the proposition. Given a node  $(t^0, t^1) \in T$  consider distinct ordinals  $\alpha, \beta \in \mathcal{A}$  such that  $(x_\alpha, h_\alpha)$  and  $(x_\beta, h_\beta)$  extend  $(t^0, t^1)$ . Since  $x_\alpha$  and  $x_\beta$  are different, we can pick a natural number  $m$  such that  $x_\alpha \upharpoonright m$  and  $x_\beta \upharpoonright m$  are different. For the second coordinate, we can find  $d \in D_0$  and  $l \geq m$  such that

$$(4.2) \quad |h_\alpha(d) - h_\beta(d)| > 1/2^l.$$

Since in  $D_0$  every element is repeated infinitely many times, we can also choose  $k > l$  such that  $d_k = d$ . We can find sequences  $t', \bar{t} \in \mathbb{N}^{k+1}$  such that  $(x_\alpha \upharpoonright (k+1), t'), (x_\beta \upharpoonright (k+1), \bar{t})$  extend  $(t^0, t^1)$  and lie in  $T$ . Moreover  $h_\alpha(d_k) \in B_{kt'(k)}$  and  $h_\beta(d_k) \in B_{k\bar{t}(k)}$ . By (4.2),  $B_{kt'(k)}$  and  $B_{k\bar{t}(k)}$  have disjoint closures, and so  $(x_\alpha \upharpoonright k, t')$  and  $(x_\beta \upharpoonright k, \bar{t})$  are the extensions of  $(t^0, t^1)$  we were looking for. ■

In the same way as in [Tod99] we will build, by induction on  $\sigma \in 2^{<\mathbb{N}}$ , nodes  $t_\sigma = (t_\sigma^0, t_\sigma^1) \in T$  and  $n$ -tuples  $(g_\sigma^0, \dots, g_\sigma^{n-1}) \in K_0^n$  which are respective approximations to  $(x_\alpha, h_\alpha)$  and  $(f_\alpha^0, \dots, f_\alpha^{n-1})$  and satisfy the following conditions:

- (6) If  $\tau \in 2^{<\mathbb{N}}$  extends  $\sigma$  then  $(t_\tau^0, t_\tau^1)$  is an extension of  $(t_\sigma^0, t_\sigma^1)$ ; that is,  $t_{(ij)\sigma}^0 \preceq t_{(ij)\tau}^0$  for all distinct  $i, j \in \{0, \dots, n-1\}$ , and  $t_\sigma^1 \preceq t_\tau^1$ .
- (7) For each  $\sigma \in 2^{<\mathbb{N}}$  the sequences  $(t_{\sigma 0}^0, t_{\sigma 0}^1)$  and  $(t_{\sigma 1}^0, t_{\sigma 1}^1)$  are extensions of  $(t_\sigma^0, t_\sigma^1)$  that properly split.
- (8) For every  $\sigma \in 2^{<\mathbb{N}}$  and distinct  $i, j \in \{0, \dots, n-1\}$  we have

$$g_\sigma^i[t_{(ij)\sigma}^0] \subset I^{ij}, \quad g_\sigma^j[t_{(ij)\sigma}^0] \subset I^{ji},$$

where  $[r] = \{x \in \mathbb{N}^{<\mathbb{N}} : r \subset x\}$  for  $r \in \mathbb{N}^{<\mathbb{N}}$ .

- (9)  $g_\sigma^i \upharpoonright D_0 \in B(t_\sigma^1)$  for every  $\sigma \in 2^{<\mathbb{N}}$  and  $i \in \{0, \dots, n-1\}$ .
- (10) If  $\tau \neq \sigma$  and  $\tau, \sigma \in 2^m$  then for all  $i, j, k, l \in \{0, \dots, n-1\}$  with  $i \neq j$ ,

$$\sup_{x \in [t_{(ij)\sigma}^0]} |g_\tau^k(x) - g_\tau^l(x)| \leq \frac{1}{2^m}.$$

This is done by induction as follows. For the purpose of this construction, if  $\sigma, \tau \in 2^{<\mathbb{N}}$  then  $\sigma < \tau$  means that both sequences have the same length and  $\sigma$  is below  $\tau$  lexicographically. Along the induction, we also build auxiliary tuples  $v(\tau, \xi) \in T$  for  $\tau, \xi \in 2^{<\mathbb{N}}$  of the same length  $m$  such that

- $t_{\tau \upharpoonright (m-1)} \preceq v(\tau, \xi) \preceq v(\tau, \xi') \preceq t_\tau$  whenever  $\xi < \xi'$ ,
- $v(\tau 0, \xi) = v(\tau 1, \xi)$  if  $\xi < \tau 0$ ,
- whenever  $i \neq j$  and  $\tau < \tau'$ , we have

$$\sup_{x \in [v_{(ij)}^0(\tau', \xi)]} |g_\tau^k(x) - g_\tau^l(x)| \leq \frac{1}{2^{m+1}}.$$

These  $v(\tau, \xi)$  are provisional values of  $t_\tau$ , so that the definite value is  $t_\tau = v(\tau, (1 \dots 1))$ .

We fix  $\sigma \in 2^{<\mathbb{N}}$  of length  $m$  and we suppose that all  $g_\tau^i, t_\tau$  have been defined when  $\text{length}(\tau) \leq m$  and  $\tau < \sigma 0$ , and the  $v(\tau, \xi)$  have been defined for all  $\tau \in 2^{m+1}$  and  $\xi < \sigma 0$ . We shall show how to define all those objects for  $\tau = \sigma 0, \sigma 1$  and for any  $\xi$  of length  $m+1$ . Let  $\sigma^-$  be the immediate lexicographical predecessor of  $\sigma$ . For notational simplicity it is convenient to consider an imaginary  $0^- = (0 \dots 0)^- \in \mathbb{N}^{m+1}$  such that  $v(\tau, 0^-) = t_{\tau \upharpoonright m}$ . The first thing is to find two nodes  $u^0, u^1 \in T$  above  $v(\sigma 0, \sigma^-) = v(\sigma 1, \sigma^-)$  that properly split. Let us first define the objects associated to  $\sigma 0$ . For every  $\tau \in 2^{m+1}$  choose  $\alpha(\tau) \in \mathcal{A}$  such that  $(x_{\alpha(\tau)}, h_{\alpha(\tau)})$  extends  $v(\tau, \sigma^-)$  for all  $\tau$ , and moreover  $(x_{\alpha(\sigma\varepsilon)}, h_{\alpha(\sigma\varepsilon)})$  extends  $u^\varepsilon$ . We can suppose that  $\alpha(\sigma 0) > \alpha(\tau)$  for all other  $\tau$ . Then, we can apply (3) and (5) for  $\alpha = \alpha(\sigma 0)$  and  $\beta = \alpha(\tau)$ . By the density of  $K_0$ , we can find  $g_{\sigma 0}^i \in K_0$  for  $i = 0, \dots, n-1$  such that

$$g_{\sigma 0}^k(x_{\alpha(\tau)}) - g_{\sigma 0}^l(x_{\alpha(\tau)}) < 2^{-m-2}, \quad \tau \neq \sigma 0, \quad g_{\sigma 0}^i(x_{\alpha(\sigma 0)}^{ij}) \in I^{ij}.$$

Using the continuity of  $g_{\sigma 0}^i$  at the points  $x_{\alpha(\tau)}^{ij}$ , we can find the nodes  $v(\tau, \sigma 0)$  large enough to satisfy all requirements. The case of  $\sigma 1$  is handled exactly the same way. This finishes the construction.

Given  $a \in 2^{\mathbb{N}}$  and different  $i, j < n$ , let  $x_a^{ij} \in \mathbb{N}^{\mathbb{N}}$  be the unique extension of  $\{t_{(ij)a \upharpoonright m}^0\}_m$ , and also let  $h_a \in \mathbb{R}^{D_0}$  be the unique function such that  $h_a(d_m) \in \overline{B_{mt_{a \upharpoonright k}^1(m)}}$  for all  $m$  and sufficiently large  $k$ . In order to make sure that such a function exists, one should check that we do not get incompatible conditions when  $d_m = d_{m'}$ ; but this is clear since every  $(t^0, t^1)$  is extended by some  $(x_\alpha, h_\alpha)$ .

PROPOSITION 4.7. *The map  $a \mapsto (x_a^{01}, \dots, x_a^{(n-1)(n-2)})$  is a homeomorphism between  $2^{\mathbb{N}}$  and the perfect subspace  $\{(x_a^{01}, \dots, x_a^{(n-1)(n-2)}) \in (\mathbb{N}^{\mathbb{N}})^{n^2-n} : a \in 2^{\mathbb{N}}\}$  of  $(\mathbb{N}^{\mathbb{N}})^{n^2-n}$ . Similarly, the map  $a \mapsto h_a$  is a homeomorphism between  $2^{\mathbb{N}}$  and the perfect subspace  $\{h_a : a \in 2^{\mathbb{N}}\}$  of  $\mathbb{R}^{D_0}$ .*

*Proof.* It is easy to see that the family  $\{\mathcal{A}_{a \upharpoonright m}\}_{m \in \mathbb{N}, a \in 2^{\mathbb{N}}}$  where

$$\mathcal{A}_{a \upharpoonright m} = \{y \in (\mathbb{N}^{\mathbb{N}})^{n^2-n} : t_{a \upharpoonright m}^0 \subset^* y\}$$

is a Cantor scheme. Moreover, the sets  $\mathcal{A}_{a \upharpoonright m}$  are clopen and for every  $a \in 2^{\mathbb{N}}$  we have  $\lim_m \text{diam}(\mathcal{A}_{a \upharpoonright m}) = 0$ . Therefore  $a \mapsto (x_a^{01}, \dots, x_a^{(n-1)(n-2)})$  is a homeomorphism.

On the other hand, it is clear that the map  $a \mapsto h_a$  is continuous because if  $a \upharpoonright (m+1) = b \upharpoonright (m+1)$  then  $|h_a(d_m) - h_b(d_m)| \leq 2^{-m}$ , and each  $d \in D_0$  is repeated infinitely many times in the sequence  $\{d_m\}$ . So if we manage to prove that the map is injective, we will be able to conclude that it is in fact a homeomorphism. So let  $a$  and  $b$  be two different sequences in the Cantor space and let  $m_0$  be the minimum coordinate where  $a(m_0) \neq b(m_0)$ . By (7),  $(t_{a \upharpoonright (m_0+1)}^0, t_{a \upharpoonright (m_0+1)}^1)$  and  $(t_{b \upharpoonright (m_0+1)}^0, t_{b \upharpoonright (m_0+1)}^1)$  properly split, so there exists  $m$  such that

$$\overline{B_{mt_{a \upharpoonright (m_0+1)}^1}(m)} \cap \overline{B_{mt_{b \upharpoonright (m_0+1)}^1}(m)} = \emptyset.$$

Hence, in particular,  $h_a(d_m) \neq h_b(d_m)$ . ■

Note that the family  $\{g_\sigma^i\}_{\sigma \in 2^{<\mathbb{N}}}$  of functions with  $i \in \{0, \dots, n-1\}$  is a relatively compact subset of  $\mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$ . By the Ramsey-like Theorem 4.1 we can ensure a uniform behaviour of the functions  $g_{a \upharpoonright m}^i$  for many  $a \in 2^{\mathbb{N}}$ . Namely, there exists a perfect Polish space  $P \subset 2^{\mathbb{N}}$  and an infinite strictly increasing sequence  $\{m_k\}_k$  of natural numbers for which the limit  $\lim_m g_{a \upharpoonright m}^i = g_a^i$  exists for every  $a \in P$  and  $i \in \{0, \dots, n-1\}$ .

PROPOSITION 4.8. *The family  $\{(g_a^0, \dots, g_a^{n-1})\}_{a \in P} \subset K^n$  has the following properties:*

(11) *For any different  $i, j \in \{0, \dots, n-1\}$  and  $a \in P$  we have*

$$g_a^i(x_a^{ij}) \in \overline{I^{ij}}, \quad g_a^j(x_a^{ij}) \in \overline{J^{ji}}.$$

(12)  *$g_a^i \upharpoonright D_0 = h_a$  for every  $i \in \{0, \dots, n-1\}$  and  $a \in P$ . In fact,  $\pi_{D_0}^{-1}(h_a) = \{g_a^0, \dots, g_a^{n-1}\}$ .*

(13) *For any  $i, j, k, l \in \{0, \dots, n-1\}$  with  $i \neq j$  and distinct  $a, b \in P$  we have*

$$g_a^k(x_b^{ij}) = g_a^l(x_b^{ij}).$$

*Proof.* Properties (11) and (13) are straightforward consequences of (8) and (10), so we only need to prove (12). If it did not hold, by the definition



of  $h_a$  we could find  $m_1 < m_0$  such that

$$g_a^i(d_{m_1}) \notin \overline{B_{m_1} t_{a|m_0}^1(m_1)}.$$

By the convergence of  $\{g_{a|m}^i\}_m$ , there exists  $m > m_0$  such that

$$g_{a|m}^i(d_{m_1}) \notin \overline{B_{m_1} t_{a|m_0}^1(m_1)}.$$

But on the other hand, by (9), we have  $g_{a|m}^i \in B(t_{a|m}^1) \subseteq B(t_{a|m_0}^1)$ , a contradiction. The last statement of (12) follows from the facts that  $g_a^i$  and  $g_a^j$  are distinct for  $i \neq j$  (see (11) and (5)) and  $\pi_{D_0}$  is at most  $n$ -to-1. ■

Now as in [Tod99], choose open intervals  $J^{ij}$  with rational endpoints such that:

(14)  $\overline{I^{ij}} \subseteq J^{ij}$  for any different  $i, j \in \{0, \dots, n - 1\}$ .

(15)  $J^{ij} \cap J^{ji} = \emptyset$  for any different  $i, j \in \{0, \dots, n - 1\}$ .

Define the following relations on  $P^2$ , where min and max refer to the natural lexicographical order of  $2^{\mathbb{N}}$ :

(16)  $(x, y) \in A_0^{ij}$  if and only if  $x \neq y$  and  $g_a^i(x_b^{ij}) \in J^{ij}$  where  $a = \min\{x, y\}$  and  $b = \max\{x, y\}$ .

(17)  $(x, y) \in A_1^{ij}$  if and only if  $x \neq y$  and  $g_a^i(x_b^{ij}) \in J^{ij}$  where  $a = \max\{x, y\}$  and  $b = \min\{x, y\}$ .

It is easy to see that the relations are symmetric and Borel, so Theorem 4.2 applies. Iterating this procedure, we obtain a perfect subset  $R \subset P$  such that for any different  $i, j \in \{0, \dots, n - 1\}$  and  $\varepsilon \in \{0, 1\}$ ,

(18) either  $R^{[2]} \subseteq A_\varepsilon^{ij}$  or  $R^{[2]} \cap A_\varepsilon^{ij} = \emptyset$ .

(Recall that  $R^{[2]} = \{(a, b) \in R^2 : a \neq b\}$ .)

Given  $a \in P$  and distinct  $i, j \in \{0, \dots, n - 1\}$ , consider the pointwise neighbourhoods

$$\mathcal{U}_j(g_a^i) = \{f \in K : f(x_a^{ij}) \in J^{ij}\}.$$

The following properties are direct consequences of (15) and (13):

(19)  $\mathcal{U}_j(g_a^i) \cap \mathcal{U}_i(g_a^j) = \emptyset$  for all  $a \in R$  and distinct  $i, j \in \{0, \dots, n - 1\}$ .

(20) If  $a, b \in R$  and  $i, j \in \{0, \dots, n - 1\}$  are different points and indices respectively, then either  $g_b^k \in \mathcal{U}_j(g_a^i)$  for every  $k \in \{0, \dots, n - 1\}$ , or  $g_b^k \notin \mathcal{U}_j(g_a^i)$  for every  $k \in \{0, \dots, n - 1\}$ .

Moreover, the Ramsey-like dichotomy (18) implies the following facts:

PROPOSITION 4.9. *Given distinct  $i, j \in \{0, \dots, n - 1\}$ , for every  $a \in R$  we have:*

(21)  $\mathcal{U}_j(g_a^i)$  either contains or is disjoint from  $\{g_b^k : k \in \{0, \dots, n - 1\}, b \in R, b < a\}$ .

(22)  $\mathcal{U}_j(g_a^i)$  either contains or is disjoint from  $\{g_b^k : k \in \{0, \dots, n - 1\}, b \in R, a < b\}$ .

*Proof.* To prove (21), suppose that the first alternative of (18) holds when we take  $\varepsilon = 0$ . Given any  $b \in R, b < a$ , we have  $(a, b) \in A_0^{ij}$ , which means that  $g_b^i \in \mathcal{U}_j(g_a^i)$ , and thus  $\{g_b^0, \dots, g_b^{n-1}\} \subset \mathcal{U}_j(g_a^i)$  by (20). As  $b$  was chosen arbitrarily, the first possibility occurs. The second case is deduced in the same way from the second alternative of (18) and (20).

(22) is proved in the same way by taking  $\varepsilon = 1$  in (18). ■

DEFINITION 4.10. Given any  $a \in R$  and  $i, k \in \{0, \dots, n - 1\}$ , we say that the function  $g_a^i$  is isolated from the functions to its left by the neighbourhood  $\mathcal{U}_k(g_a^i)$  if

$$\mathcal{U}_k(g_a^i) \cap \{g_b^l : l \in \{0, \dots, n - 1\}, b \in R, b < a\} = \emptyset.$$

In the same way we define a function isolated from the functions to its right.

PROPOSITION 4.11. For all but one  $i \in \{0, \dots, n - 1\}$  the following holds: For every  $a \in R$  the function  $g_a^i$  is isolated from the functions to its left by one of the neighbourhoods  $\mathcal{U}_k(g_a^i)$ . The analogous statement holds for the right side.

*Proof.* Suppose there were  $i, j$  two different indices failing the property stated in the proposition. By Proposition 4.9, for every  $a \in R$  and every  $l \in \{0, \dots, n - 1\}$ ,

$$\begin{aligned} \mathcal{U}_l(g_a^i) &\supset \{g_b^k : k \in \{0, \dots, n - 1\}, b < a\}, \\ \mathcal{U}_l(g_a^j) &\supset \{g_b^k : k \in \{0, \dots, n - 1\}, b < a\}, \end{aligned}$$

so in particular  $\mathcal{U}_j(g_a^i)$  and  $\mathcal{U}_i(g_a^j)$  are not disjoint for any  $a \neq \min(R)$ , contrary to (19). It remains to show that at least one index  $i$  must fail the statement of the proposition. Otherwise every function  $g_a^i$  would be isolated from the functions to its left. If we pick  $a$  which is not isolated from the left in  $R$ , then since  $b \mapsto h_b$  is continuous,  $\{h_b : b \in R, b < a\}$  accumulates to  $h_a$ . By (12) this implies that  $\{g_b^k : b \in R, b < a, k \in \{0, \dots, n - 1\}\}$  accumulates to some  $g_a^i$ , a contradiction. ■

Let  $i$  and  $j$  be the indices for which, for every  $a \in R$ , the functions  $g_a^i$  and  $g_a^j$  are not isolated from those to their left and right respectively. As a consequence of the last proposition, we have the following two possibilities:

CASE 1: The indices are equal, and thus, for every  $a \in R$ , the function  $g_a^i$  is not isolated from the others by  $\mathcal{U}_l(g_a^k)$ , neither from the right nor from the left.

CASE 2: The indices are different, and thus, for every  $a \in R$ , the functions  $g_a^i$  and  $g_a^j$  are isolated from the functions to their right and left respectively.

Let us see how Case 1 and Case 2 lead, respectively, to the Alexandroff  $n$ -plicate  $D_n(2^{\mathbb{N}})$  and the  $n$ -split interval  $S_n(I)$  inside  $K$ .

CASE 1. There exists, as we shall see immediately, a natural identification between the subspace  $\{g_a^k : k \in \{0, \dots, n-1\}, a \in R\}$  of  $K$  and the Alexandroff  $n$ -plicate of the set  $\{h_a : a \in R\}$ . First, if we prove  $D_n(\{h_a : a \in R\}) \hookrightarrow K$  we will have proved  $D_n(2^{\mathbb{N}}) \hookrightarrow K$ . Indeed,  $R$  is a perfect Polish space, so it contains copies of  $2^{\mathbb{N}}$ . Since  $a \mapsto h_a$  is a homeomorphism, there exists an embedding  $D_n(2^{\mathbb{N}}) \hookrightarrow D_n(\{h_a : a \in R\})$ , and thus  $D_n(2^{\mathbb{N}}) \hookrightarrow K$ .

Consider the map

$$\Phi : D_n(\{h_a : a \in R\}) \rightarrow \{g_a^k : k \in \{0, \dots, n-1\}, a \in R\}, \quad (h_a, k) \mapsto g_a^{\psi(k)},$$

where  $\psi : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$  is a bijection with  $\psi(0) = i$ . Since all the spaces involved are Rosenthal compacta and hence Fréchet–Urysohn, continuity can be checked sequentially. If  $\{(h_{a_m}, k_m)\}_{m \in \mathbb{N}, k_m \in \{0, \dots, n-1\}}$  is a non-trivial convergent sequence to  $(h_a, 0)$ , then  $h_{a_m}$  converges to  $h_a$ , and hence  $a_m$  converges to  $a$  since  $a \mapsto h_a$  is a homeomorphism. Then, by (12) the sequence  $\{g_{a_m}^{\psi(k_m)}\}_m$  must accumulate to some element of  $\{g_a^0, \dots, g_a^{n-1}\}$ ; but in fact it must accumulate to  $g_a^i$  since the other functions are isolated. Since  $\Phi$  is a bijection and the domain is compact, we conclude that  $\Phi$  is a homeomorphism.

CASE 2. Let us identify the perfect set  $R \subset 2^{\mathbb{N}}$  with a perfect set in the unit interval  $I$  through the standard embedding  $2^{\mathbb{N}} \subset I$  that preserves the order. Now consider the map

$$\Psi : S_n(R) \rightarrow \{g_a^k : k \in \{0, \dots, n-1\}, a \in R\}, \quad (a, k) \mapsto g_a^{\phi(k)},$$

where  $\phi : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$  is a bijection such that  $\phi(0) = i$  and  $\phi(1) = j$ . We prove the continuity of  $\Psi$  at the points of the form  $(a, 0)$ , since the reasoning for  $(a, 1)$  is analogous. Take a non-trivial sequence  $\{(a_m, k_m)\}_{m \in \mathbb{N}, k_m \in \{0, \dots, n-1\}}$  converging to  $(a, 0)$ . Then  $a_m$  is eventually lower than  $a$ . Arguing as in Case 1, we see that  $\{g_{a_m}^{\phi(k_m)}\}_m$  must accumulate to some element of  $\{g_a^0, \dots, g_a^{n-1}\}$ , and since the unique non-isolated functions are  $g_a^i$  and  $g_a^j$ , the sequence must accumulate to one of them. But  $g_a^j$  is isolated from the functions to its left, so the sequence must accumulate to  $g_a^i$ , and hence  $\Psi$  is continuous. Again, since  $\Psi$  is a bijection and the domain is compact, we conclude that  $\Psi$  is a homeomorphism. Finally, since  $S_n(R) \hookrightarrow K$ , it is enough to apply Theorem 3.8 to obtain the desired embedding  $S_n(I) \hookrightarrow K$ .

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Antonio Avilés  
 Departamento de Matemáticas  
 Universidad de Murcia  
 30100 Murcia, Spain  
 E-mail: avileslo@um.es

Alejandro Poveda  
 Departament de Matemàtiques i Informàtica  
 Universitat de Barcelona  
 Gran Via de les Corts Catalanes 585  
 08007 Barcelona, Spain  
 E-mail: alejandro.poveda@ub.edu

Stevo Todorčević  
 Department of Mathematics  
 University of Toronto  
 M5S 3G3 Toronto, Canada  
 and  
 Institut de Mathématiques de Jussieu  
 CNRS UMR 7586 Case 247  
 4 Place Jussieu  
 75252 Paris, France  
 E-mail: stevo@math.toronto.edu, stevo.todorcevic@imj-prg.fr