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NEW ISOLATED TOUGHNESS CONDITION FOR FRACTIONAL (g, f, n)-CRITICAL GRAPHS

BY

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Abstract. Let $i(G)$ be the number of isolated vertices in a graph G. The isolated toughness of G is defined as $I(G) = \infty$ if G is complete, and $I(G) = \min\{|S|/i(G-S) : S \subseteq V(G),\ i(G-S) \geq 2\}$ otherwise. We show that G is a fractional (g, f, n)-critical graph if $I(G) \geq (b^2 + bn - \Delta)/a$, where a, b are positive integers, $1 \leq a \leq b$, $b \geq 2$, and $\Delta = b - a$. Furthermore, a new isolated toughness condition for fractional (a, b, n)-critical graphs is given.

1. Introduction. The graphs considered here are finite and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we use $d_G(x)$ and $N_G(x)$ to denote the degree and the neighborhood of x in G, respectively. Let $\delta(G)$ denote the minimum degree of G. For any $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. Let $i(G-S)$ be the number of isolated vertices in $G - S$. The readers can refer to [1] for standard graph-theoretic concepts and terms used but undefined in this paper.

Let g and f be integer-valued functions on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a *(g, f)-factor* if $g(x) \leq d_F(x) \leq f(x)$ for every $x \in V(G)$. A *fractional (g, f)-factor* is a function h that assigns to each edge of G a number in [0,1] so that for each vertex x we have

$$g(x) \leq \sum_{e \in E(x)} h(e) \leq f(x).$$

If $g(x) = a$, $f(x) = b$ for all $x \in V(G)$, then a fractional (g, f)-factor is a *fractional $[a, b]$-factor*. Moreover, if $g(x) = f(x) = k$ ($k \geq 1$ is an integer throughout this paper, and we will not reiterate it again) for all $x \in V(G)$, then a fractional (g, f)-factor is just a *fractional k-factor*.

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Liu and Zhang [7] proved that there exists a fractional (g, f)-factor. Several characteristics of fractional (g, f)-factors are described in detail by Liu and Zhang [7, 8]. A graph G is called a *fractional* (g, f, n)*-critical graph* if after deleting any n vertices from G, the resulting graph still has a fractional (g, f)-factor. Similarly, a graph G is called a (g, f, n)*-critical graph* if after removing any n vertices from G, the resulting graph admits a (g, f)-factor. The reader can find some sufficient conditions for (a, b, n)-critical graphs in [12] and [13]. As an extension of the concept of fractional factor, a fractional critical graph describes fractional factors in communication networks when certain sites are damaged. More results on fractional factor can be found in [3, 4, 5].

Let $f(S) = \sum_{x\in S} f(x)$, $f(U) = \sum_{x\in U} f(x)$, $g(T) = \sum_{x\in T} g(x)$ and $d_{G-S}(T) = \sum_{x\in T} d_{G-S}(x)$. A significant result on fractional (g, f, n)-critical graphs was obtained by Liu [10]; its equivalent version can be stated as follows.

Lemma 1.1 (Liu [10]). *Let G be a graph and let g, f be two non-negative integer-valued functions defined on $V(G)$ satisfying $g(x) \leq f(x)$ for all $x \in V(G)$. Let n be a non-negative integer. Then G is a fractional (g, f, n)-critical graph if and only if*

$$f(S) - g(T) + d_{G-S}(T) \geq \max\{f(U) : U \subseteq S,\ |U| = n\} \tag{1.1}$$

for any disjoint subsets S and T of $V(G)$ with $|S| \geq n$.

The term *toughness* was first introduced by Chvátal [2] to measure the vulnerability of networks: if G is a complete graph, $t(G) = \infty$; if G is not complete, then

$$t(G) = \min\{|S|/\omega(G - S) : \omega(G - S) \geq 2\}$$

where $\omega(G-S)$ is the number of connected components of $G-S$. Gao et al. [3] established that G is fractional (g, f, n)-critical if $t(G) \geq (b^2 - 1 + bn)/a$, and G is fractional (a, b, n)-critical if $t(G) \geq (ab - b + a - 1)/b + n$.

Yang et al. [11] introduced the concept of *isolated toughness* $I(G)$ of the graph G which is described as follows. If G is not complete, then

$$I(G) = \min\{|S|/i(G - S) : S \subseteq V(G),\ i(G - S) \geq 2\}.$$

Otherwise, $I(G) = \infty$. Isolated toughness is usually regarded as a parameter to measure the strength of networks, and has been widely used in communication networks and ontology semantic structure graphs.

Recently, Gao et al. [6] found that G is a fractional (g, f, n)-critical graph if $I(G) \geq (b^2 + bn - 1)/a$ if $b > a$; and $I(G) \geq b + n$ if $a = b$. In this paper, we extend the result of [6] and determine a new bound for fractional (g, f, n)-critical graphs. Our main result to be proved in the next section can be stated as follows.

THEOREM 1.2. *Let G be a graph and g, f be two non-negative integer-valued functions defined on $V(G)$ and satisfying $a \le g(x) \le f(x) \le b$ for all $x \in V(G)$, where a, b are integers with $1 \le a \le b$ and $2 \le b$. Let n be a non-negative integer and $\Delta = b - a$. If*

$$\delta(G) \ge \frac{bn}{a} + \frac{(b+2)^2}{4a} + b - 1 \quad \text{and} \quad I(G) \ge \frac{b^2 + bn - \Delta}{a},$$

then G is a fractional (g, f, n)-critical graph.

Assume $a = g(x) = f(x) = b$ for all $x \in V(G)$. Let m be a positive integer. To demonstrate the sharpness of Theorem 1.2 in the setting where $a = b$, we construct the following graph G: $V(G) = A \cup B \cup C$ where A, B and C are disjoint with $|A| = (ma+1)(1+n)$, $|B| = ma + 1$ and $|C| = m(b^2 - b + bn)$. Both A and C are cliques in G, while B is isomorphic to $(ma+1)K_1$. Other edges in G are $\{uv : u \in B, v \in C\}$ and $\{u_1v_1, u_2v_2, \ldots, u_{ma+1}v_{ma+1}\}$, where $V(B) = \{u_1, \ldots, u_{ma+1}\}$ and $\{v_1, \ldots, v_{ma+1}\} \subseteq A$. Let $S = \{v_1, \ldots, v_{ma+1}\} \cup C$. Then $|S| = m(b^2 - b + bn) + ma + 1$ and $i(G - S) = ma + 1$. It follows that

$$I(G) = \frac{|S|}{i(G-S)} = \frac{m(b^2 - b + bn) + ma + 1}{ma + 1}.$$

Thus, $I(G)$ can be made arbitrarily close to $(b^2 - \Delta + bn)/a$ when m is large enough.

Let $V_0 \subset V(A) \setminus \{v_1, \ldots, v_{ma+1}\}$ with $|V_0| = n$, $S = C \cup V_0$ and $T = B$. We have $f(S) - f(U) = a|C|$ for any $U \subset S$ with $|U| = n$ and $d_{G-S}(x) = 1$ for each $x \in T$. Thus,

$$\begin{aligned} f(S) - g(T) + d_{G-S}(T) - \max\{f(U) : U \subseteq S,\ |U| = n\} \\ = am(b-1) - (ma+1)(b-1) < 0. \end{aligned}$$

By Lemma 1.1, G is not a fractional (g, f, n)-critical graph. In this sense, the isolated toughness bound in Theorem 1.2 is best possible if $a = b$.

The whole network can be regarded as a graph. Each site corresponds to a vertex, and each channel corresponds to an edge in the graph. In a normal network, data transmission is based on selection of the shortest path between vertices. However, in a software definition network (SDN), the data transmission relies on the computation of the network flow. It chooses the path with minimum transmission congestion at the given time. If several disjunctive sites in the network are information congested currently, we delete the corresponding vertices from the graph. Thus, the transmitting data packet is applied in the target network, and it can be taken as the fractional factor problem in the network graph: how to avoid certain special vertices. From this point of view, the model of data transmission problem in SDN is only the existence of a critical fractional factor in the corresponding graph. On the other hand, the isolated toughness is a parameter which is used to measure the stability and vulnerability of networks. As a result, Theorem 1.2 turns out quite useful in network algorithms, especially when some sites are congested or damaged.

The proof strategy is similar to the one in Liu and Zhang [9] but we need to cope with the more detailed case now and hence new methods are necessary. To prove Theorem 1.2, we need some lemmas.

Lemma 1.3 (Liu and Zhang [9]). *Let G be a graph and let $H = G[T]$ be such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \dots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and the covering set $C = V(H) - I$ satisfies*

$$\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for every $j = 1, \dots, k-1$.

The lemma below can be deduced from [9, Lemma 2.2].

Lemma 1.4 (Liu and Zhang [9]). *Let G be a graph and let $H = G[T]$ be such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. Then there exists a maximal independent set I and the covering set $C = V(H) - I$ of H satisfies*

$$|V(H)| \leq \sum_{i=1}^{k}(k-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \qquad |C| \leq \sum_{i=1}^{k}(k-i)|I^{(i)}| - \frac{|I^{(1)}|}{2},$$

where $I^{(i)} = \{x \in I : d_H(x) = k-i\}$, $1 \leq i \leq k$ and $\sum_{i=1}^{k}|I^{(i)}| = |I|$.

2. Proof of Theorem 1.2. The aim of this section is to prove our main result. We always assume that G is not complete since the result for complete graphs immediately follows from $\delta(G) \geq bn/a + (b+1)^2/(4a) + b - 1$.

Suppose that G is a counter-example to Theorem 1.2, so G satisfies the assumptions of Theorem 1.2, but $V(G)$ has disjoint subsets S and T such that

$$a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| \leq f(S) - g(T) + d_{G-S}(T) < bn. \tag{2.1}$$

We choose S and T so that $|T|$ is minimum. Obviously, $T \neq \emptyset$. If $d_{G-S}(x) \geq g(x)$ for some $x \in T$, then S and $T \setminus \{x\}$ also satisfy (2.1), contrary to the choice of S and T. Hence $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for any $x \in T$.

Let l be the number of components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{x \in V(H') : d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_b. Let S' be a set of vertices that contains exactly $b-1$ vertices in each component of K_b in H'.

If $|V(H)| = 0$, then $|S| < (b(|T_0| + l) + bn)/a$ by (2.1). Clearly, $i(G - S \cup S') \geq |T_0| + l \geq 1$. If $i(G - S \cup S') > 1$, then

$$I(G) \leq \frac{|S \cup S'|}{i(G - S - S')} < \frac{b(|T_0| + l) + bn + al(b-1)}{a(|T_0| + l)} \leq \frac{b}{a} + \frac{bn}{2a} + b - 1,$$

which contradicts $I(G) \geq (b^2 + bn - \Delta)/a$ and $b \geq 2$. If $i(G - S \cup S') = 1$, then $|T_0| + l = 1$. Hence $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq bn/a + (b+1)^2/(4a) + b - 1$. We have

$$d_{G-S}(x) \geq \frac{bn}{a} + \frac{(b+1)^2}{4a} + b - 1 - |S| > \frac{bn}{a} + \frac{b}{a} + b - 1 - \frac{b(n+1)}{a},$$

which contradicts $d_{G-S}(x) \leq b - 1$ for any $x \in T$.

Now, we consider $|V(H)| \geq 1$. Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies $d_{G-S}(x) = b - 1$ for every $x \in V(H_1)$ and $H_2 = H - H_1$. According to Lemma 1.4, with G replaced by $G - S$, there exists a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ of H_1 satisfies

$$|V(H_1)| \leq \sum_{i=1}^{b} (b - i + 1)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \tag{2.2}$$

$$|C_1| \leq \sum_{i=1}^{b} (b - i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \tag{2.3}$$

where $I^{(i)} = \{x \in I_1 : d_{H_1}(x) = b - i\}$, $1 \leq i \leq b$, and $\sum_{i=1}^{b} |I^{(i)}| = |I_1|$. Let $T_j = \{x \in V(H_2) : d_{G-S}(x) = j\}$ for $1 \leq j \leq b - 1$. Each component of H_2 has a vertex of degree at most $b - 2$ in $G - S$ by the definitions of H and H_2. From Lemma 1.3, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ satisfies

$$\sum_{j=1}^{b-1} (b - j)c_j \leq \sum_{j=1}^{b-1} (b - 2)(b - j)i_j, \tag{2.4}$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \ldots, b - 1$. Set $W = V(G) - S - T$ and $U = S \cup S' \cup C_1 \cup (N_G(I_1) \cap W) \cup C_2 \cup (N_G(I_2) \cap W)$. We infer

$$|U| \leq |S| + l(b-1) + |C_1| + \sum_{j=1}^{b-1} j i_j + \sum_{i=1}^{b} (i-1)|I^{(i)}| \tag{2.5}$$

and

$$i(G-U) \geq t_0 + l + |I_1| + \sum_{j=1}^{b-1} i_j, \tag{2.6}$$

where $t_0 = |T_0|$. Then when $i(G-U) > 1$, we get

$$|U| \geq I(G)i(G-U). \tag{2.7}$$

If $i(G-U) = 1$ then $G[T]$ is a clique with less than b vertices. By (2.1), we get

$$\begin{aligned}|S| &< \frac{bn + b|T| - d_{G-S}(T)}{a} \leq \frac{bn + b|T| - |T|(|T|-1)}{a} \\ &\leq \frac{bn + b\frac{b+1}{2} - \left(\frac{b+1}{2}\right)\left(\frac{b+1}{2} - 1\right)}{a} = \frac{bn}{a} + \frac{(b+1)^2}{4a},\end{aligned}$$

and

$$d_{G-S}(x) \geq \frac{bn}{a} + \frac{(b+1)^2}{4a} + b - 1 - |S| > \frac{bn}{a} + \frac{(b+1)^2}{4a} + b - 1 - \left(\frac{bn}{a} + \frac{(b+1)^2}{4a}\right),$$

which contradicts $d_{G-S}(x) \leq b-1$ for any $x \in T$.

In view of (2.5)–(2.7), we get

$$\begin{aligned}|S| + |C_1| \geq & \sum_{j=1}^{b-1}(I(G)-j)i_j + I(G)(t_0+l) + I(G)|I_1| \\ & - \sum_{i=1}^{b}(i-1)|I^{(i)}| - l(b-1).\end{aligned} \tag{2.8}$$

Since $b|T| - d_{G-S}(T) > a|S| - bn$, we obtain

$$bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j > a|S| - bn.$$

Combining this with (2.8), we deduce

$$\begin{aligned}& |V(H_1)| + \sum_{j=1}^{b-1}(b-j)c_j + a|C_1| \\ & > \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + (aI(G)-b)(t_0+l) + aI(G)|I_1| \\ & \quad - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn - la(b-1).\end{aligned} \tag{2.9}$$

By (2.2) and (2.3), we have

$$|V(H_1)| + a|C_1| \leq \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| - \frac{(a+1)|I^{(1)}|}{2}. \tag{2.10}$$

Using (2.4), (2.9) and (2.10), we get

$$\begin{aligned} &\sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| \\ &\quad > \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} \\ &\quad\quad - a\sum_{i=1}^{b}(i-1)|I^{(i)}| + (aI(G) - b)(t_0 + l) - bn - la(b-1). \end{aligned} \tag{2.11}$$

We now consider two cases according to the value of $t_0 + l$.

CASE 1: $t_0 + l \geq 1$. As $aI(G) \geq b^2 + bn - \Delta$, we get $(aI(G) - b)(t_0 + l) - bn - la(b-1) \geq 0$ by $b \geq 2$. Thus, (2.11) becomes

$$\begin{aligned} &\sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| \\ &\quad > \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + aI(G)|I_1| \\ &\quad\quad + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|. \end{aligned}$$

Therefore, at least one of the following two subcases must hold.

SUBCASE 1. There is at least one j such that

$$(b-2)(b-j) > aI(G) - aj - b + j,$$

hence

$$\begin{aligned} aI(G) &< b(b-2) + (a - b + 1)j + b \\ &\leq b(b-2) + (a - b + 1) + b \\ &= (b^2 - 1) + (a - b) + (2 - b) \\ &\leq b^2 - 1 - \Delta, \end{aligned}$$

which contradicts $I(G) \geq (b^2 + bn - \Delta)/a$.

Subcase 2.

$$\begin{aligned}\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}| \\ &> aI(G)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\ &\geq (b^2+bn-\Delta)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\ &\geq (b^2-\Delta)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|.\end{aligned}$$

This implies

$$\sum_{i=2}^{b}(ab+b-a-i+1-b^2+\Delta)|I^{(i)}|+\big(ab+b-\tfrac{3}{2}a-b^2-\tfrac{1}{2}+\Delta\big)|I^{(1)}|>0.$$

Let

$$h_1(b)=-b^2+(a+1)b-\tfrac{3}{2}a-\tfrac{1}{2}+\Delta.$$

From $\Delta\geq 0$, we get

$$\max\{h_1(b)\}=h_1(a+\Delta)=-\Delta^2-(a-2)\Delta-\frac{a+1}{2}<0.$$

On the other hand, $ab+b-a-i+1-b^2+\Delta\leq -b^2+(a+2)b-2a-1$ due to $i\geq 2$. Let

$$h_2(b)=-b^2+(a+2)b-2a-1.$$

We infer

$$\max\{h_2(b)\}=h_2(a)<0$$

since $b\geq a$. This is a contradiction.

Case 2: $t_0+l=0$. In this case, (2.11) becomes

$$\begin{aligned}\sum_{j=1}^{b-1}(b-2)(b-j)i_j+\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}| \\ &> \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j+aI(G)|I_1|+\frac{(a+1)|I^{(1)}|}{2} \\ &\quad -a\sum_{i=1}^{b}(i-1)|I^{(i)}|-bn.\end{aligned}$$

From what we have discussed in Subcase 1, we get

$$\sum_{j=1}^{b-1}(b-2)(b-j)i_j \le \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j.$$

If $|I_1|>0$, we deduce

$$\begin{aligned}\sum_{i=1}^{b}&(ab-ai+b-i+1)|I^{(i)}|\\ &> aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn\\ &\ge (b^2+bn-\Delta)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn\\ &\ge (b^2-\Delta)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|.\end{aligned}$$

A contradiction follows from what we discussed in Subcase 2 above.

The last situation is $|I_1|=0$ and

$$\sum_{j=1}^{b-1}(b-2)(b-j)i_j > \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j - bn.$$

Let $h_3=(b-2)(b-j)-(aI(G)-aj-b+j)+bn$. We infer

$$\begin{aligned}h_3 &= b(b-2)+(a-b+1)j+b-aI(G)+bn\\ &\le b(b-2)+(a-b+1)+b-(b^2+bn-b+a)+bn = 1-b<0,\end{aligned}$$

a contradiction.

Therefore, we can conclude that Theorem 1.2 is effective. ■

3. A new isolated toughness condition for fractional (a,b,n)-critical graphs. Let $g(x)=a$ and $f(x)=b$ for each $x\in V(G)$. A sufficient and necessary condition for being a fractional (a,b,n)-critical graph is derived from Lemma 1.1.

LEMMA 3.1. *Let G be a graph. Let a, b, n be non-negative integers such that $a\le b$. Then G is a fractional (a,b,n)-critical graph if and only if*

$$b|S|-a|T|+d_{G-S}(T)\ge bn \tag{3.1}$$

for all disjoint subsets S,T of $V(G)$ with $|S|\ge n$.

We use standard techniques similar to that of Section 2, and suppose that G is not fractional (a,b,n)-critical. We deduce $T\neq\emptyset$, and there exist

$S, T \subseteq V(G)$ such that $S \cap T = \emptyset$ and

$$b|S| - a|T| + d_{G-S}(T) < bn, \tag{3.2}$$

where $|S| \geq n$. By selecting S and T with minimum $|T|$, we get $d_{G-S}(x) \leq a - 1$ for any vertex x in T.

By using Lemma 3.1 and the techniques applied in the previous section, and considering the minor differences between (2.1) and (3.2), $d_{G-S}(x) \leq a - 1$ for each x in T here corresponds to $d_{G-S}(x) \leq b - 1$ for each x in T in Section 2. Finally, the isolated toughness condition for fractional (a, b, n)-critical graphs is stated as follows. The detailed proof is skipped.

THEOREM 3.2. *Let G be a graph and let a, b be two non-negative integers satisfying $2 \leq a \leq b$. Let n be a non-negative integer. Suppose that $|V(G)| \geq n + a + 1$ if G is complete. If*

$$I(G) \geq \frac{ab - b + a - \Delta}{b} + n,$$

then G is a fractional (a, b, n)-critical graph.

4. Open problem. Since the two examples presented in this paper only show that the results are sharp if $a = b$, we put forward the following open problem.

PROBLEM 4.1. *Is it possible to weaken the isolated toughness condition imposed in Theorems 1.2 and 3.2 for the existence of fractional (g, f, n)-critical graphs in the setting $\Delta \neq 0$?*

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REFERENCES

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, New York, 2008, 1–40.
[2] V. Chvátal, *Tough graphs and Hamiltonian circuits*, Discrete Math. 5 (1973), 215–228.
[3] W. Gao, L. Liang, T. W. Xu, and J. Zhou, *Tight toughness condition for fractional (g, f, n)-critical graphs*, J. Korean Math. Soc. 51 (2014), 55–65.
[4] W. Gao and W. Wang, *A note on fractional (g, f, m)-deleted graphs*, Ars. Combin. 113A (2014), 129–137.
[5] W. Gao and W. Wang, *Degree conditions for fractional (k, m)-deleted graphs*, Ars. Combin. 113A (2014), 273–285.

[6] W. Gao, T. W. Xu, L. Liang, and J. Zhou, *Isolated toughness condition for a graph to be a fractional (g, f, n)-critical graph*, J. Combin. Math. Combin. Comput. 91 (2014), 205–213.
[7] G. Liu and L. Zhang, *Fractional (g, f)-factors of graphs*, Acta Math. Sci. Ser. B English Ed. 21 (2001), 541–545.
[8] G. Liu and L. Zhang, *Characterizations of maximum fractional (g, f)-factors of graphs*, Discrete Appl. Math. 156 (2008), 2293–2299.
[9] G. Liu and L. Zhang, *Toughness and the existence of fractional k-factors of graphs*, Discrete Math. 308 (2008), 1741–1748.
[10] S. Liu, *On toughness and fractional (g, f, n)-critical graphs*, Inform. Process. Lett. 110 (2010), 378–382.
[11] J. Yang, Y. Ma, and G. Liu, *Fractional (g, f)-factors in graphs*, Appl. Math. J. Chinese Univ. Ser. A 16 (2001), 385–390.
[12] S. Zhou, *Independence number, connectivity and (a, b, k)-critical graphs*, Discrete Math. 309 (2009), 4144–4148.
[13] S. Zhou, *A sufficient condition for a graph to be an (a, b, k)-critical graph*, Int. J. Comput. Math. 87 (2010), 2202–2211.

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