

## Partial integrability of almost complex structures on Thurston manifolds

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**Abstract.** We prove that any left-invariant symplectic almost complex structure on a Thurston manifold which is compatible with its canonical left-invariant Riemannian metric has holomorphic type 1.

**1. Introduction.** Let  $M$  be a smooth  $2n$ -dimensional almost complex manifold with almost complex structure  $J$ . A smooth complex-valued function  $f$  on  $(M, J)$  is said to be *holomorphic* if  $\bar{\partial}f = 0$ , i.e.  $df$  is a  $(1, 0)$ -form with respect to  $J$  [8, 11]. It is well-known that  $J$  is integrable (i.e. comes from a complex structure) if and only if in a neighborhood of every point of  $M$  there exist  $n$  functionally independent holomorphic functions. By the celebrated Newlander–Nirenberg theorem [15] this is equivalent to the vanishing of the Nijenhuis tensor of  $J$ . In contrast to complex manifolds, there exist almost complex manifolds which do not admit even local holomorphic functions except constants. A typical example is the 6-sphere  $S^6$  with the almost complex structure defined by the Cayley numbers, and more generally, every isotropy irreducible homogeneous almost complex space with non-integrable almost complex structure [8]. Another class of examples is given by the compact orientable hypersurfaces of  $\mathbb{R}^7$  with the Calabi almost complex structures [3]. We refer the reader to [11, 7] for some general criteria for local existence of  $k$  ( $0 \leq k \leq n$ ) functionally independent holomorphic functions on an arbitrary almost complex manifold.

Given an almost complex manifold  $(M, J)$  and a point  $x \in M$  we denote by  $m(x)$  the maximal number of local holomorphic functions that are

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independent at  $x$ . In general  $m(x)$  is not a constant, hence we introduce the following

**DEFINITION 1.1.** The almost complex structure  $J$  is said to be *partially integrable* if  $m(x) = m = \text{const}$  for all  $x \in M$ . In this case we say that  $J$  has *holomorphic type  $m$* .

One may find various examples of partially integrable almost complex manifolds in [11, 6]. In particular, any left-invariant almost complex structure on a Lie group is partially integrable.

In 1976 Thurston [17] constructed the first example of a compact symplectic manifold  $W$  which admits no Kähler structures. It is well-known [1] that  $W$  is a 4-dimensional nilmanifold, and in [12] one of the present authors noticed that every left-invariant almost complex structure on the corresponding 2-step nilpotent Lie group has nonzero holomorphic type. Motivated by this example he introduced the class of real Lie groups that have this property (calling them Lie groups of type  $T$ ), and exhibited many examples of such groups [13]. In particular, any Lie group of dimension  $2n$  whose commutator ideal has dimension less than  $n$  is of type  $T$ . Recently the authors proved [14] that this algebraic property characterizes the 2-step nilpotent Lie groups of type  $T$ .

In this note we study the higher dimensional generalizations  $W^{2n+2}$  of the Thurston example introduced by Cordero, Fernández and de León [4] (see also [5]). More precisely, in Theorem 3.1 we describe all left-invariant symplectic almost complex structures on  $W^{2n+2}$  which are compatible with its canonical left-invariant Riemannian metric, and prove that each of these structures has holomorphic type 1. This generalizes a result of Kim [9, Theorem 2.7] (see also [13, Proposition 3.2]) for the standard symplectic structure on  $W^{2n+2}$ . Note that an almost complex structure  $J$  compatible with a Riemannian metric  $g$  is called *symplectic* if the Kähler 2-form  $F(X, Y) = g(JX, Y)$  is closed.

**2. Lie groups of type  $T$ .** Let  $G$  be a connected Lie group and let  $\mathfrak{g}$  be the Lie algebra of left-invariant vector fields on  $G$ , which we identify as usual with the tangent space to  $G$  at its unit element. We also identify the left-invariant almost complex structures on  $G$  with the endomorphisms  $J$  of  $\mathfrak{g}$  such that  $J^2 = -\text{Id}$ . The Nijenhuis tensor  $N$  of  $J$  is defined by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

for all  $X, Y \in \mathfrak{g}$ . We denote by  $\mathfrak{ln}$  the Nijenhuis space of  $J$  defined by

$$\mathfrak{ln} = \text{span}\{N(X, Y) : X, Y \in \mathfrak{g}\}.$$

It is easy to see that  $N(X, Y) = -N(Y, X)$  and  $N(JX, Y) = -JN(X, Y)$ .

Hence  $\mathfrak{ln}$  is a  $J$ -invariant subspace of  $\mathfrak{g}$ , and thus its real dimension is even. Note that the almost complex structure  $J$  is integrable if and only if  $\mathfrak{ln} = \{0\}$ .

As we already mentioned in the introduction, any left-invariant almost complex structure  $J$  on a Lie group  $G$  is partially integrable. Note that the computation of its holomorphic type can be reduced to a purely algebraic problem as follows.

DEFINITION 2.1. A Lie subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  is said to be an  $IJ$ -subalgebra if:

- (1)  $\mathfrak{h}$  is  $J$ -invariant,
- (2)  $\mathfrak{ln} \subseteq \mathfrak{h}$ ,
- (3)  $[X, Y] + J[X, JY] \in \mathfrak{h}$  whenever  $X \in \mathfrak{h}$  and  $Y \in \mathfrak{g}$ .

Denote by  $d(\mathfrak{g}, J)$  the real dimension of the intersection of all  $IJ$ -subalgebras of  $\mathfrak{g}$ . It has been proved in [13] that the holomorphic type of  $J$  is equal to  $\frac{1}{2}(\dim \mathfrak{g} - d(\mathfrak{g}, J))$ . In particular, we have the following

PROPOSITION 2.2. *If the Nijenhuis space  $\mathfrak{ln}$  of  $J$  is an  $IJ$ -subalgebra of  $\mathfrak{g}$ , then the holomorphic type of  $J$  is equal to  $\frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{ln})$ .*

*Proof.* Note that if the Nijenhuis space  $\mathfrak{ln}$  of  $J$  is an  $IJ$ -subalgebra of  $\mathfrak{g}$  then by property (2) of Definition 2.1 it follows that the intersection of all  $IJ$ -subalgebras of  $\mathfrak{g}$  is  $\mathfrak{ln}$ . Hence  $d(\mathfrak{g}, J) = \dim \mathfrak{ln}$  and the holomorphic type of  $J$  is equal to  $\frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{ln})$ . ■

Let us recall [13] that a Lie group  $G$  is said to be of type  $T$  if every left-invariant almost complex structure  $J$  on  $G$  has nonzero holomorphic type. As a consequence of the above formula for the holomorphic type we obtain the following algebraic condition for a Lie group to be of type  $T$ .

PROPOSITION 2.3 ([13]). *Every  $2n$ -dimensional Lie group whose Lie algebra has commutator ideal of dimension less than  $n$  is of type  $T$ .*

*Proof.* The given condition implies that  $[\mathfrak{g}, \mathfrak{g}] + J[\mathfrak{g}, \mathfrak{g}]$  is an  $IJ$ -subalgebra of  $\mathfrak{g}$  whose dimension is less than or equal to  $2n - 2$ . Hence  $d(\mathfrak{g}, J) \leq 2n - 2$  and by Proposition 2.2 the holomorphic type of  $J$  is at least 1. ■

Recall that a Lie group  $G$  with Lie algebra  $\mathfrak{g}$  is said to be *2-step nilpotent* if  $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$ . These Lie groups admit rich geometric structures and provide interesting examples of complex and symplectic manifolds [16]. It turns out that for them the sufficient condition in Proposition 2.3 for a Lie group to be of type  $T$  is also necessary [14].

EXAMPLE 2.4. The Thurston example mentioned in the introduction can be generalized to higher dimensions by using the so-called general-

ized Heisenberg groups  $H(q, p)$  [5]. They consist of all real matrices of the form

$$\begin{pmatrix} I_p & A & B \\ 0 & I_q & C \\ 0 & 0 & I_q \end{pmatrix}$$

where  $I_p$  denotes the identity  $p \times p$  matrix,  $A$  and  $B$  are arbitrary  $p \times q$  matrices, and  $C$  is a diagonal  $q \times q$  matrix. Note that

$$\dim H(q, p) = 2qp + q.$$

Let  $p_1, \dots, p_n$  be non-negative integers and  $q_1, \dots, q_n$  be positive integers such that  $q_1 + \dots + q_n \equiv 0 \pmod{2}$ . Consider the group

$$W(q_1, p_1, \dots, q_n, p_n) = H(q_1, p_1) \times \dots \times H(q_n, p_n).$$

Let  $\mathfrak{w}$  be the Lie algebra of  $W(q_1, p_1, \dots, q_n, p_n)$ . Then

$$\dim[\mathfrak{w}, \mathfrak{w}] = \sum_{i=1}^n q_i p_i,$$

which implies  $\dim \mathfrak{w} \geq 2 \dim[\mathfrak{w}, \mathfrak{w}] + 2$ . Hence, by Proposition 2.3, the group  $W(q_1, p_1, \dots, q_n, p_n)$  is of type  $T$ .

REMARK 2.5. It has been proved in [14] that the condition  $\dim[\mathfrak{g}, \mathfrak{g}] \leq n - 1$  is also necessary for 4- and 6-dimensional nilpotent Lie groups to be of type  $T$ . This raises the question whether the same is also true for nilpotent Lie groups of higher dimensions. Note that for general Lie groups the answer to this question is negative as shown by the following

EXAMPLE 2.6. Let  $G$  be the solvable Lie group whose Lie algebra  $\mathfrak{g}$  has a basis  $(e_1, \dots, e_{2n})$  such that

$$[e_i, e_{2n}] = e_i, \quad 1 \leq i \leq 2n - 1.$$

It has been proved in [13] (see also [16]) that every left-invariant almost complex structure on  $G$  is integrable. In particular,  $G$  is of type  $T$  but it does not satisfy the above condition since  $\dim[\mathfrak{g}, \mathfrak{g}] = 2n - 1 > n - 1$ . This shows that the answer to the question above is negative for the class of solvable Lie groups.

Let us note that J. Milnor [10] has also considered the Lie groups in this example (in odd dimensions too) and characterized them by the property that every left-invariant Riemannian metric has sectional curvature of constant sign.

**3. Partial integrability on generalized Thurston manifolds.** Let  $H^{2n+2}$  be the Lie group of matrices of the form

$$M = \begin{pmatrix} I_n & P & Q & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi is} \end{pmatrix}$$

where  $I_n$  is the unit  $n \times n$  matrix,  $P = (p_i)$  and  $Q = (q_i)$  are  $n \times 1$  real matrices, and  $r, s$  are real numbers. Then  $H^{2n+2}$  is a connected and simply-connected 2-step nilpotent Lie group of dimension  $2n + 2$ . Denote by  $W^{2n+2}$  the compact nilmanifold  $H^{2n+2}/\Gamma$ , where  $\Gamma$  is the subgroup of  $H^{2n+2}$  consisting of all matrices with integer entries. Since the group  $H^{2n+2}$  is not abelian it follows from the Benson–Gordon theorem [2] that  $W^{2n+2}$  does not admit Kähler structures (see also [4, 5]). We will see instead that it admits a large family of left-invariant symplectic almost complex structures.

Denote by  $(x_1, \dots, x_{2n+2})$  the global coordinates on  $H^{2n+2}$  defined by

$$x_i(M) = p_i, \quad x_{n+1}(M) = r, \quad x_{n+1+i}(M) = q_i, \quad x_{2n+2}(M) = s$$

for  $M \in H^{2n+2}$  and  $1 \leq i \leq n$ . The following 1-forms on  $H^{2n+2}$  are left-invariant:

$$\alpha_i = dx_i, \quad \alpha_{n+1} = dx_{n+1}, \quad \alpha_{n+1+i} = dx_{n+1+i} - x_i dx_{n+1}, \quad \alpha_{2n+2} = dx_{2n+2}.$$

It is easy to check that

$$(3.1) \quad d\alpha_i = 0, \quad d\alpha_{n+1+i} = \alpha_{n+1} \wedge \alpha_i, \quad 1 \leq i \leq n + 1.$$

Let  $X^i$  be the dual vector field of  $\alpha_i$ . Then

$$X^i = \frac{\partial}{\partial x_i}, \quad X^{n+1} = \frac{\partial}{\partial x_{n+1}} + \sum_{i=1}^n x_i \frac{\partial}{\partial x_{n+1+i}},$$

$$X^{n+1+i} = \frac{\partial}{\partial x_{n+1+i}}, \quad X^{2n+2} = \frac{\partial}{\partial x_{2n+2}}, \quad 1 \leq i \leq n.$$

Denote by  $\mathfrak{h}_{2n+2}$  the Lie algebra of  $H^{2n+2}$ . The left-invariant vector fields  $X^1, \dots, X^{2n+2}$  form a basis of  $\mathfrak{h}_{2n+2}$  and the only nonzero Lie brackets are

$$(3.2) \quad [X^i, X^{n+1}] = X^{n+1+i}, \quad 1 \leq i \leq n.$$

Denote by  $g$  the left-invariant Riemannian metric on  $H^{2n+2}$  defined by

$$g = \sum_{i=1}^{2n+2} \alpha_i \otimes \alpha_i.$$

It defines a Riemannian metric on  $W^{2n+2}$  which we denote by the same symbol. Let  $J$  be a left-invariant almost complex structure on  $H^{2n+2}$  compatible

with  $g$ . Set

$$JX^i = \sum_{j=1}^{2n+2} a_j^i X^j, \quad 1 \leq i \leq 2n+2.$$

Then the  $(2n+2) \times (2n+2)$  matrix  $J = (a_j^i)$  is orthogonal and antisymmetric. The Kähler form  $F$  of the almost Hermitian structure  $(g, J)$  is given by

$$F = \frac{1}{2} \sum_{i,j=1}^{2n+2} a_j^i \alpha_i \wedge \alpha_j.$$

The structure  $(g, J)$  is symplectic if  $dF = 0$ .

**THEOREM 3.1.** *Let  $J$  be an almost complex structure on  $W^{2n+2}$  induced by a left-invariant almost complex structure on the group  $H^{2n+2}$  and compatible with the metric  $g$ . If the almost Hermitian structure  $(g, J)$  is symplectic then  $J$  has holomorphic type 1.*

*Proof.* We first describe all left-invariant almost complex structures  $J$  on  $H^{2n+2}$  such that  $(g, J)$  is a symplectic structure.

**LEMMA 3.2.** *The almost Hermitian structure  $(g, J)$  is symplectic if and only if the matrix of  $J$  has the block-matrix form*

$$(3.3) \quad J = \begin{pmatrix} O & A \\ -{}^tA & O \end{pmatrix},$$

where  $A$  is an orthogonal  $(n+1) \times (n+1)$  matrix of the form

$$(3.4) \quad A = \begin{pmatrix} B & \vdots \\ \dots & . \end{pmatrix},$$

in which  $B$  is a symmetric  $n \times n$  matrix.

*Proof.* Using the structure equations (3.1) and the fact that the matrix  $J = (a_j^i)$  is antisymmetric we get

$$\begin{aligned} 2dF &= \sum_{i,j=1}^{2n+2} a_j^i (d\alpha_i \wedge \alpha_j - \alpha_i \wedge d\alpha_j) \\ &= \sum_{j=1}^{2n+2} \sum_{i=1}^n a_j^{n+i+1} \alpha_i \wedge \alpha_j \wedge \alpha_{n+1} + \sum_{i=1}^{2n+2} \sum_{j=1}^n a_{n+j+1}^i \alpha_i \wedge \alpha_j \wedge \alpha_{n+1} \\ &= 2 \sum_{j=1}^{2n+2} \sum_{i=1}^n a_j^{n+i+1} \alpha_i \wedge \alpha_j \wedge \alpha_{n+1}. \end{aligned}$$

Hence,  $dF = 0$  if and only if  $a_{n+j+1}^j = a_{n+j+1}^i$ ,  $1 \leq i, j \leq n$ , and  $a_{n+j+1}^{n+i+1} = 0$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n+1$ . The matrix of  $J$  is orthogonal and antisymmetric

and we see easily that it has the block-matrix form

$$J = \begin{pmatrix} C & A \\ -{}^tA & C \end{pmatrix},$$

where  $C$  and  $A$  are  $(n + 1) \times (n + 1)$  matrices and  $A$  has the form (3.4). The identity  $J^2 = -I_{2n+2}$  implies

$$C^2 - A {}^tA = -I_{n+1}, \quad CA = O, \quad {}^tAC = O, \quad {}^tAA = I_{n+1}.$$

Hence  $C = A {}^tAC = O$ , which completes the proof. ■

Now we are ready to prove the theorem.

In view of Lemma 3.2 we may assume that the matrix of  $J$  has the form (3.3). Then

$$(3.5) \quad JX^i \in \text{span}\{X^{n+2}, \dots, X^{2n+2}\}, \quad 1 \leq i \leq n + 1,$$

$$(3.6) \quad JX^i \in \text{span}\{X^1, \dots, X^{n+1}\}, \quad n + 2 \leq i \leq 2n + 2.$$

This, together with (3.2), implies that

$$N(X^i, X^{n+1}) = X^{n+i+1}, \quad 1 \leq i \leq n,$$

and

$$N(X^i, X^j) \in \text{span}\{X^{n+2}, \dots, X^{2n+1}, JX^{n+2}, \dots, JX^{2n+1}\}$$

for  $1 \leq i, j \leq 2n + 2$ . Hence, the Nijenhuis space of  $J$  is given by

$$\mathfrak{n} = \text{span}\{X^{n+2}, \dots, X^{2n+1}, JX^{n+2}, \dots, JX^{2n+1}\},$$

and it follows from (3.5) and (3.6) that  $\dim \mathfrak{n} = 2n$ . On the other hand, using (3.2), (3.5), and (3.6), one can easily check that  $\mathfrak{n}$  satisfies property (3) of Definition 2.1, while properties (1) and (2) are obvious. Thus  $\mathfrak{n}$  is an  $IJ$ -subalgebra of  $\mathfrak{h}_{2n+2}$ , and using Proposition 2.2 we conclude that the almost complex structure  $J$  has holomorphic type 1. ■

REMARK. Theorem 3.1 generalizes a result of Kim [9, Theorem 2.7] (see also [13, Proposition 3.2]) that the symplectic almost complex structure on  $W^{2n+2}$  defined by Lemma 3.2 when  $A$  is the identity  $(n + 1) \times (n + 1)$  matrix has holomorphic type 1.

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